# Endo-class and the Jacquet–Langlands correspondence

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**Abstract** Let $F$ be a non-archimedean local field. Recently, Broussous, Sécherre, and Stevens extended the notion of an endo-class, introduced by Bushnell and Henniart for $\mathrm{GL}_N(F)$ with $N \geq 1$, to an inner form of $\mathrm{GL}_N(F)$ over $F$, and conjectured that this endo-class for discrete series representations is preserved by the Jacquet–Langlands correspondence. Explicit realizations of the correspondence are given by Silberger and Zink for level-zero discrete series representations and by Bushnell and Henniart for totally ramified ones. In this paper, we show that these realizations confirm the conjecture.

## Introduction

Let $F$ be a non-archimedean local field of finite residue characteristic $p$, and let $D$ be a central division $F$-algebra of dimension $d^2$, $d \geq 1$. Let $\mathfrak{o}_F$ and $\mathfrak{o}_D$ be the rings of integers in $F$ and $D$, respectively. Let $m$ be a positive integer. The product $N = md$ being fixed, there exist bijective maps, referred to as the *Jacquet–Langlands correspondence*, between the sets of irreducible discrete series representations of $\mathrm{GL}_m(D)$ such that a character relation is preserved (see [1], [9], [12], [13]). There exist a series of works by Bushnell and Henniart (see [7], [8], [11]) and by Silberger and Zink (see [17], [18]) in which the Jacquet–Langlands correspondences were described explicitly in terms of types. The notion of an endo-class was introduced in [6], and it was proved in [5] and [8] that an endo-class is an invariant associated to an irreducible supercuspidal representation of $\mathrm{GL}_N(F)$, which is constructed as a compactly induced representation of a compact-mod-center subgroup of $\mathrm{GL}_N(F)$. Broussous, Sécherre, and Stevens [4] extended the notion of an endo-class over $F$ for $\mathrm{GL}_N(F)$ to any group of the form $\mathrm{GL}_m(D)$, that is, we can associate an endo-class over $F$ to any discrete series representation of $\mathrm{GL}_m(D)$, and it was conjectured that the Jacquet–Langlands correspondence preserves this endo-class over $F$. In this paper, we prove that the realizations of [6] and [17] confirm this conjecture.

More precisely, we give a description of the result obtained. The *simple characters* for $G = \mathrm{GL}_m(D)$ are parameterized by 4-tuples $[\mathfrak{A}, n, 0, \beta]$, which are referred to as *simple strata*, consisting of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A$ with

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$\mathfrak{P}=\mathrm{rad}(\mathfrak{A})$, a positive integer $n$, and an element $\beta \in A$ which generates a field extension $F[\beta]$ over $F$, with the technical condition $k_F(\beta)<0$ and with $\beta \in \mathfrak{P}^{-n}$. By [14], associated with a simple stratum $[\mathfrak{A},n,0,\beta]$ in $A=\mathrm{M}_m(D)$, we have a compact open subgroup $H^1(\beta,\mathfrak{A})$ of $G$ and a finite set $\mathscr{C}(\mathfrak{A},0,\beta)$ of simple characters of $H^1(\beta,\mathfrak{A})$.

From [15] and [16], it follows that every irreducible discrete series representation $\pi$ of $G$ contains a simple character $\theta \in \mathscr{C}(\mathfrak{A},0,\beta)$ attached to a simple stratum $[\mathfrak{A},n,0,\beta]$ in $A$. Neither the simple stratum nor the simple character is unique. The endo-class, denoted by $\boldsymbol{\Theta}$, for the pair $([\mathfrak{A},n,0,\beta],\theta)$ was defined by [6] and [4] so that this $\boldsymbol{\Theta}$ depends only on the representation $\pi$ of $G$ as follows. A *potential simple character* (*ps-character* for short) is an equivalence class, denoted by $\Theta$, in the set of such pairs $([\mathfrak{A},n,m,\beta],\theta)$ in $A$ as above, where $[\mathfrak{A},n,m,\beta]$ is a simple stratum in $A$ and $\theta \in \mathscr{C}(\mathfrak{A},m,\beta)$. Indeed, another pair $([\mathfrak{A}',n',m',\beta],\theta')$ in a central simple $F$-algebra $A'$ is referred to as *equivalent* to $([\mathfrak{A},n,m,\beta],\theta)$, denoted by

$$\big([\mathfrak{A},n,m,\beta],\theta\big) \sim \big([\mathfrak{A}',n',m',\beta],\theta'\big),$$

if $\theta'$ is the *transfer* of $\theta$ (see Definition 1.7). The pair $([\mathfrak{A},n,m,\beta],\theta)$ is referred to as a *realization* of $\Theta$. Two ps-characters $\Theta_1$ and $\Theta_2$ are referred to as *endo-equivalent* if, in a central simple $F$-algebra $A$, they are defined by realizations $([\mathfrak{A}_i,n_i,m_i,\beta_i],\theta_i)$, for $i=1,2$, of the same degree and normalized level, and such that the simple characters $\theta_1$ and $\theta_2$ intertwine in $A^\times$ (see Definition 1.9). Two simple characters contained in the irreducible discrete series representation $\pi$ of $G$ intertwine in $G$. Hence, the endo-class $\boldsymbol{\Theta}$ above depends only on the representation $\pi$. Write this $\boldsymbol{\Theta}$ as $\boldsymbol{\Theta}_G(\pi)$.

Let $D_{md}$ be a central division $F$-algebra of dimension $m^2d^2$, and let $\mathbf{JL}$ be the Jacquet–Langlands correspondence between the sets of isomorphism classes of irreducible discrete series representations of $G=\mathrm{GL}_m(D)$ and $H=D_{md}^\times$. Then, the equality

$$\boldsymbol{\Theta}_H \circ \mathbf{JL} = \boldsymbol{\Theta}_G$$

was conjectured by [4, Conjecture 9.5].

It was stated in [4, Introduction] that this conjecture can be seen as a generalization of the preservation of the level-zero representations through the Jacquet–Langlands correspondence, which was proved by [17]. This is explained as follows. From [10], every irreducible discrete series representation of $G=\mathrm{GL}_m(D)$ of level zero contains the trivial representation $\mathbf{1}_{U^1(\mathfrak{A})}$ for some principal hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A=\mathrm{M}_m(D)$ with $\mathfrak{P}=\mathrm{rad}(\mathfrak{A})$, where $U^1(\mathfrak{A})=1+\mathfrak{P}$. We view $[\mathfrak{A},0,0,0]$ as a simple stratum in $A$, as in [19], and view $([\mathfrak{A},0,0,0],\mathbf{1}_{U^1(\mathfrak{A})})$ as the realization of the trivial ps-character $\Theta_0$. Moreover, we have $H^1(0,\mathfrak{A})=U^1(\mathfrak{A})$ and

$$\mathscr{C}(\mathfrak{A},0,0)=\{\mathbf{1}_{U^1(\mathfrak{A})}\}.$$

Hence, by the definition of endo-class, that statement is explained.

Let $F$ be a finite extension of $\mathbb{Q}_p$ with $p \neq 2$. For a positive integer $m$, set $A = \mathrm{M}_{p^m}(F)$, and let $D$ be a central division $F$-algebra of dimension $p^{2m}$. Then, there exists the Jacquet–Langlands correspondence $\mathbf{JL}$ between the sets of isomorphism classes of irreducible discrete series representations of $G = A^\times = \mathrm{GL}_{p^m}(F)$ and $H = D^\times$. Let $\mathcal{A}_m^{wr}(F)$ be the set of isomorphism classes of irreducible supercuspidal representations $\pi$ of $G$ which are *totally ramified*: this means that $\pi$ is not isomorphic to the representation $\chi\pi : g \mapsto \chi(\det(g))\pi(g)$ for any unramified quasicharacter $\chi \neq 1$ of $F^\times$. Set $\mathcal{A}_0^{wr}(D) = \mathbf{JL}(\mathcal{A}_m^{wr}(F))$. Then, we obtain a canonical bijection, denoted again by $\mathbf{JL}$,

$$\mathbf{JL} : \mathcal{A}_m^{wr}(F) \simeq \mathcal{A}_0^{wr}(D).$$

In [6], the representations in $\mathcal{A}_m^{wr}(F)$ and $\mathcal{A}_0^{wr}(D)$ were explicitly constructed as induced representations of quasicharacters of compact-mod-center subgroups, and the correspondence $\mathbf{JL}$ was described.

Let $\pi$ be an irreducible supercuspidal representation of $G = \mathrm{GL}_{p^m}(F)$ in $\mathcal{A}_m^{wr}(F)$. Then, from the construction of $\pi$, we can choose a pair $([\mathfrak{A}, n, 0, \beta], \theta)$, as above, such that $\pi$ contains $\theta$. Set $\pi' = \mathbf{JL}(\pi)$. Then, from the realization of $\mathbf{JL}$, we can also choose a pair $([\mathfrak{o}_D, n', 0, \iota\beta], {}_D\theta)$ such that $\pi'$ contains ${}_D\theta$, where $\iota : F[\beta] \to D$ denotes an $F$-embedding. For a finite unramified extension $K/F$ of degree divisible by $p^m$, set $\mathfrak{A}_K = \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_K$ and ${}_D\mathfrak{A}_K = \mathfrak{o}_D \otimes_{\mathfrak{o}_F} \mathfrak{o}_K$. Then, through the identification $A_K = A \otimes_F K = D \otimes_F K = D_K$, we can set $\mathfrak{A}_K = {}_D\mathfrak{A}_K$ and take an element $y_0 \in \mathfrak{A}_K^\times$ such that $\iota\beta = y_0^{-1}\beta y_0 = \mathrm{Ad}(y_0^{-1})\beta$, where we identify $\beta = \beta \otimes 1$ in $A_K$. Then, we can choose simple characters $\theta(K)$ and ${}_D\theta(K)$ of $H^1(\beta, \mathfrak{A}_K)$ and $H^1(\iota\beta, {}_D\mathfrak{A}_K)$, respectively, such that

$$\theta = \theta(K) \mid H^1(\beta, \mathfrak{A}), \qquad {}_D\theta = {}_D\theta(K) \mid H^1(\iota\beta, \mathfrak{o}_D).$$

We prove that ${}_D\theta(K) = \theta(K) \circ \mathrm{Ad}(y_0)$ and that ${}_D\theta(K)$ is the transfer of $\theta(K)$. Thus, by [14, Theorem 3.53] for transfers, ${}_D\theta$ is the transfer of $\theta$, that is,

$$\big([\mathfrak{A}, n, 0, \beta], \theta\big) \sim \big([\mathfrak{o}_D, n', 0, \iota\beta], {}_D\theta\big),$$

which implies $\mathbf{\Theta}_H(\pi') = \mathbf{\Theta}_G(\pi)$.

