Trudinger's inequality and continuity for Riesz potential of functions in Orlicz spaces of two variable exponents over nondoubling measure spaces

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Abstract In this article, we consider Trudinger's inequality and continuity for Riesz potentials of functions in Orlicz spaces of two variable exponents near Sobolev's exponent over nondoubling metric measure spaces.

1. Introduction

A famous Trudinger inequality (see [39]) insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbf{R}^N$ (see also [1], [5], [29], [40]). In [23], Trudinger-type exponential integrability for Riesz potentials of functions in Orlicz spaces of two variable exponents near Sobolev's exponent was studied. Our aim in this article is to extend the result to the nondoubling metric measure setting. We also study the continuity of Riesz potentials in our setting.

For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^N$ by

$$U_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

Here it is natural to assume that $U_\alpha|f| \not\equiv \infty$, which is equivalent to (see [21, Theorem 1.1, Chapter 2])

$$\int_{\mathbf{R}^N} \bigl(1+|y|\bigr)^{\alpha-N} \bigl|f(y)\bigr|\, dy < \infty.$$

Great progress on Trudinger-type inequalities has been made for Riesz potentials of order α in the limiting case $\alpha p = N$ (see, e.g., [9]–[11], [36]). In [3], [24], and

Kyoto Journal of Mathematics, Vol. 57, No. 1 (2017), 79–96
DOI 10.1215/21562261-3759522,
Received October 6, 2015. Revised December 7, 2015. Accepted December 11, 2015.
2010 Mathematics Subject Classification: Primary 46E35; Secondary 46E30.
Ohno's work partially supported by Grant-in-Aid for Young Scientists (B), No. 16K17618, Japan Society for the Promotion of Science.
Shimomura's work partially supported by Grant-in-Aid for Scientific Research (C), No. 15K04929, Japan Society for the Promotion of Science.

[27], Trudinger-type exponential integrability was studied on Orlicz spaces as an extension of [9] and [11].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with nonstandard growth conditions. For a survey, see [7] and [8]. Trudinger-type exponential integrability was investigated on variable exponent Lebesgue spaces $L^{p(\cdot)}$ in [12]–[14].

Let $p(\cdot):\mathbf{R}^N \to (1,\infty)$ and $q(\cdot):\mathbf{R}^N \to [0,\infty)$ be variable exponents satisfying log-Hölder and loglog-Hölder conditions on G, respectively. Define

$$\Phi(x,t) = t^{p(x)}\big(\log(c_0+t)\big)^{q(x)},$$

and denote by $L^{\Phi}(G)$ the family of all measurable functions f on G such that

$$\|f\|_{L^{\Phi}(G)} = \inf\Big\{\lambda > 0 : \int_G \Phi\big(x, |f(x)/\lambda|\big)\,dx \le 1\Big\} < \infty.$$

Note that $c_0 \ge e$ is chosen so that $\Phi(x,\cdot)$ is convex on $[0,\infty)$.

Mizuta and the authors [23, Theorem 4.1] proved Trudinger-type exponential integrability for Riesz potentials of functions in Orlicz spaces $L^{\Phi}(G)$ of two variable exponents near Sobolev's exponent in the Euclidean setting. In fact we proved the following.

THEOREM A

Let $p(\cdot)$ and $q(\cdot)$ be two variable exponents on G satisfying log-Hölder and loglog-Hölder conditions on G, respectively, such that

$$p(x) \ge N/\alpha \qquad \text{and} \qquad q(x) < p(x) - 1$$

for $x \in G$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_G \exp\Big(\frac{U_\alpha f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_2(x))^{p(x)/(p(x)-q(x)-1)}}\Big)\,dx \le c_2$$

for all nonnegative measurable functions f on G with $\|f\|_{L^{\Phi}(G)} \le 1$, where

$$\gamma_2(x) = \gamma_1(x)^{-(p(x)-1)/p(x)}\big(\log\big(1/\gamma_1(x)\big)\big)^{q(x)/p(x)}$$

with $\gamma_1(x) = \min\{p(x)-q(x)-1, 1/2\}$.

In [23, Theorem 3.1], the case $p(x) < N/\alpha$ was treated. For the case $\sup_{x\in G} p(x) < N/\alpha$, see for example [17] and [22].

We denote by (X,d,μ) a metric measure space, where X is a bounded set, d is a metric on X, and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X,d,μ). For $x \in X$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with radius r, and $d_X = \sup\{d(x,y) : x, y \in X\}$. We assume that

$$\mu\big(\{x\}\big) = 0$$

for $x \in X$ and that $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. In the present article, we do not postulate on μ the so-called *doubling condition*. Recall that a Radon measure μ is said to be doubling if there exists a constant

$A_0 > 0$ such that $\mu(B(x,2r)) \le A_0\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)$ $(= X)$ and $r > 0$. Otherwise, μ is said to be nondoubling. We say that a measure μ is *lower Ahlfors Q-regular* if there exists a constant $A_1 > 0$ such that

$$\mu\big(B(x,r)\big) \ge A_1 r^Q \tag{1.1}$$

for all $x \in X$ and $0 < r < d_X$. In this article we assume that μ is lower Ahlfors Q-regular. Here note that if μ is a doubling measure and $d_X < \infty$, then μ is lower Ahlfors $\log_2 A_0$-regular since

$$\frac{\mu(B(x,r))}{\mu(B(x,d_X))} \ge A_0^{-2}\Big(\frac{r}{d_X}\Big)^{\log_2 A_0}$$

for all $x \in X$ and $0 < r < d_X$ (see, e.g., [4, Lemma 3.3]).

