On homological stability for orthogonal and special orthogonal groups

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Abstract We shall prove that the map $H_i(\mathrm{SO}_n(\mathbb{K}),\mathbb{Z}) \to H_i(\mathrm{SO}_{n+1}(\mathbb{K}),\mathbb{Z})$ is bijective for $2i < n$ and surjective for $2i \le n$. Here $\mathbb{K}$ is an arbitrary Pythagorean field and the special orthogonal group $\mathrm{SO}_n(\mathbb{K})$ is the subgroup of $\mathbb{K}$-linear automorphisms over $\mathbb{K}^n$ with determinant one which preserve the Euclidean quadratic form $\mathbf{q}(x) = x_1^2 + \cdots + x_n^2$. It is derived from the homological stability of the orthogonal groups $\mathrm{O}_n(\mathbb{K})$ with twisted coefficients $\mathbb{Z}^t$.

1. Introduction

1.1

Let $\iota_n \colon G_n \to G_{n+1}$ $(n \in \mathbb{N})$ be a sequence of groups, and let $\rho_n \colon M_n \to M_{n+1}$ $(n \in \mathbb{N})$ be a sequence of abelian groups where each M_n is a G_n-module and ρ_n is a G_n-module homomorphism through ι_n. It defines a sequence of homomorphisms on homology groups of G_n with coefficients in M_n:

$$(\iota_n)_* \colon H_i(G_n, M_n) \to H_i(G_{n+1}, M_{n+1}).$$

We say that a sequence of groups and modules (G_n, M_n) satisfies the homological stability if for any i there exists n_i such that if $n > n_i$, then $(\iota_n)_*$ is an isomorphism. There are plenty of sequences of groups and modules which have the homological stability, and we are interested in the following cases.

Let $\mathrm{O}_n(\mathbb{K})$ be the orthogonal group over a field $\mathbb{K}$. It is the subgroup of linear transformations on $\mathbb{K}^n$ preserving the Euclidean quadratic form $\mathbf{q}(x) = \sum x_i^2$ so that $\mathrm{O}_n(\mathbb{K}) = \{x \in \mathrm{GL}_n(\mathbb{K}) \mid x^t x = E_n\}$. A quadratic space which is isometric to $(\mathbb{K}^n, \mathbf{q})$ is called a Euclidean space. Now let $\mathbb{K}$ be a Pythagorean field, which means that the sum of two squares in $\mathbb{K}^n$ is always a square (see [4, Definition 8.3]), of characteristic different from 2. Quadratically closed fields and real-closed fields are typical examples of Pythagorean fields. In particular, the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$ are Pythagorean. Note that a field is Pythagorean if and only if every nondegenerate linear subspace of a Euclidean space is again Euclidean. Note also that, for any odd prime p and any positive integer f, a finite field of p^f elements has $(p^f + 1)/2$ squares. Since

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p is an odd prime, $(p^f+1)/2$ does not divide p^f. This means that a Pythagorean field of characteristic different from 2 is never finite.

There is a standard inclusion $\iota_n\colon \mathrm{O}_n(\mathbb{K}) \to \mathrm{O}_{n+1}(\mathbb{K})$. We will let $\mathbb{Z}$ be the abelian group of integers with the trivial action. We denote by $H_i(G)$ the homology group with coefficients in $\mathbb{Z}$. Let $\mathbb{Z}^t$ be the abelian group of integers with the action through the determinant. This means that an element g in $\mathrm{O}_n(\mathbb{K})$ acts on n in $\mathbb{Z}$ as $(\det g)n$. Then $(\mathrm{O}_n(\mathbb{K}), \mathbb{Z})$ and $(\mathrm{O}_n(\mathbb{K}), \mathbb{Z}^t)$ make sequences of groups and modules. The identity morphism on $\mathbb{Z}$ induces a sequence of homomorphisms on homology groups $H_i(\mathrm{O}_n(\mathbb{K})) \to H_i(\mathrm{O}_{n+1}(\mathbb{K}))$ and $H_i(\mathrm{O}_n(\mathbb{K}), \mathbb{Z}^t) \to H_i(\mathrm{O}_{n+1}(\mathbb{K}), \mathbb{Z}^t)$. Let $\mathrm{SO}_n(\mathbb{K})$ denote the special orthogonal subgroup. If we restrict to SO_n, then we get an isomorphism $\mathbb{Z} = \mathbb{Z}^t$ of SO_n-modules. It defines a sequence of homomorphisms on homology groups $H_i(\mathrm{SO}_n(\mathbb{K})) \to H_i(\mathrm{SO}_{n+1}(\mathbb{K}))$.

We will prove that the following homological stability statements hold for any Pythagorean field $\mathbb{K}$ of characteristic different from 2.

THEOREM 1.1
Let $\mathbb{K}$ be a Pythagorean field of characteristic different from 2. The induced maps on homology

$$(\iota_n)_*\colon H_i\big(\mathrm{SO}_n(\mathbb{K}), \mathbb{Z}\big) \to H_i\big(\mathrm{SO}_{n+1}(\mathbb{K}), \mathbb{Z}\big)$$

are bijective if $2i < n$ and surjective if $2i \le n$.

THEOREM 1.2
Let $\mathbb{K}$ be a Pythagorean field of characteristic different from 2. The induced maps on homology

$$(\iota_n)_*\colon H_i\big(\mathrm{O}_n(\mathbb{K}), \mathbb{Z}^t\big) \to H_i\big(\mathrm{O}_{n+1}(\mathbb{K}), \mathbb{Z}^t\big)$$

are bijective if $2i < n$ and surjective if $2i \le n$.

The theorems above extend and complement the following results, which are due to C. H. Sah and J.-L. Cathelineau.

THEOREM 1.3
(a) *The induced maps*

$$(\iota_n)_*\colon H_i\big(\mathrm{O}_n(\mathbb{K})\big) \to H_i\big(\mathrm{O}_{n+1}(\mathbb{K})\big)$$

are bijective if $i < n$ and surjective if $i \le n$ (see [5], [2]).

