On the minimality of canonically attached singular Hermitian metrics on certain nef line bundles

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Abstract   We apply Ueda theory to a study of singular Hermitian metrics of a (strictly) nef line bundle L. Especially we study minimal singular metrics of L, metrics of L with the mildest singularities among singular Hermitian metrics of L whose local weights are plurisubharmonic. In some situations, we determine a minimal singular metric of L. As an application, we give new examples of (strictly) nef line bundles which admit no smooth Hermitian metric with semipositive curvature.

1. Introduction

Our interest is the singularity of minimal singular metrics on nef line bundles over smooth projective surfaces. Minimal singular metrics of a line bundle L are metrics of L with the mildest singularities among singular Hermitian metrics of L whose local weights are plurisubharmonic. Minimal singular metrics were introduced in [DPS2, Definition 1.4] as a (weak) analytic analogue of the Zariski decomposition, and always exist when L is pseudoeffective (see [DPS2, Theorem 1.5]). In this paper, we mainly consider a topologically trivial line bundle on a surface which is defined by a smooth embedded curve with a neighborhood of nontrivial complex structure (rigorously speaking, when the curve is *of finite type* in the sense of [U, p. 589]; see Definition 3.2 here). The goal of this paper is to determine a minimal singular metric of such a line bundle. The main theorem is as follows.

THEOREM 1.1

Let X be a smooth complex surface, and let $C \subset X$ be an embedded smooth compact complex curve. Assume that $(C^2) = 0$ and that the pair (C, X) is of finite type. Then the singular Hermitian metric $|f_C|^{-2}$ on $\mathcal{O}_X(C)$ has minimal singularities, where $f_C \in H^0(X, \mathcal{O}_X(C))$ is a section whose zero divisor is C.

Especially, $\mathcal{O}_X(C)$ is nef; however, it admits no smooth Hermitian metric with semipositive curvature.

We can apply Theorem 1.1 to a line bundle defined by the section of a certain ruled surface. We are interested in such a situation since it includes the example of Demailly, Peternell, and Schneider [DPS1, Example 1.7], which is constructed as a nef line bundle which admits no smooth Hermitian metric with semipositive curvature. We determine a minimal singular metric of such a line bundle and in particular give a generalization of their result.

COROLLARY 1.2 (EXAMPLE 3.5)

Let C be a smooth projective curve, and let L be a topologically trivial line bundle on C such that $H^1(C, L) \neq 0$. For example, this condition is satisfied for all topologically trivial line bundles when C is a curve of genus greater than 1. Take a nonzero element $\xi \in H^1(C, L)$, and let E be the rank-two vector bundle on C which appears in the nonsplitting exact sequence $0 \to L \to E \to \mathcal{O}_C \to 0$ corresponding to the class ξ. Denote by X a ruled surface $\mathbb{P}(E)$ over C, and denote by D the section of the map $\mathbb{P}(E) \to C$. Then a singular Hermitian metric $|f_D|^{-2}$ on $\mathcal{O}_X(D)$ has minimal singularities, where $f_D \in H^0(X, \mathcal{O}_X(D))$ is a section whose zero divisor is D. Especially the line bundle $\mathcal{O}_X(D)$ is nef; however, it admits no smooth Hermitian metric with semipositive curvature.
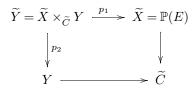
The example in [DPS1, Example 1.7] is a special case of Example 3.5, in which C is an elliptic curve and $L = \mathcal{O}_C$. We remark that they give a minimal singular metric of $\mathcal{O}_X(D)$ for this case by determining all singular Hermitian metrics on this line bundle with semipositive curvature. Our proof is based on a completely different idea.

We are also interested in minimal singular metrics of strictly nef and non-semiample line bundles, where we say a line bundle L on a variety X is *strictly nef* if the intersection number $(L.C)$ is positive for all compact curves $C \subset X$. The following examples are strictly nef and nonsemiample line bundles. Example 1.3(a) is Mumford's example, which admits a smooth Hermitian metric with semipositive curvature. Example 1.3(b) is given by Fujino. He proposed a question about whether this example admits a smooth Hermitian metric with semipositive curvature (see [F, Question 5.10]).

