Index map, σ-connections, and Connes–Chern character in the setting of twisted spectral triples

Raphaël Ponge and Hang Wang

Abstract Twisted spectral triples are a twisting of the notion of spectral triples aimed at dealing with some type III geometric situations. In the first part of the article, we give a geometric construction of the index map of a twisted spectral triple in terms of σ-connections on finitely generated projective modules. This clarifies the analogy with the indices of Dirac operators with coefficients in vector bundles. In the second part, we give a direct construction of the Connes–Chern character of a twisted spectral triple, in both the invertible and the noninvertible cases. Combining these two parts we obtain an analogue of the Atiyah–Singer index formula for twisted spectral triples.

Contents

1. Introduction

Motivated by type III geometric situations, such as, the action of an arbitrary group of diffeomorphisms on a manifold, Connes–Moscovici [CM4] introduced the notion of a twisted spectral triple. This is a modification of the usual definition of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where the boundedness of commutators $[D, a]$, $a \in \mathcal{A}$, is replaced by that of twisted commutators $[D, a]_\sigma = Da - \sigma(a)D$, where σ is a given automorphism of the algebra $\mathcal{A}$. Examples include the following:

- conformal deformations of ordinary spectral triples (see [CM4]);
- twistings of ordinary spectral triples by scaling automorphisms (see [Mo2]);
- conformal Dirac spectral triples $(C^\infty(M) \rtimes G, L_g^2(M, \$), \slashed{D}_g)_\sigma$, where $\slashed{D}_g$ is the Dirac operator acting on spinors and G is a group of conformal diffeomorphisms (see [CM4]);
- spectral triples over noncommutative tori associated with conformal weights (see [CT], [CM5]);
- twisted spectral triples associated with various quantum statistical systems, including Connes–Bost systems, graphs, and supersymmetric Riemann gas (see [GMT]);
- twisted spectral triples associated with some continuous crossed-product algebras (see [IM]).

We refer to Section 2 for a review of the first and the third examples. Connes–Moscovici [CM4] showed that, as for ordinary spectral triples, the datum of a twisted spectral $(\mathcal{A}, \mathcal{H}, D)_\sigma$ gives rise to a well-defined index map $\mathrm{ind}_D : K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z}$. Moreover, in the p-summable case, this index map is computed by the pairing of the K-theory $K_0(\mathcal{A})$ with a Connes–Chern character in ordinary cyclic cohomology.

One goal of this article is to present a geometric interpretation of the index map of a twisted spectral triple. First, instead of compressing idempotents by D and its inverse as in [CM4] (see also [FK1]), we define the index map in terms of Fredholm indices of the following operators:

$$\sigma(e)De : e\mathcal{H}^q \to \sigma(e)\mathcal{H}^q, \quad e \in M_q(\mathcal{A}), e^2 = e.$$

This definition is totally analogous to the definition of the index map of an ordinary spectral triple mentioned in [Mo1].

In the case of an ordinary spectral triple, the index map is usually defined in terms of self-adjoint idempotents, since any idempotent is equivalent to a self-adjoint idempotent. For a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ the relevant notion of self-adjointness is meant with respect to the σ-involution $a \to \sigma(a)^*$. We shall say that such an idempotent is σ-self-adjoint. In general, it is not clear that an idempotent is equivalent to a σ-self-adjoint idempotent. For this reason, it is important to define the index map for *arbitrary* idempotents. As a result, for a twisted spectral triple the index map a priori takes values in $\frac{1}{2}\mathbb{Z}$. Nevertheless, when the automorphism has a suitable square root, it can be shown that any idempotent is equivalent to a σ-self-adjoint idempotent and the index map is integer-valued (Lemma 4.7). The precise condition is called the *ribbon* condition (see Definition 4.5) and is satisfied by all the main examples of twisted spectral triples.

As it turns out, the aforementioned construction of the index map is only a special case of a more geometric construction in terms of couplings of the operator D with σ-connections on finitely generated projective modules. We refer the reader to Section 5 for the precise definition of a σ-connection. This is the twisted analogue of the usual notion of a connection. The two notions actually agree

when $\sigma = \mathrm{id}$. Given a σ-connection on a finitely generated projective module $\mathcal{E}$, the definition of the coupled operator $D_{\nabla^{\varepsilon}}$ is similar to the coupling of a Dirac operator with a connection on an auxiliary vector bundle (see Section 5 for the precise definition). In the special case $\mathcal{E} = e\mathcal{A}^q$ we recover the operator $\sigma(e)De$ by using the so-called Grassmannian σ-connection, which is the twisted analogue of the Grassmannian connection. We then show that the operator $D_{\nabla^{\varepsilon}}$ is Fredholm and we have

$$(1.1) \qquad \mathrm{ind}_{D,\sigma}[\mathcal{E}] = \mathrm{ind}\, D_{\nabla^{\varepsilon}}.$$

This provides us with a geometric interpretation of the index map of a twisted spectral triple. In the case of an ordinary spectral triple we recover the geometric interpretation of the index map mentioned in [Mo1]. The above formula exhibits a close analogy with the definition of the standard Fredholm index map of a Dirac operator (the construction of which is recalled in Section 3). In particular, we recover the latter in the special case of an ordinary Dirac spectral triple (see the discussion on this point at the end of Section 5).

Another goal of this article is to give a direct construction of the Connes–Chern character of a p-summable twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_{\sigma}$. In [CM4] the Connes–Chern character is defined as the difference of the Connes–Chern characters of a pair of bounded Fredholm modules canonically associated with the twisted spectral triple. This is the same passage as in [Co1] from the unbounded Fredholm module picture to the bounded Fredholm module picture. One advantage of our definition of the index map is the following index formula (Proposition 7.2):

$$(1.2) \qquad \mathrm{ind}\, \sigma(e)De = \frac{1}{2}\,\mathrm{Str}\big((D^{-1}[D,e]_{\sigma})^{2k+1}\big), \quad e = e^2 \in M_q(\mathcal{A}),$$

where k is any integer greater than or equal to $\frac{1}{2}(p-1)$ and D is assumed to be invertible. It is immediate that the right-hand side is the pairing of e with the cochain given by

$$\tau_{2k}^D(a^0, \ldots, a^{2k}) = c_k\,\mathrm{Str}\big(D^{-1}[D,a^0]_{\sigma} \cdots D^{-1}[D,a^{2k}]_{\sigma}\big), \quad a^j \in \mathcal{A},$$

where c_k is a normalization constant. This is the same cochain used in [CM4] to define the Connes–Chern character of a twisted spectral triple. We give a direct proof that τ_{2k}^D is a normalized cyclic cocycle whose class in periodic cyclic cohomology is independent of k (Proposition 7.6). The Connes–Chern character $\mathrm{Ch}(D)_{\sigma}$ is then defined as the class in periodic cyclic cohomology of any cocycles τ_{2k}^D.

We also use some care to define the Connes–Chern character when D is noninvertible by passing to the unital invertible double, which we define as a twisted spectral triple over the augmented unital algebra $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$. In the invertible case, we thus obtain two definitions of the Connes–Chern character, but these two definitions agree (see Proposition 7.15). This uses the homotopy invariance of the Connes–Chern character, a detailed proof of which is given in Appendix C.

With the use of the Connes–Chern character and the geometric interpretation (1.1) of the index map we obtain the following index formula: for any finitely generated projective module $\mathcal{E}$ and σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$,

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \big\langle \operatorname{Ch}(D)_{\sigma}, [\mathcal{E}] \big\rangle.$$

This is the analogue for twisted spectral triples of the Atiyah–Singer index formula.

We have attempted to give very detailed accounts of the constructions of the index map and Connes–Chern character of twisted spectral triples. It is hoped that the details of these constructions will also be helpful to readers who are primarily interested in understanding these constructions in the setting of ordinary spectral triples.

The article is organized as follows. In Section 2, we review some important definitions and examples regarding twisted spectral triples. In Section 3, we recall the construction of the Fredholm index map of a Dirac operator. In Section 4, we present the construction of the index map of a twisted spectral triple and single out a simple condition ensuring us that it is integer-valued. In Section 5, we give a geometric description of the index map of a twisted spectral triple in terms of σ-connections on finitely generated projective modules. In Section 6, we review the main definitions and properties of cyclic cohomology and periodic cyclic cohomology and their pairings with K-theory. In Section 7, we give a direct construction of the Connes–Chern character of a twisted spectral triple for both the invertible and noninvertible cases. In Appendices A and B, we present proofs of two technical lemmas from Section 5. In Appendix C, we give a detailed proof of the homotopy invariance of the Connes–Chern character of a twisted spectral triple.

2. Twisted spectral triples

In this section, we review various definitions and examples regarding twisted spectral triples.

2.1. Twisted spectral triples
We start by recalling the definition of an ordinary spectral triple.

DEFINITION 2.1

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of the following data:

(1) a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$;

(2) an involutive unital algebra $\mathcal{A}$ represented by bounded operators on $\mathcal{H}$ preserving its $\mathbb{Z}_2$-grading;

(3) a self-adjoint unbounded operator D on $\mathcal{H}$ such that, for all $a \in \mathcal{A}$,

    (a) D maps $\operatorname{dom}(D) \cap \mathcal{H}^{\pm}$ to $\mathcal{H}^{\mp}$,

    (b) the resolvent $(D+i)^{-1}$ is a compact operator,

    (c) $a \operatorname{dom}(D) \subset \operatorname{dom}(D)$ and $[D, a]$ is bounded for all $a \in \mathcal{A}$.

EXAMPLE 2.2

The paradigm of a spectral triple is given by a Dirac spectral triple,

$$\big(C^\infty(M), L_g^2(M, \$), \not{D}_g\big),$$

where (M^n, g) is a compact spin Riemannian manifold (n even) and $\not{D}_g$ is its Dirac operator acting on the spinor bundle $\$$. In this case the $\mathbb{Z}_2$-grading of $L^2(M, \$)$ arises from the $\mathbb{Z}_2$-grading $\$ = \$^+ \oplus \$^-$ of the spinor bundle in terms of positive and negative spinors.

The definition of a twisted spectral triple is similar to that of an ordinary spectral triple, except for some "twist" given by conditions (3) and (4.b) below.

DEFINITION 2.3 ([CM4])

A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of the following:

(1) a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$;
(2) an involutive unital algebra $\mathcal{A}$ represented by even bounded operators on $\mathcal{H}$;
(3) an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$;
(4) an odd self-adjoint unbounded operator D on $\mathcal{H}$ such that
    (a) the resolvent $(D + i)^{-1}$ is compact,
    (b) $a \operatorname{dom}(D) \subset \operatorname{dom}(D)$ and $[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all

$a \in \mathcal{A}$.

REMARK 2.4

The condition that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$ exactly means that the map $a \to \sigma(a)^*$ is an involutive antilinear antiautomorphism of $\mathcal{A}$.

REMARK 2.5

Throughout the article we shall further assume that the algebra $\mathcal{A}$ is closed under holomorphic functional calculus. This implies that an element $a \in \mathcal{A}$ is invertible if and only if it is invertible in $\mathcal{L}(\mathcal{H})$. This also implies that all the algebras $M_q(\mathcal{A})$, $q \in \mathbb{N}$, are closed under holomorphic functional calculus.

REMARK 2.6

The boundedness of twisted commutators naturally appears in the setting of quantum groups, but in the attempts to construct twisted spectral triples over quantum groups the compactness of the resolvent of D seems to fail (see [DA], [KS], [KW]). We also refer to [KW] for relationships between twisted spectral triples and Woronowicz's covariant differential calculi.

2.2. Conformal deformations of ordinary spectral triples

An important class of examples of twisted spectral triples arises from *conformal deformations* (i.e., inner twistings) of ordinary spectral triples.

Let us start with a Dirac spectral triple $(C^\infty(M), L_g^2(M, \$), \not{D}_g)$ associated with a compact Riemannian spin oriented manifold (M^n, g) of even dimension.

Consider a conformal change of metric,

$$\hat{g} = k^{-2}g, \quad k \in C^\infty(M), k > 0.$$

We then can form a new Dirac spectral triple $(C^\infty(M), L^2_{\hat{g}}(M, \$), \slashed{D}_{\hat{g}})$. Bearing this in mind, note that the inner product of $L^2_g(M, \$)$ is given by

$$\langle \xi, \eta \rangle_g := \int_M \big(\xi(x), \eta(x) \big) \sqrt{g(x)} \, d^n x, \quad \xi, \eta \in L^2_g(M, \$),$$

where $(\cdot, \cdot)$ is the Hermitian metric of $\$$ (and $n = \dim M$). Consider the linear isomorphism $U : L^2_g(M, \$) \to L^2_{\hat{g}}(M, \$)$ given by

$$U\xi = k^{\frac{n}{2}} \xi \quad \forall \xi \in L^2_g(M, \$).$$

We observe that U is a unitary operator since, for all $\xi \in L^2_g(M, \$)$, we have

$$\langle U\xi, U\xi \rangle_{\hat{g}} = \int_M \big(k(x)^{\frac{n}{2}} \xi(x), k(x)^{\frac{n}{2}} \xi(x) \big) \sqrt{k(x)^{-2} g(x)} \, d^n x = \langle \xi, \xi \rangle_g.$$

Moreover, the conformal invariance of the Dirac operator (see, e.g., [Hit]) means that

(2.1) $$\slashed{D}_{\hat{g}} = k^{\frac{n+1}{2}} \slashed{D}_g k^{\frac{-n+1}{2}}.$$

Thus,

(2.2) $$U^* \slashed{D}_{\hat{g}} U = k^{-\frac{n}{2}} (k^{\frac{n+1}{2}} \slashed{D}_g k^{\frac{-n+1}{2}}) k^{\frac{n}{2}} = \sqrt{k} \slashed{D}_g \sqrt{k}.$$

Therefore, we obtain the following result.

PROPOSITION 2.7
The spectral triples $(C^\infty(M), L^2_{\hat{g}}(M, \$), \slashed{D}_{\hat{g}})$ and $(C^\infty(M), L^2_g(M, \$), \sqrt{k}\slashed{D}_g\sqrt{k})$ are unitarily equivalent.

REMARK 2.8
Whereas the definition of $(C^\infty(M), L^2_{\hat{g}}(M, \$), \slashed{D}_{\hat{g}})$ requires k to be smooth, in the definition of $(C^\infty(M), L^2_g(M, \$), \sqrt{k}\slashed{D}_g\sqrt{k})$ it is enough to assume that k is a positive Lipschitz function.

More generally, let $(\mathcal{A}, \mathcal{H}, D)$ be an ordinary spectral triple, and let k be a positive element of $\mathcal{A}$. If we replace D by its conformal deformation kDk, then when $\mathcal{A}$ is noncommutative, the triple $(\mathcal{A}, \mathcal{H}, kDk)$ need not be an ordinary spectral triple. However, as the following result shows, it always gives rise to a twisted spectral triple.

PROPOSITION 2.9 ([CM4])
Consider the automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ defined by

(2.3) $$\sigma(a) = k^2 a k^{-2} \quad \forall a \in \mathcal{A}.$$

Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ is a twisted spectral triple.

REMARK 2.10

The main property to check is the boundedness of twisted commutators $[kDk, a]_\sigma$, $a \in \mathcal{A}$. This follows from the equalities

$$[kDk, a]_\sigma = (kDk)a - (k^2ak^{-2})(kDk) = k\big(D(kak^{-1}) - (kak^{-1})D\big)k$$
$$= k[D, kak^{-1}]k.$$

REMARK 2.11

We refer to [PW1] for a generalization of the above construction in terms of "pseudo-inner twistings" of ordinary spectral triples. We note that this construction also encapsulates the construction of twisted spectral triples over noncommutative tori associated with conformal weights of [CT].

2.3. Conformal Dirac spectral triple

The conformal Dirac spectral triple of [CM4] is a nice illustration of the geometric relevance of twisted spectral triples. Let Γ be the diffeomorphism group of a compact manifold M. In order to study the action of Γ on M, noncommutative geometry suggests seeking a spectral triple over the crossed-product algebra $C^\infty(M) \rtimes \Gamma$, that is, the algebra with generators $f \in C^\infty(M)$ and u_φ, $\varphi \in \Gamma$, with relations

$$u_\varphi^* = u_\varphi^{-1} = u_{\varphi^{-1}}, \qquad u_\varphi f = (f \circ \varphi)u_\varphi.$$

The first set of relations implies that any unitary representation of $C^\infty(M) \rtimes \Gamma$ induces a unitary representation of Γ.

The manifold structure is the only diffeomorphism-invariant differentiable structure on M, so in particular M does not carry a diffeomorphism-invariant metric. This prevents us from constructing a unitary representation of Γ in an L^2-space of tensors or differential forms or a first-order (pseudo-)differential operator D with a Γ-invariant principal symbol (so as to ensure the boundedness of commutators $[D, u_\varphi]$). As observed by Connes [Co2] we can bypass this issue by passing to the total space of the metric bundle $P \to M$ (seen as a ray subbundle of the bundle $T^*M \odot T^*M$ of symmetric 2-tensors). As it turns out, the metric bundle P carries a wealth of diffeomorphism-invariant structures, including a diffeomorphism-invariant Riemannian structure. The construction of a spectral triple over $C_c^\infty(P) \rtimes \Gamma$ was carried out by Connes–Moscovici [CM1], who also computed its Connes–Chern character (see [CM2], [CM3]). The passage from the base manifold M to the metric bundle P is the geometric counterpart of the well-known passage from type III factors to type II factors by taking crossed-products with the action of $\mathbb{R}$.

Even if there is a Thom isomorphism $K_*(C_c^\infty(P) \rtimes \Gamma) \simeq K_*(C^\infty(M) \rtimes \Gamma)$, it would be desirable to work directly with the base manifold. As mentioned above there are obstructions to doing so when dealing with the full group of diffeomorphisms. However, as observed by Connes–Moscovici, if we restrict our attention to a group of diffeomorphisms preserving a conformal structure, then

we are able to construct a spectral triple provided we relax the definition of an ordinary spectral triple to that of a *twisted* spectral triple. This construction can be explained as follows.

Let M^n be a compact (closed) spin oriented manifold of even dimension n equipped with a conformal structure $\mathcal{C}$, that is, a conformal class of Riemannian metrics. We denote by G (the identity component of) the group of (smooth) orientation-preserving diffeomorphisms of M preserving the conformal and spin structures. Let g be a metric in the conformal class $\mathcal{C}$ with associated Dirac operator $\not{D}_g : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ acting on the sections of the spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$. We also denote by $L_g^2(M, \not{S})$ the corresponding Hilbert space of L^2-spinors.

If $\phi : M \to M$ is a diffeomorphism preserving the conformal class $\mathcal{C}$, then there is a unique function $k_\phi \in C^\infty(M)$, $k_\phi > 0$, such that

$$(2.4) \qquad \phi_* g = k_\phi^2 g.$$

In addition, ϕ uniquely lifts to a unitary vector bundle isomorphism $\phi^{\not{S}} : \not{S} \to \phi_* \not{S}$, that is, a unitary section of $\mathrm{Hom}(\not{S}, \phi_* \not{S})$ (see [BG]). We then let $V_\phi : L_g^2(M, \not{S}) \to L_g^2(M, \not{S})$ be the bounded operator given by

$$(2.5) \qquad V_\phi u(x) = \phi^{\not{S}}\big(u \circ \phi^{-1}(x)\big) \quad \forall u \in L_g^2(M, \not{S}) \; \forall x \in M.$$

The map $\phi \to V_\phi$ is a representation of G in $L_g^2(M, \not{S})$, but this is not a unitary representation. In order to get a unitary representation we need to take into account the Jacobian $|\phi'(x)| = k_\phi(x)^n$ of $\phi \in G$. This is achieved by using the unitary operator $U_\phi : L_g^2(M, \not{S}) \to L_g^2(M, \not{S})$ given by

$$(2.6) \qquad U_\phi = k_\phi^{\frac{n}{2}} V_\phi, \quad \phi \in G.$$

Then $\phi \to U_\phi$ is a unitary representation of G in $L_g^2(M, \not{S})$. This enables us to represent the elements of the crossed-product algebra $C^\infty(M) \rtimes G$ as linear combinations of operators $f U_\phi$ on $L_g^2(M, \not{S})$, where $\phi \in G$ and $f \in C^\infty(M)$ acts by scalar multiplication. These operators are subject to the relations

$$U_{\phi^{-1}} = U_\phi^{-1} = U_\phi^* \qquad \text{and} \qquad U_\phi f = (f \circ \phi^{-1}) U_\phi.$$

We then let σ_g be the automorphism of $C^\infty(M) \rtimes G$ given by

$$(2.7) \qquad \sigma_g(f U_\phi) := k_\phi f U_\phi \quad \forall f \in C^\infty(M) \; \forall \phi \in G.$$

PROPOSITION 2.12 (SEE [CM4], [Mo2])
The triple $(C^\infty(M) \rtimes G, L_g^2(M, \not{S}), \not{D}_g)_{\sigma_g}$ is a twisted spectral triple.

REMARK 2.13
The bulk of the proof is showing the boundedness of the twisted commutators $[\not{D}_g, U_\phi]_{\sigma_g}$, $\phi \in G$. We remark that

$$U_\phi \not{D}_g U_\phi^* = k_\phi^{\frac{n}{2}} (V_\phi \not{D}_g V_\phi^{-1}) k_\phi^{-\frac{n}{2}} = k_\phi^{\frac{n}{2}} \not{D}_{\phi_* g} k_\phi^{-\frac{n}{2}} = k_\phi^{\frac{n}{2}} \not{D}_{k_\phi^2 g} k_\phi^{-\frac{n}{2}}.$$

Thus, using the conformal invariance law (2.1), we get

$$U_\phi \slashed{D}_g U_\phi^* = k_\phi^{\frac{n}{2}} (k_\phi^{-(\frac{n+1}{2})} \slashed{D}_g k_\phi^{\frac{n-1}{2}}) k_\phi^{-\frac{n}{2}} = k_\phi^{-\frac{1}{2}} \slashed{D}_g k_\phi^{-\frac{1}{2}}.$$

Using this we see that the twisted commutator $[\slashed{D}_g, U_\phi]_{\sigma_g} = \slashed{D}_g U_\phi - k_\phi U_\phi \slashed{D}_g$ is equal to

$$\big(\slashed{D}_g k_\phi^{\frac{1}{2}} - k_\phi (U_\phi \slashed{D}_g U_\phi^*) k_\phi^{\frac{1}{2}}\big) k_\phi^{-\frac{1}{2}} U_\phi = (\slashed{D}_g k_\phi^{\frac{1}{2}} - k_\phi^{\frac{1}{2}} \slashed{D}_g) k_\phi^{-\frac{1}{2}} U_\phi = [\slashed{D}_g, k_\phi^{\frac{1}{2}}] k_\phi^{-\frac{1}{2}} U_\phi.$$

This shows that $[\slashed{D}_g, U_\phi]_{\sigma_g}$ is bounded.

3. The Fredholm index map of a Dirac operator

In this section, we recall how the datum of a Dirac operator gives rise to an additive index map in K-theory. In the next two sections we shall generalize this construction to *arbitrary* twisted spectral triples.