For $\alpha > 0$ and $\tau > 0$, we define the Riesz potential of order α for a locally integrable function f on X by (see, e.g., [15], [28])

$$I_{\alpha,\tau} f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y).$$

We write $I_\alpha f = I_{\alpha,1} f$. Observe that this naturally extends the Riesz potential operator $U_\alpha f$ when (X,d) is the N-dimensional Euclidean space and $\mu = dx$.

Our main aim in this article is to give a general version of exponential integrabilities of Trudinger type for Riesz potentials $I_{\alpha,\tau} f$ of functions in Orlicz spaces $L^\Phi(X)$ of variable exponents near Sobolev's exponent on nondoubling metric measure spaces X when $p(x) \ge Q/\alpha \ge 1$ and $q(x) < p(x) - 1$ (Theorem 3.1) as an extension of Theorem A. To this end, we apply Hedberg's trick (see Hedberg [18]) by the use of the modified Hardy–Littlewood maximal operator M_λ adapted to our setting (see Section 2 for the definitions of $L^\Phi(X)$ and M_λ). What is new about this article is that we can pass our results to the nondoubling metric measure setting. In the case when $p(x) \ge Q/\alpha \ge 1$ and $q(x) \ge p(x) - 1$, we discuss double exponential integrabilities of Trudinger type (Theorem 3.6) as an extension of [23, Theorem 5.1].

On the other hand, beginning with Sobolev's embedding theorem (see, e.g., [2], [1]), continuity properties of Riesz potentials and Sobolev functions have been studied by many authors. The continuity of Riesz potentials of functions in Orlicz spaces was studied in [11], [21], [20], [25], and [27] (see also [26]). Such continuity was investigated on variable exponent Lebesgue spaces in [12], [13], and [16] and on two variable exponent Lebesgue spaces in [23].

In the final section, we consider the continuity of Riesz potentials $I_{\alpha,\tau} f$ when $p(x) \ge Q/\alpha \ge 1$ and $q(x) > p(x) - 1$ for $x \in X$ (Theorem 4.1) as an extension of [23, Theorem 7.1].

For variable exponents attaining the value 1 over nondoubling metric measure spaces, we refer the reader to [34]. For related results, see also [16], [19], [31], [33], and [35].

2. Boundedness of the maximal operator

Throughout this article, let C denote various positive constants independent of the variables in question. In this article, following Cruz-Uribe and Fiorenza [6], we consider variable exponents $p(\cdot)$ and $q(\cdot)$ such that

$$\begin{gathered}
\left|p(x)-p(y)\right| \le \frac{a}{\log(e+1/d(x,y))} \quad \text{for all } x,y\in X,\\
\left|q(x)-q(y)\right| \le \frac{b}{\log(e+\log(e+1/d(x,y)))} \quad \text{for all } x,y\in X,\\
1<p^- \equiv \inf_{x\in X} p(x) \le \sup_{x\in X} p(x) \equiv p^+ <\infty,
\end{gathered}$$

and

$$0\le q^- \equiv \inf_{x\in X} q(x) \le \sup_{x\in X} q(x) \equiv q^+ <\infty,$$

for $a,b>0$.

We say that f is a *locally integrable function* on X if f is an integrable function on all balls B in X. For $\alpha>0$ and $\tau>0$, we define the Riesz potential of order α for a locally integrable function f on X by (see, e.g., [15], [28])

$$I_{\alpha,\tau}f(x)=\int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y).$$

Define

$$\Phi(x,t)=t^{p(x)}\bigl(\log(c_0+t)\bigr)^{q(x)},$$

and denote by $L^\Phi(X)$ the family of all measurable functions f on X such that

$$\|f\|_{L^\Phi(X)}=\inf\Bigl\{\lambda>0:\int_X \Phi\bigl(x,\left|f(x)/\lambda\right|\bigr)\,d\mu(x)\le 1\Bigr\}<\infty.$$

Note that $c_0\ge e$ is chosen so that $\Phi(x,\cdot)$ is convex on $[0,\infty)$.

For a locally integrable function f on X and $\lambda\ge 1$, the Hardy–Littlewood maximal function $M_\lambda f$ is defined by

$$M_\lambda f(x)=\sup_{r>0}\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,r)\cap X}\left|f(y)\right|d\mu(y).$$

For $\lambda\ge 1$, we say that X *satisfies* $(M\lambda)$ if there exists a constant $C>0$ such that

$$\mu\bigl(\bigl\{x\in X: M_\lambda f(x)>k\bigr\}\bigr)\le \frac{C}{k}\int_X \left|f(y)\right|d\mu(y)$$

for all measurable functions $f\in L^1(X)$ and $k>0$. Nazarov, Treil, and Volberg [30] proved that X satisfies $(M3)$ if X is a separable metric space. Terasawa [38] showed that X satisfies $(M\lambda)$ for $\lambda\ge 2$ if $\mu(B(x,r))$ is continuous with the variable $r>0$ when $x\in X$ is fixed. Sawano [32] showed that X satisfies $(M\lambda)$ for $\lambda\ge 2$ if X is a separable metric space (see also [37] where $(M1)$ is true for the Poincaré disk).