EXAMPLE 1.3

(a) (see [H, Example 10.6]) Let $\widetilde{C}$ be a smooth compact curve of genus greater than 1. Mumford showed that there exists a rank-two vector bundle F on $\widetilde{C}$ such that its degree is equal to zero and its symmetric powers $S^m(F)$ are stable for all $m \geq 1$. Let Y be the total space of a projective space bundle $\mathbb{P}(F)$, and let L_Y be the relative hyperplane bundle $\mathcal{O}_Y(1)$. Then L_Y is a strictly nef and nonsemiample line bundle. In this situation, we can construct a smooth Hermitian

metric h_{L_Y} on L_Y with semipositive curvature as an induced metric from the unitary-flat metric of the stable bundle F (see Remark 4.2).

(b) (A variant of [F, Example 5.9]) Fix a smooth compact curve $\widetilde{C}$ of genus greater than 1. Since $H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \neq 0$, we can take a nonzero element $\xi \in H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$. Let E be the rank-two vector bundle on $\widetilde{C}$ which appears in the nonsplitting exact sequence $0 \to \mathcal{O}_{\widetilde{C}} \to E \to \mathcal{O}_{\widetilde{C}} \to 0$ corresponding to the class ξ. Denote by $\widetilde{X}$ the ruled surface $\mathbb{P}(E)$ over $\widetilde{C}$, denote by $\widetilde{D}$ the section of the map $\mathbb{P}(E) \to \widetilde{C}$, and denote by $\widetilde{Y}$ the fiber product $\widetilde{X} \times_{\widetilde{C}} Y$, where Y is that in Example 1.3(a):

$$
\begin{array}{ccc}
\widetilde{Y} = \widetilde{X} \times_{\widetilde{C}} Y & \xrightarrow{\ p_1\ } & \widetilde{X} = \mathbb{P}(E) \\
\downarrow{\scriptstyle p_2} & & \downarrow \\
Y & \longrightarrow & \widetilde{C}
\end{array}
$$

In this situation, it can be shown that the line bundle $\widetilde{L} := \mathcal{O}_{\widetilde{Y}}(\widetilde{D} \times_{\widetilde{C}} Y) \otimes p_2^* L_Y$ is a strictly nef and nonsemiample line bundle, where $p_2 \colon \widetilde{Y} \to Y$ is the second projection and L_Y is that in Example 1.3(a).

We also determine a minimal singular metric of $\widetilde{L}$ in Example 1.3(b) and in particular give the answer to the above question from Fujino.

COROLLARY 1.4
Let $\widetilde{L} = \mathcal{O}_{\widetilde{Y}}(\widetilde{D} \times_{\widetilde{C}} Y) \otimes p_2^* L_Y$ be that in Example 1.3(b), let f be a global section of $\mathcal{O}_{\widetilde{Y}}(\widetilde{D} \times_{\widetilde{C}} Y)$ whose zero divisor is $\widetilde{D} \times_{\widetilde{C}} Y$, and let h_{L_Y} be a smooth Hermitian metric on L_Y with semipositive curvature (see Remark 4.2 for the existence of such a metric on L_Y). Then the metric $|f|^{-2} \otimes (p_2^* h_{L_Y})$ is a minimal singular metric of $\widetilde{L}$. In particular, $\widetilde{L}$ is not semipositive.

We will prove this corollary for a more general situation (see Theorem 4.1).

The organization of the paper is as follows. In Section 2, we define some concepts on singular Hermitian metrics over possibly noncompact complex manifolds. In Section 3, we review Ueda theory, prove Theorem 1.1, and apply it to some examples. In Section 4, we prove Corollary 1.4 in a more general form (Theorem 4.1).

2. Line bundles on possibly noncompact complex manifolds

In this section, we define some concepts on singular Hermitian metrics over possibly noncompact complex manifolds. Let X be a complex manifold, and let L be a line bundle on X. (In this paper, line bundle always stands for a holomorphic line bundle.) Let h be a singular Hermitian metric of L (for the definition of the singular Hermitian metric, see [D, Definition 3.12]). Then, for each local trivialization of L on an open set of X, the inner product defined by h can be

written as

$$\langle \xi, \eta \rangle_z = e^{-\psi(z)} \xi \overline{\eta}$$

where z is a point in the open set, ξ and η are points in $\mathbb{C}$, which we regard as the z-fiber of L, and ψ is a locally integrable function defined on the open set, which we call the *local weight* of h. Here we remark that it is known that the local currents $\sqrt{-1}\partial\overline{\partial}\psi$ glue together to define the curvature current associated to h. We denote it by $\sqrt{-1}\Theta_h$.