Let (M^n, g) be a compact spin oriented Riemannian manifold of even dimension n, and let $\slashed{D}_g : C^\infty(M, \slashed{S}) \to C^\infty(M, \slashed{S})$ be the associated Dirac operator acting on sections of the spinor bundle. As n is even, the spinor bundle splits as $\slashed{S} = \slashed{S}^+ \oplus \slashed{S}^-$, where $\slashed{S}^+$ (resp., $\slashed{S}^-$) is the bundle of positive (resp., negative) spinors. The Dirac operator is odd with respect to this $\mathbb{Z}_2$-grading, and so it takes the form

$$\slashed{D}_g = \begin{pmatrix} 0 & \slashed{D}_g^- \\ \slashed{D}_g^+ & 0 \end{pmatrix}, \quad \slashed{D}_g^\pm : C^\infty(M, \slashed{S}^\pm) \to C^\infty(M, \slashed{S}^\mp).$$

Let E be a Hermitian bundle over M, and let $\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ be a Hermitian connection on E. The operator $\slashed{D}_{\nabla^E} : C^\infty(M, \slashed{S} \otimes E) \to C^\infty(M, \slashed{S} \otimes E)$ is defined by

$$\slashed{D}_{\nabla^E} = \slashed{D}_g \otimes 1_E + c(\nabla^E),$$

where $c(\nabla^E)$ is given by the composition

$$C^\infty(M, \slashed{S} \otimes E) \xrightarrow{1_{\slashed{S}} \otimes \nabla^E} C^\infty(M, \slashed{S} \otimes T^*M \otimes E) \xrightarrow{c \otimes 1_E} C^\infty(M, \slashed{S} \otimes E),$$

where $c : \slashed{S} \otimes T^*M \to \slashed{S}$ is the Clifford action of T^*M on $\slashed{S}$. With respect to the splitting $\slashed{S} \otimes E = (\slashed{S}^+ \otimes E) \oplus (\slashed{S}^- \otimes E)$, the operator $\slashed{D}_{\nabla^E}$ takes the form

$$(3.1) \qquad \slashed{D}_{\nabla^E} = \begin{pmatrix} 0 & D_{\nabla^E}^- \\ D_{\nabla^E}^+ & 0 \end{pmatrix}, \quad \slashed{D}_{\nabla^E}^\pm : C^\infty(M, \slashed{S}^\pm \otimes E) \to C^\infty(M, \slashed{S}^\mp \otimes E).$$

As ∇^E is a Hermitian connection, the operator $\slashed{D}_{\nabla^E}$ is formally self-adjoint, that is, $(\slashed{D}_{\nabla^E}^+)^* = \slashed{D}_{\nabla^E}^-$. Moreover, $\slashed{D}_{\nabla^E}$ is an elliptic differential operator and, hence, is Fredholm. We then define its Fredholm index by

$$\mathrm{ind}\, \slashed{D}_{\nabla^E} := \mathrm{ind}\, \slashed{D}_{\nabla^E}^+ = \dim \ker \slashed{D}_{\nabla^E}^+ - \dim \ker \slashed{D}_{\nabla^E}^-.$$

This index is computed by the local index formula of Atiyah–Singer [AS1], [AS2],

$$\mathrm{ind}\, \slashed{D}_{\nabla^E} = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E),$$

where $\hat{A}(R^M) = \det^{\frac{1}{2}}\left(\frac{R^M/2}{\sinh(R^M/2)}\right)$ is the (total) $\hat{A}$-form of the Riemann curvature R^M and $\mathrm{Ch}(F^E) = \mathrm{Tr}(e^{-F^E})$ is the (total) Chern form of the curvature F^E of the connection ∇^E.

We observe that, even without using the Atiyah–Singer index formula, it is not difficult to see that the value of $\mathrm{ind}\,\slashed{D}_{\nabla^E}$ depends only on the class of E in the K-theory group $K^0(M)$. First, it is immediate to see that its value is independent of the choice of the Hermitian structure of E and the Hermitian connection ∇^E, since the principal symbol of $\slashed{D}_{\nabla^E}$ does not depend on these data. Second, let $\phi : E \to E'$ be a vector bundle isomorphism. We push forward the Hermitian metric of E to a Hermitian metric on E', so that pushing forward the connection ∇^E we get a Hermitian connection on E'. Then $\slashed{D}_{\nabla^{E'}} = (1_{\slashed{S}} \otimes \phi_*)\slashed{D}_{\nabla^E}(1 \otimes \phi^*)$, so that $\ker \slashed{D}^{\pm}_{\nabla^E} \simeq \ker \slashed{D}^{\pm}_{\nabla^{E'}}$, and hence $\mathrm{ind}\,\slashed{D}_{\nabla^{E'}} = \mathrm{ind}\,\slashed{D}_{\nabla^E}$. In addition, let F be another Hermitian vector bundle equipped with a Hermitian connection ∇^F. We equip $E \oplus F$ with the connection $\nabla^{E \oplus F} = \nabla^E \oplus \nabla^F$. Then, with respect to the splitting $\slashed{S} \otimes (E \oplus F) = (\slashed{S} \otimes E) \oplus (\slashed{S} \otimes F)$, we have $\slashed{D}_{\nabla^{E \oplus F}} = \slashed{D}_{\nabla^E} \oplus \slashed{D}_{\nabla^F}$, so that $\ker \slashed{D}^{\pm}_{\nabla^{E \oplus F}} = \ker \slashed{D}^{\pm}_{\nabla^E} \oplus \ker \slashed{D}^{\pm}_{\nabla^F}$. Thus,

$$\mathrm{ind}\,\slashed{D}_{\nabla^{E \oplus F}} = \mathrm{ind}\,\slashed{D}_{\nabla^E} + \mathrm{ind}\,\slashed{D}_{\nabla^F}.$$

It follows from all these that the index $\mathrm{ind}\,\slashed{D}_{\nabla^E}$ depends only on the K-theory class of E, and there actually is a well-defined additive map,

$$(3.2) \qquad \mathrm{ind}_{\slashed{D}} : K^0(M) \to \mathbb{Z},$$

such that, for any Hermitian vector bundle E equipped with a Hermitian connection ∇^E, we have

$$(3.3) \qquad \mathrm{ind}_{\slashed{D}}[E] = \mathrm{ind}\,\slashed{D}_{\nabla^E}.$$

Let $H_{\mathrm{ev}}(M, \mathbb{C}) = \bigoplus_{i \geq 0} H_{2i}(M, \mathbb{C})$ be the even de Rham homology of M, and let $H^{\mathrm{ev}}(M, \mathbb{C}) = \bigoplus_{i \geq 0} H^{2i}(M, \mathbb{C})$ be its even de Rham cohomology. Composing the natural duality pairing between $H_{\mathrm{ev}}(M, \mathbb{C})$ and $H^{\mathrm{ev}}(M, \mathbb{C})$ with the Chern character map $\mathrm{Ch} : K^0(M) \to H^{\mathrm{ev}}(M, \mathbb{C})$, we obtain a bilinear pairing,

$$(3.4) \qquad \langle \cdot, \cdot \rangle : H_{\mathrm{ev}}(M, \mathbb{C}) \times K^0(M) \to \mathbb{C},$$

so that, for any closed even de Rham current C and any vector bundle E over M, we have

$$\langle [C], [E] \rangle = \langle C, \mathrm{Ch}(F^E) \rangle,$$

where F^E is the curvature of any connection on E. Then the Atiyah–Singer index formula can be rewritten as

$$(3.5) \qquad \mathrm{ind}\,D_{\nabla^E} = (2i\pi)^{-\frac{n}{2}} \langle \hat{A}(R^M)^{\wedge}, \mathrm{Ch}(F^E) \rangle = (2i\pi)^{-\frac{n}{2}} \langle [\hat{A}(R^M)^{\wedge}], [E] \rangle,$$

where $[\hat{A}(R^M)^{\wedge}]$ is the homology class of the Poincaré dual of the $\hat{A}$-form $\hat{A}(R^M)$.

Finally, we stress that the definition of $\slashed{D}_{\nabla^E}$ does not require the connection ∇^E to be Hermitian, and so the construction of $\slashed{D}_{\nabla^E}$ holds for any connection ∇^E on E. In this general case, the operator $\slashed{D}_{\nabla^E}$ need not be self-adjoint, but it still is Fredholm and of the form (3.1). A priori we could consider the two Fred-

holm indices $\operatorname{ind}\slashed{D}^+_{\nabla^E}$ and $\operatorname{ind}\slashed{D}^-_{\nabla^E}$ separately. When ∇^E is Hermitian, we have $\operatorname{ind}\slashed{D}^-_{\nabla^E}=\operatorname{ind}(\slashed{D}^+_{\nabla^E})^*=-\operatorname{ind}\slashed{D}^+_{\nabla^E}$. The value of these indices is independent of the choice of Hermitian connection, so we see that $\operatorname{ind}\slashed{D}^+_{\nabla^E}=-\operatorname{ind}\slashed{D}^-_{\nabla^E}$ even when ∇^E is not Hermitian. In any case, we equivalently could define the index of $\slashed{D}_{\nabla^E}$ by

$$(3.6) \qquad \operatorname{ind}\slashed{D}_{\nabla^E}=\frac{1}{2}(\operatorname{ind}\slashed{D}^+_{\nabla^E}-\operatorname{ind}\slashed{D}^-_{\nabla^E}).$$

4. The index map of a twisted spectral triple

Let $(\mathcal{A},\mathcal{H},D)_\sigma$ be a twisted spectral triple. As observed by Connes–Moscovici [CM4], the datum of $(\mathcal{A},\mathcal{H},D)_\sigma$ gives rise to a well-defined index map $\operatorname{ind}_{D,\sigma}:K_0(\mathcal{A})\to\frac{1}{2}\mathbb{Z}$. The definition of the index map in [CM4] is based on the observation that the phase $F=D|D|^{-1}$ defines an *ordinary* Fredholm module over $\mathcal{A}$ (namely, the pair $(\mathcal{H},F)$). The index map is then defined in terms of compressions of F by idempotents. As we shall now explain, we also can define the index map by using twisted versions of the compression of the operator D by idempotents. This construction is actually a special case of the coupling of D by σ-connections which will be described in the next section. We still need to deal with this special case in order to carry out the more general construction in the next section.

Let e be an idempotent in $M_q(\mathcal{A})$, $q\in\mathbb{N}$. We regard $e\mathcal{H}^q$ as a closed subspace of the Hilbert space $\mathcal{H}^q$, so that $e\mathcal{H}^q$ is a Hilbert space with the induced inner product. As the action of $\mathcal{A}$ on $\mathcal{H}$ is by *even* operators on $e\mathcal{H}^q\cap(\mathcal{H}^\pm)^q=e(\mathcal{H}^\pm)^q$, we have the orthogonal splitting $e\mathcal{H}^q=e(\mathcal{H}^+)^q\oplus e(\mathcal{H}^-)^q$. In addition, the action of $\mathcal{A}$ preserves the domain of D, so we see that $e(\operatorname{dom}(D))^q=(\operatorname{dom}(D))^q\cap e\mathcal{H}^q$. We then let $D_{e,\sigma}$ be the unbounded operator from $e\mathcal{H}^q$ to $\sigma(e)\mathcal{H}^q$ given by

$$(4.1) \qquad D_{e,\sigma}:=\sigma(e)(D\otimes 1_q), \qquad \operatorname{dom}(D_{e,\sigma})=e\big(\operatorname{dom}(D)\big)^q.$$

We note that, as D is an odd operator, with respect to the orthogonal splitting $\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-$ it takes the form

$$(4.2) \qquad D=\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}, \quad D^\pm:\operatorname{dom}(D)\cap\mathcal{H}^\pm\to\mathcal{H}^\mp.$$

Incidentally, with respect to the orthogonal splittings $e\mathcal{H}^q=e(\mathcal{H}^+)^q\oplus e(\mathcal{H}^-)^q$ and $\sigma(e)\mathcal{H}^q=\sigma(e)(\mathcal{H}^+)^q\oplus\sigma(e)(\mathcal{H}^-)^q$ the operator $D_{e,\sigma}$ takes the form

$$D_{e,\sigma}=\begin{pmatrix}0 & D^-_{e,\sigma}\\ D^+_{e,\sigma} & 0\end{pmatrix}, \quad D^\pm_{e,\sigma}=\sigma(e)(D^\pm\otimes 1_q).$$

In order to determine the adjoint of $D_{e,\sigma}$ we make the following observation.

LEMMA 4.1

Let $S_e:e\mathcal{H}^q\to e^*\mathcal{H}^q$ be the restriction to $e\mathcal{H}^q$ of e^* (which we represent as an operator on $\mathcal{H}^q$). Then S_e is a linear isomorphism from $e\mathcal{H}^q$ onto $e^*\mathcal{H}^q$ such that

$$(4.3) \qquad \langle S_e\xi_1,\xi_2\rangle=\langle\xi_1,\xi_2\rangle \quad \forall\xi_j\in e\mathcal{H}^q.$$

Proof
Let $\xi_j \in e\mathcal{H}^q$, $j = 1, 2$. Then

$$\langle S_e \xi_1, \xi_2 \rangle = \langle e^* \xi_1, \xi_2 \rangle = \langle \xi_1, e\xi_2 \rangle = \langle \xi_1, \xi_2 \rangle.$$

In particular, when $\xi_2 = \xi_1$ we get $\langle S_e \xi_1, \xi_1 \rangle = \|\xi_1\|^2$, which shows that S_e is one-to-one.

Let $\eta \in e^* \mathcal{H}^q$. Then $\langle \eta, \cdot \rangle_{|e\mathcal{H}^q}$ is a continuous linear form on $e\mathcal{H}^q$, so there exists $\tilde{\eta} \in e\mathcal{H}^q$ so that $\langle \eta, \xi \rangle = \langle \tilde{\eta}, \xi \rangle$ for all $\xi \in e\mathcal{H}^q$. Therefore, for all $\xi \in \mathcal{H}^q$,

$$\langle \eta, \xi \rangle = \langle e^* \eta, \xi \rangle = \langle \eta, e\xi \rangle = \langle \tilde{\eta}, e\xi \rangle = \langle e^* \tilde{\eta}, \xi \rangle.$$

Thus, $\eta = e^* \tilde{\eta} = S_e \tilde{\eta}$. This shows that S_e is onto. As S_e is one-to-one we then deduce that S_e is a linear isomorphism. The proof is complete.   □

The above lemma holds for the idempotent $\sigma(e)$ as well. In what follows, we denote by $S_{\sigma(e)}$ the linear isomorphism from $\sigma(e)\mathcal{H}^q$ to $\sigma(e)^* \mathcal{H}^q$ induced by $\sigma(e)^*$.

LEMMA 4.2
Let $D_{e,\sigma}^*$ be the adjoint of $D_{e,\sigma}$. Then

$$(4.4) \qquad\qquad D_{e,\sigma}^* = S_e^{-1} D_{\sigma(e)^*,\sigma} S_{\sigma(e)}.$$

Proof
Let $D_{e,\sigma}^\dagger$ be the operator given by the graph

$$G(D_{e,\sigma}^\dagger) = \left\{ (\xi, \eta) \in \sigma(e)^* \mathcal{H}^q \times e^* \mathcal{H}^q; \langle \xi, D_{e,\sigma} \zeta \rangle = \langle \eta, \zeta \rangle \ \forall \zeta \in \mathrm{dom}(D_{e,\sigma}) \right\}.$$

We note that the graph of $D_{e,\sigma}^*$ is

$$G(D_{e,\sigma}^*) = \left\{ (\xi, \eta) \in \sigma(e) \mathcal{H}^q \times e\mathcal{H}^q; \langle \xi, D_{e,\sigma} \zeta \rangle = \langle \eta, \zeta \rangle \ \forall \zeta \in \mathrm{dom}(D_{e,\sigma}) \right\}.$$

It then follows from (4.3) that a pair $(\xi, \eta) \in \sigma(e)\mathcal{H}^q \times e\mathcal{H}^q$ is contained in $G(D_{e,\sigma}^*)$ if and only if $(S_{\sigma(e)}\xi, S_e \eta)$ lies in $G(D_{e,\sigma}^\dagger)$. That is, $S_e D_{e,\sigma}^* = D_{e,\sigma}^\dagger S_{\sigma(e)}$. Therefore, showing that $D_{e,\sigma}^* = S_e^{-1} D_{\sigma(e)^*,\sigma} S_{\sigma(e)}$ is equivalent to showing that the operators $D_{e,\sigma}^\dagger$ and $D_{\sigma(e)^*,\sigma}$ agree.

Let $(\xi, \eta) \in G(D_{e^*})$. For all $\zeta \in \mathrm{dom}(D_e)$, we have

$$\langle D_{\sigma(e)^*,\sigma} \xi, \zeta \rangle = \langle \sigma(e)^* (D \otimes 1_q) e^* \xi, \zeta \rangle = \langle \xi, \sigma(e)(D \otimes 1_q) e\zeta \rangle = \langle \xi, D_{e,\sigma} \zeta \rangle.$$

Thus, $(\xi, D_{\sigma(e)^*,\sigma} \xi)$ belongs to $G(D_{e,\sigma}^\dagger)$. Then $G(D_{\sigma(e)^*,\sigma})$ is contained in $G(D_e^\dagger)$, that is, $D_e^\dagger$ is an extension of $D_{\sigma(e)^*,\sigma}$.

Let $(\xi, \eta) \in G(D_e^\dagger)$, and set $R := \sigma(e)(D \otimes 1_q)(1 - e)$. We note that

$$R = \sigma(e)(D \otimes 1_q)(1 - e)$$
$$= \sigma(e)\left\{ \left(1 - \sigma(e)\right)(D \otimes 1_q) - [D \otimes 1_q, e]_\sigma \right\}$$
$$= -\sigma(e)[D \otimes 1_q, e]_\sigma.$$

Thus, R is a bounded operator. Incidentally, its adjoint R^* is a bounded operator as well. Set $\tilde{\eta} = \eta + (1 - e^*) R^* S_{\sigma(e)}^{-1} \xi$, and let $\zeta \in (\mathrm{dom}\, D)^q$. As $e\zeta \in \mathrm{dom}(D_{e,\sigma})$

and the subspaces $e^*\mathcal{H}^q$ and $(1-e)\mathcal{H}^q$ are orthogonal to each other, we have

$$\langle\tilde{\eta},\zeta\rangle = \langle\eta,e\zeta\rangle + \langle\eta,(1-e)\zeta\rangle + \langle(1-e^*)R^*S_{\sigma(e)}^{-1}\xi,\zeta\rangle$$

$$= \langle\xi,D_{e,\sigma}(e\zeta)\rangle + \langle S_{\sigma(e)}^{-1}\xi,R((1-e)\zeta)\rangle.$$

Moreover, as $\xi \in \sigma(e)^*\mathcal{H}^q$ and $R((1-e)\zeta) \in \sigma(e)\mathcal{H}^q$, using (4.3) we see that $\langle S_{\sigma(e)}^{-1}\xi,R((1-e)\zeta)\rangle$ agrees with $\langle\xi,R((1-e)\zeta)\rangle$. Therefore, $\langle\tilde{\eta},\zeta\rangle$ is equal to

$$\langle\xi,D_{e,\sigma}(e\zeta) + R((1-e)\zeta)\rangle = \langle\xi,\sigma(e)(D\otimes 1_q)\zeta\rangle = \langle\sigma(e)^*\xi,(D\otimes 1_q)\zeta\rangle$$

$$= \langle\xi,(D\otimes 1_q)\zeta\rangle.$$

This shows that $(\xi,\tilde{\eta})$ lies in the graph of the operator $(D\otimes 1_q)^*$, which agrees with $D\otimes 1_q$ since D is self-adjoint. Thus, $(\xi,\tilde{\eta})$ lies in the graph of $D\otimes 1_q$. Therefore, we see that ξ is contained in both $(\mathrm{dom}(D))^q$ and $\sigma(e)^*\mathcal{H}^q$, so it lies in $(\mathrm{dom}(D))^q \cap \sigma(e)^*\mathcal{H}^q = \sigma(e)^*(\mathrm{dom}(D))^q = \mathrm{dom}(D_{\sigma(e)^*,\sigma})$. This shows that $\mathrm{dom}\,D_{e,\sigma}^\dagger$ is contained in $\mathrm{dom}(D_{\sigma(e)^*,\sigma})$. As $D_{e,\sigma}^\dagger$ is an extension of $D_{\sigma(e)^*,\sigma}$ we then deduce that the two operators agree. As explained above, this proves that $D_{e,\sigma}^* = S_e^{-1}D_{\sigma(e)^*,\sigma}S_{\sigma(e)}$. The proof is complete. $\qquad\square$

LEMMA 4.3

The operator $D_{e,\sigma}$ is closed and Fredholm, and we have

$$(4.5) \qquad\qquad \mathrm{ind}\,D_{e,\sigma}^\pm = \dim\ker D_{e,\sigma}^\pm - \dim\ker D_{\sigma(e)^*,\sigma}^\mp.$$

Proof
Substituting $\sigma(e)^* = \sigma^{-1}(e^*)$ for e in (4.4) shows that $D_{\sigma(e)^*,\sigma}^* = S_{\sigma(e)^*}^{-1}D_{e,\sigma}S_{e^*}$, that is, $D_{e,\sigma} = S_{\sigma(e)^*}D_{\sigma(e)^*,\sigma}^*S_{e^*}^{-1}$. As $D_{\sigma(e)^*,\sigma}^*$ is a closed operator and the operators $S_{\sigma(e)^*}$ and $S_{e^*}^{-1}$ are bounded, we see that $D_{e,\sigma}$ is a closed operator.

Let D^{-1} be the partial inverse of D, and let P_0 be the orthogonal projection onto $\ker D$. Set $Q_{e,\sigma} := e(D^{-1}\otimes 1_q)$, which we regard as a bounded operator from $\sigma(e)\mathcal{H}^q$ to $e\mathcal{H}^q$. Note that $Q_{e,\sigma}$ is a compact operator. Moreover, on $\sigma(e)\mathcal{H}^q$ we have

$$D_{e,\sigma}Q_{e,\sigma} = \sigma(e)(D\otimes 1_q)e(D^{-1}\otimes 1_q)$$

$$= \sigma(e) + \sigma(e)[D\otimes 1_q,e]_\sigma(D^{-1}\otimes 1_q) - \sigma(e)(P_0\otimes 1_q)$$

$$= 1 + e[D\otimes 1_q,e](D^{-1}\otimes 1_q) - \sigma(e)(P_0\otimes 1_q).$$

Likewise, on $e(\mathrm{dom}(D))^q$ we have

$$Q_{e,\sigma}D_{e,\sigma} = e(D^{-1}\otimes 1_q)\sigma(e)(D\otimes 1_q) = 1 - eD^{-1}[D\otimes 1_q,e]_\sigma - e(P_0\otimes 1_q)e.$$

As D^{-1},P_0 are compact operators and $[D\otimes 1_q,e]_\sigma$ is bounded, we see that $Q_{e,\sigma}$ inverts $D_{e,\sigma}$ modulo compact operators. It then follows that $D_{e,\sigma}$ is a Fredholm operator.

We note that S_e and $S_{\sigma(e)}$ are even operators, so (4.4) means that $(D_{e,\sigma}^\pm)^* = S_e^{-1}D_{\sigma(e)^*,\sigma}^\mp S_{\sigma(e)}^{-1}$. Therefore, the operator S_e induces an isomorphism

$\ker D^{\mp}_{\sigma(e)^*,\sigma} \simeq \ker(D^{\pm}_{e,\sigma})^*$. Thus,

$$\operatorname{ind} D^{\pm}_{e,\sigma} = \dim \ker D^{\pm}_{e,\sigma} - \dim \ker(D^{\pm}_{e,\sigma})^* = \dim \ker D^{\pm}_{e,\sigma} - \dim \ker D^{\mp}_{\sigma(e)^*,\sigma}.$$

The proof is complete. $\qquad\square$

We define the index of $D_{e,\sigma}$ by

$$(4.6) \qquad \operatorname{ind} D_{e,\sigma} := \frac{1}{2}(\operatorname{ind} D^+_{e,\sigma} - \operatorname{ind} D^-_{e,\sigma}).$$

Thanks to (4.5) we have

$$(4.7) \qquad \begin{aligned} \operatorname{ind} D_{e,\sigma} = \frac{1}{2}(&\dim \ker D^+_{e,\sigma} + \dim \ker D^+_{\sigma(e)^*,\sigma} \\ &- \dim \ker D^-_{e,\sigma} - \dim \ker D^-_{\sigma(e)^*,\sigma}). \end{aligned}$$

In particular, when $\sigma(e)^* = e$ we get

$$\operatorname{ind} D_{e,\sigma} = \dim \ker D^+_{e,\sigma} - \dim \ker D^-_{e,\sigma}.$$

Let $g \in \operatorname{Gl}_q(\mathcal{A})$, and set $\hat{e} = g^{-1}eg$. On $(\operatorname{dom}(D))^q$ the operator $\sigma(\hat{e})(D \otimes 1_q)\hat{e}$ agrees with

$$\begin{aligned} \sigma(g)^{-1}&\sigma(e)\sigma(g)(D \otimes 1_q)g^{-1}eg \\ &= \sigma(g)^{-1}\sigma(e)\sigma(g)\sigma(g^{-1})(D \otimes 1_q)eg + \sigma(g)^{-1}\sigma(e)\sigma(g)[D \otimes 1_q, g^{-1}]_\sigma eg \\ &= \sigma(g)^{-1}D_{e,\sigma}g + \sigma(g)^{-1}\sigma(e)\sigma(g)[D \otimes 1_q, g^{-1}]_\sigma eg. \end{aligned}$$

As $[D \otimes 1_q, g^{-1}]_\sigma$ is a bounded operator, we see that $D^{\pm}_{\hat{e},\sigma}$ and $\sigma(g)^{-1}(D^{\pm}_{e,\sigma})g$ agree up to a bounded operator. It then follows that $D^{\pm}_{\hat{e},\sigma}$ and $D^{\pm}_{e,\sigma}$ have the same Fredholm index. Thus,

$$(4.8) \qquad \operatorname{ind} D_{g^{-1}eg,\sigma} = \operatorname{ind} D_{e,\sigma} \quad \forall g \in \operatorname{Gl}_q(\mathcal{A}).$$

Moreover, if $e' \in M_{q'}(\mathcal{A})$ is another idempotent, then, with respect to the splittings $(e \oplus e')(\mathcal{H}^{\pm})^q = e(\mathcal{H}^{\pm})^q \oplus e'(\mathcal{H}^{\pm})^{q'}$ and $\sigma(e \oplus e')(\mathcal{H}^{\pm})^q = \sigma(e)(\mathcal{H}^{\pm})^q \oplus \sigma(e')(\mathcal{H}^{\pm})^{q'}$, we have $D^{\pm}_{e\oplus e',\sigma} = D^{\pm}_{e,\sigma} \oplus D^{\pm}_{e',\sigma}$. We then see that $\operatorname{ind} D^{\pm}_{e\oplus e',\sigma} = \operatorname{ind} D^{\pm}_{e,\sigma} + \operatorname{ind} D^{\pm}_{e',\sigma}$. Thus,

$$\operatorname{ind} D_{e\oplus e',\sigma} = \operatorname{ind} D_{e,\sigma} + \operatorname{ind} D_{e',\sigma}.$$

Therefore, we arrive at the following statement.

