Multiplicative formality of operads and Sinha's spectral sequence for long knots

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Abstract Lambrechts, Turchin, and Volić proved the Bousfield–Kan-type rational homology spectral sequence associated to the dth Kontsevich operad collapses at E^2-page if $d \geq 4$. The key of their proof is formality of the operad. In this paper, we simplify their proof by using a model category of operads. As by-products we obtain two new consequences. One is collapse of the spectral sequence in the case of $d = 3$ (and the coefficients being rational numbers). The other says there is no nontrivial extension for the Gerstenhaber algebra structure on the spectral sequence.

1. Introduction

The dth Kontsevich operad $\mathcal{K}_d$ is defined as a certain compactification of the configuration space of ordered points in $\mathbb{R}^d$ for each $d \geq 1$ (see [13]). It is weak equivalent to the little d-cubes operad, but it has the technical advantage that it admits a morphism of non-Σ-operads from the associative operad. So we may consider the associated cosimplicial space $\mathcal{K}_d^\bullet$ via the construction of Gerstenhaber–Voronov [2] and McClure–Smith [9]. Sinha [13] proved that the homotopy totalization of $\mathcal{K}_d^\bullet$ is weak homotopy equivalent to the space of long knots modulo immersions $\overline{\mathrm{Emb}}_d$ if $d \geq 4$ (see [13] or [7] for the definition). He also proved that the Bousfield–Kan-type homology spectral sequence associated with $\mathcal{K}_d^\bullet$ converges to the homology of $\overline{\mathrm{Emb}}_d$ if $d \geq 4$. We simply call this spectral sequence Sinha's spectral sequence.

Lambrechts, Turchin, and Volić [7] proved that Sinha's spectral sequence with rational coefficients collapses at E^2-page if $d \geq 4$. As the E^2-page is isomorphic to the Hochschild cohomology of the Poisson operad of degree $d-1$, we get a good algebraic presentation of the homology of $\overline{\mathrm{Emb}}_d$ by this collapse. The key of their proof is the formality of the Kontsevich operad.

The main purpose of this paper is to simplify their proof by using Quillen's theory of model categories. As by-products we obtain some new consequences (the case of $d = 3$ in Theorem 1.4 and Corollaries 1.6 and 1.7). To explain the situation more precisely, we prepare some notation and terminologies. In the rest of the paper, an operad means a non-Σ-operad. Let $\mathcal{CH}_{\geq 0}$ denote the category

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of nonnegatively graded chain complexes over a fixed field $\mathbf{k}$ (with differentials decreasing degree), and let $\mathcal{OPER}$ be the category of operads over $\mathcal{CH}_{\geq 0}$. Let $\mathcal{A} \in \mathcal{OPER}$ denote the associative operad.

DEFINITION 1.1

A morphism $f : \mathcal{O} \to \mathcal{P} \in \mathcal{OPER}$ is called a *weak equivalence* if the chain map $f_n : \mathcal{O}(n) \to \mathcal{P}(n)$ at each arity $n \geq 0$ is a quasi-isomorphism. For an operad $\mathcal{O} \in \mathcal{OPER}$ we define an operad $H_*(\mathcal{O})$ as follows. We put $H_*(\mathcal{O})(n) = H_*(\mathcal{O}(n))$, where the right-hand side is the usual homology group considered as a complex with the zero differential. The composition of $H_*(\mathcal{O})$ is induced by that of $\mathcal{O}$. The construction $\mathcal{O} \mapsto H_*(\mathcal{O})$ is natural for a morphism of operads. We say that a morphism $f : \mathcal{O} \to \mathcal{P} \in \mathcal{OPER}$ is *relatively formal* if there exists a chain of commutative squares in $\mathcal{OPER}$

$$\begin{tikzcd}
\mathcal{O} \arrow[d, "f"'] & \mathcal{O}_1 \arrow[l] \arrow[r] \arrow[d] & \cdots & \mathcal{O}_N \arrow[l] \arrow[r] \arrow[d] & H_*(\mathcal{O}) \arrow[d, "H_*(f)"] \\
\mathcal{P} & \mathcal{P}_1 \arrow[l] \arrow[r] & \cdots & \mathcal{P}_N \arrow[l] \arrow[r] & H_*(\mathcal{P})
\end{tikzcd}$$

where each horizontal arrow is a weak equivalence. A *multiplicative operad* is an operad $\mathcal{O}$ equipped with a morphism $\mathcal{A} \to \mathcal{O}$. A morphism of multiplicative operads is a morphism of operads under $\mathcal{A}$. We say a multiplicative operad $f : \mathcal{A} \to \mathcal{O} \in \mathcal{OPER}$ is *multiplicatively formal* if it is relatively formal and one can take a chain of commutative squares connecting f and $H_*(f)$ such that each horizontal morphism between sources is the identity (under the canonical identification $H_*(\mathcal{A}) = \mathcal{A}$).