Next, let us recall how to compare the singularity of two plurisubharmonic [psh] functions.

DEFINITION 2.1 ([DPS2, DEFINITION 1.4])

Let φ and ψ be psh functions defined on a neighborhood of $x \in X$. We say that ψ *is less singular than* φ and write $\varphi \prec_{\mathrm{sing}} \psi$ at x when there exists a constant C such that the inequality $\varphi \leq \psi + C$ holds for each point sufficiently near x. We denote $\varphi \sim_{\mathrm{sing}} \psi$ at x if $\varphi \prec_{\mathrm{sing}} \psi$ and $\varphi \succ_{\mathrm{sing}} \psi$ hold at x.

We define the minimal singular metric as follows.

DEFINITION 2.2

Let $h_{\min}$ be a singular Hermitian metric of L which satisfies $\sqrt{-1}\Theta_{h_{\min}} \geq 0$. We call $h_{\min}$ a *minimal singular metric* if $\psi \prec_{\mathrm{sing}} \varphi_{\min}$ holds at any point $x \in X$ for all singular Hermitian metrics h satisfying $\sqrt{-1}\Theta_h \geq 0$, where $\varphi_{\min}$ and ψ stands for the local weight functions of $h_{\min}$ and h, respectively.

It is known that every pseudoeffective line bundle on compact complex manifolds admits minimal singular metrics (see [DPS2, Theorem 1.5]). We remark that, although the minimal singular metric is not unique, it is unique up to the relation $\sim_{\mathrm{sing}}$ if it exists.

3. Ueda theory and proof of Theorem 1.1

First, let us review Ueda theory along [U, Section 2] for a smooth complex surface X and an embedded smooth compact complex curve $C \subset X$ with $(C^2) = 0$. Since the normal bundle $N_{C/X}$ is a topologically trivial line bundle on a compact Kähler manifold, it admits a flat structure; that is,

$$N_{C/X} = \left\{ (U_j \cap U_k, t_{jk}) \right\}_{jk}$$

holds in $H^1(C, \mathcal{O}_C^*)$ for a sufficiently fine open covering $\{U_j\}_j$ and some constants $t_{jk} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$. Let V be a sufficiently small tubular neighborhood of C in X, and let $\{V_j\}_j$ be a sufficiently fine open covering of V. Without loss of generality, we may assume that the index sets of $\{U_j\}_j$ and $\{V_j\}_j$ coincide and $V_j \cap C = U_j$ holds. We choose local coordinates (z_j, x_j) of V_j satisfying the conditions that x_j is a coordinate of U_j, that $\{z_j = 0\} = U_j$ holds on V_j, and that $z_k/z_j \equiv t_{jk}$ holds on $U_j \cap U_k$ for all j and k. Let n be a positive integer.

We call $\{(V_j, z_j, x_j)\}_j$ *a system of type* n if $\mathrm{mult}_{U_j \cap U_k}(t_{jk}z_j - z_k) \geq n+1$ holds on each $V_j \cap V_k$. When there exists a system $\{(V_j, z_j, x_j)\}_j$ of type n, the Taylor expansion of $t_{jk}z_j$ for the variable z_k on $V_j \cap V_k$ can be written in the form

$$t_{jk}z_j = z_k + f_{jk}(x_k)z_k^{n+1} + \cdots$$

for some holomorphic function f_{jk} defined on $U_j \cap U_k$. Here we remark that, for all $m > n$, a system $\{(V_j, z_j, x_j)\}_j$ of type m is also a system of type n and in this case the above f_{jk} is the constant function 0. It is known that $\{(U_j \cap U_k, f_{jk}|_{U_j \cap U_k})\}_{jk}$ satisfies the cocycle condition (see [U, p. 588]).