PROPOSITION 4.4 (SEE [CM4])
There is a unique additive map $\operatorname{ind}_{D,\sigma} : K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z}$ *such that*

$$(4.9) \qquad \operatorname{ind}_{D,\sigma}[e] = \operatorname{ind} D_{e,\sigma} \quad \forall e \in M_q(\mathcal{A}), e^2 = e.$$

As pointed out in Remark 2.4 the fact that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$ means that the map $a \to \sigma(a)^*$ is an involutive antilinear antiautomorphism of $\mathcal{A}$, which we shall call the σ-involution. An element $a \in \mathcal{A}$ is self-adjoint with respect to this involution if and only if $\sigma(a)^* = a$. As (4.7) shows, when $\sigma(e)^* = e$ the index

of $D_{e,\sigma}$ is an integer. While an idempotent in $M_q(\mathcal{A})$ is always conjugate to a self-adjoint idempotent, in general it need not be conjugate to an idempotent which is self-adjoint with respect to the σ-involution. Nevertheless, this property holds under a further assumption on the automorphism σ.

DEFINITION 4.5

The automorphism σ is called ribbon when it has a square root in the sense that there is an automorphism $\tau : \mathcal{A} \to \mathcal{A}$ such that

$$(4.10) \qquad \sigma(a) = \tau\big(\tau(a)\big) \qquad \text{and} \qquad \tau(a)^* = \tau^{-1}(a^*) \quad \text{for all } a \in \mathcal{A}.$$

REMARK 4.6

The terminology *ribbon* is used in analogy to the theory of quantum groups, where a quasitriangular Hopf algebra is called ribbon when a certain element admits a square root compatible with the quasitriangular structure (see, e.g., [Ma]).

LEMMA 4.7

Assume that the automorphism σ is ribbon. Then

    (i)   *Any idempotent $e \in M_q(\mathcal{A})$, $q \in \mathbb{N}$, is conjugate to an idempotent which is self-adjoint with respect to the σ-involution.*

    (ii) *The index map $\mathrm{ind}_{D,\sigma}$ is integer-valued.*

Proof

The second part follows by combining the first part with (4.8) and (4.5). Therefore, we only need to prove the first part. In addition, without any loss of generality, we may assume that $q = 1$ in the first part. Thus, let e be an idempotent element of $\mathcal{A}$.

Let us briefly recall how we construct a self-adjoint idempotent in $\mathcal{A}$ which is conjugate to e (see, e.g., [Bl, Proposition 4.6.2] for more details). Set $a = e - e^*$, and set $b = 1 + aa^*$. Observing that b is an invertible element of $\mathcal{A}$ which commutes with e and e^*, define $p = ee^* b^{-1}$. It can be checked that $p^2 = p^* = p$, that is, p is a self-adjoint idempotent of $\mathcal{A}$. Moreover, if we set $g = 1 - p + e$, then g has inverse $g^{-1} = 1 + p - e$ and $g^{-1}pg = e$.

We remark that the above construction holds *verbatim* if we replace the involution $a \to a^*$ by *any* other involutive antilinear antiautomorphism of $\mathcal{A}$, provided that it can be shown that the corresponding operator b is invertible (in $\mathcal{A}$). Thus, if we substitute $\sigma(e)^*$ for e^* and we assume that $b := 1 + \sigma(a)^* a$ is invertible, where $a = e - \sigma(e)^*$, then $p := e\sigma(e)^* b^{-1}$ is such that $p^2 = \sigma(p)^* = p$ and $g^{-1}pg = e$ where $g := 1 - p + e$ has inverse $g^{-1} = 1 + p - e$. Therefore, the main question at stake is to show that b is invertible.

Let τ be a square root of σ in the sense of (4.10). Then $\tau(\sigma(a)^*) = \tau \circ \sigma^{-1}(a^*) = \tau^{-1}(a^*) = \tau(a)^*$. Thus,

$$\tau(b) = 1 + \tau\big(\sigma(a)^* a\big) = 1 + \tau\big(\sigma(a)^*\big)\tau(a) = 1 + \tau(a)^* \tau(a).$$

As $\tau(a)^*\tau(a)$ is a positive element of $\mathcal{A}$ we see that $\tau(b)$ is invertible, and hence b is invertible as well. The proof is complete.   □

As we shall now see, the ribbon condition (4.10) is satisfied by the automorphisms occurring in the main examples of twisted spectral triples.

EXAMPLE 4.8

Assume that $\sigma(a) = kak^{-1}$ where k is a positive invertible element of $\mathcal{A}$. Then σ has the square root $\tau(a) = k^{\frac{1}{2}}ak^{-\frac{1}{2}}$. We note that $k^{\frac{1}{2}}$ is an element of $\mathcal{A}$ since $\mathcal{A}$ is closed under holomorphic functional calculus.

More generally, we have the following.

EXAMPLE 4.9

Suppose that σ agrees with the value at $t = -i$ of the analytic extension of a strongly continuous one-parameter group of isometric $*$-isomorphisms $(\sigma_t)_{t \in \mathbb{R}}$. This condition is called (1PG) in [CM4]. In this case σ is ribbon with square root $\tau := \sigma_{|t=-i/2}$.

REMARK 4.10

By a result of Bost [Bo] the analytic extension of a strongly continuous one-parameter group of isometric isomorphisms on an involutive Banach algebra always exists on a dense subalgebra which is closed under holomorphic functional calculus.

REMARK 4.11

Connes–Moscovici [CM4] showed that, when the condition (1PG) holds, $\operatorname{ind} D^+_{e,\sigma}$ is equal to $-\operatorname{ind} D^-_{e,\sigma}$, so that in this case the index map $\operatorname{ind}_{D,\sigma}$ is integer-valued.

EXAMPLE 4.12

The ribbon condition is also satisfied by the automorphism σ_g appearing in the construction of the conformal Dirac spectral triple. From the definition (2.7) of σ_g we see that a square root satisfies the ribbon condition (4.10). Indeed, a square root satisfying (4.10) is given by the automorphism τ_g defined by

$$\tau_g(fU_\phi) := \sqrt{k_\phi}fU_\phi \quad \forall f \in C^\infty(M) \; \forall \phi \in G.$$

In fact, σ_g satisfies the (1PG) condition with respect to the one-parameter group of isometric $*$-isomorphisms σ_t, $t \in \mathbb{R}$, given by $\sigma_t(fU_\phi) = k_\phi^{it}fU_\phi$.

5. Index map and σ-connections

In this section, we present a geometric description of the index map of a twisted spectral triple in terms of couplings by σ-connections on finitely generated projective modules (i.e., noncommutative vector bundles). As we shall explain in the next section, this description makes much more transparent the analogy to

the construction of the index map in the commutative case in terms of Dirac operators coupled with connections (see, e.g, [BGV]). We refer to [Mo1] for a similar description of the index map in the case of ordinary spectral triples.

Throughout this section we let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a twisted spectral triple. In addition, we let $\mathcal{E}$ be a finitely generated projective right module over $\mathcal{A}$, that is, $\mathcal{E}$ is the direct summand of a finite-rank free module $\mathcal{E}_0 \simeq \mathcal{A}^q$. In order to define σ-connections we need to introduce the notion of σ-translation.

DEFINITION 5.1

A σ-translation of $\mathcal{E}$ is given by a pair $(\mathcal{E}, \sigma^{\mathcal{E}})$, where

(i) $\mathcal{E}^\sigma$ is a finitely generated projective right module over $\mathcal{A}$, called the σ-translate;

(ii) $\sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma$ is a $\mathbb{C}$-linear isomorphism such that

$$(5.1) \qquad \sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \quad \text{for all } \xi \in \mathcal{E} \text{ and } a \in \mathcal{A}.$$

REMARK 5.2

The condition (5.1) means that $\sigma^{\mathcal{E}}$ is a right-module isomorphism from $\mathcal{E}$ onto $\mathcal{E}^{(\sigma)}$, where $\mathcal{E}^{(\sigma)}$ is $\mathcal{E}^\sigma$ equipped with the action $(\xi, a) \to \xi\sigma(a)$. In particular, when $\sigma = \mathrm{id}$ a σ-translation of $\mathcal{E}$ is simply given by a right-module isomorphism $\sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma$. Therefore, a canonical choice of σ-translation is to take $(\mathcal{E}, \mathrm{id})$. This will always be the choice we shall make when $\sigma = \mathrm{id}$.

REMARK 5.3

Suppose that $\mathcal{E} = e\mathcal{A}^q$, for some idempotent $e \in M_q(\mathcal{A})$, $q \geq 1$. The automorphism σ lifts to $\mathcal{A}^q$ by

$$\sigma(\xi) = \big(\sigma(\xi_j)\big) \quad \forall \xi = (\xi_j) \in \mathcal{A}^q.$$

Note that σ is a $\mathbb{C}$-linear isomorphism of $\mathcal{A}^q$ onto itself and maps $e\mathcal{A}^q$ onto $\sigma(e)\mathcal{A}^q$, and so it induces a $\mathbb{C}$-linear isomorphism $\sigma^e : e\mathcal{A}^q \to \sigma(e)\mathcal{A}^q$. Moreover, for all $\xi = (\xi_j) \in e\mathcal{A}^q$ and $a \in \mathcal{A}$,

$$(5.2) \qquad \sigma_e(\xi a) = \sigma\big((\xi_j a)\big) = \big(\sigma(\xi_j a)\big) = \big(\sigma(\xi_j)\big)\sigma(a) = \sigma_e(\xi)\sigma(a).$$

That is, σ_e satisfies (5.1). Therefore, the pair $(\sigma(e)\mathcal{A}^q, \sigma_e)$ is a σ-translation of $e\mathcal{A}^q$. This will be our canonical choice of σ-translation when $\mathcal{E}$ is of the form $e\mathcal{A}^q$, with $e^2 = e \in M_q(\mathcal{A})$, $q \geq 1$.

REMARK 5.4

In general, a σ-translation is obtained as follows. By definition, $\mathcal{E}$ is a direct summand of a free module $\mathcal{E}_0 \simeq \mathcal{A}^q$. Let $\phi : \mathcal{E}_0 \to \mathcal{A}^q$ be a right-module isomorphism. The image of $\mathcal{E}$ by ϕ is a right module of the form $e\mathcal{A}^q$ for some idempotent $e \in M_q(\mathcal{A})$. Set $\mathcal{E}^\sigma := \phi^{-1}(\sigma(e)\mathcal{A}^q)$; this is a direct summand of $\mathcal{E}_0$. The isomorphism ϕ induces isomorphisms of right modules,

$$\phi_e : \mathcal{E} \to e\mathcal{A}^q \qquad \text{and} \qquad \phi_{\sigma(e)} : \mathcal{E}^\sigma \to \sigma(e)\mathcal{A}^q.$$

Set $\sigma^{\mathcal{E}} = (\phi_{\sigma(e)})^{-1} \circ \sigma^e \circ \phi_e$, where $\sigma_e : e\mathcal{A}^q \to \sigma(e)\mathcal{A}^q$ is the σ-lift introduced in Remark 5.3. Then $\sigma^{\mathcal{E}}$ is a $\mathbb{C}$-linear isomorphism from $\mathcal{E}$ onto $\mathcal{E}^\sigma$. Moreover, using (5.1) we see that, for all $\xi \in \mathcal{E}$ and $a \in \mathcal{A}$, we have

$$\sigma^{\mathcal{E}}(\xi a) = (\phi_{\sigma(e)})^{-1} \circ \sigma_e \big(\phi(\xi)a\big) = (\phi_{\sigma(e)})^{-1}\big(\sigma_e(\xi)\sigma(a)\big) = \sigma^{\mathcal{E}}(\xi)\sigma(a).$$

This shows that $\sigma^{\mathcal{E}}$ satisfies (5.1), and so $(\mathcal{E}^\sigma, \sigma^{\mathcal{E}})$ is a σ-translation of $\mathcal{E}$.

Conversely, given a σ-translation $(\mathcal{E}, \sigma^{\mathcal{E}})$ and a right-module isomorphism $\phi : \mathcal{E} \to e\mathcal{A}^q$, the map $\phi_\sigma = \sigma^e \circ \phi \circ (\sigma^{\mathcal{E}})^{-1}$ is a $\mathbb{C}$-linear isomorphism from $\mathcal{E}^\sigma$ onto $\sigma(e)\mathcal{A}^q$ such that $\sigma^{\mathcal{E}} = (\phi_\sigma)^{-1} \circ \sigma^e \circ \phi$. We note that (5.1) implies that

$$(\sigma^{\mathcal{E}})^{-1}(\xi a) \quad \text{for all } \xi \in \mathcal{E} \text{ and } a \in \mathcal{A}.$$

Combining this with (5.2) shows that, for all ξ and $a \in \mathcal{A}$, we have

$$\phi_\sigma(\xi a) = \sigma^e \circ \phi\big((\sigma^{\mathcal{E}})^{-1}(\xi)\sigma^{-1}(a)\big) = \sigma^e\big(\phi \circ (\sigma^{\mathcal{E}})^{-1}(\xi)\sigma^{-1}(a)\big) = \phi_\sigma(\xi)a.$$

Therefore, ϕ_σ is a right-module isomorphism from $\mathcal{E}^\sigma$ onto $\sigma(e)\mathcal{A}^q$. In addition, we note that when σ is ribbon Lemma 4.7 enables us to choose the idempotent e so that $\sigma(e) = e^*$.

Following [CM4] we consider the space of twisted 1-forms

$$\Omega^1_{D,\sigma}(\mathcal{A}) = \big\{\Sigma a^i[D, b^i]_\sigma; a^i, b^i \in \mathcal{A}\big\}.$$

We note that $\Omega^1_{D,\sigma}(\mathcal{A})$ is a subspace of $\mathcal{L}(\mathcal{H})$. Moreover, it is naturally an $(\mathcal{A}, \mathcal{A})$-bimodule, since

$$a^2\big(a^1[D, b^1]_\sigma\big)b^2 = a^2 a^1[D, b^1 b^2]_\sigma - a^2 a^1 \sigma(b^1)[D, b^2]_\sigma \quad \forall a^j, b^j \in \mathcal{A}.$$

We also have a "twisted" differential $d_\sigma : \mathcal{A} \to \Omega^1_{D,\sigma}(\mathcal{A})$ defined by

$$(5.3) \qquad\qquad\qquad\qquad d_\sigma a := [D, a]_\sigma \quad \forall a \in \mathcal{A}.$$

This is a σ-derivation in the sense that

$$(5.4) \qquad\qquad\qquad d_\sigma(ab) = (d_\sigma a)b + \sigma(a)d_\sigma b \quad \forall a, b \in \mathcal{A}.$$

In what follows we let $(\mathcal{E}^\sigma, \sigma^{\mathcal{E}})$ be a σ-translation of $\mathcal{E}$.

DEFINITION 5.5

A σ-connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E}^\sigma \otimes_\mathcal{A} \Omega^1_{D,\sigma}(\mathcal{A})$ such that

$$(5.5) \qquad\qquad \nabla(\xi a) = (\nabla \xi)a + \sigma^{\mathcal{E}}(\xi) \otimes d_\sigma a \quad \forall \xi \in \mathcal{E} \ \forall a \in \mathcal{A}.$$

EXAMPLE 5.6

Suppose that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$. Then a natural σ-connection on $\mathcal{E}$ is the *Grassmannian σ-connection* $\nabla_0^{\mathcal{E}}$ defined by

$$(5.6) \qquad\qquad \nabla_0^{\mathcal{E}}\xi = \sigma(e)(d_\sigma \xi_j) \quad \text{for all } \xi = (\xi_j) \text{ in } \mathcal{E}.$$

LEMMA 5.7

The set of σ-connections on $\mathcal{E}$ is an affine space modeled on $\mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}^\sigma \otimes \Omega^1_{D,\sigma}(\mathcal{A}))$.

Proof

It follows from (5.5) that two σ-connections on $\mathcal{E}$ differ by an element of $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^{\sigma} \otimes \Omega^1_{D,\sigma}(\mathcal{A}))$. Therefore, the only issue at stake is to show that the set of σ-connections is nonempty. This is a true fact when $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$ since in this case there is always the Grassmannian σ-connection.

In general, as shown in Remark 5.4, we always can find an idempotent $e \in M_q(\mathcal{A})$, $q \geq 1$, and right-module isomorphisms $\phi : \mathcal{E} \to e\mathcal{A}^q$ and $\phi^{\sigma} : \mathcal{E}^{\sigma} \to \sigma(e)\mathcal{A}^q$ satisfying (5.1). We then can pull back to $\mathcal{E}$ any connection ∇ on $\mathcal{A}^q$ to a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{\sigma} \otimes_{\mathcal{A}} \Omega^1_{D,\sigma}(\mathcal{A})$ defined by

$$\nabla^{\mathcal{E}} = \left((\phi^{\sigma})^{-1} \otimes 1_{\Omega^1_{D,\sigma}(\mathcal{A})} \right) \circ \nabla \circ \phi.$$

For $\xi \in \mathcal{E}$ and $a \in \mathcal{A}$ we have

$$\nabla^{\mathcal{E}}(\xi a) = \left((\phi^{\sigma})^{-1} \otimes 1_{\Omega^1_{D,\sigma}(\mathcal{A})} \right) \circ \nabla \left(\phi(\xi) a \right) = (\nabla^{\mathcal{E}} \xi) a + \left((\phi^{\sigma})^{-1} \circ \sigma \circ \phi \right)(\xi) \otimes d_{\sigma} a$$

$$= (\nabla^{\mathcal{E}} \xi) a + \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma} a.$$

Thus, $\nabla^{\mathcal{E}}$ is a σ-connection on $\mathcal{E}$. The proof is complete. $\qquad \square$

In what follows we denote by $\mathcal{E}'$ the dual $\mathcal{A}$-module $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$. We also recall that $\mathcal{A}$ is a $*$-algebra represented in $\mathcal{L}(\mathcal{H})$.

DEFINITION 5.8

A Hermitian metric on $\mathcal{E}$ is a map $(\cdot, \cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that

(1) $(\cdot, \cdot)$ is $\mathcal{A}$-sesquilinear, that is, it is $\mathcal{A}$-antilinear with respect to the first variable and $\mathcal{A}$-linear with respect to the second variable;

(2) $(\cdot, \cdot)$ is positive, that is, $(\xi, \xi) \geq 0$ for all $\xi \in \mathcal{E}$;

(3) $(\cdot, \cdot)$ is nondegenerate, that is, $\xi \to (\xi, \cdot)$ is an $\mathcal{A}$-antilinear isomorphism from $\mathcal{E}$ onto $\mathcal{E}'$.

REMARK 5.9

By using (2) and a polarization argument it can be shown that $(\xi_1, \xi_2) = (\xi_2, \xi_1)^*$ for all $\xi_j \in \mathcal{A}$.

EXAMPLE 5.10

The canonical Hermitian structure on the free module $\mathcal{A}^q$ is given by

$$(5.7) \qquad (\xi, \eta)_0 = \xi_1^* \eta_1 + \cdots + \xi_q^* \eta_q \quad \text{for all } \xi = (\xi_j) \text{ and } \eta = (\eta_j) \text{ in } \mathcal{A}^q.$$

LEMMA 5.11

Suppose that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$. Then the canonical Hermitian metric of $\mathcal{A}^q$ induces a Hermitian metric on $\mathcal{E}$.

Proof

See Appendix A. $\qquad \square$

REMARK 5.12

Let $\phi : \mathcal{E} \to \mathcal{F}$ be an isomorphism of finitely generated projective modules, and assume that $\mathcal{F}$ carries a Hermitian metric $(\cdot, \cdot)_\mathcal{F}$. Then using ϕ we can pull back the Hermitian metric of $\mathcal{F}$ to the Hermitian metric on $\mathcal{E}$ given by

$$(\xi_1, \xi_2)_\mathcal{E} := \big(\phi(\xi_1), \phi(\xi_2)\big)_\mathcal{F} \quad \forall \xi_j \in \mathcal{E}.$$

In particular, if we take $\mathcal{F}$ to be of the form $e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$, then we can pull back the canonical Hermitian metric $(\cdot, \cdot)_0$ to a Hermitian metric on $\mathcal{E}$.

From now on we assume that $\mathcal{E}$ and its σ-translate carry Hermitian metrics. We denote by $\mathcal{H}(\mathcal{E})$ the pre-Hilbert space consisting of $\mathcal{E} \otimes_\mathcal{A} \mathcal{H}$ equipped with the Hermitian inner product

$$(5.8) \qquad \langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \rangle := \big\langle \zeta_1, (\xi_1, \xi_2)\zeta_2 \big\rangle, \quad \xi_j \in \mathcal{E}, \zeta_j \in \mathcal{H},$$

where $(\cdot, \cdot)$ is the Hermitian metric of $\mathcal{E}$.

LEMMA 5.13

The pre-Hilbert space $\mathcal{H}(\mathcal{E})$ is a Hilbert space, and its topology is independent of the choice of the Hermitian inner product of $\mathcal{E}$.

Proof
See Appendix B.   □

REMARK 5.14

In [Mo1] the Hilbert space $\mathcal{H}(\mathcal{E})$ is defined as the completion of $\mathcal{E} \otimes_\mathcal{A} \mathcal{H}$ with respect to the Hermitian inner product (5.8). As Lemma 5.13 shows, this pre-Hilbert space is already complete.

We note there is a natural $\mathbb{Z}_2$-grading on $\mathcal{H}(\mathcal{E})$ given by

$$(5.9) \qquad \mathcal{H}(\mathcal{E}) = \mathcal{H}^+(\mathcal{E}) \oplus \mathcal{H}^-(\mathcal{E}), \qquad \mathcal{H}^\pm(\mathcal{E}) := \mathcal{E} \otimes_\mathcal{A} \mathcal{H}^\pm.$$

We also form the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}(\mathcal{E}^\sigma)$ as above. In addition, we let $\nabla^\mathcal{E}$ be a σ-connection on $\mathcal{E}$. Regarding $\Omega^1_{D,\sigma}(\mathcal{A})$ as a subalgebra of $\mathcal{L}(\mathcal{H})$ we have a natural left-action $c : \Omega^1_{D,\sigma}(\mathcal{A}) \otimes_\mathcal{A} \mathcal{H} \to \mathcal{H}$ given by

$$c(\omega \otimes \zeta) = \omega(\zeta) \quad \text{for all } \omega \in \Omega^1_{D,\sigma}(\mathcal{A}) \text{ and } \zeta \in \mathcal{H}.$$

We denote by $c(\nabla^\mathcal{E})$ the composition $(1_{\mathcal{E}^\sigma} \otimes c) \circ (\nabla^\mathcal{E} \otimes 1_\mathcal{H}) : \mathcal{E} \otimes \mathcal{H} \to \mathcal{E}^\sigma \otimes \mathcal{H}$. Thus, for $\xi \in \mathcal{E}$ and $\zeta \in \mathcal{H}$, upon writing $\nabla^\mathcal{E}\xi = \sum \xi_\alpha \otimes \omega_\alpha$ with $\xi_\alpha \in \mathcal{E}^\sigma$ and $\omega_\alpha \in \Omega^1_{D,\sigma}(\mathcal{A})$, we have

$$(5.10) \qquad c(\nabla^\mathcal{E})(\xi \otimes \zeta) = \sum \xi_\alpha \otimes \omega_\alpha(\zeta).$$

In what follows we regard the domain of D as a left $\mathcal{A}$-module, which is possible since the action of $\mathcal{A}$ on $\mathcal{H}$ preserves $\mathrm{dom}(D)$.