Let $C_*(\mathcal{K}_d)$ denote the chain operad of the Kontsevich operad with $\mathbf{k}$-coefficients. A fixed linear embedding $\mathbb{R} \to \mathbb{R}^d$ induces a morphism $\mathcal{K}_1 \to \mathcal{K}_d$ of operads. Composing this morphism with the morphism from the (topological) associative operad $\mathcal{A}^{\mathrm{top}}$ to $\mathcal{K}_1$ which takes the unique point to the configuration whose numbering is consistent with the order of $\mathbb{R}$, we obtain a morphism $\mathcal{A}^{\mathrm{top}} \to \mathcal{K}_d$. This morphism naturally induces a morphism $\mathcal{A} \to C_*(\mathcal{K}_d)$ in $\mathcal{OPER}$ and we regard $C_*(\mathcal{K}_d)$ as a multiplicative operad with this morphism. The following theorem is a special case of the main theorem of Lambrechts and Volić [8].

THEOREM 1.2 ([8])

When $d \geq 3$ and $\mathbf{k} = \mathbb{R}$, the morphism $\mathcal{A} \to C_*(\mathcal{K}_d)$ is relatively formal.

More precisely speaking, in [8] the relative formality of the Fulton–MacPherson operad is proved but Theorem 1.2 immediately follows from it using [7, Diagram (2.5)].

The usual (absolute) formality of $C_*(\mathcal{K}_d)$ (or the little d-cubes operad) was proved first by Tamarkin [16] for $d = 2$ and later by Kontsevich [6] for general

d (see [8] and [3] for detailed descriptions of Kontsevich's proof). The relative version in the above theorem was proved by Lambrechts and Volić [8], verifying that the quasi-isomorphisms sketched by Kontsevich commute with the morphisms of operads. But their diagram does not show the multiplicative formality of Kontsevich operads as it contains the chain operad of Stasheff's associahedra.

To obtain the collapse from Theorem 1.2, Lambrechts, Turchin, and Volić [7] introduced a partial generalization of the construction of Gerstenhaber–Voronov and McClure–Smith applicable to any morphism of operads. Though it is a very general construction and should have other applications, their proof is somewhat complicated and does not work for $d = 3$. On the other hand, as pointed out in [7], if $C_*(\mathcal{K}_d)$ is multiplicatively formal, then the collapse easily follows from it. We prove that this is true. (The proof is given in Section 2.)

THEOREM 1.3
When $d \geq 3$ and $\mathbf{k} = \mathbb{R}$, $C_*(\mathcal{K}_d)$ is multiplicatively formal.

When d is equal to or greater than 4, the following theorem is the main result of [7].

THEOREM 1.4
For $d \geq 3$, Sinha's spectral sequence with rational coefficients collapses at E^2-page.

Proof
By Theorem 1.3, there is a diagram of the following form:

$$\begin{array}{ccccccccc}
\mathcal{A} & = & \mathcal{A} & = & \cdots & = & \mathcal{A} & = & \mathcal{A} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C_*(\mathcal{K}_d) & \longleftarrow & \mathcal{P}_1 & \longrightarrow & \cdots & \longleftarrow & \mathcal{P}_N & \longrightarrow & H_*(\mathcal{K}_d)
\end{array}$$

where each horizontal arrow is a weak equivalence. As the construction of Gerstenhaber–Voronov and McClure–Smith is natural for morphisms of multiplicative operads, this diagram induces a chain of termwise quasi-isomorphisms of cosimplicial chain complexes as follows:

$$C_*(\mathcal{K}_d^\bullet) \longleftarrow \mathcal{P}_1^\bullet \longrightarrow \cdots \longleftarrow \mathcal{P}_N^\bullet \longrightarrow H_*(\mathcal{K}_d^\bullet).$$