(b) *Let $\mathbb{Z}[1/2]$ be the ring of rational numbers whose denominators are powers of 2. Then on homology with $\mathbb{Z}[1/2]$-coefficients, the induced maps*

$$(\iota_n)_*\colon H_i\big(\mathrm{SO}_n(\mathbb{K}), \mathbb{Z}[1/2]\big) \to H_i\big(\mathrm{SO}_{n+1}(\mathbb{K}), \mathbb{Z}[1/2]\big)$$

are bijective if $2i < n$ and surjective if $2i \le n$ (see [2]).

(c) *The homology groups with twisted $\mathbb{Z}[1/2]$-coefficients $H_i(\mathrm{O}_{2n}(\mathbb{K}), \mathbb{Z}[1/2]^t)$ are trivial if $i < n$ (see [2]).*

(d) *For the field of real numbers* $\mathbb{R}$,

$$H_2(\mathrm{SO}_3(\mathbb{R})) \to H_2(\mathrm{SO}_n(\mathbb{R})) \to H_2(\mathrm{SO}_{n+1}(\mathbb{R}))$$

are bijective if $n \geq 5$ *(see [5]).*

Cathelineau proved that the kernel of $(\iota_n)_*$ in $H_n(\mathrm{SO}_{2n}(\mathbb{K}), \mathbb{Z}[1/2])$ is equal to $H_n(\mathrm{O}_{2n}(\mathbb{K}), \mathbb{Z}[1/2]^t)$, and if $\mathbb{K}$ is quadratically closed, then this kernel is the nth Milnor K-group of $\mathbb{K}$ tensored with $\mathbb{Z}[1/2]$, which is not zero in general (see [2, Theorem 1.5]). It is also conjectured that $H_i(\mathrm{O}_{2n}(\mathbb{K}), \mathbb{Z}^t)$ is closely connected to motivic cohomology groups of $\mathbb{K}$ if $n < i < 2n$, which is supposed to be far from zero in general. We note also that these groups play an important role in the calculation of scissors congruence groups of spheres (see [3]).

We will see in the last section that $H_i(\mathrm{SO}_n) \to H_i(\mathrm{O}_n)$ are injective in the range of stability above.

1.2. Notations

A Pythagorean field $\mathbb{K}$ is fixed. Let us denote $\mathrm{O}_n(\mathbb{K})$ just by O_n. We act similarly for SO_n.

We use a standard isometric embedding of Euclidean spaces

$$\mathbb{K}^n \to \mathbb{K}^{n+1}, \qquad v \mapsto (0, v),$$

which defines the inclusion map

$$\iota_n : \mathrm{O}_n \to \mathrm{O}_{n+1}, \qquad g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

and its restriction between special orthogonal subgroups. Note that any other isometric embeddings are conjugate to the above one by the Witt extension theorem (see [4, p. 26]); hence, ι_n induces the same map in homology.

2. Proofs of Theorems 1.2 and 1.1

2.1. Complex C.

An l-simplex is an ordered $(l+1)$-tuple of vectors $(v_0, \ldots, v_l)$ in $\mathbb{K}^{n+1}$. We assume that all v_i's are on $S(\mathbb{K}^{n+1}) = \{v \in \mathbb{K}^{n+1} \mid q(v) = 1\}$. We call each v_i a vertex of the simplex, and we call an ordered $(k+1)$-tuple $(w_0, \ldots, w_k)$ a face of the simplex if it is obtained from $(v_0, \ldots, v_l)$ by discarding some vertices. We say that an l-simplex is nondegenerate if the linear space spanned by all of its vertices is nondegenerate with respect to the quadratic form. An l-simplex $(v_0, \ldots, v_l)$ is called *geometric* if all of its faces are nondegenerate (see [2, Definition 2.1]).

In this paper we say that a geometric simplex $(v_0, \ldots, v_l)$ is *normal* if the set of vertices contains neither redundant pairs nor antipodal pairs. That is, for any different i and j, $v_i \neq v_j$ and $v_i \neq -v_j$. Notice that every face of a normal simplex is again normal. Let C_l denote the free $\mathbb{Z}$-module generated by normal l-simplices. We have that O_{n+1} acts diagonally on l-simplices:

$$g \cdot (v_0, \ldots, v_l) := (gv_0, \ldots, gv_l),$$

and this action sends any normal simplex to another normal simplex; hence, C_l is an O_{n+1}-module. We can define a homomorphism $\partial_l\colon C_l \to C_{l-1}$ as

$$\partial_l(v_0,\dots,v_l) := \sum_{i=0}^{l}(-1)^i(v_0,\dots,\hat{v}_i,\dots,v_l).$$

These define a chain complex of $\mathbb{Z}\mathrm{O}_{n+1}$-modules, and it has the augmentation homomorphism of the O_{n+1}-module, where $a\colon C_0 \to \mathbb{Z}$ is sending each 0-vertex to 1. Then

$$0 \leftarrow \mathbb{Z} \overset{a}{\leftarrow} C_0 \overset{\partial_1}{\longleftarrow} C_1 \overset{\partial_2}{\longleftarrow} C_2 \leftarrow \cdots$$

is exact. This fact is derived from the *extension property* given in [2, Proposition 2.6(ii)] and [5]. Thus we get a resolution $C.$ of $\mathbb{Z}$.

We set $C^t_. = C. \otimes \mathbb{Z}^t$, and then $C^t_. \to \mathbb{Z}^t$ is a resolution. The associated spectral sequence (filtration by rows; see [6, Definition 5.6.2]) $E^1_{p,q} := H_p(\mathrm{O}_{n+1}, C^t_q)$ strongly converges to $H_{p+q}(\mathrm{O}_{n+1},\mathbb{Z}^t)$.

2.2

We have that $C.$ is a subcomplex of the resolution associated with geometric simplices studied by Sah [5, Section 1]. We may use a variant of $C.$ consisting of geometric simplices without having antipodal pairs of vertices. Then it would be a subcomplex of $C_*(n)$ in [2, Proposition 2.5] studied by Cathelineau.