DEFINITION 5.15

The coupled operator $D_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes_{\mathcal{A}} \mathrm{dom}(D) \to \mathcal{H}(\mathcal{E}^{\sigma})$ is defined by

$$(5.11) \quad D_{\nabla^{\mathcal{E}}}(\xi \otimes \zeta) := \sigma^{\mathcal{E}}(\xi) \otimes D\zeta + c(\nabla^{\mathcal{E}})(\xi \otimes \zeta) \quad \text{for all } \xi \in \mathcal{E} \text{ and } \zeta \in \mathrm{dom}(D).$$

REMARK 5.16

As the operators $\sigma^{\mathcal{E}}$, D, and $\nabla^{\mathcal{E}}$ are not module maps, we need to check the compatibility of (5.11) with the action of $\mathcal{A}$. This is a consequence of (5.3). Indeed, if $\xi \in \mathcal{E}$ and $\zeta \in \mathrm{dom}(D)$, then for all $a \in \mathcal{A}$

$$c(\nabla^{\mathcal{E}})(\xi a \otimes \zeta) = (1 \otimes c)\big(\nabla^{\mathcal{E}}(\xi a) \otimes \zeta\big)$$
$$= (1 \otimes c)\big((\nabla^{\mathcal{E}}\xi)a \otimes \zeta + \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma}(a) \otimes \zeta\big)$$
$$= c(\nabla^{\mathcal{E}})(\xi \otimes a\zeta) + \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma}(a)\zeta.$$

Thus,

$$D_{\nabla^{\mathcal{E}}}(\xi a \otimes \zeta) - D_{\nabla^{\mathcal{E}}}(\xi \otimes a\zeta)$$
$$= \sigma^{\mathcal{E}}(\xi a) \otimes D\zeta + \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma}(a)\zeta - \sigma^{\mathcal{E}}(\xi) \otimes D(a\zeta)$$
$$= \sigma^{\mathcal{E}}(\xi)\sigma(a) \otimes D\zeta - \sigma^{\mathcal{E}}(\xi) \otimes \sigma(a)D\zeta,$$

which shows that $D_{\nabla^{\mathcal{E}}}(\xi a \otimes \zeta) = D_{\nabla^{\mathcal{E}}}(\xi \otimes a\zeta)$ in $\mathcal{E}^{\sigma} \otimes_{\mathcal{A}} \mathcal{H}$.

REMARK 5.17

With respect to the $\mathbb{Z}_2$-gradings (5.9) for $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}(\mathcal{E}^{\sigma})$ the operator $D_{\nabla^{\mathcal{E}}}$ takes the form

$$D_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & D_{\nabla^{\mathcal{E}}}^- \\ D_{\nabla^{\mathcal{E}}}^+ & 0 \end{pmatrix}, \qquad D_{\nabla^{\mathcal{E}}}^{\pm} : \mathcal{E} \otimes_{\mathcal{A}} \mathrm{dom}(D^{\pm}) \to \mathcal{H}^{\mp}(\mathcal{E}^{\sigma}).$$

That is, $D_{\nabla^{\mathcal{E}}}$ is an odd operator.

Suppose that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$. Then there is a canonical isomorphism U_e from $\mathcal{H}(\mathcal{E})$ to $e\mathcal{H}^q$ given by

$$U_e(\xi \otimes \zeta) = (\xi_j \zeta)_{1 \leq j \leq q} \quad \text{for all } \xi = (\xi_j) \in \mathcal{E} \text{ and } \zeta \in \mathcal{H},$$

where $\mathcal{E} = e\mathcal{A}^q$ is regarded as a submodule of $\mathcal{A}^q$. The inverse of U_e is given by

$$U_e^{-1}\big((\zeta_j)\big) = \sum e_j \otimes \zeta_j \quad \text{for all } (\zeta_j) \in e\mathcal{H}^q,$$

where $e\mathcal{H}^q$ is regarded as a subspace of $\mathcal{H}^q$ and $e_1, \dots, e_q$ are the column vectors of e. We also note that U_e is a graded isomorphism.

LEMMA 5.18

Suppose that $\mathcal{E} = e\mathcal{A}^q$ as above, and let $\nabla_0^{\mathcal{E}}$ be the Grassmannian σ-connection of $\mathcal{E}$. Then

$$U_{\sigma(e)} D_{\nabla_0^{\mathcal{E}}} U_e^{-1} = D_{e,\sigma},$$

where $D_{e,\sigma}$ is defined in (4.1).

Proof

The image of $\mathcal{E} \otimes_\mathcal{A} \mathrm{dom}(D)$ under U_e is $e(\mathrm{dom}(D))^q = \mathrm{dom}(D_{e,\sigma})$. Let $\zeta \in \mathrm{dom}(D)$, and let $\xi = (\xi_j)$ be in $\mathcal{E} \subset \mathcal{A}^q$. Then

$$c(\nabla_0^\mathcal{E})(\xi \otimes \zeta) = \sum \sigma(e_j) \otimes (d_\sigma \xi_j)\zeta = \sum \sigma(e_j) \otimes D(\xi_j \zeta) - \sum \sigma(e_j) \otimes \sigma(\xi_j)D\zeta.$$

The fact that $\xi \in \mathcal{E}$ means that $\xi = e\xi = \sum e_j \xi_j$. Thus,

$$\sum \sigma(e_j) \otimes \sigma(\xi_j)D\zeta = \sum \sigma(e_j\xi_j) \otimes D\zeta = \sigma^\mathcal{E}(\xi) \otimes D\zeta.$$

Therefore,

$$(5.12) \qquad D_{\nabla_0^\mathcal{E}}(\xi \otimes \eta) = \sigma^\mathcal{E}(\xi) \otimes D\eta + c(\nabla_0^\mathcal{E})(\xi \otimes \zeta) = \sum \sigma(e_j) \otimes D(\xi_j\zeta).$$

For $j = 1, \ldots, q$ set $\zeta_j = \xi_j\zeta$, so that $U_e(\xi \otimes \zeta) = (\xi_j\zeta)_{1 \le j \le q} = (\zeta_j)_{1 \le j \le q}$. From (5.12) we get

$$U_{\sigma(e)}\big(D_{\nabla_0^\mathcal{E}}(\xi \otimes \eta)\big)_i = \sum_j \sigma(e_{ij})D(\zeta_j).$$

In view of the definition of the operator $D_{e,\sigma}$ in (4.1) we see that

$$U_{\sigma(e)}D_{\nabla_0^\mathcal{E}}(\xi \otimes \eta) = \sigma(e)(D\zeta_j)_{1 \le j \le q} = D_{e,\sigma}\big((\zeta_j)_{1 \le j \le q}\big) = D_{e,\sigma}U_e(\xi \otimes \zeta).$$

This proves the lemma. $\qquad\square$

REMARK 5.19

As U_e and $U_{\sigma(e)}$ are graded isomorphisms, we further have

$$(5.13) \qquad\qquad U_{\sigma(e)}^\mp D_{\nabla_0^\mathcal{E}}^\pm (U_e^\pm)^{-1} = D_{e,\sigma}^\pm.$$

LEMMA 5.20

For any σ-connection $\nabla^\mathcal{E}$ on $\mathcal{E}$, the operator $D_{\nabla^\mathcal{E}}$ is closed and Fredholm.

Proof

Let us first assume that $\mathcal{E} = e\mathcal{A}^q$ and $\mathcal{E}^\sigma = \sigma(e)\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$. It follows from Lemmas 4.3 and 5.18 that the operator $D_{\nabla_0^\mathcal{E}}$ is closed and Fredholm. By Lemma 5.7 the difference $\nabla^\mathcal{E} - \nabla_0^\mathcal{E}$ lies in $\mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}^{(\sigma)} \otimes \Omega_{D,\sigma}^1(\mathcal{A}))$. We then deduce that the operator $D_{\nabla^\mathcal{E}}$ is closed and Fredholm.

In general, thanks to Remark 5.4 we always can find an idempotent $e \in M_q(\mathcal{A})$, $q \ge 1$, and right-module isomorphisms $\phi : \mathcal{E} \to e\mathcal{A}^q$ and $\phi^\sigma : \mathcal{E}^\sigma \to \sigma(e)\mathcal{A}^q$ satisfying (5.1). We equip $e\mathcal{A}^q$ with the Hermitian metric $(\cdot, \cdot)_\phi := (\phi^{-1}(\cdot), \phi^{-1}(\cdot))_\mathcal{E}$, where $(\cdot, \cdot)_\mathcal{E}$ is the Hermitian metric of $\mathcal{E}$. Likewise, we equip $\sigma(e)\mathcal{A}^q$ with the Hermitian metric $(\cdot, \cdot)_{\phi^\sigma} := ((\phi^\sigma)^{-1}(\cdot), (\phi^\sigma)^{-1}(\cdot))_{\mathcal{E}^\sigma}$, where $(\cdot, \cdot)_{\mathcal{E}^\sigma}$ is the Hermitian metric of $\mathcal{E}^\sigma$. We then have unitary operators $U_\phi : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(e\mathcal{A}^q)$ and $U_{\phi^\sigma} : \mathcal{H}(\mathcal{E}^\sigma) \to \sigma(e)\mathcal{A}^q$ given by

$$(5.14) \qquad\qquad U_\phi := \phi \otimes 1_\mathcal{H} \qquad \text{and} \qquad U_{\phi^\sigma} := \phi^\sigma \otimes 1_\mathcal{H}.$$

In addition, we denote by $\nabla^{\phi_*\mathcal{E}}$ the σ-connection on $e\mathcal{A}^q$ defined by

$$(5.15) \qquad\qquad \nabla^{\phi_*\mathcal{E}} := (\phi^\sigma \otimes 1_{\Omega_{D,\sigma}^1(\mathcal{A})}) \circ \nabla^\mathcal{E} \circ \phi^{-1}.$$

Let $\xi \in \mathcal{E}$, and let $\zeta \in \text{dom}(D)$. Set $\eta = \phi(\xi)$, and set $\nabla^{\mathcal{E}}\xi = \sum \xi_\alpha \otimes \omega_\alpha$ with $\xi_\alpha \in \mathcal{E}^\sigma$ and $\omega_\alpha \in \Omega^1_{D,\sigma}(\mathcal{A})$. Then $\nabla^{\phi_*\mathcal{E}}\eta = (\phi^\sigma \otimes 1_{\Omega^1_{D,\sigma}(\mathcal{A})})(\nabla^{\mathcal{E}}\xi) = \sum \phi^\sigma(\xi_\alpha) \otimes \omega_\alpha$. Thus,

$$(5.16) \qquad c(\nabla^{\phi_*\mathcal{E}})(\eta \otimes \zeta) = \sum \phi^\sigma(\xi_\alpha) \otimes \omega_\alpha(\zeta) = U_{\phi^\sigma} \circ c(\nabla^{\mathcal{E}})(\xi \otimes \zeta).$$

We also note that $\sigma(\eta) = \sigma \circ \phi(\xi) = \phi^\sigma \circ \sigma^{\mathcal{E}}(\xi)$, and hence

$$(5.17) \qquad \sigma(\eta) \otimes D\zeta = (\phi^\sigma \otimes 1_{\mathcal{H}})(\sigma^{\mathcal{E}}(\xi) \otimes D\zeta) = U_{\phi^\sigma}(\sigma^{\mathcal{E}}(\xi) \otimes D\zeta).$$

Combining (5.16) and (5.17) we get

$$D_{\nabla^{\phi_*\mathcal{E}}} U_\phi(\xi \otimes \zeta) = D_{\nabla^{\phi_*\mathcal{E}}}(\eta \otimes \zeta) = \sigma(\eta) \otimes D\zeta + c(\nabla^{\phi_*\mathcal{E}})(\eta \otimes \zeta)$$
$$= U_{\phi^\sigma} D_{\nabla^{\mathcal{E}}}(\xi \otimes \zeta).$$

This shows that

$$(5.18) \qquad\qquad D_{\nabla^{\mathcal{E}}} = U_{\phi^\sigma}^{-1} D_{\nabla^{\phi_*\mathcal{E}}} U_\phi.$$

By the first part of the proof we know that the operator $D_{\nabla^{\phi_*\mathcal{E}}}$ is closed and Fredholm. As U_ϕ and U_{ϕ^σ} are isomorphisms we then deduce that the operator $D_{\nabla^{\mathcal{E}}}$ is closed and Fredholm. The proof is complete. $\qquad\square$

In analogy with (3.6) and (4.6) we make the following definition.

DEFINITION 5.21
Given any σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, the index of $D_{\nabla^{\mathcal{E}}}$ is defined by

$$(5.19) \qquad\qquad \text{ind}\, D_{\nabla^{\mathcal{E}}} := \frac{1}{2}(\text{ind}\, D_{\nabla^{\mathcal{E}}}^+ - \text{ind}\, D_{\nabla^{\mathcal{E}}}^-).$$

We are now in a position to prove the main result of this section.

THEOREM 5.22
Let $\mathcal{E}$ be a finitely generated projective right $\mathcal{A}$-module. Then, for any σ-connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we have

$$(5.20) \qquad\qquad \text{ind}_{D,\sigma}[\mathcal{E}] = \text{ind}\, D_{\nabla^{\mathcal{E}}}.$$

Proof
Let us first assume that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$, and let $\nabla_0^{\mathcal{E}}$ be the σ-Grassmannian connection. As shown in the proof of Lemma 5.20, the Fredholm operators $D_{\nabla^{\mathcal{E}}}$ and $D_{\nabla_0^{\mathcal{E}}}$ differ by a bounded odd operator, and so $\text{ind}\, D_{\nabla^{\mathcal{E}}}^\pm = \text{ind}\, D_{\nabla_0^{\mathcal{E}}}^\pm$. Moreover, it follows from (5.13) that $\text{ind}\, D_{\nabla_0^{\mathcal{E}}}^\pm = \text{ind}\, D_{e,\sigma}^\pm$. Thus,

$$(5.21) \qquad\qquad \text{ind}\, D_{\nabla^{\mathcal{E}}}^\pm = \text{ind}\, D_{\nabla_0^{\mathcal{E}}}^\pm = \text{ind}\, D_{e,\sigma}^\pm = \text{ind}_{D,\sigma}[e].$$

In general, we can find an idempotent $e \in M_q(\mathcal{A})$ and right-module isomorphisms $\phi : \mathcal{E} \to e\mathcal{A}^q$ and $\phi^\sigma : \mathcal{E}^\sigma \to \sigma(e)\mathcal{A}^q$ satisfying (5.1). We equip $e\mathcal{A}^q$ and $\sigma(e)\mathcal{A}^q$ with Hermitian metrics as in the proof of Lemma 5.20, and let $\nabla^{\phi_*\mathcal{E}}$ be

the σ-connection on $e\mathcal{A}^q$ given by (5.15). Then from (5.18) we have

$$D_{\nabla^\mathcal{E}} = U_{\phi^\sigma}^{-1} D_{\nabla^{\phi_* \mathcal{E}}} U_\phi,$$

where $U_\phi : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(e\mathcal{A}^q)$ and $U_{\phi^\sigma} : \mathcal{H}(\mathcal{E}^\sigma) \to \mathcal{H}(\sigma(e)\mathcal{A}^q)$ are the unitary operators given by (5.14). As U_ϕ and U_{ϕ^σ} are even isomorphisms we see that $\operatorname{ind} D^\pm_{\nabla^\mathcal{E}} = \operatorname{ind} D^\pm_{\nabla^{\phi_* \mathcal{E}}}$. Combining this with (5.21) we then get

$$\operatorname{ind} D_{\nabla^\mathcal{E}} = \operatorname{ind} D_{\nabla^{\phi_* \mathcal{E}}} = \operatorname{ind}_{D,\sigma}[e] = \operatorname{ind}_{D,\sigma}[\mathcal{E}].$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We conclude this section by looking at the index formula (5.20) in the example of a Dirac spectral triple $(C^\infty(M), L^2(M, \slashed{S}), \slashed{D}_g)$, where (M^n, g) is a compact Riemannian spin oriented manifold of even dimension. Let E be a vector bundle over M, and let ∇^E be a connection on E. Then $\mathcal{E} := C^\infty(M, E)$ is a finitely generated projective module over the algebra $\mathcal{A} := C^\infty(M)$. We observe that $\mathcal{A}$ is a commutative algebra and that we can identify left and right modules over $\mathcal{A}$. It would be more convenient to work with left modules instead of right modules. This provides us with a natural identification of $\mathcal{A}$-modules $\mathcal{E}_1 \otimes_\mathcal{A} \mathcal{E}_2 \simeq \mathcal{E}_2 \otimes_\mathcal{A} \mathcal{E}_1$ for the tensor products of two modules $\mathcal{E}_1$ and $\mathcal{E}_2$; the isomorphism is given by the flip map $\xi_1 \otimes \xi_2 \to \xi_2 \otimes \xi_1$.

Let $c : \Lambda^*_\mathbb{C} T^* M \to \operatorname{End} \slashed{S}$ be the Clifford representation. Then, for all a and b in $\mathcal{A}$,

(5.22) $$a[\slashed{D}_g, b] = ac(db) = c(adb).$$

Therefore, we see that

$$\Omega^1_{\slashed{D}_g}(\mathcal{A}) = \operatorname{Span}\big\{c(\omega); \omega \in C^\infty(M, T^*_\mathbb{C} M)\big\}.$$

Note that ∇^E is a linear map from $\mathcal{E} = C^\infty(M, E)$ to $C^\infty(M, T^* M \otimes E) = C^\infty(M, T^*_\mathbb{C} M) \otimes_\mathcal{A} \mathcal{E}$. Consider the linear map $\nabla^\mathcal{E}$ from $\mathcal{E}$ to $\Omega^1_{\slashed{D}_g}(\mathcal{A}) \otimes_\mathcal{A} \mathcal{E} \simeq \mathcal{E} \otimes_\mathcal{A} \Omega^1_{\slashed{D}_g}(\mathcal{A})$ defined by

$$\nabla^\mathcal{E} := (c \otimes 1_\mathcal{E}) \circ \nabla^E.$$

Let $\xi \in \mathcal{E}$, and let $a \in \mathcal{A}$. Using (5.22) we get

$$\nabla^\mathcal{E}(a\xi) = (c \otimes 1_\mathcal{E})(da \otimes \xi + a\nabla^E \xi) = c(da) + a\nabla^\mathcal{E}\xi = [\slashed{D}_g, a]\xi + a\nabla^\mathcal{E}\xi.$$

Therefore, we see that $\nabla^\mathcal{E}$ is a connection on the finitely generated projective module $\mathcal{E}$.

As $\nabla^\mathcal{E}$ is a connection on $\mathcal{E}$, we can form the operator $\slashed{D}_{\nabla^\mathcal{E}} := (\slashed{D}_g)_{\nabla^\mathcal{E}}$. Set $\mathcal{H} = L^2_g(M, E)$. In what follows we identify $\mathcal{H}(\mathcal{E}) = \mathcal{E} \otimes_\mathcal{A} \mathcal{H}$ with $\mathcal{H} \otimes_\mathcal{A} \mathcal{E} \simeq L^2(M, \slashed{S} \otimes E)$, so that we can regard $\slashed{D}_{\nabla^\mathcal{E}}$ as an unbounded operator of $L^2(M, \slashed{S} \otimes E)$. Let $\zeta \in C^\infty(M, \slashed{S})$, and let $\xi \in \mathcal{E}$. Let us write $\nabla^E \xi = \sum \omega_\alpha \otimes \xi_\alpha$, where $\omega_\alpha \in C^\infty(M, T^*_\mathbb{C} M)$ and $\xi_\alpha \in \mathcal{E}$. For each α let us also write $\omega_\alpha = \sum a_{\alpha\beta} db_{\alpha\beta}$, with $a_{\alpha\beta}$ and $b_{\alpha\beta}$ in $C^\infty(M)$. Then, using (5.10) and (5.22), we see that $\slashed{D}_{\nabla^\mathcal{E}}(\zeta \otimes \xi)$

is equal to

$$\not{D}_g \zeta \otimes \xi + \sum_{\alpha,\beta} \big(a_{\alpha\beta}[\not{D}_g, b_{\alpha\beta}]\big)\zeta \otimes \xi_\alpha = \not{D}_g \zeta \otimes \xi + \sum_\alpha c(\omega_\alpha)\zeta \otimes \xi_\alpha$$

$$= \not{D}_{\nabla^E}(\zeta \otimes \xi).$$

Thus, under the identification $\mathcal{H}(\mathcal{E}) \simeq L^2(M, \not{S} \otimes E)$, the operators $\not{D}_{\nabla^\mathcal{E}}$ and $\not{D}_{\nabla^E}$ agree. Combining this with (3.3) and (5.20) we then deduce that, for $\sigma = \mathrm{id}$,

$$(5.23) \qquad \mathrm{ind}_{\not{D}_g,\sigma}[\mathcal{E}] = \mathrm{ind}\, D_{\nabla^\mathcal{E}} = \mathrm{ind}\, D_{\nabla^E} = \mathrm{ind}_{\not{D}_g}[E],$$

where the second index map is the Fredholm index map (5.19). Thus, under the Serre–Swan isomorphism $K^0(M) \simeq K_0(C^\infty(M))$ this Fredholm index map agrees with the index map (3.2).

6. Cyclic cohomology and pairing with K-theory

In this section, we review the main definitions and properties regarding cyclic cohomology and its pairing with K-theory. Cyclic cohomology was discovered by Connes [Co1] and Tsygan [Ts] independently. For more details on cyclic cohomology we refer to [Co4] and [Lo]. Throughout this section we let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$.

6.1. Cyclic cohomology
The Hochschild cochain complex of $\mathcal{A}$ is defined as follows. The space of m-cochains $C^m(\mathcal{A})$, $m \in \mathbb{N}_0$, consists of $(m+1)$-linear maps $\varphi : \mathcal{A}^{m+1} \to \mathbb{C}$. The Hochschild coboundary $b : C^m(\mathcal{A}) \to C^{m+1}(\mathcal{A})$, $b^2 = 0$, is given by

$$(6.1) \qquad b\varphi(a^0, \dots, a^{m+1}) = \sum_{j=0}^m (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{m+1})$$

$$(6.2) \qquad\qquad\qquad + (-1)^{m+1}\varphi(a^{m+1}a^0, \dots, a^m), \quad a^j \in \mathcal{A}.$$

A cochain $\varphi \in C^m(\mathcal{A})$ is called *cyclic* when $T\varphi = \varphi$, where the operator $T : C^m(\mathcal{A}) \to C^m(\mathcal{A})$ is defined by

$$(6.3) \qquad T\varphi(a^0, \dots, a^m) = (-1)^m \varphi(a^m, a^0, \dots, a^{m-1}), \quad a^j \in \mathcal{A}.$$

We denote by $C_\lambda^m(\mathcal{A})$ the space of cyclic m-cochains. As $b(C_\lambda^\bullet(\mathcal{A})) \subset C_\lambda^{\bullet+1}(\mathcal{A})$, we obtain a subcomplex $(C_\lambda^\bullet(\mathcal{A}), b)$, the cohomology of which is denoted by $H_\lambda^\bullet(\mathcal{A})$ and called the *cyclic cohomology* of $\mathcal{A}$.