The remainder of the present paper is organized as follows. In Section 1, we recall the notation of ps-character and endo-class defined in [5] and [4]. In Section 2, we recall the conjecture on the preservation of the endo-class of the Jacquet–Langlands correspondence given in [4]. In Section 3, we prove that the realizations of [5] and [17] confirm this conjecture.

## 1. Endo-class of ps-characters

We recall the definition of endo-class and ps-character for an inner form of $\mathrm{GL}_N(F)$ in [4], which is a generalization of the $F$-split $\mathrm{GL}_N(F)$ defined in [5].

### 1.1. Simple character

Let $F$ be a non-archimedean local field. Let $K$ be a commutative or noncommutative finite extension of $F$, let $\mathfrak{o}_K$ be the ring of integers in $K$, and let $\mathfrak{p}_K$ be the maximal ideal of $\mathfrak{o}_K$.

Let $A$ be a simple central $F$-algebra of finite dimension, and let $V$ be a simple left $A$-module. Write $D = \mathrm{End}_A(V)^{\mathrm{op}}$. Then, $D$ is a central division $F$-algebra, and $V$ can be viewed as a right $D$-vector space. There exists a canonical isomorphism $A \simeq \mathrm{End}_D(V)$.

DEFINITION 1.1

A nonempty set of right $\mathfrak{o}_D$-lattices $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ in $V$ is referred to as an *$\mathfrak{o}_D$-lattice chain* in $V$ if the following conditions are satisfied: (1) $L_i \supsetneq L_{i+1}$ for all $i \in \mathbb{Z}$, and (2) there exists a positive integer $e$ satisfying $L_{i+e} = L_i \mathfrak{p}_D$ for all $i \in \mathbb{Z}$. This integer $e$ is referred to as the *$\mathfrak{o}_D$-period* of $\mathcal{L}$ and is denoted by $e_D(\mathcal{L})$.

For $k \in \mathbb{Z}$, set

$$\mathfrak{P}_k(\mathcal{L}) = \{a \in A : aL_i \subset L_{i+k}, i \in \mathbb{Z}\}.$$

Then, $\mathfrak{A} = \mathfrak{A}(\mathcal{L}) = \mathfrak{P}_0(\mathcal{L})$ is a hereditary $\mathfrak{o}_F$-order in $A$. All such orders are obtained in this way from an $\mathfrak{o}_D$-lattice chain $\mathcal{L}$ in $V$. The set $\mathfrak{P} = \mathfrak{P}(\mathcal{L}) = \mathfrak{P}_1(\mathcal{L})$ is the Jacobson radical of $\mathfrak{A}$, and we have $\mathfrak{P}_k(\mathcal{L}) = \mathfrak{P}^k$ for all $k \in \mathbb{Z}, k \geq 0$. Thus, we have compact open subgroups of $G$ defined by

$$U(\mathfrak{A}) = U^0(\mathfrak{A}) = \mathfrak{A}^\times, \qquad U^k(\mathfrak{A}) = 1 + \mathfrak{P}^k, \quad k \in \mathbb{Z}, k > 0.$$

The $G$-centralizer $\mathfrak{K}(\mathfrak{A})$ of $\mathfrak{A}$ is defined by

$$\mathfrak{K}(\mathfrak{A}) = \{g \in G : g\mathfrak{A}g^{-1} = \mathfrak{A}\}.$$

Then, for $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, $g \in \mathfrak{K}(\mathfrak{A})$ if and only if there exists a unique $n = \nu(g) \in \mathbb{Z}$ such that $gL_i = L_{i+n}$ for all $i \in \mathbb{Z}$. We define a function $\nu_{\mathfrak{A}} : \mathfrak{K}(\mathfrak{A}) \to \mathbb{Z}$ by $\nu_{\mathfrak{A}}(g) = \nu(g)$ for $g \in \mathfrak{K}(\mathfrak{A})$. Then, we have $\mathrm{Ker}\, \nu_{\mathfrak{A}} = U(\mathfrak{A})$.

DEFINITION 1.2

(a) A *stratum* in $A$ is a 4-tuple $[\mathfrak{A}, n, m, \beta]$ made of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A$, $m, n \in \mathbb{Z}$ with $0 \leq m \leq n$ and $\beta \in \mathfrak{P}^{-n}$.

(b) Two strata $[\mathfrak{A}, n, m, \beta_i], i = 1, 2$, are referred to as *equivalent* if $\beta_2 - \beta_1 \in \mathfrak{P}^{-m}$.

Here, $[\mathfrak{A}, 0, 0, 0]$ is referred to as the *null stratum* as is defined in [19].

DEFINITION 1.3

A stratum $[\mathfrak{A}, n, m, \beta]$ in $A$ is referred to as *pure* if it satisfies the following conditions:

(a) the sub-$F$-algebra $F[\beta]$ generated by $\beta$ is a field, say, $E = F[\beta]$;
(b) $\mathfrak{A}$ is $E$-pure, that is, $E^\times \subset \mathfrak{K}(\mathfrak{A})$;
(c) $\nu_{\mathfrak{A}}(\beta) = -n$.

Let $[\mathfrak{A}, n, m, \beta]$ be a pure stratum in $A$. Let $B$ be the $A$-centralizer of $\beta$, and write $B = C_A(\beta)$. For each $k \in \mathbb{Z}$, we set $\mathfrak{n}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} : \beta x - x\beta \in \mathfrak{P}^k\}$ and define the quantity $k_0(\beta, \mathfrak{A})$ by

$$\min\{k \in \mathbb{Z} : k \geq \nu_{\mathfrak{A}}(\beta) \text{ and } \mathfrak{n}_{k+1}(\beta, \mathfrak{A}) \subset \mathfrak{A} \cap B + \mathfrak{P}\}.$$

DEFINITION 1.4

A stratum $[\mathfrak{A}, n, m, \beta]$ in $A$ is referred to as *simple* if it is pure and if $m \leq -k_0(\beta, \mathfrak{A}) - 1$.

It is convenient to view the null stratum $[\mathfrak{A}, 0, 0, 0]$ in $A$ as a simple stratum, as in [19]. Hereafter, we do so.

A simple stratum $[\mathfrak{A}, n, m, \beta]$ in $A$ gives rise to a pair

$$\mathfrak{H}(\beta, \mathfrak{A}) \subset \mathfrak{J}(\beta, \mathfrak{A}) \subset \mathfrak{A}$$

of $\mathfrak{o}_F$-orders in $A$ (see [14]). If $\beta = 0$, then we set

$$\mathfrak{H}(0, \mathfrak{A}) = \mathfrak{J}(0, \mathfrak{A}) = \mathfrak{A}.$$

We take the standard filtration subgroups of the unit groups

$$H^k(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A}),$$

$$J^k(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap U^k(\mathfrak{A}),$$

for $k \in \mathbb{Z}, k \geq 0$.

We fix a level-one additive character $\psi = \psi_F$ of $F$; that is, $\mathfrak{p}_F \subset \operatorname{Ker}\psi$ and $\psi \mid \mathfrak{o}_F \neq 1$. Through this character $\psi = \psi_F$, a finite set of characters, referred to as *simple characters*, of the compact group $H^{m+1}(\beta, \mathfrak{A})$, say, $\mathscr{C}(\mathfrak{A}, m, \beta) = \mathscr{C}(\mathfrak{A}, m, \beta, \psi)$, was defined in [14].

Associated with the null simple stratum $[\mathfrak{A}, 0, 0, 0]$ in $A$, we view $\mathscr{C}(\mathfrak{A}, 0, 0)$ as the set consisting of the single trivial character $\mathbf{1}_{U^1(\mathfrak{A})}$ of the group $H^1(0, \mathfrak{A}) = U^1(\mathfrak{A})$, that is (see [15, Remark 4.4]),

$$\mathscr{C}(\mathfrak{A}, 0, 0) = \{\mathbf{1}_{U^1(\mathfrak{A})}\}. \tag{1.1}$$

## 1.2. Ps-character and endo-class

Let $\beta$ be a nonzero element in a finite subextension of $F$ in $A$, and set $E = F[\beta]$. We denote by $\nu_E$ the normalized valuation on $E$. The set $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ is an $E$-pure $\mathfrak{o}_F$-lattice chain on the $F$-space $E$, unique up to translation. We set $A(E) = \operatorname{End}_F(E)$ and (see [8, (1.1.2)])

$$\mathfrak{A}(E) = \operatorname{End}^0_{\mathfrak{o}_F}\big(\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}\big).$$

Then, $\mathfrak{A}(E)$ is a hereditary $\mathfrak{o}_F$-order in $A(E)$. Set

$$k_F(\beta) = k_0\big(\beta, \mathfrak{A}(E)\big).$$

Then, unless $\beta \in F$, we have $k_F(\beta) \geq \nu_E(\beta)$.

DEFINITION 1.5 ([5, DEFINITION 1.5])

A *simple pair* over $F$ is a pair $(k,\beta)$ consisting of a nonzero element $\beta$ in some finite extension of $F$ and an integer $0 \le k \le -k_F(\beta) - 1$.

If $(k,\beta)$ is a simple pair over $F$, then $[\mathfrak{A}(E), -\nu_E(\beta), k, \beta]$ is a simple stratum in $A(E)$. Thus, we have a set of quasisimple characters of $H^{k+1}(\beta, \mathfrak{A}(E))$ (see [14, Section 3.3.3])

$$\mathscr{C}_F(k,\beta) = \mathscr{C}\big(\mathfrak{A}(E), k, \beta\big) = \mathscr{C}\big(\mathfrak{A}(E), k, \beta, \psi_F\big).$$

We also view the pair $(0,0)$ as a simple pair over $F$. It is referred to as the *null simple pair*. By definition, we have $\mathscr{C}_F(0,0) = \{\mathbf{1}_{U^1(\mathfrak{o}_E)}\}$, where $U^1(\mathfrak{o}_E) = 1 + \mathfrak{p}_F$.

Let $A$ be a central simple $F$-algebra, and let $V$ be a simple left $A$-module. Let $D = \mathrm{End}_A(V)^{\mathrm{op}}$. For a real number $r$, denote by $\lfloor r \rfloor$ the greatest integer that is less than or equal to $r$.