2.3

There exists a filtration $\mathcal{F}^s$ of chain complexes of O_{n+1}-modules on $C.$ (see [5, Section 1], [2, Proposition 2.6]); $\mathcal{F}^s$ is generated by simplices c having $\dim(c)$ less than or equal to $(s+1)$, where $\dim(c)$ is the dimension of the linear subspace in $\mathbb{K}^{n+1}$ spanned by the vertices of c. It is an increasing filtration of O_{n+1}-modules on $C.$, which induces a filtration $F^\bullet_p$ on $(E^1_{p,\cdot}, d^1)$ for each p as $(F^s_p)_q = H_p(\mathrm{O}_{n+1},(\mathcal{F}^s)_q)$.

2.4

We can choose a representative $(v_0,\dots,v_l)$ in the O_{n+1}-orbit of any simplex c so that all the v_i's are in the $\mathbb{K}$-linear subspace spanned by the standard orthonormal bases $e_1,\dots,e_{\dim(c)}$. We will write the orbit class which represents a simplex $(v_0,\dots,v_l)$ as $[v_0,\dots,v_l]$.

2.5

We will prove by induction on n the following statement:

$$(2.1{:}n) \qquad (\iota_n)_*\colon H_i(\mathrm{O}_n,\mathbb{Z}^t) \to H_i(\mathrm{O}_{n+1},\mathbb{Z}^t) \text{ is } \begin{cases}\text{bijective} & \text{if } 2i < n,\\ \text{surjective} & \text{if } 2i \le n.\end{cases}$$

Note that if n is an odd number $n = 2m+1$, then O_{2m+1} contains a scalar matrix -1_{2m+1} of -1, which has $\det(-1_{2m+1}) = -1$. Therefore the center kills lemma (see [3, Lemma 5.4]) tells us that

$$H_i(\mathrm{O}_{2m+1},\mathbb{Z}^t)\cong H_i(\mathrm{O}_{2m+1},\mathbb{Z}^t)\otimes\mathbb{Z}/2 \tag{2.2}$$

for every i and m. Thus, if (2.1:n) is true, the following statement holds:

$$\text{if } 2i<n, \text{ then } H_i(\mathrm{O}_n,\mathbb{Z}^t)\cong H_i(\mathrm{O}_n,\mathbb{Z}^t)\otimes\mathbb{Z}/2. \tag{2.3}$$

Because $\mathrm{O}_0=\{1\}$ and $\mathrm{O}_1=\mathbb{Z}/2$, the map

$$\mathbb{Z}=H_0(\mathrm{O}_0,\mathbb{Z}^t)\xrightarrow{H_0(\iota_0,\mathbb{Z}^t)}H_0(\mathrm{O}_1,\mathbb{Z}^t)=\mathbb{Z}/2$$

between coinvariant parts coincides with the epimorphism. We also have that

$$\mathbb{Z}/2=H_0(\mathrm{O}_1,\mathbb{Z}^t)\xrightarrow{H_0(\iota_1,\mathbb{Z}^t)}H_0(\mathrm{O}_2,\mathbb{Z}^t)=\mathbb{Z}/2$$

is bijective; hence, (2.1:0) and (2.1:1) are true. We may assume that $n\geq 2$ from now on.

Firstly we have to show that

$$E^1_{p,0}=H_p(\mathrm{O}_{n+1},C^t_0)\cong H_p(\mathrm{O}_n,\mathbb{Z}^t)\cong E^2_{p,0}. \tag{2.4}$$

From Shapiro's lemma (see [1, Proposition 6.2] or [3, Lemma 5.5]) we obtain that the first isomorphism $E^1_{p,0}\cong H_p(\mathrm{O}_n,\mathbb{Z}^t)$ for the stabilizer subgroup of 0-simplex is isomorphic to O_n. We have that

$$E^1_{p,1}=H_p(\mathrm{O}_{n+1},C^t_1)\cong\bigoplus_c H_p\big(\mathrm{Stab}(c),\mathbb{Z}^t\big)\otimes\mathbb{Z}c,$$

where the index c runs through all the O_{n+1}-orbits of simplices in C^t_1, and $\mathrm{Stab}(c)$ is the stabilizer subgroup of c in O_{n+1}, where all the groups $\mathrm{Stab}(c)$ are isomorphic to O_{n-1} in this case.

Now let $c=(v_0,v_1)$ be a normal 1-simplex, and let α be an element in $H_p(\mathrm{Stab}(c),\mathbb{Z}^t)$; then we have that

$$d^1_{p,1}\big(\alpha\otimes(v_0,v_1)\big)=\alpha\otimes(v_1)-\alpha\otimes(v_0).$$

We can find an element $g\in\mathrm{O}_{n+1}$ so that $g(v_1)=v_0$ and $\det(g)=1$ for $v_0\neq\pm v_1$ by the assumption of normality. Any such g commutes with all the elements of $\mathrm{Stab}(c)$, and g acts trivially on $H_i(\mathrm{Stab}(c),\mathbb{Z}^t)$; hence $\alpha\otimes(v_1)=\alpha\otimes g(v_0)=\alpha\otimes(v_0)$ in $H_i(\mathrm{Stab}(c),\mathbb{Z}^t)$. This induces $d^1_{p,1}(\alpha\otimes c)=0$. Thus, $d^1_{p,1}=0$ on $E^1_{p,1}$, which implies (2.4).

Secondly we have to show that

(2.5:p) $\quad E^1_{p,*}$ is $(n-2p-2)$-acyclic for $0\leq 2p<n$ augmented by $E^1_{p,0}$

under the inductive hypothesis (2.1:n') for all $n'<n$.