The operator $B : C^m(\mathcal{A}) \to C^{m-1}(\mathcal{A})$ is given by

$$(6.4) \qquad B = AB_0(1-T), \quad \text{where } A = 1 + T + \dots + T^m,$$

and the operator $B_0 : C^m(\mathcal{A}) \to C^{m-1}(\mathcal{A})$ is defined by

$$(6.5) \qquad B_0\varphi(a^0, \dots, a^{m-1}) = \varphi(1, a^0, \dots, a^{m-1}), \quad a^j \in \mathcal{A}.$$

Note that B is annihilated by cyclic cochains. Moreover, it can be checked that

$$B^2 = 0 \qquad \text{and} \qquad bB + bB = 0.$$

This enables us to define a further cochain complex $(C_\sharp^\bullet(\mathcal{A}), b+B)$, where

$$C_\sharp^i(\mathcal{A}) = \bigoplus_{k=0}^\infty C^{2k+i}(\mathcal{A}), \quad i=0,1,$$

and we regard b and B as operators between $C_\sharp^0(\mathcal{A})$ and $C_\sharp^1(\mathcal{A})$. The corresponding cohomology is called the *periodic cyclic cohomology* of $\mathcal{A}$ and is denoted by $\mathrm{HP}^\bullet(\mathcal{A})$. Note that a periodic cyclic cocycle is a *finitely supported* sequence $\varphi = (\varphi_{2k+i})$ with $\varphi_{2k+i} \in C^{2k+i}(\mathcal{A})$, $k \geq 0$, such that

$$b\varphi_{2k+i} + B\varphi_{2k+2+i} = 0 \quad \text{for all } k \geq 0.$$

As the operator B is annihilated by cyclic cochains, any cyclic m-cocycle φ is naturally identified with the periodic cyclic cocycle $(0,\ldots,0,\varphi,0,\ldots) \in C_\sharp^i(\mathcal{A})$, where i is the parity of m. This gives rise to natural morphisms

$$H_\lambda^{2k+\bullet}(\mathcal{A}) \to \mathrm{HP}^\bullet(\mathcal{A}), \quad k \geq 0.$$

Connes' periodicity operator $S : C_\lambda^m(\mathcal{A}) \to C_\lambda^{m+2}(\mathcal{A})$ is obtained from the cup product with the unique cyclic 2-cocycle on $\mathbb{C}$ taking the value 1 at $(1,1,1)$ (see [Co1], [Co4]). Equivalently,

$$S = \frac{1}{(m+1)(m+2)} \sum_{j=1}^{m+1} (-1)^j S_j,$$

where the operator $S_j : C_\lambda^m(\mathcal{A}) \to C_\lambda^{m+2}(\mathcal{A})$ is given by

$$
\begin{aligned}
S_j\varphi(a^0,\ldots,a^{m+2}) = &\sum_{0 \leq l \leq j-2} (-1)^l \varphi(a^0,\ldots,a^l a^{l+1},\ldots,a^j a^{j+1},\ldots,a^{m+2}) \\
(6.6) \qquad\qquad &+ (-1)^{j+1} \varphi(a^0,\ldots,a^{j-1}a^j a^{j+1},\ldots,a^{m+2}).
\end{aligned}
$$

Here the operator S is normalized so that, for any cochain $\varphi \in C^{m+1}(\mathcal{A})$, we have

(6.7) $\quad b\varphi$ is cyclic $\Longrightarrow B\varphi$ is a cyclic cocycle and $SB\varphi = -b\varphi$ in $H_\lambda^{m+2}(\mathcal{A})$.

Incidentally, if φ is a cyclic cocycle, then $S\varphi$ is a cyclic cocycle whose class in $\mathrm{HP}^\bullet(\mathcal{A})$ agrees with that defined by φ (cf. [Co1], [Co4]). Furthermore, Connes [Co1, Theorem II.40] proved that

(6.8) $$\varinjlim \big(H_\lambda^{2k+\bullet}(\mathcal{A}), S\big) = \mathrm{HP}^\bullet(\mathcal{A}),$$

where the left-hand side is the direct limit of the directed system defined by the maps $S : H_\lambda^{2k+\bullet}(\mathcal{A}) \to H_\lambda^{2k+2+\bullet}(\mathcal{A})$, $k \geq 0$.

It is sometimes convenient to normalize the cyclic mixed complex. More precisely, we say that a cochain $\varphi \in C^m(\mathcal{A})$ is *normalized* when

(6.9) $\qquad \varphi(a^0,\ldots,a^m) = 0 \quad \text{whenever } a^j = 1 \text{ for some } j \geq 1.$

We denote by $\overline{C}^m(\mathcal{A})$ the space of normalized m-cochains. The normalization condition (6.9) is preserved by the operators b and B. Note also that $B = AB_0$

on normalized chains. We thus obtain a subcomplex $(\overline{C}_\sharp^\bullet(\mathcal{A}), b + B)$ of the periodic cyclic complex, where $\overline{C}_\sharp^\bullet(\mathcal{A}) = \bigoplus_{k \geq 0} \overline{C}^{2k+\bullet}(\mathcal{A})$. We denote by $\overline{\mathrm{HP}}^\bullet(\mathcal{A})$ the cohomology of the normalized periodic complex. Furthermore (see [Lo] and Remark 6.1), the inclusion of $\overline{C}_0^\bullet(\mathcal{A})$ in $C^\bullet(\mathcal{A})$ gives rise to an isomorphism,

$$(6.10) \qquad \overline{\mathrm{HP}}^\bullet(\mathcal{A}) \simeq \mathrm{HP}^\bullet(\mathcal{A}).$$

REMARK 6.1

For $m \in \mathbb{N}_0$ let $\overline{C}_\lambda^m(\mathcal{A})$ be the space of normalized cyclic m-chains. We get a subcomplex $(\overline{C}_\lambda^\bullet(\mathcal{A}), b)$ of the cyclic complex. We denote by $\overline{H}_\lambda^\bullet(\mathcal{A})$ the cohomology of this complex. We observe that if $\varphi \in \overline{C}_\lambda^m(\mathcal{A})$, then the cyclicity and normalization condition (6.9) imply that

$$(6.11) \qquad \varphi(1, a^1, \ldots, a^m) = (-1)^m \varphi(a^1, \ldots, a^m, 1) = 0 \quad \forall a^j \in \mathcal{A}.$$

In fact (cf. [Lo, Section 2.2.13]), we have

$$(6.12) \qquad H_\lambda^\bullet(\mathcal{A}) \simeq H_\lambda^\bullet(\mathbb{C}) \oplus \overline{H}_\lambda^\bullet(\mathcal{A}).$$

The cyclic cohomology of $\mathbb{C}$ has dimension 1 in even degree and is zero in odd degree. Thus, given any 0-cochain $\varphi_0 \in C_\lambda^0(\mathcal{A})$ such that $\varphi_0(1) = 1$, the isomorphism (6.12) can be rewritten as

$$(6.13) \qquad H_\lambda^{2k}(\mathcal{A}) \simeq \mathbb{C}[S^k \varphi_0] \oplus \overline{H}_\lambda^{2k}(\mathcal{A}) \qquad \text{and} \qquad H_\lambda^{2k+1}(\mathcal{A}) \simeq \overline{H}_\lambda^{2k+1}(\mathcal{A}).$$

Observe that $S^{2k}\varphi_0$ is cohomologous in $\mathrm{HP}^0(\mathcal{A})$ to φ_0, which is a normalized cocycle. Therefore, combining (6.8) and (6.11) gives the isomorphism (6.10).

EXAMPLE 6.2

Let $\mathcal{A} = C^\infty(M)$, where M is a closed manifold. For $m = 0, 1, \ldots, n$, let $\Omega_m(M)$ be the space of m-dimensional currents. Any current $C \in \Omega_m(M)$ defines a cochain $\varphi_C \in C^m(\mathcal{A})$ by

$$(6.14) \qquad \varphi_C(f^0, \ldots, f^m) = \frac{1}{m!} \langle C, f^0 df^1 \wedge \cdots \wedge df^m \rangle, \quad f_j \in C^\infty(M).$$

Note that φ_C is a normalized cochain. Moreover, it can be checked that $b\varphi_C = 0$ and $B\varphi_C = \varphi_{d^t C}$, where d^t is the de Rham boundary for currents. Therefore, we obtain a morphism from the mixed complex $(\Omega_\bullet(M), 0, d^t)$ to the cyclic mixed complex of $\mathcal{A} = C^\infty(M)$. In particular, we have a natural linear map

$$\alpha^M : H_{\mathrm{ev/odd}}(M, \mathbb{C}) \longrightarrow \mathrm{HP}^\bullet\big(C^\infty(M)\big),$$

where $H_{\mathrm{ev/odd}}(M, \mathbb{C})$ is the even/odd de Rham homology of M.

6.2. Pairing with K-theory

There are two equivalent ways to define the pairing between $\mathrm{HP}^0(\mathcal{A})$ and $K_0(\mathcal{A})$ (see [Co1], [GS]). Given any even *normalized* cochain $\varphi = (\varphi_{2k})$ and an idempotent $e \in M_q(\mathcal{A})$, define

$$(6.15) \qquad \langle \varphi, e \rangle := \mathrm{tr}\,\#\varphi_0(e) + \sum_{k \geq 1}(-1)^k \frac{(2k)!}{k!}\,\mathrm{tr}\,\#\varphi_{2k}\Big(e - \frac{1}{2}, e, \ldots, e\Big),$$

where $\mathrm{tr}\,\#\varphi_{2k}$ is the $2k$-cochain on $M_q(\mathcal{A}) = M_q(\mathbb{C}) \otimes \mathcal{A}$ such that, for all $\mu^0, \ldots, \mu^{2k}$ in $M_q(\mathbb{C})$ and $a^0, \ldots, a^{2k}$ in $\mathcal{A}$, we have

$$\mathrm{tr}\,\#\varphi_{2k}(\mu^0 \otimes a^0, \ldots, \mu^{2k} \otimes a^{2k}) = \mathrm{tr}[\mu^0 \cdots \mu^k]\varphi_{2k}(a^0, \ldots, a^{2k}).$$

It can be checked that, when φ is an even normalized periodic cocycle, the value of $\langle \varphi, e \rangle$ depends only on the class of φ in $\overline{\mathrm{HP}}^0(\mathcal{A})$ and the class of e in $K_0(\mathcal{A})$. Combining this with (6.10) we then obtain a bilinear pairing

$$(6.16) \qquad\qquad\qquad \langle \cdot, \cdot \rangle : \mathrm{HP}^0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}.$$

In addition, given any cyclic $2k$-cocycle φ it can be shown (see Remark 6.4 below) that

$$(6.17) \qquad \langle \varphi, e \rangle = (-1)^k \frac{(2k)!}{k!}\,\mathrm{tr}\,\#\varphi(e, \ldots, e) \quad \forall e \in M_q(\mathcal{A}), e^2 = e.$$

In fact, the right-hand side of (6.17) depends only on the class of φ in $H_\lambda^{2q}(\mathcal{A})$ and is invariant under the periodicity operators S (see [Co1], [Co4]). Therefore, under the inductive limit (6.8) this gives rise to a pairing between $\mathrm{HP}^0(\mathcal{A})$ and $K_0(\mathcal{A})$ which agrees with the pairing (6.15).

EXAMPLE 6.3

Suppose now that $\mathcal{A} = C^\infty(M)$, where M is a closed manifold, and let $e \in M_q(C^\infty(M))$, where $e^2 = e$. Consider the vector bundle $E = \mathrm{ran}\,e$, which we regard as a subbundle of the trivial vector bundle $E_0 = M \times \mathbb{C}^q$. We note that by the Serre–Swan theorem any vector bundle over M is isomorphic to a vector bundle of this form. We equip E with the Grassmannian connection ∇^E defined by e, so that

$$\nabla_X^E \xi = e(X\xi_j) \quad \text{for all } X \in C^\infty(M, TM) \text{ and } \xi = (\xi_j) \in \mathcal{E}^q = C^\infty(M, E)^q.$$

The curvature of ∇^E is $F^E = e(de)^2 = e(de)^2 e$, and so its Chern form is given by

$$\mathrm{Ch}(F^E) = \sum (-1)^k \frac{1}{k!}\,\mathrm{tr}\big[e(de)^{2k}\big] \in \Omega^{\mathrm{ev}}(M).$$

Let $C = (C_{2k})$ be a closed even de Rham current, and denote by φ_C the associated cocycle defined by (6.14). Noting that all the cochains $\varphi_{C_{2k}}$ satisfy (6.11), we obtain

$$(6.18) \qquad \begin{aligned} \big\langle C, \mathrm{Ch}(F^E) \big\rangle &= \sum (-1)^k \frac{1}{k!}\big\langle C_{2k}, \mathrm{tr}\big[e(de)^{2k}\big]\big\rangle \\ &= \sum (-1)^k \frac{(2k)!}{k!}\,\mathrm{tr}\,\#\varphi_{C_{2k}}(e, \ldots, e) = \langle \varphi_C, e \rangle. \end{aligned}$$

Therefore, the pairing (6.17) between even periodic cyclic cohomology and K-theory reduces to the classical pairing (3.4) between de Rham homology and K-theory.

REMARK 6.4

The equality (6.17) is proved as follows. Thanks to (6.13) we know that

(6.19) $$\varphi = S^k \varphi_0 + \psi \quad \mathrm{mod}\, b\big(C_\lambda^{2k-1}(\mathcal{A})\big),$$

where $\varphi_0 \in C_\lambda^0(\mathcal{A})$ is such that $\varphi_0(1) \neq 0$ and ψ is a *normalized* cyclic $2k$-cocycle. As φ_0 and $S^k \varphi_0$ are cohomologous in $\mathrm{HP}^0(\mathcal{A})$, we see that φ is cohomologous in $\mathrm{HP}^0(\mathcal{A})$ to the *normalized* even cocycle $\varphi_0 + \psi$. Thus,

(6.20) $$\langle \varphi, e \rangle = \langle \varphi_0, e \rangle + \langle \psi, e \rangle.$$

Set $c_k = (-1)^k (k!)^{-1} (2k)!$. Using (6.6) we can check that

(6.21) $$\langle \varphi_0, e \rangle = \mathrm{tr}\, \# \varphi_0(e) = c_k \, \mathrm{tr}\, \#(S^k \varphi_0)(e, \ldots, e).$$

Moreover, as the cocycle ψ is normalized and cyclic, it follows from (6.15) and (6.11) that

$$\langle \psi, e \rangle = c_k \, \mathrm{tr}\, \# \psi\Big(e - \frac{1}{2}, e, \ldots, e\Big) = c_k \, \mathrm{tr}\, \# \psi(e, e, \ldots, e).$$

Combining this with (6.19)–(6.21) then gives

$$\langle \varphi, e \rangle = c_k \, \mathrm{tr}\, \#(S^k \varphi_0 + \psi)(e, \ldots, e) = c_k \, \mathrm{tr}\, \# \varphi(e, \ldots, e),$$

which proves (6.17).

7. Connes–Chern character and index formula

In this section, we give a direct construction of the Connes–Chern character of a twisted spectral triple. Combining this with the results of Section 5 we shall obtain a reformulation of the Atiyah–Singer index formula (3.5) for twisted spectral triples.

Throughout this section we let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a twisted spectral triple. For $p \geq 1$ we denote by $\mathcal{L}^p(\mathcal{H})$ the Schatten ideal of operators $T \in \mathcal{L}(\mathcal{H})$ such that $\mathrm{Tr}\, |T|^p < \infty$. We recall that $\mathcal{L}^p(\mathcal{H})$ is a Banach ideal with respect to the p-norm

$$\|T\|_p = \big(\mathrm{Tr}\, |T|^p\big)^{\frac{1}{p}}, \quad T \in \mathcal{L}^p(\mathcal{H}).$$

In what follows we assume that the twisted spectral triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is *p-summable*, that is,

$$D^{-1} \in \mathcal{L}^p(\mathcal{H}).$$

7.1. Invertible case

We start by assuming that D is invertible. We shall explain later how to remove this assumption. We recall the following result.

LEMMA 7.1 ([Hö, LEMMA 7.1]; SEE ALSO [Co1, P. 304])

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a Fredholm operator. Let $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be such that $1 - ST \in \mathcal{L}^p(\mathcal{H}_1)$ and $1 - TS \in \mathcal{L}^p(\mathcal{H}_2)$. Then

$$\operatorname{ind} T = \operatorname{Tr}\big((1 - ST)^q\big) - \operatorname{Tr}\big((1 - TS)^q\big) \quad \forall q \geq p.$$

The main impetus for our construction of the Connes–Chern character is the following index formula.

PROPOSITION 7.2

Let $e \in M_q(\mathcal{A})$, $q \geq 1$, be an idempotent. Then, for any integer $k \geq \frac{1}{2}p$,

$$(7.1) \qquad \operatorname{ind} D_{e,\sigma} = \frac{1}{2} \operatorname{Str}\big((D^{-1}[D, e]_\sigma)^{2k+1}\big).$$

Proof

We defined $D_{e,\sigma}$ as an unbounded operator from $e\mathcal{H}^q$ to $\sigma(e)\mathcal{H}^q$. Alternatively, let $\mathcal{H}_1$ be the Hilbert space given by the vector space $\operatorname{dom}(D)$ equipped with the Hermitian inner product

$$\langle \xi, \eta \rangle_1 := \langle \xi, \eta \rangle + \langle D\xi, D\eta \rangle, \quad \xi, \eta \in \operatorname{dom}(D).$$

We denote by $\| \cdot \|_1$ the norm of $\mathcal{H}_1$. This norm is complete since D is a closed operator. We then can regard D as an invertible bounded operator from $\mathcal{H}_1$ to $\mathcal{H}$. Let $a \in \mathcal{A}$, and let $\xi \in \operatorname{dom}(D)$. Upon writing $Da\xi = aD\xi + [D, a]\xi$ we see that

$$\|a\xi\|_1^2 = \|a\xi\|^2 + \|Da\xi\|^2$$
$$\leq \|a\xi\|^2 + 2\big(\|aD\xi\|^2 + \big\|[D, a]\xi\big\|^2\big)$$
$$\leq 2\big(\|a\|^2 + \big\|[D, a]\big\|^2\big)\|\xi\|_1^2.$$

Thus, a induces a bounded operator on $\mathcal{H}_1$. It then follows that e induces a bounded operator on $\mathcal{H}_1^q$, so that $e\mathcal{H}_1^q$ is a closed subspace of $\mathcal{H}_1^q$. We regard $D_{e,\sigma}$ as a bounded operator from $e\mathcal{H}_1^q$ to $\sigma(e)\mathcal{H}^q$. Set $Q = eD^{-1}\sigma(e) \in \mathcal{L}(\sigma(e)\mathcal{H}^q, e\mathcal{H}_1^q)$. Then the product $D_{e,\sigma}Q$ is equal to

$$(7.2) \qquad \begin{aligned} \sigma(e)DeD^{-1}\sigma(e) &= \sigma(e)\big(\sigma(e)D + [D, e]_\sigma\big)D^{-1}\sigma(e) \\ &= 1 + \sigma(e)[D, e]_\sigma D^{-1}\sigma(e), \end{aligned}$$

where we have used the fact that $e = 1$ on $e\mathcal{H}^q$. Likewise, the operator $QD_{e,\sigma}$ is equal to

$$(7.3) \qquad eD^{-1}\sigma(e)De = eD^{-1}\big(De - [D, e]_\sigma\big)e = 1 - eD^{-1}[D, e]_\sigma e.$$

We observe that $D^{-1}[D, e]_\sigma D^{-1}[D, e]_\sigma e$ is equal to

$$(7.4) \qquad \begin{aligned} \big(e - D^{-1}\sigma(e)D\big)\big(e - D^{-1}\sigma(e)D\big)e &= e - eD^{-1}\sigma(e)De \\ &= eD^{-1}[D, e]_\sigma e. \end{aligned}$$

Similarly, the operator $\sigma(e)[D,e]_\sigma D^{-1}[D,e]_\sigma D^{-1}$ is equal to

$$
\begin{aligned}
\sigma(e)\big(DeD^{-1} - \sigma(e)\big)\big(DeD^{-1} - \sigma(e)\big) &= -\sigma(e)DeD^{-1}\sigma(e) + \sigma(e) \\
(7.5) \qquad\qquad &= -\sigma(e)[D,e]_\sigma D^{-1}\sigma(e).
\end{aligned}
$$

As $D^{-1} \in \mathcal{L}^p(\mathcal{H})$ and $[D,e]_\sigma$ is bounded, using (7.4) we see that $eD^{-1}[D,e]_\sigma e$ is in the Schatten class $\mathcal{L}^{\frac{p}{2}}(e\mathcal{H}^q)$. Note also that $D^{-1}[D,e]_\sigma = D^{-1}([D,e]_\sigma D^{-1})D$. As D is an isomorphism from $\mathcal{H}_1$ onto $\mathcal{H}$, we then see that $[D,e]_\sigma D^{-1}$ induces on $\mathcal{H}^q$ an operator in $\mathcal{L}^p(\mathcal{H}^q)$. Combing this with (7.5) then shows that $\sigma(e)[D,e]_\sigma D^{-1}\sigma(e)$ induces on $\sigma(e)\mathcal{H}^q$ an operator in $\mathcal{L}^{\frac{p}{2}}(\sigma(e)\mathcal{H}^q)$.

The $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ induces a $\mathbb{Z}_2$-grading $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$ with $\mathcal{H}_1^\pm = \mathcal{H}_1 \cap \mathcal{H}^\pm$. This gives rise to splittings $e\mathcal{H}_1^q = e(\mathcal{H}_1^+)^q \oplus e(\mathcal{H}_1^-)^q$ and $\sigma(e)\mathcal{H}^q = \sigma(e)(\mathcal{H}^+)^q \oplus \sigma(e)(\mathcal{H}^-)^q$. With respect to these splittings, the operator $Q = eD\sigma(e)$ takes the form,

$$
Q = \begin{pmatrix} 0 & Q^- \\ Q^+ & 0 \end{pmatrix} \quad \text{where } Q^\pm := e(D^\mp)^{-1}\sigma(e).
$$

Therefore, (7.2) and (7.3) can be rewritten as

$$
D_{e,\sigma}^\pm Q^\mp = 1 + \sigma(e)[D^\pm,e]_\sigma(D^\pm)^{-1}\sigma(e), \qquad Q^\mp D_{e,\sigma}^\pm = 1 - e[D^\mp,e]_\sigma(D^\mp)^{-1}e,
$$

where the first equality holds in $\mathcal{L}(\sigma(e)\mathcal{H}^\pm)$ and the second holds in $\mathcal{L}(e\mathcal{H}_1^\pm)$. As shown above, the operators $\sigma(e)[D^\pm,e]_\sigma(D^\pm)^{-1}\sigma(e)$ and $e[D^\mp,e]_\sigma(D^\mp)^{-1}e$ are in the Schatten classes $\mathcal{L}^{\frac{p}{2}}(\sigma(e)(\mathcal{H}^\pm)^q)$ and $\mathcal{L}^{\frac{p}{2}}(e(\mathcal{H}_1^\pm)^q)$, respectively. Therefore, we may apply Lemma 7.1 to obtain that, for all $k \geq \frac{1}{2}p$,

$$
\operatorname{ind} D_{e,\sigma}^\pm = \operatorname{Tr}\big(e(D^\mp)^{-1}[D^\mp,e]_\sigma e\big)^k - \operatorname{Tr}\big(-\sigma(e)[D^\pm,e]_\sigma(D^\pm)^{-1}\sigma(e)\big)^k.
$$

Thus,

$$
2\operatorname{ind} D_{e,\sigma} = \operatorname{Str}\big(eD^{-1}[D,e]_\sigma e\big)^k - \operatorname{Str}\big(-\sigma(e)[D,e]_\sigma D^{-1}\sigma(e)\big)^k.
$$

Combining this with (7.4) and (7.5) we then get

$$
2\operatorname{ind} D_{e,\sigma} = \operatorname{Str}\big((D^{-1}[D,e]_\sigma)^{2k}e\big) + \operatorname{Str}\big(\sigma(e)([D,e]_\sigma D^{-1})^{2k}\big).
$$

We observe that $\operatorname{Str}(\sigma(e)([D,e]_\sigma D^{-1})^{2k})$ is equal to

$$
\operatorname{Str}\big(\sigma(e)D\big(D^{-1}[D,e]_\sigma\big)^{2k}D^{-1}\big) = -\operatorname{Str}\big(D^{-1}\sigma(e)D\big(D^{-1}[D,e]_\sigma\big)^{2k}\big).
$$

Therefore, we obtain

$$
\operatorname{ind} D_{e,\sigma} = \frac{1}{2}\operatorname{Str}\big((e - D^{-1}\sigma(e)D)\big(D^{-1}[D,e]_\sigma\big)^{2k}\big) = \frac{1}{2}\operatorname{Str}\big((D^{-1}[D,e]_\sigma)^{2k+1}\big).
$$

The lemma is thus proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

DEFINITION 7.3

For $k \geq \frac{1}{2}(p-1)$ let τ_{2k}^D be the $2k$-cochain on $\mathcal{A}$ defined by

$$
(7.6) \qquad \tau_{2k}^D(a^0, \ldots, a^{2k}) = c_k \operatorname{Str}\big(D^{-1}[D,a^0]_\sigma \cdots D^{-1}[D,a^{2k}]_\sigma\big) \quad \forall a^j \in \mathcal{A},
$$

where we have set $c_k = \frac{1}{2}(-1)^k \frac{k!}{(2k)!}$.