Here, $\mathcal{P}^\bullet$ denotes the cosimplicial chain complex associated to an operad $\mathcal{P}$, and a termwise quasi-isomorphism is a morphism which induces a quasi-isomorphism at each cosimplicial degree. In turn, this chain induces a chain of homomorphisms of spectral sequences

$$E^r_{*,*}\big(C_*(\mathcal{K}_d^\bullet)\big) \longleftarrow E^r_{*,*}(\mathcal{P}_1^\bullet) \longrightarrow \cdots \longleftarrow E^r_{*,*}(\mathcal{P}_N^\bullet) \longrightarrow E^r_{*,*}\big(H_*(\mathcal{K}_d^\bullet)\big)$$

for each $r \geq 0$. As all differentials of $H_*(\mathcal{K}_d^\bullet)$ are zero at each cosimplicial degree, the spectral sequence $\{E^r_{*,*}(H_*(\mathcal{K}_d^\bullet))\}_{r\geq 0}$ collapses at E^2-page. As a termwise quasi-isomorphism of cosimplicial complexes induces an isomorphism between E^2-pages, all arrows in the above chain are isomorphisms for each $r \geq 2$. Thus we see that the spectral sequence $\{E^r_{*,*}(C_*(\mathcal{K}_d^\bullet))\}_{r\geq 0}$ collapses at E^2-page. □

REMARK 1.5

When $d = 3$, it is *not* known whether Sinha's spectral sequence converges to the homology of $\overline{\mathrm{Emb}}_3$ but it is still worth studying as its E^2-page is isomorphic to the E^1-page of Vassiliev's spectral sequence for long knots modulo immersions in $\mathbb{R}^3$ (see [17]). In particular, its diagonal part (the part of total degree zero) is isomorphic to the space of all finite-type invariants of framed long knots. See also Volić [18] for identification between the diagonal and invariants by using Sinha's cosimplicial model and the Bott–Taubes integral.

Lambrechts, Turchin, and Volić [7] deduced the collapse of Vassiliev's spectral sequence which converges to the homology of the space of long knots in $\mathbb{R}^d$ from the collapse of Sinha's spectral sequence for each $d \geq 4$. A similar argument does not seem to work for $d = 3$ because in this case these two spectral sequences are not known to converge to the same module.

Besides simplification of the proof, Theorem 1.3 has an immediate application to the multiplicative structure on the spectral sequence. The Hochschild cohomology $HH^*(\mathcal{O})$ of a chain multiplicative operad $\mathcal{O}$ is by definition the homology of the total complex of the associated cosimplicial chain complex $\mathcal{O}^\bullet$, and $HH^*(\mathcal{O})$ carries a natural Gerstenhaber algebra structure whose product and Lie bracket are defined similarly to those on the Hochschild cohomology of an associative algebra (see [2], [12]).

COROLLARY 1.6

When $\mathbf{k} = \mathbb{R}$ *and* $d \geq 3$, *there exists an isomorphism of Gerstenhaber algebras:*

$$HH^*\big(C_*(\mathcal{K}_d)\big) \cong HH^*(\mathrm{Poiss}_{d-1}).$$

Here, Poiss_{d-1} *is the Poisson operad of degree* $d-1$ *(see [13, Definition 4.10]).*

Proof

As the construction of the Gerstenhaber algebra from a multiplicative operad is natural for morphisms of multiplicative operads, the chain of termwise quasi-isomorphisms in the proof of Theorem 1.4 induces an isomorphism of Gerstenhaber algebras $HH^*(C_*(\mathcal{K}_d)) \cong HH^*(H_*(\mathcal{K}_d))$. Since the operad $H_*(\mathcal{K}_d)$ is isomorphic to Poiss_{d-1} as a multiplicative operad, we have proved the corollary. □

This corollary says there is no extension problem in Sinha's spectral sequence as the right-hand side is isomorphic to the E^2-page with the induced operations.

The utility of formality for the extension problem was pointed out by Salvatore [12] (but a proof was not given).

McClure and Smith [9] invented a topological version of the above construction. For a topological multiplicative operad, they defined a little squares action on its (homotopy) totalization. In particular, for $d \geq 4$ the homology $H_*(\overline{\mathrm{Emb}}_d) \cong H_*(\widetilde{\mathrm{Tot}}(\mathcal{K}_d^\bullet))$ carries an induced Gerstenhaber structure whose product and bracket are given by the Pontryagin product and Browder operation. (Here $\widetilde{\mathrm{Tot}}$ denotes the homotopy totalization, that is, the homotopy limit over the category of simplices Δ.) We obtain an algebraic interpretation of this 'topological' Gerstenhaber algebra.