DEFINITION 1.6 (SEE [4])

A *realization* of a nonnull simple pair $(k,\beta)$ in $A$ is a stratum in $A$ of the form $[\mathfrak{A}, n, m, \varphi(\beta)]$ made of:

(a) a homomorphism $\varphi$ of $F$-algebras from $F[\beta]$ to $A$;
(b) a $\varphi(F[\beta])$-pure hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A$;
(c) an integer $m$ such that $k = \lfloor m/e_{F[\varphi(\beta)]}(\mathfrak{A}) \rfloor$.

It is convenient to view the null stratum $[\mathfrak{A}, 0, 0, 0]$ in $A$ as the realization of the null simple pair $(0,0)$ in $A$.

From [14, Proposition 2.5], the realization $[\mathfrak{A}, n, m, \varphi(\beta)]$ in Definition 1.6 is a simple stratum in $A$. Thus, we have a set

$$\mathscr{C}\big(\mathfrak{A}, m, \varphi(\beta)\big) = \mathscr{C}\big(\mathfrak{A}, m, \varphi(\beta), \psi_F\big)$$

of simple characters of $H^{m+1}(\varphi(\beta), \mathfrak{A})$. For a realization $[\mathfrak{A}, n, m, \varphi(\beta)]$ in $A$ of a nonnull simple pair $(k,\beta)$ over $F$, it follows from [14, Section 3.3.3] that there exists a canonical bijective map (cf. [16, Definition 2.11])

$$\tau_{\mathfrak{A},m,\varphi(\beta)} : \mathscr{C}_F(k,\beta) \to \mathscr{C}\big(\mathfrak{A}, m, \varphi(\beta)\big).$$

This map is referred to as a *transfer map*. If $(k,\beta) = (0,0)$, then it is the trivial map by definition. We denote by $\tau_{\mathfrak{A},0,0}$ the transfer map $\mathscr{C}_F(0,0) \to \mathscr{C}(\mathfrak{A}, 0, 0)$.

Given a simple pair $(k,\beta)$ over $F$, we consider a pair

$$\big([\mathfrak{A}, n, m, \varphi(\beta)], \theta\big)$$

made of a realization $[\mathfrak{A}, n, m, \varphi(\beta)]$ in $A$ and a simple character $\theta \in \mathscr{C}(\mathfrak{A}, m, \varphi(\beta))$.

DEFINITION 1.7 (SEE [4, SECTION 1.2])

Let $[\mathfrak{A}', n', m', \varphi'(\beta)]$ be another realization of the simple pair $(k,\beta)$ in some simple central $F$-algebra $A'$, and let $\theta'$ be a simple character in $\mathscr{C}(\mathfrak{A}', m', \varphi'(\beta))$.

We say that $([\mathfrak{A}, n, m, \varphi(\beta)], \theta)$ and $([\mathfrak{A}', n', m', \varphi'(\beta)], \theta')$ are *equivalent*, denoted by

$$\big([\mathfrak{A}, n, m, \varphi(\beta)], \theta\big) \sim \big([\mathfrak{A}', n', m', \varphi'(\beta)], \theta'\big),$$

if the equality $\theta' = \tau_{\mathfrak{A}', m', \varphi'(\beta)} \circ \tau^{-1}_{\mathfrak{A}, m, \varphi(\beta)}(\theta)$ is satisfied.

It is easy to see that, given a simple pair $(k, \beta)$ over $F$, it is an equivalence relation on the set of such pairs $([\mathfrak{A}, n, m, \varphi(\beta)], \theta)$, which is denoted by $\mathscr{C}_{(k,\beta)}$.

DEFINITION 1.8 (SEE [4, DEFINITION 1.5])

A *potential simple character* over $F$ (or *ps-character*) is a triple $(\Theta, k, \beta)$ made of a simple pair $(k, \beta)$ over $F$ and an equivalence class $\Theta$ in $\mathscr{C}_{(k,\beta)}$.

If a pair $([\mathfrak{A}, n, m, \varphi(\beta)], \theta)$ belongs to an equivalence class $\Theta$, we write

$$\Theta\big(\mathfrak{A}, m, \varphi(\beta)\big) = \theta.$$

DEFINITION 1.9 (SEE [4, DEFINITION 1.10])

For $i = 1, 2$, let $(\Theta_i, k_i, \beta_i)$ be a ps-character over $F$. We say that these ps-characters are *endo-equivalent*, denoted by

$$\Theta_1 \approx \Theta_2,$$

if these ps-characters satisfy the following conditions:

(a) $k_1 = k_2$;
(b) $[F[\beta_1] : F] = [F[\beta_2] : F]$;
(c) there exists a central simple $F$-algebra $A$ together with realizations $([\mathfrak{A}, n_i, m_i, \varphi_i(\beta_i)]$ of $(k_i, \beta_i)$, $i = 1, 2$, in $A$ such that $\Theta_1(\mathfrak{A}, m_1, \varphi_1(\beta_1))$ and $\Theta_2(\mathfrak{A}, m_2, \varphi_2(\beta_2))$ intertwine in $A^\times$.

## 2. The Jacquet–Langlands correspondence and endo-classes

We recall from [4, Conjecture 9.5] that an endo-class over $F$ is invariant under the Jacquet–Langlands correspondence.

### 2.1. Simple type

Let $D$ be a central division $F$-algebra of dimension $d^2$ over $F$, $d \geq 1$, and let $V$ be a right $D$-vector space of dimension $m \geq 1$. Set $A = \mathrm{End}_D(V)$. Through a $D$-basis of $V$, we identify $A = M_m(D)$ and set $G = A^\times = \mathrm{GL}_m(D)$.

Associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, we have the compact open subgroups $J(\beta, \mathfrak{A}) \supset J^1(\beta, \mathfrak{A}) = J(\beta, \mathfrak{A}) \cap U^1(\mathfrak{A})$, as defined in Section 1.1. Let $E = F[\beta]$, let $B = C_A(E)$, and let $\mathfrak{B} = \mathfrak{A} \cap B$. Then, there exists a canonical isomorphism

$$J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \simeq U(\mathfrak{B})/U^1(\mathfrak{B}),$$

and there exist a central $E$-algebra $D_E$ of dimension $d_E^2$ and a positive integer $m_E$ such that $B \simeq \mathrm{M}_{m_E}(D_E)$.

DEFINITION 2.1 ([10, SECTION 0.6], [15, 2.5.1])
A simple type of *level zero* in $G$ is a pair $(U, \tau)$, where

(a) $U = U(\mathfrak{A})$ for a principal hereditary $\mathfrak{o}_F$-order in $A$ with $r = e_F(\mathfrak{A})$;

(b) $\tau$ is an irreducible representation of $U = U(\mathfrak{A})$, trivial on $U^1(\mathfrak{A})$ and inflated from a representation $\overline{\sigma}_0^{\otimes r}$ of the quotient group $U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq \mathrm{GL}_s(k_D)^r$, where $\overline{\sigma}_0$ is an irreducible cuspidal representation of $\mathrm{GL}_s(k_D)$ and $r, s$ are positive integers satisfying $rs = m$.

We say that a simple type $(U, \tau) = (U(\mathfrak{A}), \tau)$ of level zero in $G$ is *attached to* the null simple stratum $[\mathfrak{A}, 0, 0, 0]$ in $A$ (see [15, Remark 4.1]).

DEFINITION 2.2
A simple type of *positive level* in $G$ is a pair $(J, \lambda)$, attached to a nonnull simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, given as follows:

(a) there exists a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$ such that $J = J^0(\beta, \mathfrak{A})$ and that if $E = F[\beta]$, $B = C_A(E)$ and $\mathfrak{B} = \mathfrak{A} \cap B$, $\mathfrak{B}$ is a principal hereditary $\mathfrak{o}_E$-order in $B$ with $r = e_E(\mathfrak{B})$;

(b) there exist a simple character $\theta \in \mathscr{C}(\mathfrak{A}, 0, \beta, \psi_F)$ and a simple type $(U(\mathfrak{B}), \tau)$ of level zero in $B^\times$ such that $\lambda$ is a representation of $J$ of the form

$$\lambda = \kappa \otimes \sigma,$$

where

(1) $\kappa$ is a $\beta$-extension of $\eta_\theta$;

(2) $\sigma$ is the representation of $J$, trivial on $J^1$, deduced from $\tau$ via the isomorphism $J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B})$ and $\tau$ is an irreducible representation of $U = U(\mathfrak{B})$, trivial on $U^1(\mathfrak{B})$ and inflated from a representation $\overline{\sigma}_0^{\otimes r}$ of the quotient group $U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_{m_E/r}(k_{D_E})^r$, where $\overline{\sigma}_0$ is an irreducible cuspidal representation of $\mathrm{GL}_{m_E/r}(k_{D_E})$.

## 2.2. Conjecture about preservation of the endo-class

Let $A = \mathrm{M}_m(D)$, and let $G = A^\times$ be as defined in Section 2.1. Let $\mathrm{Nrd}_A : A \to F$ be the reduced norm.

An irreducible smooth representation $\pi$ of $G$ is referred to as *essentially square-integrable* (or *discrete series*) if there exists an unramified character $\chi$ of $F^\times$ such that $(\chi \circ \mathrm{Nrd}_A) \otimes \pi$ is square-integrable modulo $F^\times$. Let $\mathcal{A}^2(G)$ be the set of isomorphism classes of irreducible essentially square-integrable representations of $G$, and let $\mathcal{E}(F)$ be the set of endo-classes of ps-characters over $F$ (see [4, Section 9.3]).

THEOREM 2.3
*For each* $\pi \in \mathcal{A}^2(G)$*, there exist a simple type* $(J,\lambda)$ *in* $G$ *attached to a simple stratum* $[\mathfrak{A},n,0,\beta]$ *in* $A$ *such that* $\pi \mid J$ *contains* $\lambda$*.*

*Proof*
This follows from [2] and [16]. □

From Theorem 2.3, for each $\pi \in \mathcal{A}^2(G)$, a pair $([\mathfrak{A},n,0,\beta],\theta)$ is given such that the character $\theta$ occurs in $\pi \mid H^1(\beta,\mathfrak{A})$. Let $(\Theta,0,\beta)$ be the ps-character defined by the pair $([\mathfrak{A},n,0,\beta],\theta)$ and denote by $\boldsymbol{\Theta}$ its endo-class. This endo-class $\boldsymbol{\Theta}$ depends only on the representation $\pi$, as in the Introduction. Thus, we write this endo-class $\boldsymbol{\Theta}$ as $\boldsymbol{\Theta}_G(\pi)$. Hence, we get a map

$$\boldsymbol{\Theta}_G : \mathcal{A}^2(G) \to \mathcal{E}(F).$$

For $\pi \in \mathcal{A}^2(G)$, we denote by $\chi_\pi$ the character function of $\pi$.