If a geometric simplex c has $\dim(c)\leq n-2p$, then by the hypothesis of induction (2.1:p), we get that

$$H_p\big(\mathrm{Stab}(c),\mathbb{Z}^t\big)\cong H_p(\mathrm{O}_{n+1-\dim(c)},\mathbb{Z}^t)\cong H_p(\mathrm{O}_{2p+1},\mathbb{Z}^t).$$

Thus, if $q \le n-2p-1$, then it holds that

$$E^1_{p,q} = H_p(\mathrm{O}_{n+1}, C^t_q) \cong \bigoplus_c H_p\big(\mathrm{Stab}(c), \mathbb{Z}^t\big) \otimes \mathbb{Z}c \\ \cong H_p(\mathrm{O}_{2p+1}, \mathbb{Z}^t) \otimes \bigoplus_c \mathbb{Z}c. \tag{2.5}$$

In particular, as we saw in (2.2) we have that the elements in (2.5) are annihilated by 2. (Notice that, through the isomorphism of Shapiro's lemma (2.5), $d^1_{p,*}$ may not equal $\mathrm{id}_{H_p(\mathrm{O}_{2p+1},\mathbb{Z}^t)} \otimes \partial_*$, because the action of O_{n+1} on $C.$ is twisted in $H_p(\mathrm{O}_{n+1}, C^t_q)$ by the determinant and these data may cause a change of sign on the fixed representatives of O_{n+1}-orbits. But this problem can be ignored because of (2.3) and the induction hypothesis in this case.)

We take l arbitrarily for $0 < l \le n-2p-2$. Let $\gamma \in E^1_{p,l}$ satisfy $d^1_{p,l}(\gamma) = 0$. Apply (2.5), so

$$\gamma = \sum_j \alpha_j \otimes [v^j_0, \dots, v^j_l],$$

where each α_j is in $H_p(\mathrm{O}_{2p+1}, \mathbb{Z}^t)$ and $(v^j_0, \dots, v^j_l) \in C_l$ is a representative chosen as in Section 2.4. Then we have $\mathrm{Span}_{\mathbb{K}}(v^j_0, \dots, v^j_l) \perp e_{l+2}$ ($\mathrm{Span}_{\mathbb{K}}$ means the linear span of vectors), and the inclusion $\mathrm{O}_{2p+1} \hookrightarrow \mathrm{Stab}(v^j_0, \dots, v^j_l)$ factors through $\mathrm{Stab}(v^j_0, \dots, v^j_l, e_{l+2})$ for $l \le n-2p-2$. Since $\mathbb{K}$ is Pythagorean, $[v_0, \dots, v_l, e_{l+2}]$ has a representative of geometric and thus normal simplex. Define $\gamma\#e$ as follows. For each orbit class of a normal l-simplex $\gamma = (v_0, \dots, v_l)$, we set $\gamma\#e = [v_0, \dots, v_l, e_{l+2}]$. Then $\gamma\#e$ is normal and we extend this linearly: $\gamma\#e = \sum_j \alpha_j \otimes [v^j_0, \dots, v^j_l, e_{l+2}]$, which is contained in $E^1_{p,l+1}$. (This construction is called *orthogonal join construction* by Sah in [5, proof of (1.5)].) From Witt's extension theorem, we see that

$$d^1_{p,l+1}(\gamma\#e) = d^1_{p,l}(\gamma)\#e + (-1)^{l+1}\gamma.$$

Since $d^1_{p,l}(\gamma) = 0$, we obtain that $d^1_{p,l+1}(\gamma\#e) = (-1)^{l+1}\gamma$.

Finally we have to extend the acyclicity of $E^1_{p,*}$ one more degree above:

(2.6:p) $E^1_{p,*}$ is $(n-2p-1)$-acyclic for $0 \le 2p < n$.

Again we have that

$$E^1_{p,n-2p} = \bigoplus_c H_p\big(\mathrm{Stab}(c), \mathbb{Z}^t\big) \otimes \mathbb{Z}c \\ = \bigoplus_c H_p(\mathrm{O}_{2p}, \mathbb{Z}^t) \otimes \mathbb{Z}c \oplus \bigoplus_{c'} H_p(\mathrm{O}_{2p+1}, \mathbb{Z}^t) \otimes \mathbb{Z}c', \tag{2.6}$$

where the index c in the first sum runs through O_{n+1}-orbits of simplices in C_{n-2p} which satisfy $\dim(c) = n-2p+1$, and the index c' in the second sum runs through O_{n+1}-orbits of simplices which satisfy $\dim(c') \le n-2p$, that is, the second sum is in the associated filtration F^{n-2p-2}_p.

Let $\gamma \in E^1_{p,n-2p-1}$ be such that $d^1_{p,n-2p-1}(\gamma) = 0$. If $\gamma \in F_p^{n-2p-2}$, then the orthogonal join $\gamma \# e$ constructed as before is contained in the second component in (2.6), and it is a boundary element.

If $\gamma \notin F_p^{n-2p-2}$, then we may assume that γ is homologous to an element $\sum_j \alpha_j \otimes c_j$, where α_j is in $H_p(\mathrm{O}_{2p+1}, \mathbb{Z}^t)$ and c_j is an O_{n+1}-orbit of an $(n-2p-1)$-simplex. Since $\max\{\dim(c_j)\} = n-2p$, we have that $\max\{\dim(c_j \# e_{n-2p+1})\} = n-2p+1$. The map $H_p(\mathrm{O}_{2p}, \mathbb{Z}^t) \to H_p(\mathrm{O}_{2p+1}, \mathbb{Z}^t)$ is surjective by the induction hypothesis (2.1:p), so we can find $\beta_j \in H_p(\mathrm{O}_{2p}, \mathbb{Z}^t)$ such that $H_p(\iota_{2p}, \mathbb{Z}^t)(\beta_j) = \alpha_j$ for each j. Using these β_j's, we obtain that

$$\begin{aligned} d^1_{p,n-2p}\Big(\sum_j \beta_j \otimes (c_j \# e_{n-2p+1})\Big) \\ &= \pm \sum_j \alpha_j \otimes (\partial c_j) \# e_{n-2p+1} + (-1)^{n-2p} \sum_j \alpha_j \otimes c_j \\ &= \big(d^1_{p,n-2p-1}(\gamma)\big) \# e + (-1)^{n-2p}\gamma \\ &= (-1)^{n-2p}\gamma, \end{aligned}$$

and therefore we have proved that γ is a boundary, which implies (2.6:p).