COROLLARY 1.7

When $\mathbf{k} = \mathbb{R}$ and $d \geq 4$, there exists an isomorphism of Gerstenhaber algebras:

$$H_*(\overline{\mathrm{Emb}}_d) \cong HH^*(\mathrm{Poiss}_{d-1}).$$

Proof

Combine Corollary 1.6 with [11, Theorem 4.6] or [12, Proposition 22]. □

REMARK 1.8

Songhafouo Tsopméné [14] also obtained the results stated above independently and simultaneously.

2. Proof of Theorem 1.3

Besides Theorem 1.2, the other key to the proof of Theorem 1.3 is the following.

THEOREM 2.1 ([4], [15], [1], [10])

The category $\mathcal{OPER}$ of non-Σ-operads over $\mathcal{CH}_{\geq 0}$ admits a left proper model category structure where

- *weak equivalences are those defined in Definition 1.1, and*
- *fibrations are those morphisms $f : \mathcal{O} \to \mathcal{P}$ such that, for each $n \geq 0$ and $k \geq 1$, the linear map $f_{n,k} : \mathcal{O}(n)_k \to \mathcal{P}(n)_k$ at arity n and degree k is an epimorphism.*

Our notion of a model category is that of Hovey [5]. Recall that a model category $\mathcal{M}$ is said to be *left proper* if a pushout of a weak equivalence by a cofibration is also a weak equivalence. Theorem 2.1 is not new. For the case of Σ-operads, the existence of a (semi-) model category structure was proved first by Hinich [4] for chain complexes and later by Spitzweck [15] and Berger–Moerdijk [1] for general model categories, and left properness was also proved in [15] and [1]. For the case of non-Σ-operads, Muro [10] proved the existence of a model category structure for general model categories. In our simple case the proof is somewhat shorter. For the reader's convenience we give a proof of Theorem 2.1 in Section 3.

For a model category $\mathcal{M}$ and a morphism $f : X \to Y \in \mathcal{M}$, a Quillen adjoint pair

$$P_f : X/\mathcal{M} \rightleftarrows Y/\mathcal{M} : U_f$$

between under categories (with the comma model structures; see [5, the paragraph below Proposition 1.1.8]) is defined by $P_f(Z) = Y \cup_X Z$ and $U_f(Z)$ is the composition $X \to Y \to Z$. The following proposition is well known and can be easily proved using [5, Corollary 1.3.16].

PROPOSITION 2.2
Under the above notations, if $\mathcal{M}$ is left proper, then for any weak equivalence f the induced adjunction (P_f, U_f) is a Quillen equivalence. In particular, the derived adjunction $(\mathbb{L}P_f, U_f)$ induces an equivalence between the homotopy category.

An operad weak equivalent to $\mathcal{A}$ is called an A_∞-operad. Let $\mathbf{Ho}(\mathcal{M})$ denote the homotopy category of a model category $\mathcal{M}$.

LEMMA 2.3
(a) *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be two A_∞-operads, and suppose that $\mathcal{B}_0$ is cofibrant. There exists a bijection $[\mathcal{B}_0, \mathcal{B}_1] \cong \mathbf{k}^\times$. Here, $[\cdot,\cdot]$ denotes the set of (left or right) homotopy classes of morphisms. If one fixes a morphism $f : \mathcal{B}_0 \to \mathcal{B}_1$, the bijection is given by $\mathbf{k}^\times \ni a \mapsto a * f \in [\mathcal{B}_0, \mathcal{B}_1]$, $(a * f)_n = a^{n-1} f_n$.*
(b) *Let*

$$\begin{tikzcd}
\mathcal{B}_0 \arrow[d, "f_1"] & \mathcal{B}_0 \arrow[d, "f_2"] \\
\mathcal{B}_1 \arrow[r, "\alpha"] \arrow[d, "g_1"] & \mathcal{B}_2 \arrow[d, "g_2"] \\
\mathcal{O}_1 \arrow[r, "\beta"] & \mathcal{O}_2
\end{tikzcd}$$

be a commutative diagram in $\mathcal{OPER}$ where $\mathcal{B}_0$, $\mathcal{B}_1$, and $\mathcal{B}_2$ are A_∞-operads, $\mathcal{B}_0$ is cofibrant, and β is a weak equivalence. Then the compositions $g_1 \circ f_1$ and $g_2 \circ f_2$ are isomorphic as objects of $\mathbf{Ho}(\mathcal{B}_0/\mathcal{OPER})$.