THEOREM 2.4 ([1], [9], [12], [13])
*Let* $D'$ *be another central simple* $F$*-algebra of dimension* $d'^2$*,* $d' \geq 1$*, and let* $G' = \mathrm{GL}_{m'}(D')$ *for a positive integer* $m'$ *with* $m'd' = md$*. Then, there exists a canonical bijection, referred to as the Jacquet–Langlands correspondence,*

$$\mathbf{JL} : \mathcal{A}^2(G) \to \mathcal{A}^2(G') \tag{2.1}$$

*such that, if* $\pi' = \mathbf{JL}(\pi)$ *for* $\pi \in \mathcal{A}^2(G)$*, then we have that*

$$(-1)^m \chi_\pi(g) = (-1)^{m'} \chi_{\pi'}(g'),$$

*where* $g$ *and* $g'$ *are regular elliptic elements of* $G$ *and* $G'$*, respectively, whose characteristic polynomials over* $F$ *are the same.*

In [4, 9.3], the following conjecture is given:

$$\boldsymbol{\Theta}_{G'}\big(\mathbf{JL}(\pi)\big) = \boldsymbol{\Theta}_G(\pi), \tag{2.2}$$

for any $\pi \in \mathcal{A}^2(G)$.

REMARK 2.5
Moreover, it is probable that there exists a single simple pair $(0,\beta)$ over $F$ such that, as representatives, $\boldsymbol{\Theta}_{G'}(\mathbf{JL}(\pi))$ and $\boldsymbol{\Theta}_G(\pi)$ have ps-characters over $F$ $(\Theta',0,\beta)$ and $(\Theta,0,\beta)$, respectively.

## 3. Some examples for the conjecture

We shall see that the Jacquet–Langlands correspondences given by Bushnell and Henniart [6] and Silberger and Zink [17] satisfy the equality (2.2).

### 3.1. An example for level-zero representations

Let $A = \mathrm{M}_m(D)$, and let $G = A^\times = \mathrm{GL}_m(D)$ be as above.

DEFINITION 3.1 ([10, SECTION 0.6])

An irreducible smooth representation $\pi$ of $G$ is referred to as *level zero* if there exists a principal hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A$ such that its representation space $\mathcal{V}$ has a nonzero $U^1(\mathfrak{A})$-fixed vector.

Let $\mathcal{A}_0^2(G)$ be the subset of level-zero representations in $\mathcal{A}^2(G)$. If a smooth representation $\pi$ of $G$ belongs to $\mathcal{A}_0^2(G)$, then, from [10, Theorem 5.5(i)], it contains a simple type $(J,\lambda)=(U(\mathfrak{A}),\tau)$ of level zero in $G$. Thus, we obtain a ps-character $(\Theta,0,0)$ with $([\mathfrak{A},0,0,0],\mathbf{1}_{U^1(\mathfrak{A})})\in\Theta$, that is,

$$\Theta(\mathfrak{A},0,0)=\mathbf{1}_{U^1(\mathfrak{A})}\in\mathscr{C}(\mathfrak{A},0,0),$$

and consequently the endo-class, denoted by $\boldsymbol{\Theta}_G(\pi)$, of this $(\Theta,0,0)$.

We now let $D'$ be a central division $F$-algebra of dimension $m^2d^2$, and let $G'=\mathrm{GL}_1(D')$. Then, from Theorem 2.4, we have the Jacquet–Langlands correspondence $\mathbf{JL}$: $\mathcal{A}^2(G')\to\mathcal{A}^2(G)$.

PROPOSITION 3.2 ([17, PROPOSITION 3.2])

*The Jacquet–Langlands correspondence* $\mathbf{JL}$ *induces a canonical bijection* $\mathcal{A}_0^2(G')\to\mathcal{A}_0^2(G)$.

We again denote by

$$\mathbf{JL}:\mathcal{A}_0^2(G')\to\mathcal{A}_0^2(G)$$

the bijection of Proposition 3.2.

THEOREM 3.3

*Let* $\mathbf{JL}$ *be the correspondence defined above. Then, for* $\pi\in\mathcal{A}_0^2(G')$, *we have*

$$\boldsymbol{\Theta}_G\big(\mathbf{JL}(\pi)\big)=\boldsymbol{\Theta}_{G'}(\pi).$$

*Proof*

Suppose that a class $\Theta'$ belongs to the endo-class $\boldsymbol{\Theta}_{G'}(\pi)$ and that a class $\Theta$ belongs to the endo-class $\boldsymbol{\Theta}_G(\mathbf{JL}(\pi))$. Then, we have the realizations $([\mathfrak{A}',0,0,0],\mathbf{1}_{U^1(\mathfrak{A}')})\in\Theta'$ and $([\mathfrak{A},0,0,0],\mathbf{1}_{U^1(\mathfrak{A})})\in\Theta$. Since, by definition, we have $\mathscr{C}(\mathfrak{A}',0,0)=\{\mathbf{1}_{U^1(\mathfrak{A}')}\}$ and $\mathscr{C}(\mathfrak{A},0,0)=\{\mathbf{1}_{U^1(\mathfrak{A})}\}$, we obtain

$$\mathbf{1}_{U^1(\mathfrak{A})}=\tau_{\mathfrak{A},0,0}\circ\tau_{\mathfrak{A}',0,0}^{-1}(\mathbf{1}_{U^1(\mathfrak{A}')}),$$

where, for example, $\tau_{\mathfrak{A},0,0}$ is the transfer $\mathscr{C}_F(0,0)\to\mathscr{C}(\mathfrak{A},0,0)$ defined in Section 1.2. Hence, by Definition 1.7, we have

$$\big([\mathfrak{A},0,0,0],\mathbf{1}_{U^1(\mathfrak{A})}\big)\sim\big([\mathfrak{A}',0,0,0],\mathbf{1}_{U^1(\mathfrak{A}')}\big)$$

and so $\Theta=\Theta'$. This shows the equality of this theorem and the proof is complete. $\square$

### 3.2. An example for totally ramified representations

In this section, we shall show that the explicit Jacquet–Langlands correspondence realized by Bushnell and Henniart [6] also satisfies the conjecture (2.2). This is never trivial. We first recall the realization of the correspondence.

Let $F$ be a finite extension of $\mathbb{Q}_p$ with $p \neq 2$, and let $D$ be a central division $F$-algebra of dimension $p^m$, $m \geq 1$. Set $G = \mathrm{GL}_{p^m}(F)$ and $G' = \mathrm{GL}_1(D) = D^\times$.

Let $\pi$ be an irreducible smooth representation of an inner form of $G$. Denote by $t(\pi)$ the cardinality of the unramified characters $\chi$ of $F^\times$ such that $(\chi \circ \mathrm{Nrd}) \otimes \pi \simeq \pi$, where Nrd denotes the reduced norm. This is referred to as the *inertial degree* of $\pi$. The representation $\pi$ is referred to as *totally ramified* if $t(\pi) = 1$ is satisfied.

From Theorem 2.4, there exists the Jacquet–Langlands correspondence

$$\mathbf{JL} : \mathcal{A}^2(G) \to \mathcal{A}^2(G').$$

Denote by $\mathcal{A}_m^{wr}(F)$ the set of isomorphism classes of irreducible totally ramified supercuspidal representations of $G = \mathrm{GL}_{p^m}(F)$, as in [6]. Then, this is a subset of $\mathcal{A}^2(G)$. We can define a subset $\mathcal{A}_0^{wr}(D)$ of $\mathcal{A}^2(G')$ by

$$\mathcal{A}_0^{wr}(D) = \mathbf{JL}\big(\mathcal{A}_m^{wr}(F)\big).$$

Thus, we get a canonical bijection, denoted again by $\mathbf{JL}$,

$$\mathbf{JL} : \mathcal{A}_m^{wr}(F) \to \mathcal{A}_0^{wr}(D).$$

In [6], this correspondence is explicitly described. From [7, (1.4.4)], we have $t(\mathbf{JL}(\pi)) = t(\pi)$, for $\pi \in \mathcal{A}^2(G)$. Thus, every $\pi \in \mathcal{A}_0^{wr}(D)$ is totally ramified.

We prepare notation to describe $\mathbf{JL}$. Set $A = \mathrm{M}_{p^m}(F)$. Let $\mathfrak{A}$ be the minimal hereditary $\mathfrak{o}_F$-order in $A$, and denote by $\mathscr{S}^{wr}(\mathfrak{A})$ the set of elements $\alpha$ of $\mathfrak{K}(\mathfrak{A})$ satisfying the following conditions (see [6, Section 1.1]):

(1) $[\mathfrak{A}, n, 0, \alpha]$ is a simple stratum in $A$, where $n = -\nu_{\mathfrak{A}}(\alpha)$;
(2) the field extension $F[\alpha]/F$ is of degree $p^m$.

Then, since $\mathfrak{A}$ is minimal, the extension $F[\alpha]/F$ is totally ramified.

We fix a level-one character $\psi_F$ of $F^\times$ as before. Let $\beta \in \mathscr{S}^{wr}(\mathfrak{A})$. Then, associated with the simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, we have compact open subgroups of $G = A^\times$

$$H^1(\beta, \mathfrak{A}) \subset J^1(\beta, \mathfrak{A})$$

as in Section 1.1. In order to indicate the base field, we write them as follows:

$$H^1_F(\beta, \mathfrak{A}) \subset J^1_F(\beta, \mathfrak{A}).$$

We have a certain open subgroup $I^1_F(\beta, \mathfrak{A})$ of $G$ that is normalized by $F[\beta]^\times$ and satisfies

$$H^1_F(\beta, \mathfrak{A}) \subset I^1_F(\beta, \mathfrak{A}) \subset J^1_F(\beta, \mathfrak{A}).$$

See [6, Section 6.4] for the definition. This group depends only on the simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$. We can define the subgroup $I_F(\beta, \mathfrak{A})$ of $G$ by

$$I_F(\beta, \mathfrak{A}) = F[\beta]^\times I_F^1(\beta, \mathfrak{A}).$$

We denote by

$$\mathscr{D}(\mathfrak{A}, \beta, \psi_F) = \mathscr{D}_F(\beta, \psi_F)$$

the group of certain quasicharacters of the group $I_F(\beta, \mathfrak{A})$ defined in [6, Section 8.4]. To simplify, we will write $I_F(\beta, \mathfrak{A})$ as $I_F(\beta)$.