LEMMA 2.1

Let $1<p_0<\infty$, and let $\lambda\ge 1$. Suppose that X satisfies $(M\lambda)$. Then there exists a constant $C>0$ such that

$$\int_X \{M_\lambda f(x)\}^{p_0}\,d\mu(x) \le C$$

for all measurable functions f *on* X *with* $\|f\|_{L^{p_0}(X)} \le 1$.

LEMMA 2.2 ([31, LEMMA 2.3])

Let $\lambda \ge 1$, *and let* f *be a nonnegative measurable function on* X *with* $\|f\|_{L^\Phi(X)} \le 1$ *such that* $f(x) \ge 1$ *or* $f(x) = 0$ *for each* $x \in X$. *Set*

$$I = I_\lambda(x,r,f) = \frac{1}{\mu(B(x,\lambda r))}\int_{B(x,r)\cap X} f(y)\,d\mu(y)$$

and

$$J = J_\lambda(x,r,f) = \frac{1}{\mu(B(x,\lambda r))}\int_{B(x,r)\cap X} g(y)\,d\mu(y),$$

where $g(y) = \Phi(y, f(y))$. *Then there exists a constant* $C > 0$ *such that*

$$I \le C J^{1/p(x)}\big(\log(e+J)\big)^{-q(x)/p(x)}$$

for all $x \in X$.

Proof

Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in X$. First, consider the case when $J \ge 1$. Set

$$k = J^{1/p(x)}\big(\log(e+J)\big)^{-q(x)/p(x)}.$$

Then we have

$$I \le k + \frac{1}{\mu(B(x,\lambda r))}\int_{B(x,r)\cap X} f(y)\Big(\frac{f(y)}{k}\Big)^{p(y)-1}\Big(\frac{\log(c_0+f(y))}{\log(c_0+k)}\Big)^{q(y)}\,d\mu(y).$$

Since $\|f\|_{L^\Phi(X)} \le 1$, we find by (1.1) that

$$J \le \frac{1}{\mu(B(x,\lambda r))}\int_X g(y)\,d\mu(y) \le \frac{1}{\mu(B(x,\lambda r))} \le A_1^{-1}\lambda^{-Q} r^{-Q}.$$

Hence, we obtain, for $y \in B(x,r)$,

$$\begin{aligned} k^{-p(y)} &\le C\big\{J^{1/p(x)}\big(\log(e+J)\big)^{-q(x)/p(x)}\big\}^{-p(x)+\frac{a}{\log(e+1/r)}} \\ &\le C\big\{J^{1/p(x)}\big(\log(e+J)\big)^{-q(x)/p(x)}\big\}^{-p(x)+\frac{a}{\log(e+1/(CJ^{-1/Q}))}} \\ &\le CJ^{-1}\big(\log(e+J)\big)^{q(x)} \end{aligned}$$

and

$$\begin{aligned} \big(\log(c_0+k)\big)^{-q(y)} &\le C\big\{\log(e+J)\big\}^{-q(x)+\frac{b}{\log(e+\log(e+1/(CJ^{-1/Q})))}} \\ &\le C\big(\log(e+J)\big)^{-q(x)}. \end{aligned}$$

Consequently, it follows that

$$I \le C J^{1/p(x)}\big(\log(e+J)\big)^{-q(x)/p(x)}.$$

In the case $J \le 1$, we find that

$$I \le J \le CJ^{1/p(x)}\big(\log(e+J)\big)^{-q(x)/p(x)}.$$

Now the result follows. □

Now we are ready to show the boundedness of the maximal operator M_λ.

THEOREM 2.3 ([31, THEOREM 2.4])
Let $\lambda \ge 1$. Suppose that X satisfies $(M\lambda)$. Then there exists a constant $c_M > 0$ such that

$$\int_X \Phi\big(x, M_\lambda f(x)\big)\, d\mu(x) \le c_M$$

for all measurable functions f on X with $\|f\|_{L^\Phi(X)} \le 1$.

Proof
Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Write

$$f = f\chi_{\{y\in X: f(y)\ge 1\}} + f\chi_{\{y\in X: f(y)<1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E. Then, since $M_\lambda f_2 \le 1$ on X, we see from Lemma 2.2 that

$$\Phi\big(x, M_\lambda f(x)\big) \le C\big\{1 + M_\lambda g(x)\big\},$$

where $g(y) = \Phi(y, f(y))$. Now take p_1 such that $1 < p_1 < p^-$. Then, applying the above inequality with $p(x)$ replaced by $p(x)/p_1$, we obtain

$$\Phi\big(x, M_\lambda f(x)\big) \le C\big\{1 + \big\{M_\lambda g_1(x)\big\}^{p_1}\big\},$$

where $g_1(y) = \Phi(y, f(y))^{1/p_1} = g(y)^{1/p_1}$. By Lemma 2.1, we see that

$$\int_X \Phi\big(x, M_\lambda f(x)\big)\, d\mu(x) \le c_M,$$

as required. □

3. Trudinger's exponential integrability

This section concerns the exponential integrability of Trudinger's type. Our main result is the following, which is an extension of [23, Theorem 4.1].

