Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras

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Abstract Let $A, B$ be square irreducible matrices with entries in $\{0, 1\}$. We will show that if the one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent, then the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, and hence $\det(\text{id} - A) = \det(\text{id} - B)$. As a result, the one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent if and only if the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$.

1. Introduction

The interplay between the orbit equivalence of topological dynamical systems and the theory of $C^*$-algebras has been studied by many authors. Giordano, Putnam, and Skau [7] have proved that two minimal homeomorphisms on a Cantor set are strongly orbit equivalent if and only if the associated $C^*$-crossed products are isomorphic. Boyle and Tomiyama [3] and Tomiyama [20] have studied relationships between orbit equivalence and $C^*$-crossed products for topologically free homeomorphisms on compact Hausdorff spaces.

In this paper, we classify one-sided irreducible topological Markov shifts up to continuous orbit equivalence and show that there exists a close connection with the Cuntz–Krieger algebras. The class of one-sided topological Markov shifts is an important class of topological dynamical systems on Cantor sets, though they are not homeomorphisms but local homeomorphisms. The first author [11] introduced the notion of continuous orbit equivalence for one-sided topological Markov shifts (see Definition 2.1) and proved that one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ for irreducible matrices $A$ and $B$ with entries in $\{0, 1\}$ are continuously orbit equivalent if and only if there exists an isomorphism between the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ preserving their canonical Cartan sub-algebras $\mathcal{D}_A$ and $\mathcal{D}_B$. The second author in [15] and [16] studied the associated étale groupoids $G_A$ and their homology groups $H_n(G_A)$ and topological full groups $\lbrack G_A \rbrack$. In fact, the two shifts are continuously orbit equivalent if and only
if $G_A$ is isomorphic to $G_B$ (see Theorem 2.3). In [12] it was also shown that if $O_A$ is isomorphic to $O_B$ and $\det(id - A) = \det(id - B)$, then there exists an isomorphism $\Psi : O_A \rightarrow O_B$ such that $\Psi(D_A) = D_B$, and hence the one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent. Since there were no known examples of irreducible matrices $A, B$ such that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent and $\det(id - A) \neq \det(id - B)$, the first author [12, Section 6] presented the following conjecture: the determinant $\det(1 - A)$ is an invariant for the continuous orbit equivalence class of $(X_A, \sigma_A)$. In the present article we confirm this conjecture. In other words, we show that $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent if and only if $O_A$ is isomorphic to $O_B$ and $\det(id - A) = \det(id - B)$ (see Theorem 3.6).

Our proof is closely related to another notion of equivalence for shifts, namely, flow equivalence for two-sided topological Markov shifts. Two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are said to be flow equivalent if there exists an orientation-preserving homeomorphism between their suspension spaces (see [17]). Two characterizations of the flow equivalence are known. One is due to Boyle and Handelman [2] and the other is due to Parry and Sullivan [17], Bowen and Franks [1], and Franks [6] (see Theorems 2.4 and 2.6). By using the former characterization and the groupoid approach, we show that if $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuously orbit equivalent, then $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent (see Theorem 3.5). This, together with the second characterization, implies that $\det(id - A) = \det(id - B)$, and so the conjecture is confirmed.

It is known that flow equivalence has a close relationship to stable isomorphisms of Cuntz–Krieger algebras (see [4], [5], [6], [8], [9], [19]). As a corollary of the main result, we also prove that two-sided irreducible topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent if and only if there exists an isomorphism between the stable Cuntz–Krieger algebras $O_A \otimes K$ and $O_B \otimes K$ preserving their canonical maximal abelian subalgebras (see Corollary 3.8).

2. Preliminaries

We write $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The transpose of a matrix $A$ is written $A^t$. The characteristic function of a set $S$ is denoted by $1_S$. We say that a subset of a topological space is clopen if it is both closed and open. A topological space is said to be totally disconnected if its topology is generated by clopen subsets. By a Cantor set, we mean a compact, metrizable, totally disconnected space with no isolated points. It is known that any two such spaces are homeomorphic. A good introduction to symbolic dynamics can be found in the standard textbook [10] by Lind and Marcus.