We define the subset $\mathscr{S}^{wr}(\mathfrak{o}_D)$ of $G' = D^\times$ like $\mathscr{S}^{wr}(\mathfrak{A}) \subset G = \mathrm{GL}_{p^m}(F)$. Let $\alpha \in \mathscr{S}^{wr}(\mathfrak{o}_D)$. Then, associated with the simple stratum $[\mathfrak{o}_D, -\nu_D(\alpha), 0, \alpha]$ in $D$, we similarly have the compact open subgroups $H^1(\alpha, \mathfrak{o}_D) \subset J^1(\alpha, \mathfrak{o}_D)$ (see [3], [14]) and the group ${}_D I^1(\alpha, \mathfrak{o}_D)$, defined in [6, Section 6.4], that is normalized by $F[\alpha]^\times$ and satisfies

$$H^1(\alpha, \mathfrak{o}_D) \subset {}_D I^1(\alpha, \mathfrak{o}_D) \subset J^1(\alpha, \mathfrak{o}_D).$$

We define the open subgroup ${}_D I(\alpha, \mathfrak{o}_D)$ of $G' = D^\times$ by

$${}_D I(\alpha, \mathfrak{o}_D) = F[\alpha]^\times {}_D I^1(\alpha, \mathfrak{o}_D).$$

We also write

$${}_D H_F^1(\alpha) = H^1(\alpha, \mathfrak{o}_D), \qquad {}_D J_F^1(\alpha) = J^1(\alpha, \mathfrak{o}_D), \qquad {}_D I_F^1(\alpha) = {}_D I^1(\alpha, \mathfrak{o}_D).$$

We denote by

$$\mathscr{D}(\mathfrak{o}_D, \alpha, \psi_F) = {}_D\mathscr{D}_F(\alpha, \psi_F)$$

the group of certain quasicharacters of the group ${}_D I_F(\alpha) = {}_D I(\alpha, \mathfrak{o}_D)$ (see [6, Comment 8.4]).

Write $G_F = G = \mathrm{GL}_{p^m}(F)$ and $G'_F = G' = D^\times$ to indicate the base field. Now we can describe the Jacquet–Langlands correspondence **JL** as follows.

THEOREM 3.4 ([6, COROLLARIES 2–4 TO THEOREM 3.1])
*For $\pi \in \mathcal{A}_m^{wr}(F)$, there exist $\beta \in \mathscr{S}^{wr}(\mathfrak{A})$ and $\lambda \in \mathscr{D}_F(\beta, \psi_F)$ such that*

$$\pi \simeq \text{c-Ind}_{I_F(\beta)}^{G_F} \lambda,$$

*and there exist $\iota\beta \in D^\times$ and ${}_D\lambda \in {}_D\mathscr{D}_F(\iota\beta, \psi_F)$ such that*

$$\mathbf{JL}(\pi) \simeq \mathrm{Ind}_{{}_D I_F(\iota\beta)}^{G'_F} {}_D\lambda.$$

*Here, the element $\iota\beta \in \mathfrak{o}_D$ is conjugate to $\beta = \beta \otimes 1$ in $A \otimes_F K = D \otimes_F K$ for some finite unramified extension $K/F$ (see below).*

In Theorem 3.4, we write

$$\pi_F(\lambda) = \text{c-Ind}_{I_F(\beta)}^{G_F} \lambda, \qquad \pi_D({}_D\lambda) = \mathrm{Ind}_{{}_D I_F(\iota\beta)}^{G'_F} {}_D\lambda.$$

### 3.3. Realizations for the endo-classes

Assume that a smooth representation $\pi$ of $G = \mathrm{GL}_{p^m}(F)$ belongs to $\mathcal{A}_m^{wr}(F)$. From Theorem 3.4, we have $\pi \simeq \pi_F(\lambda)$ for some $\lambda \in \mathscr{D}_F(\beta, \psi_F)$. We may identify $\pi = \pi_F(\lambda)$. Since $H_F^1(\beta) \subset I_F^1(\beta)$, by the definition of the quasicharacter $\lambda$ in [6, Section 8.4], we get that

$$\theta = \lambda \mid H_F^1(\beta) \in \mathscr{C}(\mathfrak{A}, 0, \beta, \psi_F).$$

Thus, $\pi = \pi_F(\lambda)$ contains the simple character $\theta$. Hence, we can associate $\pi$ with a pair $([\mathfrak{A}, n, 0, \beta], \theta)$, where $n = -\nu_{\mathfrak{A}}(\beta)$. Let $(\Theta, 0, \beta)$ be the ps-character over $F$ defined by the pair $([\mathfrak{A}, n, 0, \beta], \theta)$. Hence, we can associate $\pi$ with the endo-class of $(\Theta, 0, \beta)$. We denote this endo-class as $\boldsymbol{\Theta}_G(\pi)$.

Set $\pi' = \mathbf{JL}(\pi) \in \mathcal{A}_0^{wr}(D)$. Then, again from Theorem 3.4, we have $\pi' \simeq \pi_D({}_D\lambda)$ for some ${}_D\lambda \in {}_D\mathscr{D}_F(\iota\beta, \psi_F)$. We also identify $\pi' = \pi_D({}_D\lambda)$. Then, we obtain

$$ {}_D\theta = {}_D\lambda \mid {}_DH_F^1(\iota\beta) \in \mathscr{C}(\mathfrak{o}_D, 0, \iota\beta, \psi_F) \tag{3.1}$$

and consequently a pair $([\mathfrak{o}_D, n', 0, \iota\beta], {}_D\theta)$, where $n' = -\nu_D(\iota\beta)$. Let $({}_D\Theta, 0, \iota\beta)$ be the ps-character over $F$ defined by the pair $([\mathfrak{o}_D, n', 0, \iota\beta], {}_D\theta)$. Thus, we can associate $\pi'$ with the endo-class of $({}_D\Theta, 0, \iota\beta)$. We denote this endo-class as $\boldsymbol{\Theta}_{G'}(\pi')$.

In order to show the conjecture (2.2) that $\boldsymbol{\Theta}_{G'}(\pi') = \boldsymbol{\Theta}_G(\pi)$, we shall show that

$$ \big([\mathfrak{A}, n, 0, \beta], \theta\big) \sim \big([\mathfrak{o}_D, n', 0, \iota\beta], {}_D\theta\big) \tag{3.2}$$

in the sense of Definition 1.7.

### 3.4. Relationship between the quasicharacters

We retain the notation and assumptions of Section 3.2. We observe the relationship between the quasicharacters $\lambda$ and ${}_D\lambda$ in Theorem 3.4.

Assume that $K$ is a finite unramified extension of $F$ of degree divisible by $p^m$. Set $A_K = A \otimes_F K$ and $D_K = D \otimes_F K$. For the hereditary $\mathfrak{o}_F$-orders $\mathfrak{A}$ and $\mathfrak{o}_D$ in $A = \mathrm{M}_{p^m}(F)$ and $D$, respectively, we also set

$$\mathfrak{A}_K = \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_K, \qquad {}_D\mathfrak{A}_K = \mathfrak{o}_D \otimes_{\mathfrak{o}_F} \mathfrak{o}_K.$$

Then, from [6, Lemma 2.5], there exists an isomorphism of $K$-algebras $\iota : A_K \to D_K$ such that

$$\iota\beta \in \mathscr{S}^{wr}(\mathfrak{o}_D), \qquad \iota(\mathfrak{A}_K) = {}_D\mathfrak{A}_K.$$

We remark that $\iota\beta \in G' = D^\times$. For the simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{M}_{p^m}(F)$, from [6, Proposition 5.1], the stratum $[\mathfrak{A}_K, n, 0, \beta \otimes 1]$ in $A_K = \mathrm{M}_{p^m}(K)$ is simple. We identify $\beta = \beta \otimes 1$. The open subgroup of $G_K = A_K^\times$

$$I_K(\beta) = K[\beta]^\times I_K^1(\beta)$$

is defined in the same way as that of $I_F(\beta)$. Here, since the extension $K/F$ is unramified and the extension $F[\beta]/F$ is totally ramified, $K[\beta] = K \cdot F[\beta]$ is a totally ramified extension field of $K$ of degree $p^m$. Let $\zeta$ be a level-one additive character of $K$ such that $\zeta \mid F = \psi_F$. Then, we denote by $\mathscr{D}(\mathfrak{A}_K, \beta, \zeta) = \mathscr{D}_K(\beta, \zeta)$ the set of certain quasicharacters of $I_K(\beta)$ with respect to $\zeta$, as above.

We obtain $I_K(\beta) \cap A^\times = I_K(\beta)$ from [6, Proposition 1.5].

Let $F_{nr}/F$ be a maximal unramified extension, and let $\tilde{F}$ be the completion of $F_{nr}$ with respect to the discrete valuation $\nu$. Hereafter, we fix a level-one character $\Psi$ of $\tilde{F}$ such that $\Psi \mid F = \psi_F$. For $K/F$ finite and contained in $F_{nr}$, we set $\Psi^K = \Psi \mid K$. From this character $\Psi^K$, we obtain the sets of quasicharacters $\mathscr{D}_K(\beta, \Psi^K)$ and ${}_D\mathscr{D}_K(\iota\beta, \Psi^K)$. Then, it follows from [6, Section 1.3.2] that, through the $K$-isomorphism $\iota$ above, the map $\mu \mapsto \mu \circ \iota$ induces a bijection

$$\mathscr{D}_K(\beta, \Psi^K) \simeq {}_D\mathscr{D}_K(\iota\beta, \Psi^K),$$

denoted again by $\iota$.