2.6

On the spectral sequence $E^1_{p,q} = H_p(\mathrm{O}_{n+1}, C_q^t) \Rightarrow H_{p+q}(\mathrm{O}_{n+1}, \mathbb{Z}^t)$, we know that, under the inductive assumption, $E^2_{p,0} \cong H_p(\mathrm{O}_n, \mathbb{Z}^t)$ (see (2.4)) and $E^2_{p,q} = 0$ for $0 < q \leq n-2p-1$ (see (2.6)). Therefore, the edge homomorphism coincides with the $(\iota_n)_*$:

$$(\iota_n)_* \colon H_i(\mathrm{O}_n, \mathbb{Z}^t) \to H_i(\mathrm{O}_{n+1}, \mathbb{Z}^t),$$

which is bijective for $2i < n$ and surjective for $2i \leq n$. This ends the proof of Theorem 1.2.

2.7. Bockstein exact sequences

The group ring $\mathbb{Z}[\mathbb{Z}/2]$ of $\mathbb{Z}/2 = \{\epsilon, \sigma \mid \sigma^2 = \epsilon\}$ admits the action of O_n through the determinant:

$$\begin{aligned} g\cdot\epsilon &= \epsilon, & g\cdot\sigma &= \sigma & &\text{if } \det(g) = 1, \\ g\cdot\epsilon &= \sigma, & g\cdot\sigma &= \epsilon & &\text{if } \det(g) = -1, \end{aligned}$$

for $g \in \mathrm{O}_n$. There exist an inclusion

$$\mathbb{Z}^t \to \mathbb{Z}[\mathbb{Z}/2], \quad 1 \mapsto \epsilon - \sigma$$

and a projection

$$\mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}[\mathbb{Z}/2]/(\epsilon - \sigma) \cong \mathbb{Z}$$

of (left) $\mathbb{Z}\mathrm{O}_n$-modules for $n \geq 0$. It makes a short exact sequence of $\mathbb{Z}\mathrm{O}_n$-modules

$$0 \to \mathbb{Z}^t \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z} \to 0, \tag{2.7}$$

and we see that $H_i(\mathrm{O}_n, \mathbb{Z}[\mathbb{Z}/2]) \cong H_i(\mathrm{SO}_n)$. (Use Shapiro's lemma and the fact that the stabilizer of O_n on $\mathbb{Z}[\mathbb{Z}/2]$ is SO_n.) We get a homology Bockstein exact sequence

$$\cdots \to H_{i+1}(\mathrm{O}_n) \to H_i(\mathrm{O}_n, \mathbb{Z}^t) \to H_i(\mathrm{SO}_n) \to H_i(\mathrm{O}_n) \to H_{i-1}(\mathrm{O}_n, \mathbb{Z}^t) \to \cdots. \tag{2.8}$$

The inclusion $\iota_n \colon \mathrm{O}_n \to \mathrm{O}_{n+1}$ induces a homomorphism between exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \to & H_{i+1}(\mathrm{O}_n) & \to & H_i(\mathrm{O}_n, \mathbb{Z}^t) & \to & H_i(\mathrm{SO}_n) & \to & H_i(\mathrm{O}_n) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & H_{i+1}(\mathrm{O}_{n+1}) & \to & H_i(\mathrm{O}_{n+1}, \mathbb{Z}^t) & \to & H_i(\mathrm{SO}_{n+1}) & \to & H_i(\mathrm{O}_{n+1}) & \to & \cdots, \end{array}$$

where columns are exact and maps in the vertical maps are induced from group inclusions $\iota_n \colon \mathrm{SO}_n \to \mathrm{SO}_{n+1}$ and $\iota_n \colon \mathrm{O}_n \to \mathrm{O}_{n+1}$. If we adapt Theorem 1.3(a) and (2.1:n) in the above diagram, then, using the five lemma, we obtain Theorem 1.1.

REMARK 2.1

We have another short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}^t \to 0 \tag{2.9}$$

consisting of

$$\mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/2], \qquad 1 \mapsto \epsilon + \sigma$$

and

$$\mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}[\mathbb{Z}/2]/(\epsilon + \sigma) \cong \mathbb{Z}^t.$$

REMARK 2.2

If we use the unmodified complex $(\mathcal{C}_*, \partial_*)$ used in [2, Proposition 2.5] (this may contain antipodal pairs but not contain simplices which have $v_{i-1} = v_i$ for some i), then

$$\mathcal{E}^2_{p,0} = H_p(\mathrm{O}_{n+1}, \mathcal{C}^t_0) \cong H_p(\mathrm{O}_n, \mathbb{Z}^t) \otimes \mathbb{Z}/2.$$

This is because $\mathcal{C}_1$ admits the simplex $(v, -v)$ and the reflection that maps v to $-v$ has determinant -1. The spectral sequence defined by $\mathcal{E}^1_{p,q} = H_p(\mathrm{O}_{n+1}, \mathcal{C}^t_q)$ is also strongly convergent to $H_{p+q}(\mathrm{O}_{n+1}, \mathbb{Z}^t)$. Thus we can see that

$$H_i(\iota_n, \mathbb{Z}^t) \colon H_i(\mathrm{O}_n, \mathbb{Z}^t) \to H_i(\mathrm{O}_{n+1}, \mathbb{Z}^t)$$

factors through $H_i(\mathrm{O}_n, \mathbb{Z}^t) \otimes \mathbb{Z}/2$ for all n and i. This implies that, though it is contained in an unstable range, $\operatorname{Im} H_i(\iota_n, \mathbb{Z}^t)$ is annihilated by 2.

2.8. $\mathbb{Z}/2$-coefficients

We can improve the range of homological stability of special orthogonal groups with coefficients in $\mathbb{Z}/2$. We use only Theorem 1.3 and the Bockstein exact sequence.