Let $A = [A(i, j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we assume that $A$ has no rows or columns identically equal to zero. Define

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N} \mid A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{N}\}. $$
It is a compact Hausdorff space with natural product topology on \( \{1, \ldots, N\}^\mathbb{Z} \).

The shift transformation \( \sigma_A \) on \( X_A \) defined by \( \sigma_A((x_n)_n) = (x_{n+1})_n \) is a continuous surjective map on \( X_A \). The topological dynamical system \( (X_A, \sigma_A) \) is called the (right) one-sided topological Markov shift for \( A \). We henceforth assume that \( A \) satisfies condition (I) in the sense of [5]. The matrix \( A \) satisfies condition (I) if and only if \( X_A \) has no isolated points, that is, \( X_A \) is a Cantor set.

We let \((\bar{X}_A, \bar{\sigma}_A)\) denote the two-sided topological Markov shift. Namely,
\[
\bar{X}_A = \{(x_n)_{n \in \mathbb{Z}} \in \{1, \ldots, N\}^\mathbb{Z} \mid A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}\}
\]
and \(\bar{\sigma}_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}\).

A subset \( S \) in \( X_A \) (resp., in \( \bar{X}_A \)) is said to be \( \sigma_A \)-invariant (resp., \( \bar{\sigma}_A \)-invariant) if \( \sigma_A(S) = S \) (resp., \( \bar{\sigma}_A(S) = S \)).

### 2.1. Continuous orbit equivalence

For \( x = (x_n)_{n \in \mathbb{N}} \in X_A \), the orbit \( \text{orb}_{\sigma_A}(x) \) of \( x \) under \( \sigma_A \) is defined by
\[
\text{orb}_{\sigma_A}(x) = \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)).
\]

**Definition 2.1 ([11, Section 5])**

Let \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) be two one-sided topological Markov shifts. If there exists a homeomorphism \( h : X_A \to X_B \) such that \( h(\text{orb}_{\sigma_A}(x)) = \text{orb}_{\sigma_B}(h(x)) \) for \( x \in X_A \), then \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are said to be topologically orbit equivalent.

In this case, there exist \( k_1, l_1 : X_A \to \mathbb{Z}_+ \) such that
\[
\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \forall x \in X_A.
\]

Similarly there exist \( k_2, l_2 : X_B \to \mathbb{Z}_+ \) such that
\[
\sigma_A^{k_2(x)}(h^{-1}(\sigma_B(x))) = \sigma_A^{l_2(x)}(h^{-1}(x)) \quad \forall x \in X_B.
\]

Furthermore, if we may choose \( k_1, l_1 : X_A \to \mathbb{Z}_+ \) and \( k_2, l_2 : X_B \to \mathbb{Z}_+ \) as continuous maps, then the topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are said to be continuously orbit equivalent.

If two one-sided topological Markov shifts are topologically conjugate, then they are continuously orbit equivalent. For the two matrices
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},
\]
the topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent, but not topologically conjugate (see [11, Section 5]).

Let \([\sigma_A]\) denote the set of all homeomorphisms \( \tau \) of \( X_A \) such that \( \tau(x) \in \text{orb}_{\sigma_A}(x) \) for all \( x \in X_A \). It is called the full group of \((X_A, \sigma_A)\). Let \( \Gamma_A \) be the set of all \( \tau \) in \([\sigma_A]\) such that there exist continuous functions \( k, l : X_A \to \mathbb{Z}_+ \) satisfying \( \sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x) \) for all \( x \in X_A \). The set \( \Gamma_A \) is a subgroup of \([\sigma_A]\) and is called the continuous full group for \((X_A, \sigma_A)\). We note that the group \( \Gamma_A \)
has been written as $[\sigma_A]_c$ in the earlier paper [11]. It has been proved in [14] that the isomorphism class of $\Gamma_A$ as an abstract group is a complete invariant of the continuous orbit equivalence class of $(X_A, \sigma_A)$ (see [16] for more general results and further studies).