THEOREM 3.1
Let $\tau > \lambda \ge 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \text{and} \qquad q(x) < p(x) - 1$$

for $x \in X$. Assume that X satisfies $(M\lambda)$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_X \exp\Big(\frac{I_{\alpha,\tau} f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_2(x))^{p(x)/(p(x)-q(x)-1)}}\Big)\, d\mu(x) \le c_2$$

for all nonnegative measurable functions f on X with $\|f\|_{L^\Phi(X)} \le 1$, where

$$\gamma_2(x) = \gamma_1(x)^{-(p(x)-1)/p(x)} \big(\log(1/\gamma_1(x))\big)^{q(x)/p(x)}$$

with $\gamma_1(x) = \min\{p(x) - q(x) - 1, 1/2\}$.

COROLLARY 3.2

Let $\tau > \lambda \ge 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \text{and} \qquad q(x) < p(x) - 1$$

for $x \in X$. Assume that X satisfies $(M\lambda)$. Then there exists a constant $c_3 > 0$ such that

$$\int_X \Big\{\exp\Big(\frac{I_{\alpha,\tau} f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_3\gamma_2(x))^{p(x)/(p(x)-q(x)-1)}}\Big) - 1\Big\}\, d\mu(x) \le 1$$

for all nonnegative measurable functions f on X with $\|f\|_{L^\Phi(X)} \le 1$.

REMARK 3.3 ([23, REMARK 4.3])

Let $\mathbf{B} = B(0,1) \subset \mathbf{R}^N$. For $0 < \delta < 1$, we can find $f \in L^\Phi(\mathbf{B})$ such that

$$\int_{\mathbf{B}} \exp\big(\big(\gamma_2(x)^{-\delta} U_\alpha f(x)\big)^{p(x)/(p(x)-q(x)-1)}\big)\, dx = \infty.$$

This implies that the weight $\gamma_2(x)^{-p(x)/(p(x)-q(x)-1)}$ in Theorem 3.1 is needed.

Before proving Theorem 3.1, we prepare the following result.

LEMMA 3.4

Let $\tau > 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \text{and} \qquad q(x) < p(x) - 1$$

for $x \in X$. Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Then there exists a constant $C > 0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\gamma_2(x)\big(\log(1/\delta)\big)^{(p(x)-q(x)-1)/p(x)}$$

for all $x \in X$ and $0 < \delta < 1/2$.

Proof

Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$.

First we consider the case $\gamma_1(x)^{1/Q}/\tau \le \delta$. We have

$$\begin{aligned}\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) &\le A_1^{-1}\tau^{-Q} \int_{X\setminus B(x,\delta)} d(x,y)^{\alpha-Q} f(y)\, d\mu(y)\\ &\le A_1^{-1}\tau^{-Q}\delta^{\alpha-Q} \int_{X\setminus B(x,\delta)} f(y)\, d\mu(y)\\ &\le A_1^{-1}\tau^{-Q}\delta^{\alpha-Q} \int_X \{1 + g(y)\}\, d\mu(y)\end{aligned}$$

$$
\begin{aligned}
&\le C\gamma_1(x)^{(\alpha-Q)/Q} \\
&\le C\gamma_2(x)\big(\log(1/\delta)\big)^{(p(x)-q(x)-1)/p(x)},
\end{aligned}
$$

where $g(y) = \Phi(y, f(y))$.

Next we consider the case $\gamma_1(x)^{1/Q}/\tau > \delta$. Note that

$$
\begin{aligned}
&\int_{X\setminus B(x,\gamma_1(x)^{1/Q}/\tau)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
&\le C\gamma_1(x)^{(\alpha-Q)/Q} \int_{X\setminus B(x,\gamma_1(x)^{1/Q}/\tau)} f(y)\, d\mu(y) \\
&\le C\gamma_2(x)\big(\log(1/\delta)\big)^{(p(x)-q(x)-1)/p(x)}.
\end{aligned}
$$

Setting

$$
\eta(x) = \gamma_1(x)^{1/p(x)}\big(\log\big(1/\gamma_1(x)\big)\big)^{q(x)/p(x)}
$$

and

$$
N(x,y) = d(x,y)^{-Q/p(x)}\big(\log\big(1/d(x,y)\big)\big)^{-(q(x)+1)/p(x)},
$$

we have

$$
\begin{aligned}
&\int_{B(x,\gamma_1(x)^{1/Q}/\tau)\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
&\le \int_{B(x,\gamma_1(x)^{1/Q}/\tau)\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} \big\{\eta(x)N(x,y)\big\}\, d\mu(y) \\
&\quad + \int_{B(x,\gamma_1(x)^{1/Q}/\tau)\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))} \\
&\quad \times \Big(\frac{f(y)}{\eta(x)N(x,y)}\Big)^{p(y)-1}\Big(\frac{\log(c_0+f(y))}{\log(c_0+\eta(x)N(x,y))}\Big)^{q(y)}\, d\mu(y) \\
&= J_1 + J_2.
\end{aligned}
$$