PROPOSITION 3.5 ([6, SECTION 2.5])

*Let $\lambda \in \mathscr{D}_F(\beta, \psi_F)$ and ${}_D\lambda \in {}_D\mathscr{D}_F(\iota\beta, \psi_F)$ be the quasicharacters in Theorem 3.4. Then, there exists a quasicharacter $\lambda(K) \in \mathscr{D}_K(\beta, \Psi^K)$ such that*

$$\lambda(K) \mid I_F(\beta) = \lambda, \qquad {}_D\lambda = \lambda(K) \circ \iota^{-1} \mid {}_D I_F(\iota\beta).$$

*Proof*

The quasicharacters $\lambda(K)$ and ${}_D\lambda$ are replaced by $\tilde{\lambda}^K$ satisfying $\tilde{\lambda}^K \mid I_F(\beta) = \lambda$ and $\tilde{\lambda}^K \circ \iota^{-1} \mid {}_D I_F(\iota\beta) = \iota(\tilde{\lambda}^K)^F$ in [6, Section 2.5], respectively. Thus, the equalities of this proposition follow and the proof is complete. □

In the proof of Proposition 3.5, we remark that the representation $\pi_D({}_D\lambda)$ defined in Section 3.2 is replaced by ${}_D\pi(\lambda)$ in [6, Section 2.5]. By the proof of [6, Section 3.3 Lemma 2], we can identify

$$D_K = A_K, \qquad {}_D\mathfrak{A}_K = \mathfrak{A}_K$$

and find a $K$-automorphism $\iota$ of $D_K = A_K$ satisfying the conditions: (1) $\iota(\mathfrak{A}_K) = \mathfrak{A}_K$ and (2) $\iota(F[\beta]) \subset D$. Thus, we have $\iota = \mathrm{Ad}(y_0)$ for some $y_0 \in U(\mathfrak{A}_K) = \mathfrak{A}_K^\times$.

PROPOSITION 3.6

*The group ${}_D I_F(\iota\beta)$ and the quasicharacter ${}_D\lambda$ in Proposition 3.5 may be replaced by*

$${}_D I_F(y_0^{-1}\beta y_0) = y_0^{-1} I_K(\beta) y_0 \cap D^\times, \qquad \lambda(K) \circ \mathrm{Ad}(y_0) \mid {}_D I_F(y_0^{-1}\beta y_0).$$

*Proof*

This follows from the proof of [6, Section 3.3 Lemma 2]. The proof is complete. □

Since we have $y_0 \in \mathfrak{A}_K^\times$, we obtain

$$I_K(\iota\beta) = I_K(y_0^{-1}\beta y_0) = y_0^{-1} I_K(\beta) y_0.$$

### 3.5. Simple and quasisimple characters

Let $\mathfrak{A}$ be the minimal hereditary $\mathfrak{o}_F$-order in $A = \mathrm{M}_{p^m}(F)$, and let $\beta \in \mathscr{S}^{wr}(\mathfrak{A})$. Then, the pair $(0,\beta)$ is a simple pair over $F$. Set $E = F[\beta]$. Then, the field $E$ is a totally ramified extension of $F$ of degree $p^m$. Let $A(E)$ and $\mathfrak{A}(E)$ be the objects defined in Section 1.2. Then, through a basis of $E$ as an $F$-vector space, we identify

$$A(E) = \mathrm{M}_{p^m}(F) = A.$$

Then, we may set $\mathfrak{A}(E) = \mathfrak{A}$. Thus, in $A = A(E)$, we identify

$$\left[\mathfrak{A}(E), n, 0, \beta\right] = [\mathfrak{A}, n, 0, \beta],$$

and

$$\mathscr{C}_F(0,\beta) = \mathscr{C}\left(\mathfrak{A}(E), 0, \beta, \psi_F\right) = \mathscr{C}(\mathfrak{A}, 0, \beta, \psi_F), \tag{3.3}$$

with respect to the fixed level-one additive character $\psi_F$ of $F$.

Let $K/F$ be an unramified extension of degree divisible by $p^m$, and let $\Psi^K$ be a character of $K$ as before such that $\Psi^K \mid F = \psi_F$. Set $A_K = A \otimes_F K, \mathfrak{A}_K = \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_K$, and $\tilde{E} = E \otimes_F K$. Then, we have $\tilde{E} = E \cdot K = K[\beta]$ and this is a totally ramified extension of $K$ of degree $p^m$, as seen in Section 3.3. Thus, we can identify

$$A_K(\tilde{E}) = \mathrm{End}_K(\tilde{E}) = A_K, \qquad \mathfrak{A}_K(\tilde{E}) = \mathrm{End}^0_{\mathfrak{o}_K}\left(\{\mathfrak{p}^i_{\tilde{E}} : i \in \mathbb{Z}\}\right) = \mathfrak{A}_K.$$

Hence, we have $[\mathfrak{A}_K(\tilde{E}), n, 0, \beta] = [\mathfrak{A}_K, n, 0, \beta]$ and

$$\mathscr{C}_K(0,\beta) = \mathscr{C}\left(\mathfrak{A}_K(\tilde{E}), 0, \beta, \Psi^K\right) = \mathscr{C}(\mathfrak{A}_K, 0, \beta, \Psi^K). \tag{3.4}$$

### 3.6. Descent of transfers

We come back to Section 3.4 and investigate the representation $\pi_D({}_D\lambda)$ of $G' = D^\times$. From Proposition 3.5, we can set

$$\theta(K) = \lambda(K) \mid H^1_K(\beta) \in \mathscr{C}(\mathfrak{A}_K, 0, \beta, \Psi^K)$$

as in Section 3.3. Then, we have $\theta(K) \mid H^1_F(\beta) = \theta$. Hereafter, set $\iota\beta = y_0^{-1}\beta y_0$. We can also set

$${}_D\theta(K) = \lambda(K) \circ \mathrm{Ad}(y_0) \mid {}_DH^1_K(\iota\beta) \in \mathscr{C}({}_D\mathfrak{A}_K, 0, \iota\beta, \Psi^K).$$

Since ${}_D\mathfrak{A}_K = \mathfrak{A}_K$, we have

$$H^1_K(\iota\beta) = H^1(\iota\beta, \mathfrak{A}_K) = H^1(\iota\beta, {}_D\mathfrak{A}_K) = {}_DH^1_K(\iota\beta)$$

and

$${}_DH^1_K(\iota\beta) \cap D^\times = {}_DH^1_F(\iota\beta) = H^1(\iota\beta, \mathfrak{o}_D).$$

Hence, from the equality (3.1), we obtain

$${}_D\theta = {}_D\theta(K) \mid {}_DH^1_F(\iota\beta) \in \mathscr{C}(\mathfrak{o}_D, 0, \iota\beta, \psi_F).$$

To prove the equivalence (3.2), it is enough to prove the following condition.

*Condition C1.* ${}_D\theta$ *is the transfer of* $\theta$.

From (3.3), (3.4), and the definition [14, Section 3.3], there exist canonical bijections, referred to as the *transfers*,

$$\tau_F = \tau_{\mathfrak{A},0,\beta} : \mathscr{C}_F(0,\beta) = \mathscr{C}(\mathfrak{A},0,\beta,\psi_F) \to \mathscr{C}(\mathfrak{o}_D,0,\iota\beta,\psi_F)$$

and

$$\tau_K = \tau_{\mathfrak{A}_K,0,\beta} : \mathscr{C}_K(0,\beta) = \mathscr{C}(\mathfrak{A}_K,0,\beta,\Psi^K) \to \mathscr{C}({}_D\mathfrak{A}_K,0,\iota\beta,\Psi^K).$$

From [14, Theorem 3.53], we get the following commutative diagram:

$$\begin{array}{ccc} \mathscr{C}(\mathfrak{A}_K,0,\beta,\Psi^K) & \xrightarrow{\tau_K} & \mathscr{C}({}_D\mathfrak{A}_K,0,\iota\beta,\Psi^K) \\ {\scriptstyle \mathrm{res}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \mathscr{C}(\mathfrak{A},0,\beta,\psi_F) & \xrightarrow[\tau_F]{} & \mathscr{C}(\mathfrak{o}_D,0,\iota\beta,\psi_F), \end{array}$$

where the vertical maps are the restrictions. Hence, to prove Condition C1, it is enough to prove the following condition.

*Condition C2.* $\tau_K(\theta') = \theta' \circ \mathrm{Ad}(y_0)$, *for* $\theta' \in \mathscr{C}(\mathfrak{A}_K,0,\beta,\Psi^K)$.

In fact, if Condition C2 is satisfied, then by setting $\theta' = \theta(K)$, we obtain that

$$\tau_K\big(\theta(K)\big) = \theta(K) \circ \mathrm{Ad}(y_0) = {}_D\theta(K).$$

Thus, by the commutative diagram above, we obtain that

$$\tau_F(\theta) = \tau_F\big(\mathrm{res}\big(\theta(K)\big)\big) = \mathrm{res}\big(\tau_K\big(\theta(K)\big)\big) = \mathrm{res}\big({}_D\theta(K)\big) = {}_D\theta,$$

which means Condition C1 holds.

Since $[\mathfrak{A}_K,n,0,\beta]$ is a simple stratum in $A_K = \mathrm{M}_{p^m}(K)$ and $K[\beta]/K$ is a totally ramified extension of degree $p^m$, we have

$$\beta = \beta \otimes 1 \in \mathscr{S}^{wr}(\mathfrak{A}_K).$$

Moreover, we have $\iota\beta = y_0^{-1}\beta y_0$ for the element $y_0 \in \mathfrak{A}_K^\times$ defined above.

Finally, in order to prove Condition C2, by replacing the base field $K$ of Condition C2 by the field $F$, it is enough to prove the following.

PROPOSITION 3.7

*Let* $\mathfrak{A}$ *be the minimal hereditary* $\mathfrak{o}_F$*-order in* $A = \mathrm{M}_{p^m}(F)$ *and let* $\beta \in \mathscr{S}^{wr}(\mathfrak{A})$. *Let* $y_0$ *be an element of* $\mathfrak{A}^\times$ *and let* $\iota : F[\beta] \to A$ *be an* $F$*-embedding defined by* $\iota\beta = y_0^{-1}\beta y_0$. *Then, the transfer*

$$\tau_F = \tau_{\mathfrak{A},0,\beta} : \mathscr{C}_F(0,\beta) = \mathscr{C}(\mathfrak{A},0,\beta,\psi_F) \to \mathscr{C}(\mathfrak{A},0,\iota\beta,\psi_F)$$

*satisfies*

$$\tau_F(\theta) = \theta \circ \mathrm{Ad}(y_0), \quad \theta \in \mathscr{C}(\mathfrak{A},0,\beta,\psi_F).$$

We devote the next section to a proof of this proposition.

### 3.7. A proof of the auxiliary proposition

Hereafter, let $V$ be an $F$-vector space of dimension $p^m$, $m \geq 1$, let $A = \mathrm{End}_F(V)$, and let $G = A^\times$. If necessary, through an $F$-basis of $V$, we identify $A = \mathrm{M}_{p^m}(F)$ and $G = \mathrm{GL}_{p^m}(F)$.