DEFINITION 3.1

Suppose that there exists a system of type n. Then the cohomology class

$$u_n(C, X) := \left\{(U_j \cap U_k, f_{jk}|_{U_j \cap U_k})\right\}_{jk} \in H^1(C, N_{C/X}^{-n})$$

is called the nth Ueda class of the pair (C, X).

The nth Ueda class does not depend on the choice of the local coordinate system (see [N, Remark 1.3]). It is known that $u_n(C, X) = 0$ if and only if there exists a system of type $n+1$. Thus, only one phenomenon of the following occurs.

(1) There exists an integer $n \in \mathbb{Z}_{>0}$ such that $u_m(C, X)$ can be defined only when $m \leq n$, $u_m(C, X) = 0$ holds for all $m < n$, and $u_n(C, X) \neq 0$ holds.

(2) For every integer $n \in \mathbb{Z}_{>0}$, $u_n(C, X)$ can be defined and it is equal to zero.

DEFINITION 3.2 ([U, P. 589])

We denote $\mathrm{type}(C, X) = n$ and say that the pair (C, X) is of finite type when (1) above occurs. In the other case, we denote $\mathrm{type}(C, X) = \infty$ and say that the pair (C, X) is of infinite type.

THEOREM 3.3 ([U, THEOREM 2])

Let V be a neighborhood of C in X, and let Ψ be a psh function on $V \setminus C$. Assume that $\mathrm{type}(C, X) = n$ holds and that there exists a positive real number $a < n$ such that $\Psi(p) = o(\mathrm{dist}(p, C)^{-a})$ as p approaches C, where $\mathrm{dist}(p, C)$ is the Euclidean distance between p and C. Then there exists a neighborhood V_0 of C such that Ψ is constant on $V_0 \setminus C$.

Now we prove Theorem 1.1 as an application of Theorem 3.3.

LEMMA 3.4

Let X be a smooth complex surface, let $C \subset X$ be an embedded smooth compact complex curve, let $f_C \in H^0(X, \mathcal{O}_X(C))$ be a section whose zero divisor is C, and let h be a singular Hermitian metric of $\mathcal{O}_X(C)$ with semipositive curvature. Assume that $(C^2) = 0$, that $\mathrm{type}(C, X) < \infty$, and that the local weight functions

of h are less singular than the psh function $\log |f_C|^2$. Then there exists a positive number M such that $h = M|f_C|^{-2}$ holds on a neighborhood of C in X.

Proof

Let us fix a sufficiently small neighborhood V of C in X, and consider a function $\Psi := -\log |f_C|_h^2$ defined on $V \setminus C$. In the following, we sometimes restrict ourselves to a sufficiently small open neighborhood of each point of V. On the open neighborhood, we fix a local trivialization of $\mathcal{O}_X(C)$, and by using a local weight function φ of h, we denote $|f_C|_h^2$ locally by $|f_C|^2 e^{-\varphi}$, where we regard f_C as a locally defined holomorphic function corresponding to the section f_C.

First we check that this function is psh. By using the above notation, it is clear that the equation $\Psi = \varphi - \log |f_C|^2$ holds locally. Since $\log |f_C|^2$ is harmonic on $V \setminus C$ and φ is psh by assumption, we can conclude that Ψ is a psh function.

Next, we evaluate the divergence of Ψ near C. Since $\log |f_C|^2 \prec_{\text{sing}} \varphi$, Ψ is bounded from below. Thus all we have to do is to evaluate the divergence of Ψ to $+\infty$. We use the local coordinate system (z, x) of V such that x is a local coordinate of C and the equation $\{z = 0\} = C$ holds locally. Without loss of generality, we may assume that $f_C(z, x) = z$ holds on this locus. Since φ is a psh function, it is locally bounded from above. Thus

$$\Psi(z, x) = \varphi(z, x) - \log |z|^2 \le C - \log |z|^2 = o\big(1/|z|^{1/2}\big) \quad \text{as } |z| \to 0$$

holds. Therefore we can apply Theorem 3.3, and thus, after making the neighborhood V smaller, we can conclude that Ψ is a constant map, which proves the lemma. $\qquad\square$