### 2.2. Étale groupoids

By an étale groupoid we mean a second countable locally compact Hausdorff groupoid such that the range map is a local homeomorphism. We refer the reader to [18] for background material on étale groupoids. For an étale groupoid $G$, we let $G^{(0)}$ denote the unit space, and we let $s$ and $r$ denote the source and range maps, respectively. For $x \in G^{(0)}$, $r(Gx)$ is called the $G$-orbit of $x$. When every $G$-orbit is dense in $G^{(0)}$, $G$ is said to be minimal. For $x \in G^{(0)}$, we write $G_x = r^{-1}(x) \cap s^{-1}(x)$ and call it the isotropy group of $x$. The isotropy bundle is $G' = \{g \in G \mid r(g) = s(g)\} = \bigcup_{x \in G^{(0)}} G_x$. We say that $G$ is principal if $G' = G^{(0)}$. When the interior of $G'$ is $G^{(0)}$, we say that $G$ is essentially principal. A subset $U \subset G$ is called a $G$-set if $r[U, s|U]$ are injective. For an open $G$-set $U$, we let $\pi_U$ denote the homeomorphism $r \circ (s|U)^{-1}$ from $s(U)$ to $r(U)$.

We would like to recall the notion of topological full groups for étale groupoids.

**Definition 2.2 ([15, Definition 2.3])**

Let $G$ be an essentially principal étale groupoid whose unit space $G^{(0)}$ is compact.

(a) The set of all $\alpha \in \text{Homeo}(G^{(0)})$ such that for every $x \in G^{(0)}$ there exists $g \in G$ satisfying $r(g) = x$ and $s(g) = \alpha(x)$ is called the full group of $G$ and is denoted by $[G]$.

(b) The set of all $\alpha \in \text{Homeo}(G^{(0)})$ for which there exists a compact open $G$-set $U$ satisfying $\alpha = \pi_U$ is called the topological full group of $G$ and is denoted by $[[G]]$.

Obviously $[G]$ is a subgroup of $\text{Homeo}(G^{(0)})$ and $[[G]]$ is a subgroup of $[G]$.

For $\alpha \in [[G]]$ the compact open $G$-set $U$ as above uniquely exists, because $G$ is essentially principal. Since $G$ is second countable, it has countably many compact open subsets, and so $[[G]]$ is at most countable. For minimal groupoids on Cantor sets, it is known that the isomorphism class of $[[G]]$ is a complete invariant of $G$ (see [16, Theorem 3.10]).

Let $(X_A, \sigma_A)$ be a topological Markov shift. The étale groupoid $G_A$ for $(X_A, \sigma_A)$ is given by

$$G_A = \{ (x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{Z}_+, n = k - l, \sigma^k_A(x) = \sigma^l_A(y) \}.$$

The topology of $G_A$ is generated by the sets

$$\{ (x, k - l, y) \in G_A \mid x \in V, y \in W, \sigma^k_A(x) = \sigma^l_A(y) \},$$

where $V, W \subset X_A$ are open and $k, l \in \mathbb{Z}_+$. Two elements $(x, n, y)$ and $(x', n', y')$ in $G_A$ are composable if and only if $y = x'$, and the multiplication and the inverse
are
\[(x, n, y) \cdot (y, n', y') = (x, n + n', y'), \quad (x, n, y)^{-1} = (y, -n, x)\].

The range and source maps are given by \( r(x, n, y) = (x, 0, x) \) and \( s(x, n, y) = (y, 0, y) \), respectively. We identify \( X_A \) with the unit space \( G_A^{(0)} \) via \( x \mapsto (x, 0, x) \). The groupoid \( G_A \) is essentially principal. The groupoid \( G_A \) is minimal if and only if \((X_A, \sigma_A)\) is irreducible. It is easy to see that the topological full group \([G_A]\) is canonically isomorphic to the continuous full group \(\Gamma_A\).