Let $\mathfrak{A}$ be the minimal hereditary $\mathfrak{o}_F$-order in $A$, and let $\beta \in \mathscr{S}^{wr}(\mathfrak{A})$. Set $E = F[\beta]$. Then, $E$ is a totally ramified extension of $F$ of degree $p^m$, and $\mathfrak{A}$ is $E$-pure. Thus, $V$ is a one-dimensional $E$-vector space. Identifying $V = E$, we have $A = \mathrm{End}_F(V) = \mathrm{End}_F(E) = A(E)$ and $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}) = \mathfrak{A}(E)$, as in Section 3.5. We set $\mathcal{L} = \{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ and write $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. We remark that the element $y_0 \in \mathfrak{A}(\mathcal{L})^\times$ satisfies

$$y_0^{-1}\mathfrak{A}(\mathcal{L})y_0 = \mathfrak{A}(\mathcal{L}).$$

We prove Proposition 3.7 by the method of [8, (3.6.14)]. Set $B = C_A(E)$ and $\mathfrak{B} = B \cap \mathfrak{A}$. Then, we may identify $B = E$ and $\mathfrak{B} = \mathfrak{o}_E$. Set

$$\tilde{V} = V \oplus V = E \oplus E.$$

Then, $\tilde{V}$ is a $2p^m$-dimensional $F$-vector space, and it can be viewed as a two-dimensional $E$-vector space. Set

$$\tilde{A} = \mathrm{End}_F(\tilde{V}).$$

We distinguish the factors $V$ of $\tilde{V}$ as follows: $\tilde{V} = V \oplus V = V_1 \oplus V_2$. Set $A_i = \mathrm{End}_F(V_i)$, $i = 1, 2$. We view $\mathfrak{A}$ as the $\mathfrak{o}_F$-order in $A_1$. Then, the elements $\beta$ and $y_0$ belong to $A_1$, and $\mathcal{L}$ is the $\mathfrak{o}_F$-lattice chain in $V_1 = V$. This $\mathcal{L}$ can be also viewed as the $\mathfrak{o}_F$-lattice chain in $V_2 = V$. In the $F$-space $V_1$, we set

$$\mathcal{L}_1 = y_0^{-1}\mathcal{L} = \{y_0^{-1}\mathfrak{p}_E^i : i \in \mathbb{Z}\}.$$

Since $y_0 \in \mathfrak{A}(\mathcal{L})^\times = \mathrm{Ker}\,\nu_{\mathfrak{A}}$, we have $\mathcal{L}_1 = \mathcal{L}$ and so

$$y_0^{-1}\mathfrak{P}(\mathcal{L})^k y_0 = \mathfrak{P}(\mathcal{L})^k, \quad k \geq 0. \tag{3.5}$$

For $i = 1, 2$, we set

$$L_j^i = \mathfrak{p}_E^j, \quad j \in \mathbb{Z},$$

and $\mathcal{L}_i = \{L_j^i : j \in \mathbb{Z}\} = \mathcal{L}$.

We define $\mathfrak{o}_F$-lattices in $\tilde{V} = V_1 \oplus V_2$ by

$$M_j = L_j^1 \oplus L_j^2, \quad j \in \mathbb{Z},$$

and set $\mathcal{M} = \{M_j : j \in \mathbb{Z}\}$. Then, $\mathcal{M}$ is an $\mathfrak{o}_F$-lattice chain in $\tilde{V}$ of $\mathfrak{o}_F$-period $p^m$, and also an $\mathfrak{o}_E$-lattice chain in $\tilde{V}$ of $\mathfrak{o}_E$-period one. Set

$$\tilde{\mathfrak{A}} = \mathfrak{A}(\mathcal{M}) = \{x \in \tilde{A} : xM_j \subset M_j, j \in \mathbb{Z}\}$$

and $\tilde{\mathfrak{P}} = \mathfrak{P}(\mathcal{M})$. Then, $\tilde{\mathfrak{A}}$ is a hereditary $\mathfrak{o}_F$-order in $\tilde{A}$, and $\tilde{\mathfrak{P}}$ is the Jacobson radical of $\tilde{\mathfrak{A}}$. For $i = 1, 2$, let $\boldsymbol{e}_i$ be the canonical projection $\tilde{V} = V_1 \oplus V_2 \to V_i$. Then, we have

$$\tilde{A} = \coprod_{i,j} \boldsymbol{e}_i \tilde{A} \boldsymbol{e}_j.$$

In particular, we identify $A_i = \mathrm{End}_F(V_i) = \boldsymbol{e}_i \tilde{A} \boldsymbol{e}_i$, $i = 1, 2$. Then, there exists a canonical embedding $A_1 \times A_2 \hookrightarrow \tilde{A}$. For $\beta \in A$, set

$$\varphi(\beta) = (\iota\beta, \beta) = (y_0^{-1}\beta y_0, \beta) \in A_1 \times A_2 \subset \tilde{A}.$$

Then, the map $\beta \mapsto \varphi(\beta)$ defines an $F$-embedding $E = F[\beta] \to \tilde{A}$, denoted again by $\varphi$. We identify $E = F[\beta] = F[\varphi(\beta)] = \varphi(E) \subset \tilde{A}$. Thus, we can view $\tilde{V} = V_1 \oplus V_2$ as an $E$-vector space. By the definition of $\tilde{\mathfrak{A}}$, we have $E^\times \subset \mathfrak{K}(\tilde{\mathfrak{A}})$. Let $\tilde{B} = C_{\tilde{A}}(\varphi(\beta))$, let $B_1 = C_{A_1}(\iota\beta)$, and let $B_2 = C_{A_2}(\beta)$. Then, through the identification $A_1 = A_2 = A$, we have $B_1 = y_0^{-1} B y_0$ and $B_2 = B$. In $A_i = \mathrm{End}_F(V_i)$, set

$$\mathfrak{A}_i = \mathfrak{A}(\mathcal{L}_i) = \mathfrak{A}(\mathcal{L}),$$

for $i = 1, 2$. We have

$$E^\times \simeq \boldsymbol{e}_i E^\times \boldsymbol{e}_i \subset \mathfrak{K}(\mathfrak{A}_i).$$

Set $\tilde{\mathfrak{B}} = \tilde{\mathfrak{A}} \cap \tilde{B}$ and $\mathfrak{B}_i = \mathfrak{A}_i \cap B_i$, for $i = 1, 2$. From (3.5), we obtain that

$$\mathfrak{B}_1 = \mathfrak{A}_1 \cap B_1 = y_0^{-1}\mathfrak{A}(\mathcal{L}) y_0 \cap y_0^{-1} B y_0 = y_0^{-1}\mathfrak{B} y_0$$

and $\mathfrak{B}_2 = \mathfrak{B}$. Since $\mathfrak{H}^k(\varphi(\beta), \tilde{\mathfrak{A}})$ is a $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}})$-bimodule, by [8, (3.6.15)], we obtain

$$\mathfrak{H}^k\big(\varphi(\beta), \tilde{\mathfrak{A}}\big) \cap A_i = \boldsymbol{e}_i \mathfrak{H}^k\big(\varphi(\beta), \tilde{\mathfrak{A}}\big) \boldsymbol{e}_i, \quad k \geq 0,$$

for $i = 1, 2$. In fact, for $k \geq 0$, we prove that

$$\text{(3.6)} \qquad \begin{cases} \mathfrak{H}^k(\varphi(\beta), \tilde{\mathfrak{A}}) \cap A_1 = \mathfrak{H}^k(\iota\beta, \mathfrak{A}(\mathcal{L})) = y_0^{-1}\mathfrak{H}^k(\beta, \mathfrak{A}(\mathcal{L})) y_0, \\ \mathfrak{H}^k(\varphi(\beta), \tilde{\mathfrak{A}}) \cap A_2 = \mathfrak{H}^k(\beta, \mathfrak{A}(\mathcal{L})). \end{cases}$$

It is enough to prove this for the case $k = 0$. We proceed by induction along $\beta$. Assume that $\beta$ is minimal over $F$. Then, we have $\mathfrak{H}(\varphi(\beta), \tilde{\mathfrak{A}}) = \tilde{\mathfrak{B}} + \tilde{\mathfrak{P}}^{\lfloor -\nu/2 \rfloor + 1}$, where $\nu = \nu_E(\beta)$. From [8, (3.6.15)], we obtain that

$$\mathfrak{H}\big(\varphi(\beta), \tilde{\mathfrak{A}}\big) \cap A_i = \boldsymbol{e}_i \tilde{\mathfrak{B}} \boldsymbol{e}_i + \boldsymbol{e}_i \tilde{\mathfrak{P}}^{\lfloor -\nu/2 \rfloor + 1} \boldsymbol{e}_i = \mathfrak{B}_i + \mathfrak{P}_i^{\lfloor -\nu/2 \rfloor + 1}.$$

Moreover, we have that

$$\mathfrak{B}_1 + \mathfrak{P}_1^{\lfloor -\nu/2 \rfloor + 1} = y_0^{-1}\mathfrak{B} y_0 + y_0^{-1}\mathfrak{P}^{\lfloor -\nu/2 \rfloor + 1} y_0 = y_0^{-1}\mathfrak{H}\big(\beta, \mathfrak{A}(\mathcal{L})\big) y_0$$

and $\mathfrak{B}_2 + \mathfrak{P}_2^{\lfloor -\nu/2 \rfloor + 1} = \mathfrak{B} + \mathfrak{P}^{\lfloor -\nu/2 \rfloor + 1} = \mathfrak{H}(\beta, \mathfrak{A}(\mathcal{L}))$. Thus, (3.6) is proved.