Proof of Theorem 1.1

Let h be a singular Hermitian metric of $\mathcal{O}_X(C)$ with semipositive curvature. We denote by $\widetilde{h}$ the new metric $\min\{h, |f_C|^{-2}\}$. Clearly we obtain that the local weight function of $\widetilde{h}$ is $\max\{\log |f_C|^2, \varphi\}$, where φ is the local weight function of h. Since this function is the maximum of two psh functions, $\widetilde{h}$ also has semipositive curvature. For $\log |f_C|^2 \prec_{\text{sing}} \max\{\log |f_C|^2, \varphi\}$, we can apply Lemma 3.4 and thus we obtain that $\log |f_C|^2 \sim_{\text{sing}} \max\{\log |f_C|^2, \varphi\} \succ_{\text{sing}} \varphi$ holds. $\qquad\square$

In the rest of this section, we give some examples of nef but not semipositive line bundles over smooth projective varieties as applications of Theorem 1.1.

EXAMPLE 3.5

Let C be a smooth projective curve of positive genus, and let E be the rank-two vector bundle on C which appears in the exact sequence $0 \to L \to E \to \mathcal{O}_C \to 0$, where L is a topologically trivial line bundle on C such that $H^1(C, L) \ne 0$. Let us define X as the ruled surface $\mathbb{P}(E)$ over C, and let $D := \mathbb{P}(\mathcal{O}_C) \subset \mathbb{P}(E)$ be the section of the map $\mathbb{P}(E) \to C$. We denote by $\pi \colon X \to C$ the canonical surjection. From simple computations, we can conclude that the normal bundle

$N_{D/X}$ is isomorphic to $(\pi|_D)^* L^{-1}$ and thus it is topologically trivial. Therefore we can define Ueda classes for the pair (D, X). In our case, we obtain that the first Ueda class $u_1(D, X) \in H^1(D, N_{D/X}^{-1})$ coincides with cohomology class $\{E\} \in \mathrm{Ext}^1(\mathcal{O}_C, L) = H^1(C, L)$ via the isomorphism $\pi|_D \colon D \to C$ (see [N, Proposition 7.6]). Thus the first Ueda class $u_1(D, X)$ vanishes if and only if $E = \mathcal{O}_C \oplus L$, and in this case, the line bundle $\mathcal{O}_X(D)$ is semipositive.

Assume that the exact sequence $0 \to L \to E \to \mathcal{O}_C \to 0$ is nonsplitting. In this case, the first Ueda class $u_1(D, X)$ does not vanish, and thus $\mathrm{type}(D, X) = 1 < \infty$ and we can apply Theorem 1.1. As a result, we can conclude that $|f_D|^{-2}$ defines a minimal singular metric of $\mathcal{O}_X(D)$, where $f_D \in H^0(X, \mathcal{O}_X(D))$ is a section whose zero divisor is D.

When C is a smooth elliptic curve and $L = \mathcal{O}_C$, the minimal singularity of the above singular Hermitian metric $|f_D|^{-2}$ is known by Demailly, Peternell, and Schneider [DPS1, Example 1.7]. We also remark that, in this case, they more precisely show that, for every singular Hermitian metric h of $\mathcal{O}_X(D)$ with semipositive curvature, there exists a positive constant M such that $h = M|f_D|^{-2}$ holds.

EXAMPLE 3.6

Let C be a smooth compact curve of genus 2, and let Y be its Jacobian. Fix two points $p, q \in C$ conjugate to each other by the hyperelliptic involution. We denote by X the blowup of Y at p and q and denote by D the strict transformation of C. According to Neeman [N, Proposition 10.5], $\mathrm{type}(D, X) = 1 < \infty$ holds for this D and X. Therefore we can apply Theorem 1.1 and conclude that $|f_D|^{-2}$ defines a minimal singular metric of $\mathcal{O}_X(D)$, where $f_D \in H^0(X, \mathcal{O}_X(D))$ is a section whose zero divisor is D.

4. Strictly nef line bundle which admits no smooth Hermitian metric with semipositive curvature

In this section, we prove the following.