### 2.3. Cuntz–Krieger algebras

Let \( A = [A(i, j)]_{i, j=1}^{N} \) be an \( N \times N \) matrix with entries in \( \{0, 1\} \), and let \((X_A, \sigma_A)\) be the one-sided topological Markov shift. The Cuntz–Krieger algebra \( O_A \), introduced in [5], is the universal \( C^* \)-algebra generated by \( N \) partial isometries \( S_1, \ldots, S_N \) subject to the relations
\[
\sum_{j=1}^{N} S_j S_j^* = 1 \quad \text{and} \quad S_i^* S_i = \sum_{j=1}^{N} A(i, j) S_j S_j^*.
\]

The subalgebra \( D_A \) of \( O_A \) generated by elements \( S_{i_1} S_{i_2} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* \) is naturally isomorphic to \( C(X_A) \), and is a Cartan subalgebra in the sense of [18]. It is also well known that the pair \((O_A, D_A)\) is isomorphic to the pair \((C^*_r(G_A), C(X_A))\), where \( C^*_r(G_A) \) denotes the reduced groupoid \( C^* \)-algebra and \( C(X_A) \) is regarded as a subalgebra of it. Thus, there exists an isomorphism \( \Psi : O_A \to C^*_r(G) \) such that \( \Psi(D_A) = C(X_A) \).

**Theorem 2.3**

Let \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) be two irreducible one-sided topological Markov shifts. The following conditions are equivalent.

- (a) \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent.
- (b) The \(\epsilon\)tale groupoids \( G_A \) and \( G_B \) are isomorphic.
- (c) There exists an isomorphism \( \Psi : O_A \to O_B \) such that \( \Psi(D_A) = D_B \).

**Proof**

The equivalence between (a) and (c) follows from [11, Theorem 1.1]. The equivalence between (b) and (c) follows from [18, Proposition 4.11] (see also [15, Theorem 5.1]).

### 2.4. Flow equivalence

In this section, we would like to recall Boyle–Handelman’s theorem, which says that the ordered cohomology group is a complete invariant for flow equivalence between irreducible shifts of finite type.

Let \( A = [A(i, j)]_{i, j=1}^{N} \) be an \( N \times N \) matrix with entries in \( \{0, 1\} \), and consider the two-sided topological Markov shift \((\bar{X}_A, \bar{\sigma}_A)\). Set
\[
\bar{H}^A = C(\bar{X}_A, \mathbb{Z})/\{\xi - \xi \circ \bar{\sigma}_A | \xi \in C(\bar{X}_A, \mathbb{Z})\}.
\]
The equivalence class of a function \( \xi \in C(\bar{X}_A,\mathbb{Z}) \) in \( \bar{H}^A \) is written \( [\xi] \). We define the positive cone \( \bar{H}^A_+ \) by

\[
\bar{H}^A_+ = \{ [\xi] \in \bar{H}^A \mid \xi(x) \geq 0 \ \forall x \in \bar{X}_A \}.
\]

The pair \((\bar{H}^A, \bar{H}^A_+)\) is called the ordered cohomology group of \((\bar{X}_A, \bar{σ}_A)\) (see [2, Section 1.3]). Boyle and Handelman proved the following theorem, which plays a key role in this paper.

**THEOREM 2.4 ([2, THEOREM 1.12])**

Suppose that \((\bar{X}_A, \bar{σ}_A)\) and \((\bar{X}_B, \bar{σ}_B)\) are irreducible two-sided topological Markov shifts. Then the following are equivalent.

(a) \((\bar{X}_A, \bar{σ}_A)\) and \((\bar{X}_B, \bar{σ}_B)\) are flow equivalent.

(b) The ordered cohomology groups \((\bar{H}^A, \bar{H}^A_+)\) and \((\bar{H}^B, \bar{H}^B_+)\) are isomorphic; that is, there exists an isomorphism \( Φ: \bar{H}^A \to \bar{H}^B \) such that \( Φ(\bar{H}^A_+) = \bar{H}^B_+ \).