Proof
(a) As any object of $\mathcal{OPER}$ is fibrant, by homotopy invariance of the set of homotopy classes, we may replace $\mathcal{B}_1$ with the associative operad $\mathcal{A}$. As a morphism $f : \mathcal{B}_0 \to \mathcal{A}$ uniquely factors through the morphism $H_*(f) : H_*(\mathcal{B}_0) \to H_*(\mathcal{A}) \cong \mathcal{A}$, the set $[\mathcal{B}_0, \mathcal{A}]$ is bijective to the set of endomorphisms on $\mathcal{A}$. This latter set is bijective to $\mathbf{k}^\times$ since an endomorphism of $\mathcal{A}$ is determined by its image of a generator of $\mathcal{A}(2)$.

(b) We shall consider the case where α and β are the identities. If f_1 and f_2 are homotopic, by standard properties of left and right homotopies, $g_1 \circ f_1$ and $g_2 \circ f_2$ $(= g_1 \circ f_2)$ are right homotopic. This implies the claim by definition. So we may assume that $f_2 = a * f_1$ for some $a \in \mathbf{k}^\times$ by (a). Define a morphism $\phi_a : \mathcal{O}_1 \to \mathcal{O}_1$ as $\phi_{a,n} = a^{n-1} : \mathcal{O}_1(n) \to \mathcal{O}_1(n)$. Clearly ϕ_a is an isomorphism between $g_1 \circ f_1$ and $g_2 \circ f_2$ in $\mathcal{B}_0/\mathcal{OPER}$ and hence in $\mathbf{Ho}(\mathcal{B}_0/\mathcal{OPER})$.

We shall consider the general case. Clearly $g_1 \circ f_1$ and $g_2 \circ \alpha \circ f_1$ are isomorphic in $\mathbf{Ho}(\mathcal{B}_0/\mathcal{OPER})$. By applying the above case to f_2 and $\alpha \circ f_1$, we get the claim in the general case. □

Proof of Theorem 1.3
By relative formality (Theorem 1.2), there exists a commutative diagram in $\mathcal{OPER}$:

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "g"] & \mathcal{B}_1 \arrow[l] \arrow[r] \arrow[d, "g_1"] & \cdots & \mathcal{B}_N \arrow[l] \arrow[r] \arrow[d, "g_N"] & \mathcal{A} \arrow[d, "H_*(g)"] \\
C_*(\mathcal{K}_d) & \mathcal{O}_1 \arrow[l] \arrow[r] & \cdots & \mathcal{O}_N \arrow[l] \arrow[r] & H_*(\mathcal{K}_d)
\end{tikzcd}$$

where all horizontal arrows are weak equivalences. Let $f : \mathcal{B}_0 \to \mathcal{A}$ be a cofibrant replacement of the associative operad. We can pick a morphism $f_i : \mathcal{B}_0 \to \mathcal{B}_i \in \mathcal{OPER}$ for each $i = 1, \ldots, N$ as each B_i is a (fibrant) A_∞-operad. So we obtain the following diagram:

$$\begin{tikzcd}
\mathcal{B}_0 \arrow[d, "f"] & \mathcal{B}_0 \arrow[d, "f_1"] & \cdots & \mathcal{B}_0 \arrow[d, "f_N"] & \mathcal{B}_0 \arrow[d, "f"] \\
\mathcal{A} \arrow[d, "g"] & \mathcal{B}_1 \arrow[l] \arrow[r] \arrow[d, "g_1"] & \cdots & \mathcal{B}_N \arrow[l] \arrow[r] \arrow[d, "g_N"] & \mathcal{A} \arrow[d, "H_*(g)"] \\
C_*(\mathcal{K}_d) & \mathcal{O}_1 \arrow[l] \arrow[r] & \cdots & \mathcal{O}_N \arrow[l] \arrow[r] & H_*(\mathcal{K}_d)
\end{tikzcd}$$