THEOREM 4.1

Let $\widetilde{C}$, Y, and L_Y be those in Example 1.3(a), let $\widetilde{X} \to \widetilde{C}$ be a smooth holomorphic fiber bundle over $\widetilde{C}$ ($\widetilde{X}$ need not to be that in Example 1.3(b) and also need not to be compact), and let $L_{\widetilde{X}}$ be a pseudoeffective line bundle on $\widetilde{X}$. Denote by $\widetilde{Y}$ the fiber product $\widetilde{X} \times_{\widetilde{C}} Y$, and denote by $\widetilde{L}$ the line bundle $p_1^* L_{\widetilde{X}} \otimes p_2^* L_Y$, where $p_1 \colon \widetilde{Y} \to \widetilde{X}$ and $p_2 \colon \widetilde{Y} \to Y$ are the first and second projections, respectively. Assume that $L_{\widetilde{X}}$ admits a minimal singular metric $h_{L_{\widetilde{X}}}$. Then the metric $(p_1^* h_{L_{\widetilde{X}}}) \otimes (p_2^* h_{L_Y})$ is a minimal singular metric of $\widetilde{L}$, where h_{L_Y} is that in Example 1.3(a) (see Remark 4.2).

Here a smooth holomorphic fiber bundle over $\widetilde{C}$ means a holomorphic map from a smooth variety onto $\widetilde{C}$ which satisfies that, for each point $x \in \widetilde{C}$, there exists

an open neighborhood $\widetilde{U} \subset \widetilde{C}$ of x such that the restriction to the preimage of $\widetilde{U}$ can be regarded as the natural projection from the product space of $\widetilde{U}$ and some manifold to $\widetilde{U}$. We remark that if $L_{\widetilde{X}}$ in the above theorem is nef, then $\widetilde{L}$ is strictly nef. From this, Theorem 4.1, and Corollary 1.2, we can deduce Corollary 1.4.

We first review the metric of Mumford's example (Example 1.3(a)).

REMARK 4.2

The line bundle L_Y in Example 1.3(a) admits a smooth Hermitian metric with semipositive curvature. Indeed, we can construct such a metric as follows. First we remark that, by the Narasimhan–Seshadri theorem [NS, Theorem 2], there exists a representation $\rho \colon \pi_1(\widetilde{C}) \to U(2)$ such that the dual F^* of F is isomorphic to the quotient $(\mathbb{H} \times \mathbb{C}^2)/\pi_1(\widetilde{C})$, where $\pi_1(\widetilde{C})$ acts on $\mathbb{H} \times \mathbb{C}^2$ via ρ. Here we denote by $\mathbb{H}$ an upper half-plane $\{x \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Thus, the fiberwise Euclidean metric induces a flat metric h_{F^*} of F^* and, on every open set $U \subset \widetilde{C}$, we can choose a nice local frame (s_1^*, s_2^*) of F^* such that

$$|s_1^*|^2_{h_{F^*}} \equiv 1, \qquad |s_2^*|^2_{h_{F^*}} \equiv 1, \qquad \langle s_1^*, s_2^* \rangle_{h_{F^*}} \equiv 0$$

holds on U. We use the function

$$(w, x) \mapsto \left[s_1^*(x) + w s_2^*(x) \right] \in \mathbb{P}(F)$$

as a local coordinate system of $\mathbb{P}(F)$. Then the fiberwise Fubini–Study metric defines a smooth Hermitian metric h_{L_Y}, whose curvature tensor can be computed as

$$\sqrt{-1}\Theta_{h_{L_Y}} = \sqrt{-1}\partial\overline{\partial}\log\big|s_1^*(x) + w s_2^*(x)\big|^2_{F^*} = \frac{\sqrt{-1}\,dw \wedge d\overline{w}}{(1 + |w|^2)^2}.$$

Therefore, this smooth Hermitian metric h_{L_Y} clearly has a semipositive curvature tensor.

Proof of Theorem 4.1

We fix a singular Hermitian metric $h_{\widetilde{L}}$ of $\widetilde{L}$ with semipositive curvature and show the existence of a constant $M_{\widetilde{W}}$ for each sufficiently small open set $\widetilde{W} \subset \widetilde{Y}$ such that $(p_1^* h_{L_{\widetilde{X}}}) \otimes (p_2^* h_{L_Y}) \leq M_{\widetilde{W}} h_{\widetilde{L}}$ holds on $\widetilde{W}$.