We also recall the following from [2] for later use.

**PROPOSITION 2.5 ([2, PROPOSITION 3.13(A)])**

Let \((\bar{X}_A, \bar{σ}_A)\) be a two-sided topological Markov shift, and let \( ξ \in C(\bar{X}_A,\mathbb{Z}) \). Then \( [ξ] \) is in \( \bar{H}^A_+ \) if and only if

\[
\sum_{x \in O} ξ(x) \geq 0
\]

holds for any finite \( \bar{σ}_A \)-invariant set \( O \subset \bar{X}_A \).

In the same way as above, we introduce \((H^A, H^A_+)\) for the one-sided topological Markov shift \((X_A, σ_A)\) as follows:

\[
H^A = C(X_A,\mathbb{Z}) / \{ ξ - ξ \circ σ_A \mid ξ \in C(X_A,\mathbb{Z}) \}
\]

and

\[
H^A_+ = \{ [ξ] \in H^A \mid ξ(x) \geq 0 \ \forall x \in X_A \}.
\]

We will show that \((\bar{H}^A, \bar{H}^A_+)\) and \((H^A, H^A_+)\) are actually isomorphic (see Lemma 3.1).

**2.5. The Bowen–Franks group**

Let \( A = [A(i,j)]^N_{i,j=1} \) be an \( N \times N \) matrix with entries in \( \{0,1\} \). The Bowen–Franks group BF(A) is the abelian group \( \mathbb{Z}^N / (\text{id} - A)\mathbb{Z}^N \). Bowen and Franks [1] have proved that the Bowen–Franks group is an invariant of flow equivalence. Parry and Sullivan [17] have proved that the determinant of \( \text{id} - A \) is also an invariant of flow equivalence. Evidently, if BF(A) is an infinite group, then \( \det(\text{id} - A) \) is zero. If BF(A) is a finite group, then \( |\det(\text{id} - A)| \) is equal to the cardinality of BF(A). Therefore it is sufficient to know the Bowen–Franks group
and the sign of the determinant in order to find the determinant. The following theorem by Franks shows that these invariants are complete.

**THEOREM 2.6 ([6, THEOREM])**

Suppose that \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are irreducible two-sided topological Markov shifts. Then \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent if and only if \(BF(A) \cong BF(B)\) and \(\text{sgn}(\det(\text{id} - A)) = \text{sgn}(\det(\text{id} - B))\).

In what follows, we consider \(BF(A^t) = \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N\). Although \(BF(A^t)\) is isomorphic to \(BF(A)\) as an abelian group, there does not exist a canonical isomorphism between them, and so we must distinguish them carefully.

We denote the equivalence class of \((1, 1, \ldots, 1) \in \mathbb{Z}^N\) in \(BF(A^t)\) by \(u_A\). By [4, Proposition 3.1], \(K_0(O_A)\) is isomorphic to \(BF(A^t)\) and the class of the unit of \(O_A\) maps to \(u_A\) under this isomorphism. And \(K_1(O_A)\) is isomorphic to \(\text{Ker}(\text{id} - A^t)\) on \(\mathbb{Z}^N\). In [15], it has been shown that these groups naturally arise from the homology theory of étale groupoids.

Let \(G\) be an étale groupoid whose unit space \(G(0)\) is a Cantor set. One can associate the homology groups \(H_n(G)\) with \(G\) (see [15, Section 3] for the precise definition). The homology group \(H_0(G)\) is the quotient of \(C(G^{(0)}, \mathbb{Z})\) by the subgroup generated by \(1_{r(U)} - 1_{s(U)}\) for compact open \(G\)-sets \(U\). We denote the equivalence class of \(\xi \in C(G^{(0)}, \mathbb{Z})\) in \(H_0(G)\) by \([\xi]\). For the étale groupoid \(G_A\), we have the following.