We note that τ_{2k}^D is a normalized cyclic cochain. Moreover, using (6.17), for $k \geq \frac{1}{2}p$, we can rewrite the index formula (7.1) in the form

$$(7.7) \qquad\qquad \operatorname{ind} D_{e,\sigma} = \langle \tau_{2k}^D, e \rangle \quad \forall e \in M_q(\mathcal{A}), e^2 = e.$$

For $m \geq p$, we let φ_m and ψ_m be the normalized m-cochains on $\mathcal{A}$ defined by

$$(7.8) \quad \varphi_m(a^0, \ldots, a^m) = \operatorname{Str}\big(a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^m]_\sigma\big),$$

$$(7.9) \quad \psi_m(a^0, \ldots, a^m) = \operatorname{Str}\big(\sigma(a^0)[D, a^1]_\sigma D^{-1} \cdots [D, a^m]_\sigma D^{-1}\big), \quad a^j \in \mathcal{A}.$$

We observe that $\psi_m(a^0, \ldots, a^m)$ is equal to

$$(7.10) \qquad \begin{aligned} &-\operatorname{Str}\big(D^{-1}\sigma(a^0)DD^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^m]_\sigma\big) \\ &= -\varphi_m\big(D^{-1}\sigma(a^0)D, a^1, \ldots, a^m\big). \end{aligned}$$

Using the equality $D^{-1}[D, a^0]_\sigma = a^0 - D^{-1}\sigma(a^0)D$, we then see that, for $k \geq \frac{1}{2}p$,

$$(7.11) \qquad \begin{aligned} c_k^{-1}\tau_{2k}^D(a^0, \ldots, a^{2k}) &= \operatorname{Str}\big\{a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^{2k}]_\sigma\big\} \\ &\quad - \operatorname{Str}\big\{D^{-1}\sigma(a^0)DD^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^{2k}]_\sigma\big\} \\ &= \varphi_{2k}(a^0, \ldots, a^{2k}) + \psi_{2k}(a^0, \ldots, a^{2k}). \end{aligned}$$

LEMMA 7.4

Let $k \geq \frac{1}{2}(p-1)$. Then

$$B\varphi_{2k+1} = -B\psi_{2k+1} = (2k+1)c_k^{-1}\tau_{2k}^D.$$

Proof
As φ_{2k+1} and ψ_{2k+1} are normalized cochains, we know that $B\varphi_{2k+1} = AB_0\varphi_{2k+1}$ and $B\psi_{2k+1} = AB_0\psi_{2k+1}$. Moreover, it follows from (7.8) and (7.9) that

$$B_0\varphi_{2k+1} = -B_0\psi_{2k+1} = c_k^{-1}\tau_{2k}^D.$$

As τ_{2k}^D is a cyclic cochain, we then deduce that

$$B\varphi_{2k+1} = -B\psi_{2k+1} = c_k^{-1}A\tau_{2k}^D = (2k+1)c_k^{-1}\tau_{2k}^D.$$

This proves the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

LEMMA 7.5

Let $k \geq \frac{1}{2}(p+1)$. Then

$$(7.12) \qquad b\varphi_{2k-1} = \varphi_{2k} \qquad and \qquad b\psi_{2k-1} = -\psi_{2k}.$$

Proof
For $j = 1, \ldots, 2k$ let θ'_j and θ''_j be the $2k$-cochains on $\mathcal{A}$ defined by

$$\theta'_j(a^0, \ldots, a^{2k}) = \operatorname{Str}\big(a^0 D^{-1}[D, a^1]_\sigma \cdots a^j \cdots D^{-1}[D, a^{2k}]_\sigma\big),$$

$$\theta''_j(a^0, \ldots, a^{2k}) = \operatorname{Str}\big(a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}\sigma(a^j)D^{-1} \cdots D^{-1}[D, a^{2k}]_\sigma\big), \quad a^l \in \mathcal{A}.$$

We note that

$$\theta'_j(a^0,\ldots,a^{2k}) - \theta''_j(a^0,\ldots,a^{2k})$$

$$= \mathrm{Str}\big(a^0 D^{-1}[D,a^1]_\sigma \cdots \big(a^j - D^{-1}\sigma(a^j)D\big) \cdots D^{-1}[D,a^{2k}]_\sigma\big)$$

$$= \varphi_{2k}(a^0,\ldots,a^{2k}).$$

Using the equality $D^{-1}[D,a^j a^{j+1}]_\sigma = D^{-1}[D,a^j]_\sigma a^{j+1} + D^{-1}\sigma(a^j)D \cdot D^{-1}[D,a^{j+1}]_\sigma$ we also find that

$$b_j \varphi_{2k-1}(a^0,\ldots,a^{2k}) = \mathrm{Str}\big(a^0 D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^j a^{j+1}]_\sigma \cdots D^{-1}[D,a^{2k}]_\sigma\big)$$

$$= \theta'_{j+1}(a^0,\ldots,a^{2k}) + \theta''_j(a^0,\ldots,a^{2k}).$$

Thus, $\sum_{j=1}^{2k-1}(-1)^j b_j \varphi_{2k-1}$ is equal to

$$\sum_{j=1}^{2k-1}(-1)^j\big(\theta'_{j+1} + \theta''_j\big) = -\theta''_1 + \sum_{j=2}^{2k-1}(-1)^{j-1}\big(\theta'_{j-1} - \theta''_j\big) - \theta'_{2k}$$

(7.13)
$$= -\theta''_1 + \sum_{j=2}^{2k-1}(-1)^{j-1}\varphi_{2k} - \theta'_{2k}$$

$$= -\theta''_1 - \theta'_{2k}.$$

We also note that

$$b_0 \varphi_{2k-1}(a^0,\ldots,a^{2k}) = \mathrm{Str}\big(a^0 a^1 D^{-1}[D,a^2]_\sigma \cdots D^{-1}[D,a^{2k}]_\sigma\big) = \theta'_1(a^0,\ldots,a^{2k}).$$

Moreover, the cochain $b_{2k}\varphi_{2k-1}(a^0,\ldots,a^{2k})$ is equal to

$$\mathrm{Str}\big(a^{2k}a^1 D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma\big)$$

$$= \mathrm{Str}\big(a^0 D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma a^{2k}\big)$$

$$= \theta'_{2k}(a^0,\ldots,a^{2k}).$$

Therefore, we find that

$$b\varphi_{2k-1} = \sum_{j=0}^{2k}(-1)^j b_j \varphi_{2k-1} = b_0\varphi_{2k-1} - \theta''_1 - \theta'_{2k} + b_{2k}\varphi_{2k-1} = \theta'_1 - \theta''_1 = \varphi_{2k}.$$

As $\psi_{2k-1}(a^0,\ldots,a^{2k-1}) = -\varphi_{2k-1}(D^{-1}\sigma(a^0)D,a^1,\ldots,a^{2k-1})$, using (7.13) we get

$$\sum_{j=1}^{2k-1}(-1)^j b_j \psi_{2k-1}$$

$$= \theta''_1\big(D^{-1}\sigma(a^0)D,a^1,\ldots,a^{2k}\big) + \theta'_{2k}\big(D^{-1}\sigma(a^0)D,a^1,\ldots,a^{2k}\big).$$

We observe that $b_0\psi_{2k-1}(a^0,\ldots,a^{2k})$ is equal to

$$-\mathrm{Str}\big(D^{-1}a^0 a^1 D^{-1}[D,a^2]_\sigma \cdots D^{-1}[D,a^{2k}]_\sigma\big) = -\theta''_1\big(D^{-1}\sigma(a^0)D,a^1,\ldots,a^{2k-1}\big).$$

Moreover, we have

$$b_{2k}\psi_{2k-1}(a^0,\ldots,a^{2k})$$
$$= -\mathrm{Str}\big(D^{-1}\sigma(a^{2k})\sigma(a^0)D \cdot D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma\big)$$
$$= -\mathrm{Str}\big(D^{-1}a^0 D \cdot D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma D^{-1}\sigma(a^{2k})D\big)$$
$$= -\theta''_{2k}\big(D^{-1}\sigma(a^0)D, a^1,\ldots,a^{2k}\big).$$

Therefore, we see that $b\psi_{2k-1}(a^0,\ldots,a^{2k}) = \sum_{j=0}^{2k}(-1)^j b_j\psi_{2k-1}(a^0,\ldots,a^{2k})$ is equal to

$$\theta'_{2k}\big(D^{-1}\sigma(a^0)D, a^1,\ldots,a^{2k}\big) - \theta''_{2k}\big(D^{-1}\sigma(a^0)D, a^1,\ldots,a^{2k}\big)$$
$$= \varphi_{2k}\big(D^{-1}\sigma(a^0)D, a^1,\ldots,a^{2k}\big)$$
$$= -\psi_{2k}(a^0, a^1,\ldots,a^{2k}).$$

The proof is complete. $\qquad\square$

PROPOSITION 7.6 ([CM4])

Let $k \geq \frac{1}{2}(p-1)$.

(1) *The cochain τ_{2k}^D in (7.6) is a normalized cyclic cocycle.*
(2) *The class of τ_{2k}^D in $\mathrm{HP}^0(\mathcal{A})$ is independent of the value of k.*

Proof
We already know that τ_{2k}^D is a cyclic normalized cochain. We also note that

$$c_{k+1} = \frac{1}{2}(-1)^{k+1}\frac{(k+1)!}{(2k+2)!} = -\frac{c_k}{2(2k+1)}.$$

Combining this with Lemma 7.4 we get

$$(7.14) \qquad \tau_{2k}^D = \frac{c_k}{2(2k+1)}B(\varphi_{2k+1} - \psi_{2k+1}) = -c_{k+1}B(\varphi_{2k+1} - \psi_{2k+1}).$$

Using (7.14) and the fact that $bB = -Bb$ we then see that

$$b\tau_{2k}^D = -c_{k+1}bB(\varphi_{2k+1} - \psi_{2k+1}) = c_{k+1}Bb(\varphi_{2k+1} - \psi_{2k+1}).$$

Moreover, using (7.11) and Lemma 7.5 we get

$$(7.15) \qquad \tau_{2k+2}^D = c_{k+1}(\varphi_{2k+2} + \psi_{2k+2}) = c_{k+1}b(\varphi_{2k+1} - \psi_{2k+1}).$$

As B is annihilated by cyclic cochains we then deduce that

$$b\tau_{2k}^D = B\big(c_{k+1}b(\varphi_{2k+1} - \psi_{2k+1})\big) = B\tau_{2k+2}^D = 0.$$

That is, τ_{2k}^D is a cocycle. We also see that

$$\tau_{2k+2}^D - \tau_{2k}^D = c_{k+1}b(\varphi_{2k+1} - \psi_{2k+1}) + c_{k+1}B(\varphi_{2k+1} - \psi_{2k+1}).$$

This shows that τ_{2k+2}^D and τ_{2k}^D define the same class in $\mathrm{HP}^0(\mathcal{A})$. It then follows that the class of τ_{2k}^D in $\mathrm{HP}^0(\mathcal{A})$ is independent of the value of k. The proof is complete. $\qquad\square$

REMARK 7.7

The proof of Lemma 7.4 uses the fact that the unit of $\mathcal{A}$ is represented by the identity of $\mathcal{H}$. Otherwise, the equalities $B_0\varphi_{2k+1} = -B_0\psi_{2k+1} = c_k^{-1}\tau_{2k}^D$ need not hold. Therefore, the unitality of $\mathcal{A}$ is a crucial ingredient of the proof of Proposition 7.6.

REMARK 7.8

As in [Co1], we can get a more precise relationship between the cocycles τ_{2k}^D and τ_{2k+2}^D by using the S-operator. Indeed, using (6.7) and (7.12) we get

$$SB(\varphi_{2k+1} - \psi_{2k+1}) = -b(\varphi_{2k+1} - \psi_{2k+1}) = -(\varphi_{2k+2} + \psi_{2k+2}) \quad \text{in } H_\lambda^{2k}(\mathcal{A}).$$

Combining this with (7.14) and (7.15) we then deduce that

$$S\tau_{2k}^D = -c_{k+1}SB(\varphi_{2k+1} - \psi_{2k+1}) = c_{k+1}(\varphi_{2k+2} + \psi_{2k+2}) = \tau_{2k+2}^D \quad \text{in } H_\lambda^{2k}(\mathcal{A}).$$

As τ_{2k}^D and $S\tau_{2k}^D$ are cohomologous in $\mathrm{HP}^0(\mathcal{A})$, this provides us with an alternative argument showing that τ_{2k}^D and τ_{2k+2}^D define the same class in $\mathrm{HP}^0(\mathcal{A})$.

DEFINITION 7.9 ([CM4])

Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a p-summable twisted spectral triple with D invertible. Then its Connes–Chern character, denoted by $\mathrm{Ch}(D)_\sigma$, is the class in $\mathrm{HP}^0(\mathcal{A})$ of any of the cocycles τ_{2k}^D, $k \geq \frac{1}{2}(p-1)$.

We are now in a position to reformulate the Atiyah–Singer index formula (3.5) for twisted spectral triples in the invertible case.

THEOREM 7.10

Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a p-summable twisted spectral triple with D invertible. Then, for any finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ and any σ-connection on $\mathcal{E}$, we have

$$\mathrm{ind}\, D_{\nabla^\mathcal{E}} = \langle \mathrm{Ch}(D)_\sigma, [\mathcal{E}] \rangle.$$

Proof

Thanks to Theorem 5.22 we know that $\mathrm{ind}\, D_{\nabla^\mathcal{E}} = \mathrm{ind}_{D,\sigma}[\mathcal{E}]$. Let e be an idempotent in some matrix algebra $M_q(\mathcal{A})$ such that $\mathcal{E} \simeq e\mathcal{A}^q$. Then (7.7) shows that, for $k \geq \frac{1}{2}p$,

$$\mathrm{ind}_{D,\sigma}[e] = \mathrm{ind}\, D_{e,\sigma} = \langle \tau_{2k}^D, e \rangle = \langle \mathrm{Ch}(D)_\sigma, [e] \rangle.$$

As $\mathcal{E}$ and e define the same class in $K_0(\mathcal{A})$ we then deduce that

$$\mathrm{ind}\, D_{\nabla^\mathcal{E}} = \mathrm{ind}_{D,\sigma}[e] = \langle \mathrm{Ch}(D)_\sigma, [\mathcal{E}] \rangle.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7.2. General case

The assumption on the invertibility of D can be removed by passing to the unital invertible double as follows. Consider the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, which we

equip with the $\mathbb{Z}_2$-grading given by

$$\tilde{\gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix},$$

where γ is the grading operator of $\mathcal{H}$. On $\tilde{\mathcal{H}}$ consider the self-adjoint operators,

$$\tilde{D}_0 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{D} = \tilde{D}_0 + J = \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix},$$

where the domain of $\tilde{D}_0$ and $\tilde{D}$ is $\mathrm{dom}(D) \oplus \mathrm{dom}(D)$. As $\tilde{D}_0 J + J\tilde{D}_0 = 0$ and $J^2 = 1$ we get

$$(7.16) \qquad \tilde{D}^2 = \tilde{D}_0^2 + 1 = \begin{pmatrix} D^2 + 1 & 0 \\ 0 & D^2 + 1 \end{pmatrix}.$$

It then follows that $\tilde{D}$ is invertible. Moreover,

$$(7.17) \qquad \mathrm{Tr}\,|\tilde{D}|^{-p} = \mathrm{Tr}(\tilde{D}^2)^{-\frac{p}{2}} = 2\,\mathrm{Tr}(D^2 + 1)^{-\frac{p}{2}} \leq 2\,\mathrm{Tr}\,|D|^{-p} < \infty.$$

That is, $\tilde{D}^{-1} \in \mathcal{L}^p(\tilde{\mathcal{H}})$.

Let $\pi : \mathcal{A} \to \mathcal{L}(\tilde{\mathcal{H}})$ be the linear map given by

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \forall a \in \mathcal{A}.$$

We note that π is multiplicative, but as $\pi(1) \neq 0$ this is not a representation of the unital algebra $\mathcal{A}$. As mentioned in Remark 7.7 the representation of the unit 1 by the identity of $\mathcal{H}$ is essential to the construction of the Connes–Chern character in the invertible case. To remedy the lack of unitality of π we pass to the $*$-algebra $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ with product and involution given by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu), \qquad (a, \lambda)^* = (a^*, \overline{\lambda}), \quad a, b \in \mathcal{A}, \lambda, \mu \in \mathbb{C}.$$

The unit of $\tilde{\mathcal{A}}$ is $1_{\tilde{\mathcal{A}}} = (0, 1)$. Thus, by identifying any element $a \in \mathcal{A}$ with $(a, 0)$, any element $\tilde{a} = (a, \lambda) \in \tilde{\mathcal{A}}$ can be uniquely written as $(a, \lambda) = a + \lambda 1_{\tilde{\mathcal{A}}}$. We extend π into the (unital) representation $\tilde{\pi} : \tilde{\mathcal{A}} \to \mathcal{L}(\tilde{\mathcal{H}})$ given by

$$\tilde{\pi}(a + \lambda 1_{\tilde{\mathcal{A}}}) = \pi(a) + \lambda \quad \forall (a, \lambda) \in \mathcal{A} \times \mathbb{C}.$$

We also extend the automorphism σ into the automorphism $\tilde{\sigma} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ given by

$$\tilde{\sigma}(a + \lambda 1_{\tilde{\mathcal{A}}}) = \sigma(a) + \lambda 1_{\tilde{\mathcal{A}}} \quad \forall (a, \lambda) \in \mathcal{A} \times \mathbb{C}.$$

For $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the twisted commutator $[\tilde{D}, \tilde{\pi}(a + \lambda 1_{\tilde{\mathcal{A}}})]_{\tilde{\sigma}}$ is equal to

$$\begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix} \begin{pmatrix} a + \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} \sigma(a) + \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix}$$

$$= \begin{pmatrix} [D, a]_\sigma & -\sigma(a) \\ a & 0 \end{pmatrix} \in \mathcal{L}(\tilde{\mathcal{H}}).$$

Combining all these we obtain the following statement.

PROPOSITION 7.11

The triple $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})_{\tilde{\sigma}}$ is a p-summable twisted spectral triple.

As $\tilde{D}$ is invertible, we can form the normalized cyclic cocycles $\tau_{2k}^{\tilde{D}}$, $k \geq \frac{1}{2}(p-1)$, as in Definition 7.3. We note that if φ is a cyclic m-cochain on $\tilde{\mathcal{A}}$, then its restriction $\overline{\varphi}$ to $\mathcal{A}^{m+1}$ is a cyclic cochain on $\mathcal{A}$. Moreover, by using (6.1) and (6.6) we see that

$$(7.18) \qquad \overline{b\varphi} = b\overline{\varphi} \qquad \text{and} \qquad \overline{S\varphi} = S\overline{\varphi}.$$

If in addition φ is normalized, then the normalization condition and (6.11) imply that

$$(7.19) \qquad \tilde{\varphi}(a^0 + \lambda^0 1_{\tilde{\mathcal{A}}}, \ldots, a^m + \lambda^m 1_{\tilde{\mathcal{A}}}) = \varphi(a^0, \ldots, a^m) \quad \forall a^j \in \mathcal{A} \ \forall \lambda^j \in \mathbb{C}.$$

Thus, φ is uniquely determined by its restriction $\overline{\varphi}$ to $\mathcal{A}^{m+1}$. Conversely, any cyclic m-cochain φ on $\mathcal{A}$ uniquely extends to a normalized cyclic m-cochain $\tilde{\varphi}$ on $\tilde{\mathcal{A}}$ satisfying (7.19).

DEFINITION 7.12
Let $k \geq \frac{1}{2}(p-1)$. Then $\overline{\tau}_{2k}^D$ is the cyclic $2k$-cochain on $\mathcal{A}$ given by the restriction of $\tau_{2k}^{\tilde{D}}$ to $\mathcal{A}^{2k+1}$.

PROPOSITION 7.13
Let $k \geq \frac{1}{2}(p-1)$.

(1) The $2k$-cochain $\overline{\tau}_{2k}^D$ is a cyclic cocycle whose class in $\mathrm{HP}^0(\mathcal{A})$ is independent of k.

(2) For any idempotent $e \in M_q(\mathcal{A})$, we have

$$\mathrm{ind}\, D_{\sigma,e} = \langle \overline{\tau}_{2k}^D, e \rangle.$$

Proof
It follows from (7.18) that $b\overline{\tau}_{2k}^D = \overline{b\tau_{2k}^{\tilde{D}}} = 0$, so $\overline{\tau}_{2k}^D$ is a cyclic cocycle. By Remark 7.8 the cocycles $S\tau_{2k}^{\tilde{D}}$ and $\tau_{2k+2}^{\tilde{D}}$ are cohomologous in $H_\lambda^{2k+2}(\tilde{\mathcal{A}})$. Therefore, by (7.18) their respective restrictions to $\mathcal{A}$, namely, $S\overline{\tau}_{2k}^D$ and $\overline{\tau}_{2k+2}^D$, are cohomologous in $H_\lambda^{2k+2}(\mathcal{A})$. As $S\overline{\tau}_{2k}^D$ and $\overline{\tau}_{2k}^D$ define the same class in $\mathrm{HP}^0(\mathcal{A})$, we deduce that the classes of $\overline{\tau}_{2k}^D$ and $\overline{\tau}_{2k+2}^D$ in $\mathrm{HP}^0(\mathcal{A})$ agree. It then follows that the class of $\overline{\tau}_{2k}^D$ in $\mathrm{HP}^0(\mathcal{A})$ is independent of k.

Let $e \in M_q(\mathcal{A})$, $e^2 = e$. Regarding e as an idempotent in $M_q(\tilde{\mathcal{A}})$, we have

$$\tilde{\pi}(e)\tilde{\mathcal{H}}^q = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} (\mathcal{H}^q \oplus \mathcal{H}^q) = e(\mathcal{H}^q) \oplus \{0\} \simeq e\mathcal{H}^q.$$

Likewise, $\tilde{\pi}(\sigma(e))\tilde{\mathcal{H}}^q = \sigma(e)(\mathcal{H}^q) \oplus \{0\} \simeq \sigma(e)\mathcal{H}^q$. Moreover,

$$\sigma(e)\tilde{D}e = \begin{pmatrix} \sigma(e) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma(e)De & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, under the identifications $\tilde{\pi}(e)\tilde{\mathcal{H}}^q \simeq e\mathcal{H}^q$ and $\tilde{\pi}(\sigma(e))\tilde{\mathcal{H}}^q \simeq \sigma(e)\mathcal{H}^q$ above, the operators $\tilde{D}_{e,\sigma}$ and $D_{e,\sigma}$ agree. Therefore, using (7.7) we get $\mathrm{ind}\, D_{e,\sigma} =$

ind $\tilde{D}_{e,\sigma} = \langle \tau_{2k}^{\tilde{D}}, e \rangle$. As (7.19) implies that $\langle \tau_{2k}^{\tilde{D}}, e \rangle = \langle \overline{\tau}_{2k}^{D}, e \rangle$, we then deduce that ind $D_{e,\sigma} = \langle \overline{\tau}_{2k}^{D}, e \rangle$. The proof is complete.   □

DEFINITION 7.14

Let $(\mathcal{A}, \mathcal{H}, D)_{\sigma}$ be a p-summable twisted spectral triple. Then its Connes–Chern character, denoted by $\mathrm{Ch}(D)_{\sigma}$, is the class in $\mathrm{HP}^0(\mathcal{A})$ of any of the cocycles $\overline{\tau}_{2k}^{D}$, $k \geq \frac{1}{2}(p-1)$.