By applying Lemma 2.3(b) to each part of this diagram, we see that $g \circ f, g_1 \circ f_1, g_2 \circ f_2, \ldots, H_*(g) \circ f$ are all isomorphic in $\mathbf{Ho}(\mathcal{B}_0/\mathcal{OPER})$. In other words, $U_f(C_*(\mathcal{K}_d))$ and $U_f(H_*(\mathcal{K}_d))$ are isomorphic in $\mathbf{Ho}(\mathcal{B}_0/\mathcal{OPER})$. By this and Proposition 2.2 we have isomorphisms $C_*(\mathcal{K}_d) \cong \mathbb{L}P_f U_f(C_*(\mathcal{K}_d)) \cong \mathbb{L}P_f U_f(H_*(\mathcal{K}_d)) \cong H_*(\mathcal{K}_d)$ in $\mathbf{Ho}(\mathcal{A}/\mathcal{OPER})$. This implies that $C_*(\mathcal{K}_d)$ and $H_*(\mathcal{K}_d)$ can be connected by a chain of weak equivalences under $\mathcal{A}$, which means that $C_*(\mathcal{K}_d)$ is multiplicatively formal. □

3. Proof of Theorem 2.1

To prove Theorem 2.1, we use [5, Theorem 2.1.19]. So we need two sets I and J of morphisms of $\mathcal{OPER}$, which play the roles of sets of generating cofibrations and trivial cofibrations, respectively. To define I and J, we use the free construction in $\mathcal{OPER}$, and to describe this construction, we shall recall the language of trees.

A *tree* is a finite connected acyclic graph. Let T be a tree, and let $\phi : |T| \to \mathbb{R} \times [0,1]$ be an embedding of the geometric realization of T such that $\mathrm{Im}(\phi) \cap \mathbb{R} \times \{0\}$ consists of only one vertex (or 0-cell), which we call the *root* of T and $\mathrm{Im}(\phi) \cap \mathbb{R} \times \{1\}$ consists of univalent vertices. Let $n \geq 0$ be an integer, and let $\alpha : \{1, \ldots, n\} \to \mathrm{Im}(\phi) \cap \mathbb{R} \times \{1\}$ be an order-preserving monomorphism, where the linear order on $\mathrm{Im}(\phi) \cap \mathbb{R} \times \{1\}$ is induced by the usual order on $\mathbb{R} \times \{1\} = \mathbb{R}$. We call a vertex in $\mathrm{Im}(\alpha)$ a *leaf* of T and a vertex in $\mathrm{Im}(\phi) \cap \mathbb{R} \times \{1\} - \mathrm{Im}(\alpha)$ a *null vertex* of T. For an edge e of T, the vertex of e farther from the root is called the *source* of e, and the other is called the *target*.

DEFINITION 3.1

For each $n \geq 0$ consider isotopy classes of triples (T, ϕ, α) which satisfy the above conditions, where an isotopy is assumed to respect the map α. We call such an isotopy class $\{(T, \phi, \alpha)\}$ a *regular planer n-tree* if each vertex in $\mathrm{Im}(\phi) \cap \mathbb{R} \times (0,1)$ is at least bivalent. By abuse of notations, a regular planar n-tree is denoted by the same notation as the underlying tree.

Let T be a regular planar n-tree, and let v be a vertex of T. We define a number $\mathrm{In}(v)$ as 0 if v is a null vertex, and as the number of the edges whose targets are v otherwise. The set of vertices which are not leaves is denoted by $V_{\mathrm{in}}(T)$. The *level* of a vertex v is one less than the number of vertices on the shortest path connecting the root and v. For example, the level of the root is 0. We put

$$V^0_{\mathrm{in}}(T) = \{ v \in V_{\mathrm{in}}(T) \mid \text{the level of } v \text{ is even} \}, \qquad V^1_{\mathrm{in}}(T) = V_{\mathrm{in}}(T) - V^0_{\mathrm{in}}(T).$$

We say that T is *odd* if the level of each vertex in $\mathrm{Im}(\alpha)$ is odd. Let $\mathcal{T}_n$ (resp., $\mathcal{T}^1_n$) denote the set of all regular planar n-trees (resp., odd regular planar n-trees).

REMARK 3.2

For each $n \geq 0$, $\mathcal{T}_n$ is bijective to the set of all isomorphism classes of planted planar trees with n leaves defined in [10, Definition 3.4].