Let j_0 be the smallest positive integer such that $\tau^{j_0}\delta \ge \gamma_1(x)^{1/Q}/\tau$. We obtain

$$
\begin{aligned}
J_1 &\le \sum_{j=1}^{j_0} \int_{X\cap(B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta))} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}\big\{\eta(x)N(x,y)\big\}\, d\mu(y) \\
&\le \eta(x) \\
&\quad \times \sum_{j=1}^{j_0} \int_{X\cap(B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta))} \frac{(\tau^j\delta)^{\alpha-Q/p(x)}(\log(1/(\tau^j\delta)))^{-(q(x)+1)/p(x)}}{\mu(B(x,\tau^j\delta))}\, d\mu(y) \\
&\le \eta(x)\sum_{j=1}^{j_0}(\tau^j\delta)^{\alpha-Q/p(x)}\big(\log\big(1/(\tau^j\delta)\big)\big)^{-(q(x)+1)/p(x)} \\
&\le C\eta(x)\int_\delta^{2d_X} t^{\alpha-Q/p(x)}\big(\log(1/t)\big)^{-(q(x)+1)/p(x)}\,\frac{dt}{t}.
\end{aligned}
$$

Since $p(x) \ge Q/\alpha$, we have

$$J_1 \le C\eta(x) \int_\delta^{2d_X} \bigl(\log(1/t)\bigr)^{-(q(x)+1)/p(x)} \frac{dt}{t}$$
$$\le C\eta(x)\gamma_1(x)^{-1}\bigl(\log(1/\delta)\bigr)^{(p(x)-q(x)-1)/p(x)}$$
$$\le C\gamma_2(x)\bigl(\log(1/\delta)\bigr)^{(p(x)-q(x)-1)/p(x)}.$$

Next we estimate J_2. If $y \in B(x, \gamma_1(x)^{1/Q}/\tau)$, then $\gamma_1(x)^{-p(y)} \le C\gamma_1(x)^{-p(x)}$, so that $\eta(x)^{-p(y)} \le C\eta(x)^{-p(x)}$. Hence,

$$\bigl\{\eta(x)N(x,y)\bigr\}^{-p(y)} \le C\eta(x)^{-p(x)} d(x,y)^Q \bigl(\log\bigl(1/d(x,y)\bigr)\bigr)^{q(x)+1},$$

and by the fact that $\log(c_0 + st) \le \log(c_0 + s)\log(c_0 + t)$ when $s, t > 0$,

$$\bigl\{\log\bigl(c_0 + \eta(x)N(x,y)\bigr)\bigr\}^{-q(y)} \le C\bigl(\log\bigl(1/\gamma_1(x)\bigr)\bigr)^{q(x)}\bigl(\log\bigl(1/d(x,y)\bigr)\bigr)^{-q(x)}$$

for $y \in B(x, \gamma_1(x)^{1/Q}/\tau)$. Therefore, we have by (1.1)

$$J_2 \le A_1^{-1}\tau^{-Q} \int_{B(x,\gamma_1(x)^{1/Q}/\tau)\setminus B(x,\delta)} d(x,y)^{\alpha-Q}\Bigl(\frac{1}{\eta(x)N(x,y)}\Bigr)^{p(y)-1}$$
$$\times \Bigl(\frac{1}{\log(c_0 + \eta(x)N(x,y))}\Bigr)^{q(y)} g(y)\, d\mu(y)$$
$$\le C\eta(x)^{1-p(x)}\bigl(\log\bigl(1/\gamma_1(x)\bigr)\bigr)^{q(x)} \int_{B(x,\gamma_1(x)^{1/Q}/\tau)\setminus B(x,\delta)} d(x,y)^{\alpha-Q/p(x)}$$
$$\times \bigl(\log\bigl(1/d(x,y)\bigr)\bigr)^{(p(x)-q(x)-1)/p(x)} g(y)\, d\mu(y)$$
$$\le C\gamma_2(x)\bigl(\log(1/\delta)\bigr)^{(p(x)-q(x)-1)/p(x)},$$

where $g(y) = \Phi(y, f(y))$ as before.

Consequently, it follows that

$$\int_{G\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\gamma_2(x)\bigl(\log(1/\delta)\bigr)^{(p(x)-q(x)-1)/p(x)}$$

for $0 < \delta < 1/2$, which gives the lemma. $\square$

Proof of Theorem 3.1

Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Note that

$$\int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le \frac{\delta^\alpha}{1-(\tau/\lambda)^{-\alpha}} M_\lambda f(x),$$

since $\tau > \lambda \ge 1$ (see [31, proof of Theorem 3.1]). Lemma 3.4 gives

$$I_{\alpha,\tau} f(x) \le C\bigl\{\delta^\alpha M_\lambda f(x) + \gamma_2(x)\bigl(\log(1/\delta)\bigr)^{(p(x)-q(x)-1)/p(x)}\bigr\}$$