Assume now that D is invertible. We then have two definitions of the Connes–Chern character: one in terms of the cocycles τ_{2k}^{D} and the other in terms of the cocycles $\overline{\tau}_{2k}^{D}$. We shall now show that these definitions are equivalent.
    Consider the homotopy of operators

$$\tilde{D}_t = \tilde{D}_0 + tJ, \quad 0 \leq t \leq 1.$$

In the same way as in (7.16) we have

$$\tilde{D}_t^2 = \tilde{D}_0^2 + t^2 = \begin{pmatrix} D^2 + t^2 & 0 \\ 0 & D^2 + t^2 \end{pmatrix},$$

which shows that $\tilde{D}_t$ is invertible for all $t \in [0,1]$. Moreover, as in (7.17) we have

$$\mathrm{Tr}\,|\tilde{D}_t|^{-p} = 2\,\mathrm{Tr}(D^2 + t^2)^{-\frac{p}{2}} \leq 2\,\mathrm{Tr}\,|D|^{-p}.$$

Thus, $(\tilde{D}_t^{-1})_{0 \leq t \leq 1}$ is a bounded family in $\mathcal{L}^p(\tilde{\mathcal{H}})$. Therefore, the family $(\tilde{D}_t)_{0 \leq t \leq 1}$ satisfies the assumption of Proposition C.1 in Appendix C on the homotopy invariance of the Connes–Chern character. We then deduce that $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D}_t)_{\tilde{\sigma}}$ is a p-summable twisted spectral triple for all $t \in [0,1]$ and, for $k \geq \frac{1}{2}(p+1)$, the cocycles $\tau_{2k}^{\tilde{D}_0}$ and $\tau_{2k}^{\tilde{D}_1} = \tau_{2k}^{\tilde{D}}$ are cohomologous in $H_{\lambda}^{2k}(\tilde{\mathcal{A}})$. Denote by $\overline{\tau}_{2k}^{\tilde{D}_0}$ the restriction to $\mathcal{A}^{2k+1}$ of $\tau_{2k}^{\tilde{D}_0}$. Then (7.18) shows that $\overline{\tau}_{2k}^{\tilde{D}_0}$ and $\overline{\tau}_{2k}^{D}$ are cohomologous in $H_{\lambda}^{2k}(\mathcal{A})$.
    Bearing this in mind, we note that, for $a \in \mathcal{A}$, we have

$$\left[\tilde{D}_0, \tilde{\pi}(a)\right]_{\tilde{\sigma}} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sigma(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} = \begin{pmatrix} [D,a]_{\sigma} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$\tilde{D}_0^{-1}\left[\tilde{D}_0, \tilde{\pi}(a)\right]_{\tilde{\sigma}} = \begin{pmatrix} D^{-1}[D,a]_{\sigma} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and}$$

$$\tilde{\gamma}\tilde{D}_0^{-1}\left[\tilde{D}_0, \tilde{\pi}(a)\right]_{\tilde{\sigma}} = \begin{pmatrix} \gamma D^{-1}[D,a]_{\sigma} & 0 \\ 0 & 0 \end{pmatrix}.$$

It then follows that, for $a^0, \ldots, a^{2k}$ in $\mathcal{A}$, we have

$$\tau_{2k}^{\tilde{D}_0}(a^0, \ldots, a^{2k}) = \mathrm{Tr}\big(\gamma D^{-1}[D,a^0]_{\sigma} \cdots D^{-1}[D,a^{2k}]_{\sigma}\big) = \tau_{2k}^{D}(a^0, \ldots, a^{2k}).$$

That is, the restriction of $\tau_{2k}^{\tilde{D}_0}$ to $\mathcal{A}^{2k+1}$ is precisely the cocycle τ_{2k}^{D}. Therefore, we arrive at the following statement.

PROPOSITION 7.15

If D is invertible, then for any $k \geq \frac{1}{2}(p+1)$ the cyclic cocycles τ_{2k}^D and $\overline{\tau}_{2k}^D$ are cohomologous in $H_\lambda^{2k}(\mathcal{A})$ and, hence, define the same class in $\mathrm{HP}^0(\mathcal{A})$.

It follows from this that, when D is invertible, Definitions 7.9 and 7.14 provide us with equivalent definitions of the Connes–Chern character of $(\mathcal{A}, \mathcal{H}, D)_\sigma$.

Finally, using Proposition 7.13 and arguing as in the proof of Theorem 7.10 enable us to remove the invertibility assumption in Theorem 7.10. We thus obtain the following index formula.

THEOREM 7.16

Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a p-summable twisted spectral triple. Then, for any finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ and any σ-connection on $\mathcal{E}$, we have

$$\mathrm{ind}\, D_{\nabla \varepsilon} = \big\langle \mathrm{Ch}(D)_\sigma, [\mathcal{E}] \big\rangle.$$

REMARK 7.17

The cocycles τ_{2k}^D and $\overline{\tau}_{2k}^D$ may be difficult to compute in practice, even in the case of a Dirac spectral triple (see [Co1, Part I, Theorem 6.5] and [BF]). In the ordinary case, a representation of the Connes–Chern character in entire cyclic cohomology is given by JLO cocycle of Jaffe–Lesniewski–Osterwalder (see [JLO], [Co3]), the existence of which only requires θ-summability. We refer to the paper of Quillen [Qu] for interpretations of the Connes–Chern character and the JLO cocycle in terms of Chern characters of superconnections on cochains.

REMARK 7.18

Under further assumptions, a representative in periodic cyclic cohomology is given by the CM cocycle of Connes–Moscovici ([CM1]; see also [Hi]). The components of the CM cocycle are given by formulas that are *local* in the sense that they involve universal linear combinations of functionals of the form

$$\int a^0 [D, a^1]^{[\alpha_1]} \cdots [D, a^{2k}]^{[\alpha_{2k}]} D^{-2(|\alpha|+k)}, \quad a^j \in \mathcal{A},$$

where $T^{[j]}$ is the jth iterated commutator of T with D^2 and $\int$ is an analogue of the noncommutative residue trace of Guillemin [Gu] and Wodzicki [Wo1]. This thus expresses the index pairing as a linear combination of residues of various zeta functions, in the spirit of the index formula of Atiyah–Bott [At]. We refer to [Wo2], [Ka], [MN], and [MoN] for other types of residue index formulas.

REMARK 7.19

Let (M^n, g) be a compact Riemannian manifold. The Connes–Chern character of the Dirac spectral triple $(C^\infty(M), L^2(M, \slashed{S}), \slashed{D}_g)$ is represented by the CM

cocycle. This CM cocycle can be computed by heat kernel techniques (see [CM1], [Po]). We obtain the even cocycle $\varphi = (\varphi_{2k})$ given by

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!} \int_M \hat{A}(R^M) \wedge f^0 \, df^1 \wedge \cdots \wedge df^k.$$

In other words $\varphi = (2i\pi)^{-\frac{n}{2}} \varphi_{\hat{A}(R^M)^\wedge}$ in the sense of (6.14), where $\hat{A}(R^M)^\wedge$ is the Poincaré dual current of the $\hat{A}$-form $\hat{A}(R^M)$. Let us explain how this enables us to recover the Atiyah–Singer index formula. Let $e \in M_q(C^\infty(M))$, $e^2 = e$, and form the vector bundle $E = \operatorname{ran} e$, which we equip with its Grassmannian connection ∇^E. Then by (5.23) we have

$$\operatorname{ind} \not{D}_{\nabla^E} = \operatorname{ind}_{\not{D}_g, \sigma}[e].$$

As $(2i\pi)^{-\frac{n}{2}} \varphi_{\hat{A}(R^M)^\wedge}$ represents the Connes–Chern character, by Theorem 7.16 we have

$$\operatorname{ind}_{\not{D}_g, \sigma}[e] = (2i\pi)^{-\frac{n}{2}} \langle \varphi_{\hat{A}(R^M)^\wedge}, e \rangle.$$

Moreover, using (6.18) we have

$$\langle \varphi_{\hat{A}(R^M)^\wedge}, e \rangle = \langle \hat{A}(R^M)^\wedge, \operatorname{Ch}(F^E) \rangle = \int_M \hat{A}(R^M) \wedge \operatorname{Ch}(F^E),$$

where F^E is the curvature of ∇^E. Therefore, we obtain

$$\operatorname{ind} \not{D}_{\nabla^E} = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \operatorname{Ch}(F^E),$$

which is the Atiyah–Singer index formula.

REMARK 7.20

It remains an open question to construct a version of the CM cocycle for twisted spectral triples. Moscovici [Mo2] devised an ansatz for such a cocycle and verified it in the case of twistings of ordinary spectral triples by scaling automorphisms. To date, this seems to be the only known example of twisted spectral triples satisfying Moscovici's ansatz. It would be interesting to have a version of Connes's [Co4] Hochschild character formula. We refer to [FK1] for a Hochschild character formula in the special case of twistings of ordinary spectral triples by scaling automorphisms.

REMARK 7.21

We refer to [PW2] and [PW3] for the computation of the Connes–Chern character of the conformal Dirac spectral triple of [CM4] (the construction of which is recalled in Section 2).

Appendix A: Proof of Lemma 5.11

It is immediate that the restriction of $(\cdot, \cdot)_0$ is $\mathcal{A}$-sesquilinear and positive. The only issue at stake is nondegeneracy.

LEMMA A.1

Set $\mathcal{E}^* = e^*\mathcal{A}^q$. Then the restriction of $(\cdot, \cdot)$ to $\mathcal{E}^* \times \mathcal{E}$ is nondegenerate.

Proof of Lemma A.1
We need to show that $\Phi : \mathcal{E}^* \ni \xi \to (\xi, \cdot)_0|_{\mathcal{E}} \in \mathcal{E}'$ is an $\mathcal{A}$-antilinear isomorphism. Let $\xi = (\xi_j) \in \mathcal{E}^*$. Then

$$(\xi, e\xi)_0 = (e^*\xi, \xi)_0 = (\xi, \xi)_0 = \sum \xi_j^* \xi_j.$$

It then follows that if $(\xi, \cdot)_0$ vanishes on $\mathcal{E}$, then all the positive operators $\xi_j^* \xi_j$ vanish on $\mathcal{H}$ and hence $\xi = 0$. This shows that Φ is injective.

Let $\varphi \in \mathcal{E}'$, and let $\tilde{\varphi} \in (\mathcal{A}^q)'$ be defined by $\tilde{\varphi}(\xi) = \varphi(e\xi)$ for all $\xi \in \mathcal{A}^q$. The nondegeneracy of $(\cdot, \cdot)_0$ implies that there is $\tilde{\eta} \in \mathcal{A}^q$ such that $\tilde{\varphi}(\xi) = (\tilde{\eta}, \xi)_0$ for all $\xi \in \mathcal{A}^q$. Set $\eta = e^*\eta \in \mathcal{E}^*$. Then, for all $\xi \in \mathcal{E}$,

$$\varphi(\xi) = \tilde{\varphi}(e\xi) = (\tilde{\eta}, e\xi)_0 = (e^*\tilde{\eta}, \xi)_0 = (\eta, \xi)_0.$$

Thus, $\varphi = \Phi(\eta)$. This shows that Φ is surjective. Therefore, Φ is an $\mathcal{A}$-antilinear isomorphism. Likewise, $\Psi : \mathcal{E} \ni \eta \to (\cdot, \eta)_0|_{\mathcal{E}^*} \in (\mathcal{E}^*)'$ is an $\mathcal{A}$-linear isomorphism. This completes the proof of the lemma. $\square$

LEMMA A.2

Denote by $\mathfrak{t} : \mathcal{E} \to \mathcal{E}^*$ the $\mathcal{A}$-linear map defined by

$$\mathfrak{t}\xi = e^*\xi \quad \forall \xi \in \mathcal{E}.$$

Then $\mathfrak{t}$ is an $\mathcal{A}$-linear isomorphism from $\mathcal{E}$ onto $\mathcal{E}^*$.

Proof of Lemma A.2
If $\mathcal{F}$ is a right submodule of $\mathcal{A}^q$, then we shall denote by $\mathcal{F}^\perp$ its orthogonal complement with respect to the canonical Hermitian metric of $\mathcal{A}^q$. For $a \in M_q(\mathcal{A})$ we shall identify a with the associated $\mathcal{A}$-linear map $\mathcal{A}^q \to \mathcal{A}^q$. We observe that with this convention a^* is identified with the adjoint of a with respect to $(\cdot, \cdot)_0$, that is,

$$(a^*\xi, \eta)_0 = (\xi, a\eta)_0 \quad \forall \xi, \eta \in \mathcal{A}^q.$$

We deduce from this that, for any idempotent $f \in M_q(\mathcal{A})$,

(A.1) $\qquad (f\mathcal{A}^q)^\perp = (\operatorname{ran} f)^\perp = \ker f^* = \operatorname{ran}(1 - f^*) = (1 - f^*)\mathcal{A}^q.$

We note this implies that $((f\mathcal{A}^q)^\perp)^\perp = f\mathcal{A}^q$.

Using (A.1) we get

$$\ker \mathfrak{t} = \ker e^* \cap \operatorname{ran} e = (\operatorname{ran} e)^\perp \cap \operatorname{ran} e = \{0\},$$

which shows that $\mathfrak{t}$ is one-to-one. Moreover, as $\mathcal{A}$ is closed under holomorphic functional calculus, there is $g \in \mathrm{GL}_q(\mathcal{A})$ such that $f := e^*eg$ is a self-adjoint idempotent which is similar to e^* (cf. [Bl]). Thus,

$$\operatorname{ran} e^*e = \operatorname{ran} f = ((f\mathcal{A}^q)^\perp)^\perp = ((\operatorname{ran} ee^*)^\perp)^\perp = (\ker e^*e)^\perp.$$

Obviously $\ker e \subset \ker e^* e$. As $(e\xi, e\xi)_0 = (e^* e\xi, \xi)_0$ for all $\xi \in \mathcal{A}^q$, we see that $\ker e^* e$ is contained in $\ker e$, and so the two submodules agree. Thus,

$$\operatorname{ran} e^* e = (\ker e^* e)^\perp = (\ker e)^\perp = \operatorname{ran} e^*.$$

This shows that $\mathfrak{t}(\mathcal{E}) = e^*(\operatorname{ran} e^*) = \operatorname{ran} e^* e = \operatorname{ran} e^* = \mathcal{E}^*$, that is, $\mathfrak{t}$ is onto. Therefore, the $\mathcal{A}$-linear map $\mathfrak{t}$ is an isomorphism. $\qquad\square$

Let us go back to the proof of Lemma 5.11. For all ξ_1 and ξ_2 in $\mathcal{E}$, we have

$$(\xi_1, \xi_2)_0 = (\xi_1, e\xi_2)_0 = (e^*\xi_1, \xi_2)_0 = (\mathfrak{t}\xi_1, \xi_2)_0.$$

As $(\cdot, \cdot)_0$ is nondegenerate on $\mathcal{E}^* \times \mathcal{E}$ by Lemma A.1 and $\mathfrak{t}$ is an $\mathcal{A}$-linear isomorphism by Lemma A.2, we then deduce that $(\cdot, \cdot)_0$ is nondegenerate on $\mathcal{E} \times \mathcal{E}$. This completes the proof of Lemma 5.11.

Appendix B: Proof of Lemma 5.13

Let us first assume that $\mathcal{E} = \mathcal{A}^q$ for some $q \in \mathbb{N}_0$. Let us denote by $\mathcal{H}(\mathcal{A}^q)_0$ the pre-Hilbert space associated with the canonical Hermitian metric $(\cdot, \cdot)_0$ on $\mathcal{A}^q$. There is a canonical isomorphism $U_0 : \mathcal{A}^q \otimes_{\mathcal{A}} \mathcal{H} \to \mathcal{H}^q$ given by

$$U_0(\xi \otimes \zeta) = (\xi_1 \zeta, \ldots, \xi_q \zeta) \quad \text{for all } \xi = (\xi_j) \in \mathcal{A}^q \text{ and } \zeta \in \mathcal{H}.$$

The inverse of U_0 is given by

$$U^*(\zeta_1, \ldots, \zeta_q) = \varepsilon_1 \otimes \zeta_1 + \cdots + \varepsilon_q \otimes \zeta_q, \quad \zeta_j \in \mathcal{H},$$

where $\varepsilon_1, \ldots, \varepsilon_q$ form the canonical basis of $\mathcal{A}^q$. We also observe that, for $\xi \in \mathcal{A}^q$ and $\zeta \in \mathcal{H}$,

$$\left\| U_0(\xi \otimes \zeta) \right\|^2 = \sum_i \langle \xi_j \zeta, \xi_j \zeta \rangle = \sum_i \langle \zeta, \xi_j^* \xi_j \zeta \rangle = \langle \zeta, (\xi, \xi)_0 \zeta \rangle = \| \xi \otimes \zeta \|_0^2,$$

where $\| \cdot \|_0$ is the norm of $\mathcal{H}(\mathcal{A}^q)_0$. This shows that U_0 is an isometric isomorphism from $\mathcal{H}(\mathcal{A}^q)_0$ onto $\mathcal{H}^q$. As $\mathcal{H}^q$ is complete, we then deduce that $\mathcal{H}(\mathcal{A}^q)_0$ is a Hilbert space.

Let $(\cdot, \cdot)$ be a Hermitian metric on $\mathcal{A}^q$. We denote by $\mathcal{H}(\mathcal{E})$ the associated pre-Hilbert space and by $\langle \cdot, \cdot \rangle$ its inner product. The nondegeneracy of $(\cdot, \cdot)$ and $(\cdot, \cdot)_0$ implies there is a self-adjoint element $g \in \mathrm{GL}_q(\mathcal{A})$ such that

$$(\xi, \eta) = (g\xi, \eta)_0 \quad \forall \xi, \eta \in \mathcal{A}^q.$$

We also denote by g the representation of g as a self-adjoint bounded operator of $\mathcal{H}^q$. Let $\xi = (\xi_j)$ and $\xi' = (\xi_j')$ be in $\mathcal{A}^q$, and let ζ and ζ' be in $\mathcal{H}$. Then

$$\langle \xi \otimes \zeta, \xi' \otimes \zeta' \rangle = \langle \zeta, (\xi, \xi') \zeta' \rangle = \langle \zeta, (g\xi, \xi')_0 \zeta' \rangle = \sum_{i,j} \langle \zeta, (\xi_j^* g_{ij}^* \xi_i') \zeta' \rangle$$

$$= \sum_{i,j} \langle g_{ij}(\xi_j \zeta), \xi_i' \zeta' \rangle = \langle g U_0(\xi \otimes \zeta), U_0(\xi' \otimes \zeta') \rangle.$$

By bilinearity it then follows that $\langle \eta, \eta' \rangle = \langle gU_0\eta, U_0\eta \rangle$ for all η and η' in $\mathcal{H}(\mathcal{A}^q)$. Thus, for all $\eta \in \mathcal{H}(\mathcal{A}^q)$ and $\zeta \in \mathcal{H}^q$,

$$(\text{B.1}) \qquad \|\eta\|^2 = \langle gU_0\eta, U_0\eta \rangle \qquad \text{and} \qquad \langle g\zeta, \zeta \rangle = \|U_0^{-1}\zeta\|^2.$$

The second equality in (B.1) shows that g is a positive operator of $\mathcal{H}^q$. As g is invertible, we see that its spectrum is contained in an interval $[c^{-1}, c]$ for some $c > 1$, and so, for all $\zeta \in \mathcal{H}^q$,

$$(\text{B.2}) \qquad c^{-1}\|\zeta\|^2 \leq \langle g\zeta, \zeta \rangle \leq c\|\zeta\|^2.$$

Combining (B.1) and the fact that U_0 is an isometry from $\mathcal{H}(\mathcal{A}^q)_0$ onto $\mathcal{H}^q$ we deduce that, for all $\eta \in \mathcal{A}^q \otimes_{\mathcal{A}} \mathcal{H}$, we have

$$\|\eta\|^2 = \langle gU_0\eta, U_0\eta \rangle \in \left[c^{-1}\|U_0\eta\|^2, c\|U_0\eta\|^2 \right] = \left[c^{-1}\|\eta\|_0^2, c\|\eta\|_0^2 \right].$$

This shows that the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent on $\mathcal{A}^q \otimes_{\mathcal{A}} \mathcal{H}$. Therefore, $\mathcal{H}(\mathcal{A}^q)$ has the same topology as $\mathcal{H}(\mathcal{A}^q)_0$. In particular, $\mathcal{H}(\mathcal{A}^q)$ is complete and, hence, is a Hilbert space. This proves Lemma 5.13 in the special case $\mathcal{E} = \mathcal{A}^q$.

Let us now assume that $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$. By Lemma 5.11 the canonical Hermitian metric of $\mathcal{A}^q$ induces a Hermitian metric on $\mathcal{E}$. We denote by $\langle \cdot, \cdot \rangle$ and $\mathcal{H}(\mathcal{E})_0$ the associated inner product and pre-Hilbert space. We also denote by e the representation of e as a bounded operator on $\mathcal{H}^q$. We note that as e is idempotent, $e\mathcal{H}^q$ is a closed subspace of $\mathcal{H}^q$.

Let $\xi = (\xi_j) \in \mathcal{A}^q$, and let $\zeta \in \mathcal{H}$. For $i = 1, \ldots, q$, we have

$$U_0\big((e\xi) \otimes \zeta\big)_i = (e\xi)_i \zeta = \sum_j e_{ij}\xi_j\zeta = \sum_j e_{ij}U_0(\xi \otimes \zeta)_j = \big(eU_0(\xi \otimes \zeta)\big)_i.$$

That is,

$$U_0\big((e\xi) \otimes \zeta\big) = eU_0(\xi \otimes \zeta).$$

As U_0 is an isometric isomorphism from $\mathcal{H}(\mathcal{A}^q)_0$ onto $\mathcal{H}^q$ we see that U_0 induces an isometric isomorphism from $\mathcal{H}(\mathcal{E})_0$ onto $e\mathcal{H}^q$. As $e\mathcal{H}^q$ is complete (since this is a closed subspace of $\mathcal{H}^q$) we deduce that $\mathcal{H}(\mathcal{E})_0$ is a Hilbert space.

Let $(\cdot, \cdot)$ be a Hermitian metric on $\mathcal{E}$. Thanks to the nondegeneracy of $(\cdot, \cdot)_0$ and $(\cdot, \cdot)$ there is a unique $\mathcal{A}$-linear isomorphism $a : \mathcal{E} \to \mathcal{E}$ such that

$$(\xi, \eta) = (a\xi, \eta)_0 \quad \text{for all } \xi \text{ and } \eta \text{ in } \mathcal{E}.$$

We then extend $(\cdot, \cdot)$ into the $\mathcal{A}$-sesquilinear form on $\mathcal{A}^q$ defined by

$$(\text{B.3}) \qquad (\xi, \eta) := (ae\xi, \eta)_0 + \big((1-e)\xi, (1-e)\eta\big)_0 \quad \text{for all } \xi \text{ and } \eta \text{ in } \mathcal{A}^q.$$

We note that $(\cdot, \cdot)$ is positive on $\mathcal{A}^q$, and

$$(\xi, \eta) = (g\xi, \eta)_0 \quad \text{for all } \xi \text{ and } \eta \text{ in } \mathcal{A}^q,$$

where we have set $g = e^*ae + (1-e)^*(1-e)$.

By Lemma A.2 we know that e^* induces an $\mathcal{A}$-linear isomorphism from $e\mathcal{A}^q$ onto $e^*\mathcal{A}^q$ and $(1 - e^*)$ induces an isomorphism from $(1-e)\mathcal{A}^q$ onto $(1-e^*)\mathcal{A}^q$. As a is an isomorphism from $\mathcal{E} = e\mathcal{A}^q$ onto itself, we deduce that g is a right-

module isomorphism from $\mathcal{A}^q$ onto itself. Combining this with (B.3) we then see that $(\cdot, \cdot)$ is nondegenerate on $\mathcal{A}^q \times \mathcal{A}^q$. Thus, $(\cdot, \cdot)$ is a Hermitian metric on $\mathcal{A}^q$. Therefore, by the first part of the proof, the associated norm on $\mathcal{A}^q \otimes_{\mathcal{A}} \mathcal{H}$ is equivalent to the norm of $\mathcal{H}(\mathcal{A}^q)$. As these norms restrict to the norms of $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}(\mathcal{E})_0$ on $\mathcal{E} = e\mathcal{A}^q$, we then deduce that the norms of $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}(\mathcal{E})_0$ are equivalent. This proves Lemma 5.13 in the special case $\mathcal{E} = e\mathcal{A}^q$, $e = e^2 \in M_q(\mathcal{A})$.