Let $\mathcal{SEQ}$ be the category of sequences in $\mathcal{CH}_{\geq 0}$. An object of $\mathcal{SEQ}$ is a sequence $\mathcal{S}(0), \mathcal{S}(1), \ldots$ of chain complexes, and a morphism is a sequence of chain maps. The free construction (or free functor) $\mathcal{F} : \mathcal{SEQ} \longrightarrow \mathcal{OPER}$; that is, the left adjoint of the forgetful functor $\mathcal{U} : \mathcal{OPER} \longrightarrow \mathcal{SEQ}$ is defined by

$$\mathcal{F}(\mathcal{S})(n) = \mathbf{k} \cdot \delta_{1,n} \oplus \bigoplus_{T \in \mathcal{T}_n} \bigotimes_{v \in V_{\mathrm{in}}(T)} \mathcal{S}\big(\mathrm{In}(v)\big).$$

Here, $\mathbf{k} \cdot \delta_{1,n}$ is the module generated by a formal unit if $n = 1$ and the zero module otherwise.

To define the sets of morphisms I and J, we shall recall a set of generating (trivial) cofibrations of $\mathcal{CH}_{\geq 0}$. For $p \geq 1$ we define a complex D^p as follows: $D^p_l = \mathbf{k}$ if $l = p, p-1$; $D^p_l = 0$ otherwise. The differential d_p is the identity. For $p \geq 0$ we define another complex S^p by $S^p_p = \mathbf{k}$ and $S^p_l = 0$ for $l \neq p$. Let $i^p : S^{p-1} \to D^p$

be the chain map which is the identity on S^{p-1}_{p-1}, and let $j^p : 0 \to D^p$ be the unique chain map. Put $I_0 = \{i^p \mid p \geq 1\}$, and put $J_0 = \{j^p \mid p \geq 1\}$. The following proposition is well known and can be proved in a way analogous to the proof of [5, Theorem 2.3.11].

PROPOSITION 3.3

We have that $\mathcal{CH}_{\geq 0}$ admits a cofibrantly generated model category structure with I_0 (resp., J_0) being a set of generating cofibrations (resp., trivial cofibrations), where

- *weak equivalences are quasi-isomorphisms,*
- *fibrations are those morphisms $f : C \to D$ such that the linear map $f_k : C_k \to D_k$ is an epimorphism for each $k \geq 1$.*

Let $D^{p,q}$ and $S^{p,q}$ be two objects of $\mathcal{SEQ}$ defined by $D^{p,q}(q) = D^p$, $D^{p,q}(n) = 0$ for $n \neq q$, and $S^{p,q}(q) = S^p$, $S^{p,q}(n) = 0$ for $n \neq q$, respectively. Let $i^{p,q} : S^{p-1,q} \to D^{p,q}$ and $j^{p,q} : 0 \to D^{p,q}$ be the morphisms induced by i^p and j^p, respectively. Put $I_1 = \{i^{p,q} \mid p \geq 1, q \geq 0\}$, and put $J_1 = \{j^{p,q} \mid p \geq 1, q \geq 0\}$.

We define two sets of morphisms I and J as the image of I_1 and J_1 by $\mathcal{F}$, respectively.

Proof of Theorem 2.1

We apply [5, Theorem 2.1.19] to the sets I and J defined above and the class of weak equivalences given in Theorem 2.1. We must verify the six conditions stated in the theorem. The first condition (2-out-of-3 and closedness under retraction of $\mathcal{W}$) is clear. The second and third conditions (smallness of the domains of I and J) follow from [5, Lemma 2.3.2] and the adjointness of the pair $(\mathcal{F}, \mathcal{U})$. By adjointness, the class I-inj (resp., J-inj) is equal to the class of morphisms $f : \mathcal{O} \to \mathcal{O}'$ such that $\mathcal{U}(f) : \mathcal{U}(\mathcal{O}) \to \mathcal{U}(\mathcal{O}')$ is I_1-inj (resp., J_1-inj). This and Proposition 3.3 imply the fifth and sixth conditions (I-inj $\subset \mathcal{W} \cap J$-inj and $\mathcal{W} \cap J$-inj $\subset I$-inj).