for $0 < \delta < 1/2$. Here, considering

$$\delta = C\bigl(\gamma_2(x)^{-1}M_\lambda f(x)\bigr)^{-1/\alpha}\bigl(\log\bigl(\gamma_2(x)^{-1}M_\lambda f(x)\bigr)\bigr)^{(p(x)-q(x)-1)/(\alpha p(x))}$$

when $\gamma_2(x)^{-1}M_\lambda f(x) \ge e$, we find that

$$I_{\alpha,\tau}f(x) \le C\{\gamma_2(x)\big(\log\big(c_0+\gamma_2(x)^{-1}M_\lambda f(x)\big)\big)^{(p(x)-q(x)-1)/p(x)} + \gamma_2(x)\}$$
$$\le C\gamma_2(x)\big(\log\big(c_0+M_\lambda f(x)\big)\big)^{(p(x)-q(x)-1)/p(x)}.$$

Hence, it follows that

$$c_1^{-1}\gamma_2(x)^{-1}I_{\alpha,\tau}f(x) \le \big(\log\big(c_0+M_\lambda f(x)\big)\big)^{(p(x)-q(x)-1)/p(x)},$$

so that

$$\exp\Big(\frac{I_{\alpha,\tau}f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_2(x))^{p(x)/(p(x)-q(x)-1)}}\Big) \le c_0 + M_\lambda f(x) \le C\{\Phi\big(x, M_\lambda f(x)\big)+1\}.$$

By Theorem 2.3, we have

$$\int_X \exp\Big(\frac{I_{\alpha,\tau}f(x)^{p(x)/(p(x)-q(x)-1)}}{(c_1\gamma_2(x))^{p(x)/(p(x)-q(x)-1)}}\Big)\,d\mu(x)$$
$$\le C\Big(\int_X \Phi\big(x, M_\lambda f(x)\big)\,d\mu(x) + 1\Big) \le C,$$

as required. □

Finally, we consider the case in which $p(x) \ge Q/\alpha \ge 1$ and $q(x) \ge p(x)-1$.

LEMMA 3.5

Let $\tau > 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \text{and} \qquad q(x) \ge p(x)-1$$

for $x \in X$. Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Then there exists a constant $C > 0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \le C\big(\log\big(\log(1/\delta)\big)\big)^{(p(x)-1)/p(x)}$$

for all $x \in X$ and $0 < \delta < 1/4$.

Proof

Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. First note that

$$\int_{X\setminus B(x,1/(4\tau))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)$$
$$\le A_1^{-1}\tau^{-Q}\int_{X\setminus B(x,1/(4\tau))} d(x,y)^{\alpha-Q} f(y)\,d\mu(y) \le C.$$

Next, setting

$$N(x,y) = d(x,y)^{-Q/p(x)}\big(\log\big(1/d(x,y)\big)\big)^{-1}\big(\log\big(\log\big(1/d(x,y)\big)\big)\big)^{-1/p(x)},$$

we have

$$
\begin{aligned}
&\int_{B(x,1/(4\tau))\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
&\quad \le \int_{B(x,1/(4\tau))\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} N(x,y)\, d\mu(y) \\
&\qquad + \int_{B(x,1/(4\tau))\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))} \Big(\frac{f(y)}{N(x,y)}\Big)^{p(y)-1} \\
&\qquad \times \Big(\frac{\log(c_0+f(y))}{\log(c_0+N(x,y))}\Big)^{q(y)} d\mu(y) \\
&\quad \le C\Big\{ \big(\log\big(\log(1/\delta)\big)\big)^{(p(x)-1)/p(x)} + \int_{B(x,1/(4\tau))\setminus B(x,\delta)} d(x,y)^{\alpha-Q/p(x)} \\
&\qquad \times \big(\log\big(1/d(x,y)\big)\big)^{p(x)-q(x)-1} \big(\log\big(\log\big(1/d(x,y)\big)\big)\big)^{(p(x)-1)/p(x)} g(y)\, d\mu(y)\Big\} \\
&\quad \le C\big(\log\big(\log(1/\delta)\big)\big)^{(p(x)-1)/p(x)},
\end{aligned}
$$

where $g(y) = \Phi(y, f(y))$, as required. □

As in the proof of Theorem 3.1, we establish the following double exponential integrability for $f \in L^\Phi(X)$ in view of Lemma 3.5 and Theorem 2.3.

THEOREM 3.6

Let $\tau > \lambda \ge 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \text{and} \qquad q(x) \ge p(x) - 1$$

for $x \in X$. Assume that X satisfies $(M\lambda)$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_X \exp\Big(\exp\Big(\frac{I_{\alpha,\tau} f(x)^{p(x)/(p(x)-1)}}{c_1^{p(x)/(p(x)-1)}}\Big)\Big)\, d\mu(x) \le c_2$$

for all nonnegative measurable functions f on X with $\|f\|_{L^\Phi(X)} \le 1$.

4. Continuity of Riesz potentials

In this section, we discuss the continuity of Riesz potentials under the condition that there are constants $\theta > 0$, $\iota > 1$, and $C_0 > 0$ such that

$$\Big|\frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))}\Big| \le C_0 \Big(\frac{d(x,z)}{d(x,y)}\Big)^\theta \frac{d(x,y)^\alpha}{\mu(B(x,\iota d(x,y)))} \tag{4.1}$$

whenever $d(x,z) \le d(x,y)/2$.