Let us now prove Lemma 5.13 when $\mathcal{E}$ is an arbitrary finitely generated projective module, that is, it is the direct summand of a free module $\mathcal{E}_0$. Let $\phi : \mathcal{E}_0 \to \mathcal{A}^q$ be an $\mathcal{A}$-linear isomorphism. Then $\phi(\mathcal{E}) = e\mathcal{A}^q$ for some idempotent $e \in M_q(\mathcal{A})$. If $(\cdot, \cdot)$ is a Hermitian metric on $\mathcal{E}$, then we define a Hermitian metric $(\cdot, \cdot)_\phi$ on $e\mathcal{A}^q$ by

$$(\xi, \eta)_\phi = \left(\phi^{-1}(\xi), \phi^{-1}(\eta) \right) \quad \text{for all } \xi \text{ and } \eta \text{ in } e\mathcal{A}^q.$$

We denote by $\langle \cdot, \cdot \rangle_\phi$ and $\mathcal{H}(e\mathcal{A}^q)_\phi$ the associated inner product and Hilbert space, respectively.

Set $U_\phi := \phi \otimes 1_{\mathcal{H}}$. This a vector bundle isomorphism from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ onto $(e\mathcal{A}^q) \otimes_{\mathcal{A}} \mathcal{H}$. Let ξ and ξ' be in $\mathcal{E}$, and let ζ and ζ' be in $\mathcal{H}$. Then

$$\left\langle U_\phi(\xi \otimes \zeta), U_\phi(\xi' \otimes \zeta') \right\rangle_\phi = \left\langle \zeta, \left(\phi(\xi), \phi(\xi') \right)_\phi \zeta' \right\rangle = \left\langle \zeta, (\xi, \xi') \zeta' \right\rangle$$

$$= \left\langle \xi \otimes \zeta, \xi' \otimes \zeta' \right\rangle.$$

Thus, U_ϕ is an isometric isomorphism from $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}(e\mathcal{A}^q)_\phi$. As $\mathcal{H}(e\mathcal{A}^q)_\phi$ is a Hilbert space, we then deduce that $\mathcal{H}(\mathcal{E})$ is a Hilbert space as well.

Finally, we observe that pushing forward norms by U_ϕ gives rise to a one-to-one correspondence between norms on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ and $(e\mathcal{A})^q \otimes \mathcal{H}$ arising from Hermitian metrics on $\mathcal{E}$ and $e\mathcal{A}^q$. As all those norms on $e\mathcal{A}^q$ are equivalent to each other, we then deduce that the same result holds on $\mathcal{E}$. That is, the topology of $\mathcal{H}(\mathcal{E})$ is independent of the choice of the Hermitian metric. The proof of Lemma 5.13 is complete.

Appendix C: Homotopy invariance of the Connes–Chern character

In this appendix, we give a proof of the homotopy invariance of the Connes–Chern character in the following form.

PROPOSITION C.1

Let $(\mathcal{A}, \mathcal{H}, D)_\sigma$ be a p-summable twisted spectral triple. Consider an operator homotopy of the form

$$D_t = D + V_t, \quad 0 \le t \le 1,$$

where $(V_t)_{0 \le t \le 1}$ is a C^1 self-adjoint family in $\mathcal{L}(\mathcal{H})$ such that D_t is invertible for all $t \in [0, 1]$ and $(D_t^{-1})_{0 \le t \le 1}$ is a bounded family in $\mathcal{L}^p(\mathcal{H})$.

(1) $(\mathcal{A}, \mathcal{H}, D_t)_\sigma$ *is a p-summable twisted spectral triple for all $t \in [0, 1]$.*
(2) *For any $k \ge \frac{1}{2}(p-1)$, the cocycles $\tau_{2k}^{D_0}$ and $\tau_{2k}^{D_1}$ are cohomologous in $H_\lambda^{2k}(\mathcal{A})$.*

(3) *The twisted spectral triples $(\mathcal{A}, \mathcal{H}, D_0)_\sigma$ and $(\mathcal{A}, \mathcal{H}, D_1)_\sigma$ have the same Connes–Chern character in $\mathrm{HP}^0(\mathcal{A})$.*

By assumption the resolvent D_t^{-1} lies in $\mathcal{L}^p(\mathcal{H})$. Moreover, for all $a \in \mathcal{A}$,

(C.1) $$[D_t, a]_\sigma = [D, a]_\sigma + \big(V_t a - \sigma(a) V_t\big) \in \mathcal{L}(\mathcal{H}).$$

Therefore, $(\mathcal{A}, \mathcal{H}, D_t)_\sigma$ is a p-summable twisted spectral triple, and so, for any integer $k \geq \frac{1}{2}(p-1)$, we can form the cyclic $2k$-cocycle

$$\tau_{2k}^{D_t}(a^0, \ldots, a^{2k}) = c_k \operatorname{Str}\big(D_t^{-1}[D_t, a^0]_\sigma \cdots D_t^{-1}[D_t, a^{2k}]_\sigma\big), \quad a^j \in \mathcal{A}.$$

The rest of the proof is devoted to comparing the cocycles $\tau_{2k}^{D_1}$ and $\tau_{2k}^{D_0}$.

In what follows, we set $\dot{V}_t = \frac{d}{dt} V_t$, and for $a \in \mathcal{A}$, we define

$$\delta_t(a) = D_t^{-1}\big[\dot{V}_t D_t^{-1}, \sigma(a)\big] D_t.$$

We note that

$$\delta_t(a) = \big[D_t^{-1}\dot{V}_t, D_t^{-1}\sigma(a) D_t\big] = \big[D_t^{-1}\dot{V}_t, a\big] - \big[D_t^{-1}\dot{V}_t, [D_t, a]_\sigma\big].$$

As (C.1) shows that $([D_t, a]_\sigma)_{0 \leq t \leq 1}$ is a continuous family in $\mathcal{L}(\mathcal{H})$ and $(D_t^{-1}\dot{V}_t)_{0 \leq t \leq 1}$ is a continuous family in $\mathcal{L}^p(\mathcal{H})$, we then see that $(\delta_t(a))_{0 \leq t \leq 1}$ is a continuous family in $\mathcal{L}^p(\mathcal{H})$. For $j = 1, \ldots, 2k+1$, we let η_j^t be the $(2k+1)$-cochain on $\mathcal{A}$ defined by

$$\eta_j^t(a^0, \ldots, a^{2k+1}) = \operatorname{Str}\big(\alpha_j(a^0) D_t^{-1}[D_t, a^1]_\sigma \cdots \delta_t(a^j) \cdots D_t^{-1}[D_t, a^{2k+1}]_\sigma\big),$$

$$a^l \in \mathcal{A},$$

where $\alpha_j(a) = a$ if j is even and $\alpha_j(a) = D_t^{-1}\sigma(a) D_t$ if j is odd. Note that η_j^t is a normalized cochain.

In what follows we shall say that a family $(\varphi^t)_{0 \leq t \leq 1} \subset C^m(\mathcal{A})$ is C^α, $\alpha \geq 0$, when, for all $a^0, \ldots, a^m$, the function $t \to \varphi^t(a^0, \ldots, a^m)$ is C^α on $[0, 1]$. Given a C^1-family $(\varphi^t)_{0 \leq t \leq 1}$ of m-cochains, we define m-cochains $\frac{d}{dt}\varphi^t$, $t \in [0, 1]$, by

$$\Big(\frac{d}{dt}\varphi^t\Big)(a^0, \ldots, a^m) := \frac{d}{dt}(\varphi^t)(a^0, \ldots, a^m), \quad a^j \in \mathcal{A}.$$

Given a C^0-family $(\psi^t)_{0 \leq t \leq 1}$ of m-cochains we define the integral $\int \psi^t$ as the m-cochain given by

$$\Big(\int_0^1 \psi^t \, dt\Big)(a^0, \ldots, a^m) := \int_0^1 \psi^t(a^0, \ldots, a^m) \, dt, \quad a^j \in \mathcal{A}.$$

If F is any of the operators b, A, B_0, or B, then

(C.2) $$F\Big(\frac{d}{dt}\varphi^t\Big) = \frac{d}{dt}(F\varphi^t) \qquad \text{and} \qquad F\Big(\int_0^1 \psi^t \, dt\Big) = \int_0^1 F\psi^t \, dt.$$

Moreover, we have

(C.3) $$\int_0^1 \Big(\frac{d}{dt}\varphi^t\Big) dt = \varphi^1 - \varphi^0.$$

LEMMA C.2

The family $(\tau_{2k}^{D_t})_{0 \le t \le 1}$ is a C^1-family of $2k$-cochains and we have

$$\frac{d}{dt}\tau_{2k}^{D_t} = \frac{c_k}{2k+1} \sum_{j=1}^{2k+1} B\eta_j^t.$$

Proof
It follows from (C.1) that $([D_t, a]_\sigma)_{0 \le t \le 1}$ is a C^1-family in $\mathcal{L}(\mathcal{H})$ and we have

(C.4) $$\frac{d}{dt}[D_t, a]_\sigma = \dot{V}_t a - \dot{V}_t \sigma(a).$$

By assumption the family $(D_t^{-1})_{0 \le t \le 1}$ is bounded in $\mathcal{L}^p(\mathcal{H})$. Moreover,

$$D_{t+s}^{-1} - D_t^{-1} = -D_{t+s}^{-1}(D_{t+s} - D_t)D_t^{-1} = -D_{t+s}^{-1}(V_{t+s} - V_t)D_t^{-1}.$$

We then deduce that $(D_t^{-1})_{0 \le t \le 1}$ is a continuous family in $\mathcal{L}^p(\mathcal{H})$. Combining this with the above equality then shows that $(D_t^{-1})_{0 \le t \le 1}$ is a differentiable family in $\mathcal{L}^p(\mathcal{H})$ with

(C.5) $$\frac{d}{dt}D_t^{-1} = -D_t^{-1}\dot{V}_t D_t^{-1}.$$

As the above right-hand side is a continuous family in $\mathcal{L}^p(\mathcal{H})$ we eventually see that $(D_t^{-1})_{0 \le t \le 1}$ is a C^1-family in $\mathcal{L}^p(\mathcal{H})$. The product in $\mathcal{L}(\mathcal{H})$ induces a continuous bilinear map from $\mathcal{L}^p(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ to $\mathcal{L}^p(\mathcal{H})$. Therefore, we deduce that $(D_t^{-1}[D_t, a]_\sigma)_{0 \le t \le 1}$ is a C^1-family in $\mathcal{L}^p(\mathcal{H})$, and using (C.4) and (C.5) we obtain

$$\frac{d}{dt}D_t^{-1}[D_t, a]_\sigma = -D_t^{-1}\dot{V}_t D_t^{-1}[D_t, a]_\sigma + D_t^{-1}\big(\dot{V}_t a - \sigma(a)\dot{V}_t\big)$$

(C.6) $$= D_t^{-1}\big(-\dot{V}_t a + \dot{V}_t D_t^{-1}\sigma(a) + \dot{V}_t a - \sigma(a)\dot{V}_t\big)$$

$$= \delta_t(a).$$

Let $a^0, \ldots, a^{2k}$ be in $\mathcal{A}$. As $2k + 1 \ge p$ the product of $\mathcal{L}(\mathcal{H})$ induces a continuous $(2k + 1)$-linear map from $\mathcal{L}^p(\mathcal{H})^{2k+1}$ to $\mathcal{L}^1(\mathcal{H})$. Therefore, the map $t \to D_t^{-1}[D_t, a^0]_\sigma \cdots D_t^{-1}[D_t, a^{2k+1}]_\sigma$ is a C^1-map from $[0, 1]$ to $\mathcal{L}^1(\mathcal{H})$. Composing it with the supertrace on $\mathcal{L}^1(\mathcal{H})$ we then deduce that the function $t \to \tau_{2k}^{D_t}(a^0, \ldots, a^{2k})$ is C^1 on $[0, 1]$. Moreover, using (C.2) we get

$$\frac{d}{dt}\tau_{2k}^{D_t}(a^0, \ldots, a^{2k})$$

$$= c_k \sum_{j=0}^{2k} \text{Str}\big(D_t^{-1}[D_t, a^0]_\sigma \cdots \delta_t(a^j) \cdots D_t^{-1}[D_t, a^{2k}]_\sigma\big).$$

Noting that $\alpha_j(1) = 1$ we see that

$$\text{Str}\big(D_t^{-1}[D_t, a^0]_\sigma \cdots \delta_t(a^j) \cdots D_t^{-1}[D_t, a^{2k}]_\sigma\big) = B_0\eta_{j+1}(a^0, \ldots, a^{2k}).$$

Therefore, we see that $(\tau_{2k}^{D_t})_{0 \leq t \leq 1}$ is a C^1-family of cochains and $\frac{d}{dt}\tau_{2k}^{D_t} = c_k \sum_{j=1}^{2k+1} B_0\eta_j^t$. As the η_j^t's are normalized cochains and $\tau_{2k}^{D_t}$ is a cyclic cocycle, using (C.2) we get

$$\frac{d}{dt}\tau_{2k}^{D_t} = \frac{1}{2k+1}\frac{d}{dt}A\tau_{2k}^{D_t} = \frac{1}{2k+1}A\left(\frac{d}{dt}\tau_{2k}^{D_t}\right) = \frac{c_k}{2k+1}\sum_{j=1}^{2k+1}AB_0\eta_j^t$$

$$= \frac{c_k}{2k+1}\sum_{j=1}^{2k+1}B\eta_j^t.$$

The proof is complete. □

LEMMA C.3

For $t \in [0,1]$ and $j = 1, \ldots, 2k+1$ the cochain η_j^t is a Hochschild cocycle, that is, $b\eta_j^t = 0$.

Proof
Let β and γ be the $(2k+2)$-cochains on $\mathcal{A}$ given by

$$\beta(a^0, \ldots, a^{2k+2}) = \mathrm{Str}\left(\alpha_j(a^0)D^{-1}[D,a^1]_\sigma \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D,a^{2k+2}]_\sigma\right),$$

$$\gamma(a^0, \ldots, a^{2k+2}) = \mathrm{Str}\left(\alpha_j(a^0)D^{-1}[D,a^1]_\sigma \cdots \delta_t(a^j) \cdots D^{-1}[D,a^{2k+2}]_\sigma\right),$$

$$a^j \in \mathcal{A}.$$

For $l = 1, \ldots, j$ we let β_l' and β_l'' be the $(2k+2)$-cochains defined by

$$\beta_l'(a^0, \ldots, a^{2k+2})$$
$$= \mathrm{Str}\left(\alpha_j(a^0)D^{-1}[D,a^1]_\sigma \cdots a^l \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D,a^{2k+2}]_\sigma\right),$$
$$\beta_l''(a^0, \ldots, a^{2k+2})$$
$$= \mathrm{Str}\left(\alpha_j(a^0)D^{-1}[D,a^1]_\sigma \cdots D^{-1}\sigma(a^l)D \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D,a^{2k+2}]_\sigma\right).$$

We note that $\beta_l'(a^0, \ldots, a^{2k+2}) - \beta_l''(a^0, \ldots, a^{2k+2})$ is equal to

$$\mathrm{Str}\left(\alpha_j(a^0)D^{-1}[D,a^1]_\sigma \cdots \left(a^l - D^{-1}\sigma(a^l)D\right) \cdots\right.$$
(C.7)
$$\left. \times \delta_t(a^{j+1}) \cdots D^{-1}[D,a^{2k+2}]_\sigma\right)$$
$$= \beta(a^0, \ldots, a^{2k+2}).$$

Moreover, from the equality $D^{-1}[D,a^l a^{l+1}]_\sigma = D^{-1}[D,a^l]_\sigma a^{l+1} + D^{-1}\sigma(a^l)D \cdot D^{-1}[D,a^{l+1}]_\sigma$ we deduce that

(C.8) $$b_l\eta_j^t = \beta_{l+1}' + \beta_l''.$$

For $l = j+1, \ldots, 2k+1$ we let γ_l' and γ_l'' be the $(2k+2)$-cochains on $\mathcal{A}$ defined by

(C.9)
$$\gamma_l'(a^0, \ldots, a^{2k+2})$$
$$= \mathrm{Str}\left(\alpha_j(a^0)D^{-1}[D,a^1]_\sigma \cdots \delta_t(a^j) \cdots a^l \cdots D^{-1}[D,a^{2k+2}]_\sigma\right),$$

(C.10)
$$\gamma_l''(a^0, \ldots, a^{2k+2})$$
$$= \mathrm{Str}\big(\alpha_j(a^0) D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^j) \cdots D^{-1}\sigma(a^l) D \cdots D^{-1}[D, a^{2k+2}]_\sigma\big).$$

As in (C.7) and (C.8) we have

(C.11) $$\gamma_l' - \gamma_l'' = \gamma \qquad \text{and} \qquad b_l\eta = \gamma_{l+1}' + \gamma_l.$$

In addition, using the equality $\delta_t(a^j a^{j+1}) = \delta_t(a^j) D^{-1}\sigma(a^j) D + D^{-1} a^j D \delta_t(a^{j+1})$ we find that

(C.12)
$$b_j\eta_j^t(a^0, \ldots, a^{2k+2})$$
$$= \mathrm{Str}\big(\alpha_j(a^0) D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^j a^{j+1}) \cdots D^{-1}[D, a^{2k+1}]_\sigma\big)$$
$$= \gamma_{j+1}''(a^0, \ldots, a^{2k+2}) + \beta_j''(a^0, \ldots, a^{2k+2}).$$

Using (C.7)–(C.12) we obtain

$$\sum_{l=1}^{2k+1} b\eta_j^t = \sum_{l=1}^{j-1} (-1)^l (\beta_{l+1}' + \beta_l'') + (-1)^j (\beta_{j+1}'' + \beta_j'') + \sum_{l=j+1}^{2k+1} (-1)^l (\beta_{l+1}' + \beta_l'')$$

$$= -\beta_1'' + \sum_{l=2}^{j} (-1)^l (\beta_l'' - \beta_l') + \sum_{l=j+2}^{2k+1} (-1)^l (\gamma_l'' - \gamma_l') - \gamma_{2k+1}'$$

$$= -\beta_1'' + \sum_{l=2}^{j} (-1)^{l-1}\beta + \sum_{l=j+2}^{2k+1} (-1)^{l-1}\gamma - \gamma_{2k+1}'.$$

Noting that $\sum_{l=2}^{j}(-1)^{l-1} = -\frac{1}{2}(1 + (-1)^j)$ and $\sum_{l=j+2}^{2k+1}(-1)^{l-1} = \frac{1}{2}(1 - (-1)^j)$ we see that

(C.13)
$$b\eta_j^t = \sum_{l=0}^{2k+2} b\eta_j^t$$
$$= b_0\eta - \beta_1'' - \frac{1}{2}\big(1 + (-1)^j\big)\beta + \frac{1}{2}\big(1 - (-1)^j\big)\gamma + b_{2k_2}\eta - \gamma_{2k+1}'.$$

We note that $b_0\eta_j^t(a^0, \ldots, a^{2k+2}) - \beta_1''(a^0, \ldots, a^{2k+2})$ is equal to

(C.14)
$$\mathrm{Str}\big(\alpha_j(a^0)\big(\alpha_j(a^1) - D^{-1}\sigma(a^1)D\big)$$
$$\times D^{-1}[D, a^2]_\sigma \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma\big).$$

We also observe that

$$b_{2k+2}\eta_j^t(a^0, \ldots, a^{2k+2})$$
$$= \mathrm{Str}\big(\alpha_j(a^{2k+2})\alpha_j(a^0) D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma\big)$$
$$= \mathrm{Str}\big(\alpha_j(a^0) D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma \alpha_j(a^{2k+2})\big).$$

Thus, $b_{2k+2}\eta_j^t(a^0, \ldots, a^{2k+2}) - \beta_{2k+1}(a^0, \ldots, a^{2k+2})$ is equal to

(C.15) $\mathrm{Str}\big(\alpha_j(a^0) D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma\big(\alpha_j(a^{2k+2}) - a^{2k+2}\big)\big).$

Suppose that j is even, so that $\alpha_j(a) = a$. Then (C.15) shows that $b_{2k+2}\eta - \beta_{2k+1} = 0$. Moreover, $\alpha_j(a) - D^{-1}\sigma(a)D = D^{-1}[D,a]_\sigma$, and so using (C.14) we see that $b_0\eta - \beta_1'' = \beta$. Therefore, in this case (C.13) gives

$$b\eta_j^t = \beta - \frac{1}{2}(1+1)\beta + \frac{1}{2}(1-1)\gamma + 0 = 0.$$

When j is odd, $\alpha_j(a) = D^{-1}\sigma(a)D$, and we similarly find that $b_0\eta - \beta_1'' = 0$ and $b_{2k+2}\eta - \beta_{2k+1} = -\gamma$. Thus, in this case (C.13) gives

$$b\eta_j^t = 0 + -\frac{1}{2}(1-1)\beta + \frac{1}{2}(1+1)\gamma - \gamma = 0.$$

In any case, η_j^t is a Hochschild cocycle. The proof is complete. $\qquad\square$

Let us go back to the proof of Proposition C.1. In the same way as in the proof of Lemma C.2 it can be shown that each family $(\eta_j^t)_{0 \le t \le 1}$ is a continuous family of cochains. Note also that these cochains are normalized. Let η be the $(2k+1)$-cochain defined by

$$\eta = \sum_{j=1}^{2k+1} \int_0^1 \eta_j^t \, dt.$$

It follows from (C.2)–(C.3) and Lemma C.2 that

$$\begin{aligned}
B\eta &= \int_0^1 \Big(\sum_{j=1}^{2k+1} B\eta_j^t \Big) \, dt = (2k+1)c_k^{-1} \int_0^1 \Big(\frac{d}{dt} \tau_{2k}^{D_t} \Big) \, dt \\
&= (2k+1)c_k^{-1}(\tau_{2k}^{D_1} - \tau_{2k}^{D_0}).
\end{aligned}$$

(C.16)

Moreover, using (C.2) and Lemma C.3 we get

$$b\eta = \sum_{j=1}^{2k+1} \int_0^1 b\eta_j^t \, dt = 0.$$

In particular, as η is a normalized cochain and $b\eta$ is cyclic, we may apply (6.7) to get

$$(2k+1)c_k^{-1}(S\tau_{2k}^{D_1} - S\tau_{2k}^{D_0}) = SB\eta = -b\eta = 0 \quad \text{in } H_\lambda^{2k+2}(\mathcal{A}).$$

As by Remark 7.8 we know that $\tau_{2k+2}^{D_j}$ and $S\tau_{2k}^{D_j}$ are cohomologous in $H_\lambda^{2k+2}(\mathcal{A})$, and we then deduce that $\tau_{2k+2}^{D_0}$ and $\tau_{2k+2}^{D_1}$ define the same class in $H_\lambda^{2k+2}(\mathcal{A})$. This proves the second part of Proposition C.1. This also implies that $\tau_k^{D_0}$ and $\tau_k^{D_1}$ define the same class in $\mathrm{HP}^0(\mathcal{A})$, and so the twisted spectral triples $(\mathcal{A}, \mathcal{H}, D_0)_\sigma$ and $(\mathcal{A}, \mathcal{H}, D_1)_\sigma$ have the same Connes–Chern character in $\mathrm{HP}^0(\mathcal{A})$. This completes the proof of Proposition C.1.

REMARK C.4

By using the bounded Fredholm module pairs associated with a twisted spectral triple in [CM4], we also can deduce Proposition C.1 from the homotopy invariance

of the Connes–Chern character of a bounded Fredholm module in [Co1, Part I, Section 5].

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Ponge: Department of Mathematical Sciences, Seoul National University, Seoul, South Korea; ponge.snu@gmail.com

Wang: School of Mathematical Sciences, University of Adelaide, Adelaide, Australia; hang.wang01@adelaide.edu.au