We shall prove the fourth condition (J-cell $\subset \mathcal{W} \cap I$-cof). First, J-cell $\subset$ I-cof is clear by adjointness. To prove J-cell $\subset \mathcal{W}$, as quasi-isomorphisms are closed under transfinite composition, it is enough to prove that a pushout by a morphism in J is in $\mathcal{W}$. Take a morphism $\mathcal{F}(j^{p,q}) : \mathcal{F}(0) \to \mathcal{F}(D^{p,q}) \in J$ and an operad $\mathcal{O}$. The pushout $\mathcal{P} = \mathcal{F}(D^{p,q}) \bigcup_{\mathcal{F}(0)} \mathcal{O}$ $(= \mathcal{F}(D^{p,q}) \sqcup \mathcal{O})$ has the following presentation:

$$\mathcal{P}(n) = \bigoplus_{T \in \mathcal{T}^1_n} \Big(\bigotimes_{v \in V^0_{\mathrm{in}}(T)} \mathcal{O}(\mathrm{In}(v)) \otimes \bigotimes_{v \in V^1_{\mathrm{in}}(T)} D^{p,q}(\mathrm{In}(v)) \Big).$$

(In this presentation, the unit of $\mathcal{O}$ serves as the unit of $\mathcal{P}$, and for $x, y \in D^{p,q}$, a composition $x \circ_i y$ is equal to $((1 \circ x \circ_i 1) \circ_i y) \circ 1^{\otimes m}$ for some m, so we do not need even trees or other partitions of the set $V_{\mathrm{in}}(T)$.) As the tensor product over a field preserves quasi-isomorphisms, we see that the pushout morphism $\mathcal{O} \to \mathcal{P}$ is in the class $\mathcal{W}$ from this presentation.

We shall prove left properness. As any cofibration is a retract of a relative I-cell, it is enough to prove that a pushout by a generating cofibration preserves weak equivalences. Take a morphism $\mathcal{F}(i^{p,q}) : \mathcal{F}(S^{p-1,q}) \to \mathcal{F}(D^{p,q})$. Let $f : \mathcal{O} \to \mathcal{O}'$ be a weak equivalence, and let $g : \mathcal{F}(S^{p-1,q}) \to \mathcal{O}$ be a morphism. As a graded operad, that is, if we forget the differentials, the pushout $\tilde{\mathcal{P}}_{\mathcal{O}} = \mathcal{O} \bigcup_{\mathcal{F}(S^{p-1,q})} \mathcal{F}(D^{p,q})$ has a presentation analogous to the above presentation of $\mathcal{P}$. It is given by replacing $D^{p,q}$ with $S^{p,q}$ in the above one. By the Leipniz rule, the differential is determined by its restrictions to $\mathcal{O}$ and to $S^{p,q}$. On $\mathcal{O}$ it is equal to the original differential of $\mathcal{O}$, and on $S^{p,q}$ it is given by the composition $S^{p,q}(q)_p = D^p_p \xrightarrow{d^p} D^p_{p-1} = S^{p-1,q}(q)_{p-1} \xrightarrow{g} \mathcal{O}(q)_{p-1}$. What we have to prove is that the induced morphism $\tilde{\mathcal{P}}_{\mathcal{O}} \to \tilde{\mathcal{P}}_{\mathcal{O}'}$ is a weak equivalence. For each $l \geq 0$ let $F^l_{\mathcal{O}} \subset \tilde{\mathcal{P}}_{\mathcal{O}}$ be the subsequence which is spanned by the summands corresponding to regular planer trees with $\sharp V^1_{\mathrm{in}}(T) \leq l$. As $F^{l+1}_{\mathcal{O}}(n)/F^l_{\mathcal{O}}(n)$ is isomorphic to a sum of tensors of $\mathcal{O}(m)$'s and S^p's (as chain complexes), the induced morphism $F^{l+1}_{\mathcal{O}}(n)/F^l_{\mathcal{O}}(n) \to F^{l+1}_{\mathcal{O}'}(n)/F^l_{\mathcal{O}'}(n)$ is a quasi-isomorphism. By an inductive argument, using a long exact sequence, we see that the morphism $F^l_{\mathcal{O}}(n) \to F^l_{\mathcal{O}'}(n)$ is a quasi-isomorphism for each $l \geq 0$. As $\tilde{\mathcal{P}}_{\mathcal{O}} = \bigcup_l F^l_{\mathcal{O}}$, we obtain the desired claim. □

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