**THEOREM 2.7 ([15, THEOREM 4.14])**

Let \((X_A, \sigma_A)\) be a one-sided topological Markov shift. Then

\[
H_n(G_A) \cong \begin{cases} 
BF(A^t) = \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N, & n = 0, \\
\text{Ker}(\text{id} - A^t), & n = 1, \\
0, & n \geq 2.
\end{cases}
\]

Moreover, there exists an isomorphism \(\Phi : H_0(G_A) \to BF(A^t)\) such that \(\Phi([1_{X_A}]) = u_A\).

In particular, it follows from Theorem 2.3 that the pair \((BF(A^t), u_A)\) is an invariant for continuous orbit equivalence of one-sided topological Markov shifts (see also [13, Theorem 1.3]). Thus, if \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent, then there exists an isomorphism \(\Phi : BF(A^t) \to BF(B^t)\) such that \(\Phi(u_A) = u_B\).

### 3. Classification up to continuous orbit equivalence

Let \((X_A, \sigma_A)\) be an irreducible one-sided topological Markov shift. As in the previous section, \((\bar{X}_A, \bar{\sigma}_A)\) denotes the two-sided topological Markov shift corresponding to \((X_A, \sigma_A)\). Define \(\rho : \bar{X}_A \to X_A\) by \(\rho((x_n)_{n \in \mathbb{Z}}) = (x_n)_{n \in \mathbb{N}}\). Clearly we have that \(\sigma_A \circ \rho = \rho \circ \bar{\sigma}_A\).
LEMMA 3.1
The map \( C(X_A, \mathbb{Z}) \ni \xi \mapsto \xi \circ \rho \in C(\tilde{X}_A, \mathbb{Z}) \) gives rise to an isomorphism \( \tilde{\rho} \) from \( H^A \) to \( H^A \) satisfying \( \tilde{\rho}(H^A_+) = H^A_+ \).

Proof
For any \( \eta \in C(X_A, \mathbb{Z}) \), one has that \( (\eta - \eta \circ \sigma_A) \circ \rho = \eta \circ \rho - \eta \circ \rho \circ \sigma_A \), and so \( [\xi] \mapsto [\xi \circ \rho] \) is a well-defined homomorphism \( \tilde{\rho} \) from \( H^A \) to \( H^A \).

Let \( \zeta \in C(\tilde{X}_A, \mathbb{Z}) \). Then \( \zeta(x) \) depends only on finitely many coordinates of \( x \in \tilde{X}_A \). Hence, for sufficiently large \( n \in \mathbb{N} \), there exists \( \xi \in C(X_A, \mathbb{Z}) \) such that \( \zeta \circ \sigma^n_A = \xi \circ \rho \). Thus \( \tilde{\rho} \) is surjective.

Clearly \( \tilde{\rho}(H^A_+) \subset H^A_+ \). It follows from the argument above that \( H^A_+ \) is contained in \( \tilde{\rho}(H^A_+) \).

It remains for us to show the injectivity. Let \( \xi \in C(X_A, \mathbb{Z}) \). Suppose that there exists \( \zeta \in C(\tilde{X}_A, \mathbb{Z}) \) such that \( \xi \circ \rho = \zeta \circ \sigma_A \). In the same way as above, for sufficiently large \( n \in \mathbb{N} \), there exists \( \eta \in C(X_A, \mathbb{Z}) \) such that \( \zeta \circ \sigma^n_A = \eta \circ \rho \). Then

\[
\xi \circ \sigma^n_A \circ \rho = \xi \circ \rho \circ \sigma^n_A = \zeta \circ \sigma^n_A - \zeta \circ \sigma^{n+1}_A = (\eta - \eta \circ \sigma_A) \circ \rho.
\]

Hence \( \xi \circ \sigma^n_A = \eta - \eta \circ \sigma_A \). Thus \( [\xi] = [\xi \circ \sigma^n_A] = 0 \) in \( H^A \).

LEMMA 3.2
For \( \xi \in C(X_A, \mathbb{Z}) \), \([\xi]\) is in \( H^A_+ \) if and only if \( \sum_{x \in O} \xi(x) \geq 0 \) holds for every finite \( \sigma_A \)-invariant set \( O \subset X_A \).