We have $(\mathbb{Z}/2)^t \cong \mathbb{Z}/2$ as O_n-modules. Thus we have the same short exact sequence of O_n-modules

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2[\mathbb{Z}/2] \to \mathbb{Z}/2 \to 0.$$

In the same way if we construct the Bockstein exact sequence from (2.8), then we get a long exact sequence

$$\cdots \to H_{i+1}(\mathrm{O}_n,\mathbb{Z}/2) \to H_i(\mathrm{O}_n,\mathbb{Z}/2) \to H_i(\mathrm{SO}_n,\mathbb{Z}/2) \to H_i(\mathrm{O}_n,\mathbb{Z}/2) \to \cdots.$$

From the universal coefficient theorem, Theorem 1.3(a) means that

$$H_i(\mathrm{O}_n,\mathbb{Z}/2) \to H_i(\mathrm{O}_{n+1},\mathbb{Z}/2) \text{ is bijective for } i < n \text{ and surjective for } i \le n.$$

Thus as in Section 2.7 we get the following result.

PROPOSITION 2.3

The map $H_i(\mathrm{SO}_n,\mathbb{Z}/2) \to H_i(\mathrm{SO}_{n+1},\mathbb{Z}/2)$ is bijective for $i < n$ and surjective for $i \le n$.

3. Variants

We consider a semidirect product of groups

$$1 \to \mathrm{SO}_n \to \mathrm{O}_n \xrightarrow{\det} \mathbb{Z}/2 \to 1 \quad \text{for } n \ge 1 \tag{3.1}$$

with a section

$$s_n : \mathbb{Z}/2 \to \mathrm{O}_n \tag{3.2}$$

as $s_n(-1) = \mathrm{diag}(-1,1,1,\dots,1)$.

In the case $n = 2m+1$, it becomes the direct product of groups

$$\mathrm{O}_{2m+1} \cong \mathrm{SO}_{2m+1} \times \mathbb{Z}/2.$$

Thus there is the Künneth short exact sequence

$$\oplus H_p(\mathrm{SO}_{2m+1}) \otimes H_q(\mathbb{Z}/2) \hookrightarrow H_i(\mathrm{O}_{2m+1}) \twoheadrightarrow \oplus \mathrm{Tor}_1^{\mathbb{Z}}\big(H_p(\mathrm{SO}_{2m+1}), H_q(\mathbb{Z}/2)\big), \tag{3.3}$$

and it is comparable with ι_{2m+1}. It is true that

$$H_i(\mathrm{O}_{2m+1}) \to H_i(\mathrm{O}_{2m+3}) \text{ is bijective for } i < 2m+1 \text{ and surjective for } i \le 2m+1. \tag{3.4}$$

When $i \le 2m+1$, ι_{2m+1} induces an isomorphism on the Tor terms by Section 1.1. We obtain the following result.

PROPOSITION 3.1

We have that $H_i(\mathrm{SO}_{2m+1}) \to H_i(\mathrm{SO}_{2m+3})$ is bijective for $i < 2m+1$ and surjective for $i \le 2m+1$.

On the other hand, (2.9) implies that

$$\begin{array}{ccccccccc} \cdots \to & H_{i+1}(\mathrm{O}_{2m+1}, \mathbb{Z}^t) & \to & H_i(\mathrm{O}_{2m+1}) & \to & H_i(\mathrm{SO}_{2m+1}) & \to & H_i(\mathrm{O}_{2m+1}, \mathbb{Z}^t) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_{i+1}(\mathrm{O}_{2m+3}, \mathbb{Z}^t) & \to & H_i(\mathrm{O}_{2m+3}) & \to & H_i(\mathrm{SO}_{2m+3}) & \to & H_i(\mathrm{O}_{2m+3}, \mathbb{Z}^t) & \to \cdots \end{array} \tag{3.5}$$

Thus we obtain that

$$\begin{array}{c} H_i(\mathrm{O}_{2m+1}, \mathbb{Z}^t) \to H_i(\mathrm{O}_{2m+3}, \mathbb{Z}^t) \\ \text{is bijective for } i < 2m+1 \text{ and surjective for } i \le 2m+1. \end{array} \tag{3.6}$$

Notice that, using (2.1:n) and (3.6), we have that the sequence

$$H_i(\mathrm{O}_{2m-1}, \mathbb{Z}^t) \to H_i(\mathrm{O}_{2m}, \mathbb{Z}^t) \to H_i(\mathrm{O}_{2m+1}, \mathbb{Z}^t)$$

splits as

$$H_i(\mathrm{O}_{2m}, \mathbb{Z}^t) \cong H_i(\mathrm{O}_{2m-1}, \mathbb{Z}^t) \oplus K_{m,i}$$

for $i < 2m$, where $K_{m,i} = \mathrm{Ker}\{H_i(\mathrm{O}_{2m}, \mathbb{Z}^t) \to H_i(\mathrm{O}_{2m+1}, \mathbb{Z}^t)\}$.

As we mentioned in (2.2), we have that $H_i(\mathrm{O}_{2m+1}, \mathbb{Z}^t) \cong H_i(\mathrm{O}_{2m+1}, \mathbb{Z}^t) \otimes \mathbb{Z}/2$. Thus we get that, for $m \le i < 2m$,

$$\begin{aligned} H_i(\mathrm{O}_{2m}, \mathbb{Z}^t) &\cong \operatorname*{colim}_n H_i(\mathrm{O}_n, \mathbb{Z}^t) \oplus K_{m,i} \\ &\cong H_i(\mathrm{O}_\infty, \mathbb{Z}^t) \otimes \mathbb{Z}/2 \oplus K_{m,i}. \end{aligned}$$

4. $(\mathbb{Z}/2)$-action on $H_*(\mathrm{SO}_n)$

There is a $(\mathbb{Z}/2)$-action on $H_i(\mathrm{SO}_n)$ induced from the group extension (3.1). Let σ denote the involution induced by $\sigma \in \mathbb{Z}/2 = \{\epsilon, \sigma\}$. The structure of this involution is important to apply the homological result to the problem of scissors congruence.

4.1. Involution σ

PROPOSITION 4.1

The involution σ on $H_i(\mathrm{SO}_n)$ is trivial if $2i < n$.