In the general case, let

$$r_0 = -k_0\big(\beta, \mathfrak{A}(\mathcal{L})\big) = -k_0\big(\iota\beta, \mathfrak{A}(\mathcal{L})\big) = -k_0\big(\beta, \mathfrak{A}(E)\big).$$

Then, there exists a simple stratum $[\mathfrak{A}(E), -\nu, r_0, \gamma]$ in $A(E)$ that is equivalent to $[\mathfrak{A}(E), -\nu, r_0, \beta]$. Since $\gamma$ belongs to $A(E) = A$, we can define an $F$-embedding $\varphi : F[\gamma] \to \tilde{A}$ by

$$\varphi(\gamma) = (\iota\gamma, \gamma) = (y_0^{-1}\gamma y_0, \gamma).$$

The stratum $[\mathfrak{A}_1, -\nu, r_0, \iota\gamma]$ is simple in $A_1 = A = A(E)$ and is equivalent to $[\mathfrak{A}_1, -\nu, r_0, \iota\beta]$. Similarly, the stratum $[\mathfrak{A}_2, -\nu, r_0, \gamma]$ is simple in $A_2 = A = A(E)$

and is equivalent to $[\mathfrak{A}_2, -\nu, r_0, \beta]$. Thus, we obtain

$$\mathfrak{H}\big(\varphi(\beta), \tilde{\mathfrak{A}}\big) = \tilde{\mathfrak{B}} + \mathfrak{H}^{\lfloor r_0/2 \rfloor + 1}\big(\varphi(\gamma), \tilde{\mathfrak{A}}\big).$$

Moreover, by induction, we obtain

$$\begin{aligned}\mathfrak{H}\big(\varphi(\beta), \tilde{\mathfrak{A}}\big) \cap A_1 &= \mathfrak{B}_1 + \mathfrak{H}^{\lfloor r_0/2 \rfloor + 1}(\iota\gamma, \mathfrak{A}_1) \\ &= y_0^{-1} \mathfrak{B} y_0 + y_0^{-1} \mathfrak{H}^{\lfloor r_0/2 \rfloor + 1}\big(\gamma, \mathfrak{A}(\mathcal{L})\big) y_0 \\ &= y_0^{-1} \mathfrak{H}\big(\beta, \mathfrak{A}(\mathcal{L})\big) y_0,\end{aligned}$$

and similarly $\mathfrak{H}(\varphi(\beta), \tilde{\mathfrak{A}}) \cap A_2 = \mathfrak{H}(\beta, \mathfrak{A}(\mathcal{L}))$. Hence, the proof of (3.5) is finished and we have

$$y_0^{-1} H^k\big(\beta, \mathfrak{A}(\mathcal{L})\big) y_0 \times H^k\big(\beta, \mathfrak{A}(\mathcal{L})\big) \subset H^k\big(\varphi(\beta), \tilde{\mathfrak{A}}\big),$$

for $k \geq 0$. Given $\theta \in \mathscr{C}(\tilde{\mathfrak{A}}, 0, \varphi(\beta), \psi_F)$, we set

$$\theta_1 = \theta \mid H^1\big(\iota\beta, \mathfrak{A}(\mathcal{L})\big), \qquad \theta_2 = \theta \mid H^1\big(\beta, \mathfrak{A}(\mathcal{L})\big).$$

We shall prove

$$\theta_1 = \theta_2 \circ \mathrm{Ad}(y_0). \tag{3.7}$$

We again proceed by induction along $\beta$. For the fixed additive character $\psi_F$ of $F$, we set

$$\psi = \psi_{\tilde{A}} = \psi_F \circ \mathrm{tr}_{\tilde{A}/F}, \qquad \psi_i = \psi_{A_i} = \psi_F \circ \mathrm{tr}_{A_i/F}, \quad i = 1, 2.$$

Then, we have

$$\psi \mid A_i = \psi_i, \quad i = 1, 2.$$

For $a \in \tilde{A}$, define the character $\psi_a$ of $\tilde{A}$ by $\psi_a(x) = \psi(a(x-1))$, $x \in \tilde{A}$. If $a = (a_1, a_2)$, $a_i \in A_i$, then we have $\psi_a \mid A_i = \psi_{i,a_i}$, $i = 1, 2$. We identify

$$\beta = \varphi(\beta) = (\iota\beta, \beta) = (y_0^{-1} \beta y_0, \beta) \in A_1 \oplus A_2 \subset \tilde{A}.$$

Assume that $\beta$ is minimal over $F$. Let $\chi_0$ be a unique character of $U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{o}_E)$ such that

$$\psi_\beta \mid U^{\lfloor -\nu/2 \rfloor + 1}(\tilde{\mathfrak{B}}) = \chi_0 \circ \det_{\tilde{B}}.$$

Then, we also have

$$\begin{cases} \psi_{1,\iota\beta} \mid U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{B}_1) = \chi_0 \circ \det_{B_1}, \\ \psi_{2,\beta} \mid U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{B}_2) = \chi_0 \circ \det_{B_2}. \end{cases}$$

For $\mathfrak{B} = \mathfrak{A}(\mathcal{L}) \cap B$ in $A = A(E)$ as before, we can identify

$$\mathfrak{B}_1 = y_0^{-1} \mathfrak{B} y_0 = y_0^{-1} \mathfrak{B}_2 y_0.$$

Thus, we have

$$U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{B}_1) = y_0^{-1} U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{B}_2) y_0$$

and so

$$\psi_{1,\iota\beta} \mid U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{B}_1) = \psi_{2,\beta} \circ \mathrm{Ad}(y_0) \mid U^{\lfloor -\nu/2 \rfloor + 1}(\mathfrak{B}_1). \tag{3.8}$$

In fact, for $z \in U^{\lfloor -\nu/2 \rfloor+1}(\mathfrak{B}_1)$, we obtain

$$\begin{aligned}\psi_{1,\iota\beta}(z) &= \psi_F \circ \mathrm{tr}_{A_1}\big(\iota\beta(z-1)\big) = \psi_F \circ \mathrm{tr}_{A_1}\big(y_0^{-1}\beta y_0(z-1)\big)\\ &= \psi_F \circ \mathrm{tr}_{A_1}\big(\beta(y_0 z y_0^{-1}-1)\big) = \psi_{2,\beta}(y_0 z y_0^{-1})\end{aligned}$$

and hence obtain (3.8). Take $\theta \in \mathscr{C}(\tilde{\mathfrak{A}}, 0, \varphi(\beta), \psi_F)$. When $0 \geq \lfloor -\nu/2 \rfloor$, we have $\theta = \psi_{\varphi(\beta)}$ and so

$$\theta_1 = \psi_{1,\iota\beta}, \qquad \theta_2 = \psi_{2,\beta}.$$

Moreover, we have $\theta_1 \in \mathscr{C}(\mathfrak{A}(\mathcal{L}), 0, \iota\beta, \psi_F)$, $\theta_2 \in \mathscr{C}(\mathfrak{A}(\mathcal{L}), 0, \beta, \psi_F)$, and the map $\theta \mapsto \theta_i$ is bijective. Since (3.8) implies (3.7), we obtain the bijection

$$\theta_2 \mapsto \theta_1 = \theta_2 \circ \mathrm{Ad}(y_0)$$

from $\mathscr{C}(\mathfrak{A}(\mathcal{L}), 0, \beta, \psi_F)$ to $\mathscr{C}(\mathfrak{A}(\mathcal{L}), 0, \iota\beta, \psi_F)$. When $\lfloor -\nu/2 \rfloor > 0$, we can choose a character $\chi_\theta$ of $U^1(\mathfrak{o}_E)$ such that

$$\theta \mid U^1(\tilde{\mathfrak{B}}) = \chi_\theta \circ \det_{\tilde{B}/E}.$$

Then, as in the proof of [8, (3.6.1)], we obtain the bijection $\theta \mapsto \chi_\theta$ from $\mathscr{C}(\tilde{\mathfrak{A}}, 0, \varphi(\beta), \psi_F)$ to the set of characters $\chi$ of $U^1(\mathfrak{o}_E)$ such that $\chi \mid U^{\lfloor -\nu/2 \rfloor+1}(\mathfrak{o}_E) = \chi_\theta$. Since $\theta_i \mid U^1(\mathfrak{B}_i) = \chi_\theta \circ \det_{B_i}$, we thus obtain the bijection $\theta \mapsto \theta_i$, $i = 1, 2$. From the equality

$$\det_B(x) = \det_{B_1}(y_0^{-1} x y_0), \quad x \in B$$

together with (3.8), we obtain (3.7) by [8, (3.2.1)], and hence obtain the bijection $\theta_2 \mapsto \theta_1 = \theta_2 \circ \mathrm{Ad}(y_0)$ as above.

In the general case, we set $r_0 = -k_0(\iota\beta, \mathfrak{A}(\mathcal{L})) = -k_0(\beta, \mathfrak{A}(\mathcal{L}))$ and take an element $\gamma \in A = A_1 = A_2$ and an $F$-embedding $\varphi : F[\gamma] \to \tilde{A}$, as before. Set

$$c = \varphi(\beta) - \varphi(\gamma) = (\iota\beta, \beta) - (\iota\gamma, \gamma) = \big(y_0^{-1}(\beta-\gamma)y_0, \beta-\gamma\big).$$

Suppose that $0 \geq \lfloor r_0/2 \rfloor$. Take $\theta \in \mathscr{C}(\tilde{\mathfrak{A}}, 0, \varphi(\beta), \psi_F)$. Then, this character can be written in the form $\theta = \theta_0 \cdot \psi_c, \theta_0 \in \mathscr{C}(\tilde{\mathfrak{A}}, 0, \varphi(\beta), \psi_F)$, and we have

$$\begin{cases}\theta_1 = (\theta_0 \mid H^1(\iota\beta, \mathfrak{A}(\mathcal{L}))) \cdot \psi_{1,\iota\beta-\iota\gamma},\\ \theta_2 = (\theta_0 \mid H^1(\gamma, \mathfrak{A}(\mathcal{L}))) \cdot \psi_{2,\beta-\gamma}.\end{cases}$$

In this case, by induction and by [8, (3.3.18)], we see that $\theta \mapsto \theta_i$ is bijective. We also obtain

$$\begin{aligned}\theta_1 &= \big(\theta_0 \mid H^1\big(\iota\gamma, \mathfrak{A}(\mathcal{L})\big)\big) \cdot \psi_{1,\iota\beta-\iota\beta}\\ &= \big[\big(\theta_0 \mid H^1\big(\gamma, \mathfrak{A}(\mathcal{L})\big)\big) \circ \mathrm{Ad}(y_0)\big] \cdot \big[\psi_{2,\beta-\gamma} \circ \mathrm{Ad}(y_0)\big]\\ &= \theta_2 \circ \mathrm{Ad}(y_0).\end{aligned}$$

Hence, $\theta_2 \mapsto \theta_1 = \theta_2 \circ \mathrm{Ad}(y_0)$ is the bijection from $\mathscr{C}(\mathfrak{A}(\mathcal{L}), 0, \beta, \psi_F)$ to $\mathscr{C}(\mathfrak{A}(\mathcal{L}), 0, \iota\beta, \psi_F)$. The case $\lfloor r_0/2 \rfloor > 0$ follows in a way quite similar to that of the proof in the case where $\beta$ is minimal over $F$. The assertion of Proposition 3.7 follows from the uniqueness of the transfer $\tau_F$ by [14, Theorem 3.53]. The proof is complete.

Finally, Proposition 3.7 confirms the conjecture of Remark 2.5.

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