Let us fix a smooth Hermitian metric h_∞ of $L_{\widetilde{X}}$. We here remark that the tensor product $(p_1^* h_\infty) \otimes (p_2^* h_{L_Y})$ (or $p_1^* h_\infty p_2^* h_{L_Y}$, for simplicity) defines a smooth Hermitian metric of $\widetilde{L}$. Therefore, there exists a quasi-psh function $\chi \colon \widetilde{Y} \to \mathbb{R}$ such that

$$h_{\widetilde{L}} = p_1^* h_\infty p_2^* h_{L_Y} e^{-\chi}$$

holds. Let us describe our situations in the words of local weight functions. Here we use a local coordinate system (z, w, x) of $\widetilde{Y}$, where x is a local coordinate of an open set $U \subset \widetilde{C}$, (z, x) is a local coordinate system of $\widetilde{X}$ where z is a fiber coordinate, and (w, x) is a local coordinate system of Y just as in Remark 4.2. Let φ_∞ be the local weight function of h_∞, and let $\varphi_{\widetilde{L}}$ be the local weight function

of $h_{\widetilde{L}}$. Then the equation

$$\varphi_{\widetilde{L}}(z, w, x) = \varphi_{\infty}(z, x) + \log\left(1 + |w|^2\right) + \chi(z, w, x)$$

holds. Since $\sqrt{-1}\partial\overline{\partial}\varphi_{\widetilde{L}} \geq 0$ holds from the assumption, the inequality

$$\sqrt{-1}\partial_{z,x}\overline{\partial}_{z,x}\varphi_{\widetilde{L}}(z, w_0, x) \geq 0$$

also holds for each point $w_0 \in p^{-1}(U)$, where $p\colon Y \to \widetilde{C}$ is the natural projection and $\partial_{z,x}$ (resp., $\overline{\partial}_{z,x}$) is the operator ∂ (resp., $\overline{\partial}$) with w_0 fixed. Thus, it can be said that the function $\chi|_{\widetilde{\pi}^{-1}(U)\times\{w_0\}}\colon \widetilde{\pi}^{-1}(U) \to \mathbb{R}$, $(z, x) \mapsto \chi(z, w_0, x)$ is a φ_{∞}-psh function (i.e., $\sqrt{-1}\partial_{z,x}\overline{\partial}_{z,x}(\chi|_{\widetilde{\pi}^{-1}(U)\times\{w_0\}} + \varphi_{\infty}) \geq 0$ holds) on $\widetilde{\pi}^{-1}(U)$, where $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{C}$ is the natural surjection. Therefore, the function

$$\widetilde{\chi}(z, x) := \max_{w \in p^{-1}(x)} \chi(z, w, x)$$

is also a φ_{∞}-psh function on $\widetilde{\pi}^{-1}(U)$. (Since χ is quasi-psh and thus is locally bounded from above, we can use the same argument as in the proof of [DPS2, Theorem 1.5].) As this function $\widetilde{\chi}$ can be regarded as a global function on $\widetilde{X}$, it is clear that the singular Hermitian metric $h_2 := h_{\infty}e^{-\widetilde{\chi}}$ of $L_{\widetilde{X}}$ is a well-defined singular Hermitian metric with semipositive curvature. Therefore, from the assumption, there exists a positive constant M_W for each sufficiently small open set $W \subset \widetilde{X}$ such that $M_W h_{\infty}e^{-\widetilde{\chi}} \geq h_{L_{\widetilde{X}}}$ holds on W. Thus we obtain an inequality

$$p_1^* h_{L_{\widetilde{X}}} p_2^* h_{L_Y} \leq M_W p_1^*(h_{\infty}e^{-\widetilde{\chi}})p_2^* h_{L_Y} \leq M_W p_1^* h_{\infty} p_2^* h_{L_Y} e^{-\chi} = M_W h_{\widetilde{L}}$$

that holds on $p_1^{-1}(W)$, and therefore we can conclude that the metric $p_1^* h_{L_{\widetilde{X}}} p_2^* h_{L_Y}$ is a minimal singular metric of $\widetilde{L}$. $\qquad\square$

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