THEOREM 4.1

Let $\tau > 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \text{and} \qquad q(x) > p(x) - 1$$

for $x \in X$. If f is a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$, then $I_{\alpha,\tau} f(x)$ is continuous for all $x \in X$ and there exists a constant $C > 0$ such that

$$\big|I_{\alpha,\tau} f(z) - I_{\alpha,\tau} f(x)\big| \le C\gamma_4(x)\big(\log\big(1/d(x,z)\big)\big)^{-(q(x)-p(x)+1)/p(x)}$$

as $z \to x$ for each $x \in X$, where

$$\gamma_4(x) = \gamma_3(x)^{-(p(x)-1)/p(x)}\big(\log\big(1/\gamma_3(x)\big)\big)^{q(x)/p(x)}$$

with $\gamma_3(x) = \min\{q(x) - p(x) + 1, 1/2\}$.

For a proof of Theorem 4.1, we prepare two lemmas.

LEMMA 4.2

Let $\tau > 1$. Suppose that

$$p(x) \ge Q/\alpha \ge 1 \qquad \textit{and} \qquad q(x) > p(x) - 1$$

for $x \in X$. Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Then there exists a constant $C > 0$ such that

$$\int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\gamma_4(x,\delta)\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)}$$

for all $x \in X$ and $0 < \delta < 1/2$, where

$$\gamma_4(x,t) = \gamma_3(x)^{-(p(x)-1)/p(x)-a/(p(x)\log(1/t))} \times \big(\log\big(1/\gamma_3(x)\big)\big)^{q(x)/p(x)-aq(x)/(p(x)\log(1/t))+b/\log(\log(1/t))}.$$

Proof

Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Setting

$$\eta(x) = \gamma_3(x)^{1/p(x)}\big(\log\big(1/\gamma_3(x)\big)\big)^{q(x)/p(x)}$$

and

$$N(x,y) = d(x,y)^{-Q/p(x)}\big(\log\big(1/d(x,y)\big)\big)^{-(q(x)+1)/p(x)},$$

we have

$$\begin{aligned}
&\int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
&\quad \le C\Big\{\int_{B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}\eta(x)N(x,y)\, d\mu(y) \\
&\qquad + \int_{B(x,\delta)} d(x,y)^{\alpha-Q} f(y)\Big(\frac{f(y)}{\eta(x)N(x,y)}\Big)^{p(y)-1} \\
&\qquad \times \Big(\frac{\log(c_0 + f(y))}{\log(c_0 + \eta(x)N(x,y))}\Big)^{q(y)}\, d\mu(y)\Big\}.
\end{aligned}$$

Note that

$$\{\eta(x)N(x,y)\}^{-p(y)} \le \eta(x)^{-p(x)-a/\log(1/\delta)} d(x,y)^Q \big(\log\big(1/d(x,y)\big)\big)^{q(x)+1}$$

and

$$\begin{aligned}&\big\{\log\big(c_0+\eta(x)N(x,y)\big)\big\}^{-q(y)}\\ &\quad\le C\big(\log\big(1/\gamma_3(x)\big)\big)^{q(x)+b/\log(\log(1/\delta))}\big(\log\big(1/d(x,y)\big)\big)^{-q(x)}\end{aligned}$$

for $y \in B(x,\delta)$. Consequently, it follows that

$$\begin{aligned}&\int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\ &\quad\le C\Big\{\eta(x)\gamma_3(x)^{-1}\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)}\\ &\qquad+\eta(x)^{-p(x)+1-a/\log(1/\delta)}\big(\log\big(1/\gamma_3(x)\big)\big)^{q(x)+b/\log(\log(1/\delta))}\\ &\qquad\times\int_{B(x,\delta)} d(x,y)^{\alpha-Q/p(x)}\big(\log\big(1/d(x,y)\big)\big)^{-(q(x)-p(x)+1)/p(x)} g(y)\,d\mu(y)\Big\}\\ &\quad\le C\gamma_4(x,\delta)\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)}\Big(1+\int_{B(x,\delta)} g(y)\,d\mu(y)\Big)\\ &\quad\le C\gamma_4(x,\delta)\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)},\end{aligned}$$

where $g(y)=\Phi(y,f(y))$, as required. □

LEMMA 4.3

Let $\tau>1$. Suppose that

$$p(x)\ge Q/\alpha\ge 1 \qquad \text{and} \qquad q(x)>p(x)-1$$

for $x\in X$. Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)}\le 1$. Then there exists a constant $C>0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\le C\delta^{-\theta}\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)}$$

for all $x\in X$ and $0<\delta<1/2$.