We can write the action of the involution σ on the bar resolution of $H_i(\mathrm{SO}_n)$ (see [1, Chapter I, Section 5]) as

$$[g_1 \mid \cdots \mid g_i] \mapsto \big[s_n(-1) g_1 s_n(-1)^{-1} \bigm| \cdots \bigm| s_n(-1) g_i s_n(-1)^{-1}\big].$$

For convenience, we write $\iota_n(g) = (1, g)$ for $\iota_n\colon \mathrm{SO}_n \to \mathrm{SO}_{n+1}$, and let g^σ denote the image $s_n(-1)gs_n(-1)^{-1}$. We get that

$$\begin{aligned}
&\bigl(-1, s_n(-1)\bigr)\iota_n(g^\sigma)\bigl(-1, s_n(-1)\bigr)^{-1} \\
&\quad = \bigl(-1, s_n(-1)\bigr)\bigl(1, s_n(-1)gs_n(-1)^{-1}\bigr)\bigl(-1, s_n(-1)\bigr)^{-1} \\
&\quad = (-1, 1_n)(1, g)(-1, 1_n)^{-1} = (1, g).
\end{aligned}$$

Since $(-1, s_n(-1)) = \mathrm{diag}(-1, -1, 1, \dots, 1)$ is contained in SO_{n+1}, $H_i(\iota_n) \circ \sigma = H_i(\iota_n)$. We obtain the following lemma.

LEMMA 4.2

We have that $H_i(\iota_n)\colon H_i(\mathrm{SO}_n) \to H_i(\mathrm{SO}_{n+1})$ factors through the σ-coinvariant part $H_i(\mathrm{SO}_n)_\sigma$:

$$\begin{array}{ccc}
H_i(\mathrm{SO}_n) & \xrightarrow{H_i(\iota_n)} & H_i(\mathrm{SO}_{n+1}) \\
\big\downarrow & \overset{\rho_n}{\nearrow} & \\
H_i(\mathrm{SO}_n)_\sigma & &
\end{array}$$

where the vertical map in the above diagram is the projection

$$H_i(\mathrm{SO}_n) \to H_i(\mathrm{SO}_n)/(1 - \sigma) = H_i(\mathrm{SO}_n)_\sigma.$$

On the other hand, Theorem 1.1 claims that if $2i < n$, then $H_i(\iota_n)$ must be an isomorphism; thus we have that

$$H_i(\mathrm{SO}_n)^\sigma \cong H_i(\mathrm{SO}_n) \cong H_i(\mathrm{SO}_n)_\sigma,$$

and this implies Proposition 4.1.

4.2. The edge homomorphism of the Lyndon–Hochschild–Serre spectral sequence

The group extension (3.1) induces the Lyndon–Hochschild–Serre spectral sequence (see [1, Chapter VII, Theorem 6.8] or [6, Section 6.8])

$$E^2_{p,q} = H_p\bigl(\mathbb{Z}/2, H_q(\mathrm{SO}_n)\bigr) \Rightarrow H_{p+q}(\mathrm{O}_n) \tag{4.1}$$

for $n \geq 0$. We will study the edge homomorphism

$$e_q\colon H_q(\mathrm{SO}_n)_\sigma = E^2_{0,q} \to E^\infty_{0,q} \to H_q(\mathrm{O}_n).$$

PROPOSITION 4.3

We have that $e_q\colon H_q(\mathrm{SO}_n)_\sigma \to H_q(\mathrm{O}_n)$ is injective for $q \geq 0$.

REMARK 4.4

As we can see in [6, Section 6.8], e_q is compatible with the map

$$H_i(u)\colon H_i(\mathrm{SO}_n) \to H_i(\mathrm{O}_n)$$

induced by the natural inclusion $u\colon \mathrm{SO}_n \to \mathrm{O}_n$. We know that $H_i(u)\circ\sigma = H_i(u)$; thus, $H_i(u)$ factors through the σ-coinvariant part $H_i(\mathrm{SO}_n)_\sigma$, which is the edge homomorphism e_q.

The compositions with transfer maps

$$H_i(\mathrm{SO}_n) \xrightarrow{H_i(u)} H_i(\mathrm{O}_n) \xrightarrow{\mathrm{tr}} H_i(\mathrm{SO}_n)$$

and

$$H_i(\mathrm{SO}_n) \xrightarrow{H_i(u,\mathbb{Z}^t)} H_i(\mathrm{O}_n,\mathbb{Z}^t) \xrightarrow{\mathrm{tr}^t} H_i(\mathrm{SO}_n)$$

are the norm maps $(1+\sigma)$ and $(1-\sigma)$, respectively (see [1, Chapter III, Proposition 9.5]). Thus we have that

$$\mathrm{Im}(\mathrm{tr}) \supseteq (1+\sigma)H_i(\mathrm{SO}_n) \tag{4.2}$$

and

$$\mathrm{Im}(\mathrm{tr}^t) \supseteq (1-\sigma)H_i(\mathrm{SO}_n),$$

where the maps tr and tr^t are identified as

$$H_i(\mathrm{O}_n) \to H_i\big(\mathrm{O}_n,\mathbb{Z}[\mathbb{Z}/2]\big) \xrightarrow{\cong} H_i(\mathrm{SO}_n) \tag{4.3}$$

and

$$H_i(\mathrm{O}_n,\mathbb{Z}^t) \to H_i\big(\mathrm{O}_n,\mathbb{Z}[\mathbb{Z}/2]\big) \xrightarrow{\cong} H_i(\mathrm{SO}_n) \tag{4.4}$$

in the Bockstein exact sequences (2.9) and (2.7), respectively. The map tr coincides with the trace map, and so does tr^t (see [1, Chapter III, Section 9]). Notice that the later map in (4.3) and (4.4) is an inverse of the map in Shapiro's lemma. It is induced from a map of chain complexes; namely,

$$\rho\colon [g_1 \mid g_2 \mid \cdots \mid g_i] \otimes g \otimes x \mapsto [\widehat{g}^{-1} g_1 \widehat{z_1} \mid \widehat{z_1}^{-1} g_2 \widehat{z_2} \mid \cdots \mid \widehat{z_{i-1}}^{-1} g_i \widehat{z_i}] \otimes (\widehat{g}^{-1} g)x,$$