Proof

Let f be a nonnegative measurable function on X with $\|f\|_{L^\Phi(X)}\le 1$. First note that

$$\begin{aligned}&\int_{X\setminus B(x,1/(2\tau))} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\ &\qquad\le C\int_{X\setminus B(x,1/(2\tau))} d(x,y)^{\alpha-Q-\theta} f(y)\,d\mu(y)\le C.\end{aligned}$$

Setting

$$N(x,y)=d(x,y)^{-Q/p(x)}\big(\log\big(1/d(x,y)\big)\big)^{-(q(x)+1)/p(x)},$$

we have

$$
\begin{aligned}
&\int_{B(x,1/(2\tau))\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y)\\
&\le C\Big\{\int_{B(x,1/(2\tau))\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta}}{\mu(B(x,\tau d(x,y)))} N(x,y)\, d\mu(y)\\
&\quad + \int_{B(x,1/(2\tau))\setminus B(x,\delta)} d(x,y)^{\alpha-Q-\theta} f(y)\Big(\frac{f(y)}{N(x,y)}\Big)^{p(y)-1}\\
&\quad \times \Big(\frac{\log(c_0+f(y))}{\log(c_0+N(x,y))}\Big)^{q(y)}\, d\mu(y)\Big\}.
\end{aligned}
$$

Since

$$
\{N(x,y)\}^{-p(y)} \le C d(x,y)^{Q}\big(\log(1/d(x,y))\big)^{q(x)+1}
$$

and

$$
\{\log(c_0+N(x,y))\}^{-q(y)} \le C\big(\log(1/d(x,y))\big)^{-q(x)}
$$

for $y \in B(x,1/(2\tau))$, it follows that

$$
\begin{aligned}
&\int_{B(x,1/(2\tau))\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y)\\
&\le C\Big\{\delta^{-\theta}\big(\log(1/\delta)\big)^{-(q(x)+1)/p(x)}\\
&\quad + \int_{B(x,1/(2\tau))\setminus B(x,\delta)} d(x,y)^{\alpha-Q/p(x)-\theta}\big(\log(1/d(x,y))\big)^{-(q(x)-p(x)+1)/p(x)}\\
&\quad \times g(y)\, d\mu(y)\Big\}\\
&\le C\delta^{-\theta}\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)}\Big(1+\int_{B(x,1/(2\tau))\setminus B(x,\delta)} g(y)\, d\mu(y)\Big)\\
&\le C\delta^{-\theta}\big(\log(1/\delta)\big)^{-(q(x)-p(x)+1)/p(x)},
\end{aligned}
$$

where $g(y)=\Phi(y,f(y))$, as required. □

Proof of Theorem 4.1
Let f be a nonnegative measurable function on X with $\|f\|_{L^{\Phi}(X)} \le 1$.
Write

$$
\begin{aligned}
&I_{\alpha,\tau}f(x) - I_{\alpha,\tau}f(z)\\
&\quad = \int_{B(x,2d(x,z))} \frac{d(x,y)^{\alpha} f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y)\\
&\qquad - \int_{B(x,2d(x,z))} \frac{d(z,y)^{\alpha} f(y)}{\mu(B(z,\tau d(z,y)))}\, d\mu(y)\\
&\qquad + \int_{X\setminus B(x,2d(x,z))} \Big(\frac{d(x,y)^{\alpha}}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^{\alpha}}{\mu(B(z,\tau d(z,y)))}\Big) f(y)\, d\mu(y)
\end{aligned}
$$

for $x, z \in X$. By Lemma 4.2, we have

$$\int_{B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\gamma_4\big(x, 2d(x,z)\big)\big(\log\big(1/d(x,z)\big)\big)^{-(q(x)-p(x)+1)/p(x)}$$

and

$$\begin{aligned}\int_{B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\, d\mu(y) &\le \int_{B(z,3d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\, d\mu(y)\\ &\le C\gamma_4\big(z, 3d(x,z)\big)\big(\log\big(1/d(x,z)\big)\big)^{-(q(z)-p(z)+1)/p(z)}\end{aligned}$$

for $0 < d(x,z) < 1/6$. On the other hand, we have by (4.1) and Lemma 4.3

$$\begin{aligned}&\Big|\int_{X\setminus B(x,2d(x,z))} \Big(\frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))}\Big) f(y)\, d\mu(y)\Big|\\ &\le Cd(x,z)^\theta \int_{X\setminus B(x,2d(x,z))} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\iota d(x,y)))}\, d\mu(y)\\ &\le C\big(\log\big(1/d(x,z)\big)\big)^{-(q(x)-p(x)+1)/p(x)}.\end{aligned}$$

Now we establish

$$\begin{aligned}\big|I_{\alpha,\tau}f(x) - I_{\alpha,\tau}f(z)\big| \le C\Big\{&\gamma_4\big(x, 2d(x,z)\big)\big(\log\big(1/d(x,z)\big)\big)^{-(q(x)-p(x)+1)/p(x)}\\ &+ \gamma_4\big(z, 3d(x,z)\big)\big(\log\big(1/d(x,z)\big)\big)^{-(q(z)-p(z)+1)/p(z)}\Big\}\end{aligned}$$

for $0 < d(x,z) < 1/6$, which implies

$$\big|I_{\alpha,\tau}f(z) - I_{\alpha,\tau}f(x)\big| \le C\gamma_4(x)\big(\log\big(1/d(x,z)\big)\big)^{-(q(x)-p(x)+1)/p(x)}$$

as $z \to x$ for each $x \in X$. □

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