where $\widehat{h} = \mathrm{diag}(\det(h),1,\dots,1)$ and $z_j = g_j^{-1}\cdots g_1^{-1} g$, gives an isomorphism $H_i(\mathrm{O}_n,\mathbb{Z}[\mathrm{O}_n]\otimes_{\mathbb{Z}[\mathrm{SO}_n]}\mathbb{Z}) \cong H_i(\mathrm{O}_n,\mathbb{Z}[\mathbb{Z}/2]) \cong H_i(\mathrm{SO}_n)$ (see [3, Remark after Lemma 5.5]). We can write the inverse direction

$$H_i(\mathrm{O}_n) \to H_i\big(\mathrm{O}_n,\mathbb{Z}[\mathbb{Z}/2]\big) \cong H_i(\mathrm{O}_n,\mathbb{Z}\mathrm{O}_n \otimes_{\mathbb{Z}\mathrm{SO}_n} \mathbb{Z})$$

as

$$\begin{aligned}[g_1 \mid \cdots \mid g_i] &\mapsto [g_1 \mid \cdots \mid g_i] \otimes \epsilon + [g_1 \mid \cdots \mid g_i] \otimes \sigma \\ &\mapsto [g_1 \mid \cdots \mid g_i] \otimes 1_n \otimes 1 + [g_1 \mid \cdots \mid g_i] \otimes s_n(-1) \otimes 1;\end{aligned}$$

hence, the composition with ρ is

$$\begin{aligned}&[\widehat{1_n}^{-1} g_1 \widehat{z_1} \mid \widehat{z_1}^{-1} g_2 \widehat{z_2} \mid \cdots \mid \widehat{z_{i-1}}^{-1} g_i \widehat{z_i}] \otimes (\widehat{1_n}^{-1} 1_n) \cdot 1 \\ &\quad + \big[\widehat{s_n(-1)}^{-1} g_1 \widehat{z'_1} \mid \widehat{z'_1}^{-1} g_2 \widehat{z'_2} \mid \cdots \mid \widehat{z'_{i-1}}^{-1} g_i \widehat{z'_i}\big] \otimes \big(\widehat{s_n(-1)}^{-1} s_n(-1)\big) \cdot 1 \\ &= [\widehat{1_n}^{-1} g_1 \widehat{z_1} \mid \widehat{z_1}^{-1} g_2 \widehat{z_2} \mid \cdots \mid \widehat{z_{i-1}}^{-1} g_i \widehat{z_i}] \otimes 1\end{aligned}$$

$$+ \big[s_n(-1)^{-1} g_1 \widehat{z_1} \mid s_n(-1)^{-1} \widehat{z_1}^{\,-1} g_2 \widehat{z_2} s_n(-1) \mid$$
$$\cdots \mid s_n(-1)^{-1} \widehat{z_{i-1}}^{\,-1} g_i \widehat{z_i} s_n(-1)\big] \otimes 1,$$

where we set $z_j = g_j^{-1} \cdots g_1^{-1} 1_n$ and $z'_j = g_j^{-1} \cdots g_1^{-1} s_n(-1)$. Now there is a chain homotopy (see [3, Lemma 5.4]) between

$$\big[s_n(-1)^{-1} g_1 \widehat{z_1} \mid s_n(-1)^{-1} \widehat{z_1}^{\,-1} g_2 \widehat{z_2} s_n(-1) \mid \cdots \mid s_n(-1)^{-1} \widehat{z_{i-1}}^{\,-1} g_i \widehat{z_i} s_n(-1)\big] \otimes 1$$

and

$$\sigma\big([\widehat{1_n}^{\,-1} g_1 \widehat{z_1} \mid \widehat{z_1}^{\,-1} g_2 \widehat{z_2} \mid \cdots \mid \widehat{z_{i-1}}^{\,-1} g_i \widehat{z_i}] \otimes 1\big).$$

Thus, from the above calculation, we get that

$$\mathrm{Im}(\mathrm{tr}) \subseteq (1+\sigma) H_i(\mathrm{SO}_n). \tag{4.5}$$

We can prove Proposition 4.3 by the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_i(\mathrm{O}_n, \mathbb{Z}^t) & \xrightarrow{\mathrm{tr}^t} & H_i(\mathrm{SO}_n) & \longrightarrow & H_i(\mathrm{O}_n) & \longrightarrow & \cdots \\ & & & & \downarrow & & \uparrow & & \\ & & & & \dfrac{H_i(\mathrm{SO}_n)}{(1-\sigma)H_i(\mathrm{SO}_n)} & = & H_i(\mathrm{SO}_n)_\sigma & & \end{array}$$

obtained by combining the exact sequence

$$H_i(\mathrm{O}_n, \mathbb{Z}^t) \xrightarrow{\mathrm{tr}^t} H_i(\mathrm{SO}_n) \to H_i(\mathrm{SO}_n)/(1-\sigma)H_i(\mathrm{SO}_n) \to 0$$

and the Bockstein exact sequence (2.8).

In the same way, we can see that, in the Lyndon–Hochschild–Serre spectral sequence

$${}^{t}E^2_{p,q} = H_p\big(\mathbb{Z}/2, H_q(\mathrm{SO}_n)^t\big) \Rightarrow H_{p+q}(\mathrm{O}_n, \mathbb{Z}^t),$$

the edge homomorphism

$${}^{t}e_q \colon H_q(\mathrm{SO}_n)_{-\sigma} \to H_q(\mathrm{O}_n, \mathbb{Z}^t)$$

is an injection.

COROLLARY 4.5

If $2i < n$, then σ on $H_i(\mathrm{SO}_n)$ is trivial as we saw in Proposition 4.1. Hence we obtain that $\mathrm{tr}^t \colon H_i(\mathrm{O}_n, \mathbb{Z}^t) \to H_i(\mathrm{SO}_n)$ is a zero map in this range.

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