Proof
Suppose that \([\xi]\) is in \( H^A_+ \). By the lemma above, \( \tilde{\rho}([\xi]) = [\xi \circ \rho] \) is in \( H^A_+ \). Let \( O \subset X_A \) be a finite \( \sigma_A \)-invariant set. There exists a finite \( \sigma_A \)-invariant set \( \tilde{O} \subset \tilde{X}_A \) such that \( \rho | O \) is a bijection from \( \tilde{O} \) to \( O \). It follows from Proposition 2.5 that \( \sum_{x \in \tilde{O}} \xi(\rho(x)) \geq 0 \). Hence \( \sum_{x \in O} \xi(x) \geq 0 \).

Suppose that \( \sum_{x \in O} \xi(x) \geq 0 \) holds for every finite \( \sigma_A \)-invariant set \( O \subset X_A \). For any finite \( \sigma_A \)-invariant set \( \tilde{O} \subset \tilde{X}_A \), \( O = \rho(\tilde{O}) \subset X_A \) is a finite \( \sigma_A \)-invariant set and \( \rho | \tilde{O} \) is injective. Therefore \( \sum_{x \in \tilde{O}} \xi(\rho(x)) = \sum_{x \in O} \xi(x) \geq 0 \). By Proposition 2.5, \([\xi \circ \rho]\) is in \( H^A_+ \). By the lemma above, \([\xi]\) is in \( H^A_+ \) as desired.

Let \( G \) be an étale groupoid. We denote by \( \text{Hom}(G, \mathbb{Z}) \) the set of continuous homomorphisms \( \omega : G \to \mathbb{Z} \). We think of \( \text{Hom}(G, \mathbb{Z}) \) as an abelian group by pointwise addition. For \( \xi \in C(G^{(0)}, \mathbb{Z}) \), we can define \( \partial(\xi) \in \text{Hom}(G, \mathbb{Z}) \) by \( \partial(\xi)(g) = \xi(r(g)) - \xi(s(g)) \). The cohomology group \( H^1(G) = H^1(G, \mathbb{Z}) \) is the quotient of \( \text{Hom}(G, \mathbb{Z}) \) by \{ \( \partial(\xi) \mid \xi \in C(G^{(0)}, \mathbb{Z}) \) \}. The equivalence class of \( \omega : G \to \mathbb{Z} \) is written \([\omega] \in H^1(G) \).

Let \( g \in G \) be such that \( r(g) = s(g) \), that is, \( g \in G' \). Since \( \partial(\xi)(g) = 0 \) for any \( \xi \in C(G^{(0)}, \mathbb{Z}) \), \([\omega] \mapsto \omega(g) \) is a well-defined homomorphism from \( H^1(G) \) to \( \mathbb{Z} \). We say that \( g \in G \) is attracting if there exists a compact open \( G \)-set \( U \) such that \( g \in U \),
then $r(U) \subset s(U)$ and

$$\lim_{n \to +\infty} (\pi_U)^n(y) = r(g)$$

holds for any $y \in s(U)$.

Let $(X_A, \sigma_A)$ be a one-sided topological Markov shift, and consider the étale groupoid $G_A$ (see Section 2.2 for the definition). We say that $x \in X_A$ is \textit{eventually periodic} if there exist $k, l \in \mathbb{Z}_+$ such that $k \neq l$ and $\sigma_A^k(x) = \sigma_A^l(x)$. This is equivalent to saying that $\{\sigma_A^n(x) \in X_A \mid n \in \mathbb{Z}_+\}$ is a finite set. When $x$ is eventually periodic, we call

$$\min\{k - l \mid k, l \in \mathbb{Z}_+, k > l, \sigma_A^k(x) = \sigma_A^l(x)\}$$

the \textit{period} of $x$.

**Lemma 3.3**

Let $x \in X_A$.

(a) If $x$ is not eventually periodic, then the isotropy group $(G_A)_x$ is trivial.

(b) If $x$ is eventually periodic, then $(G_A)_x = \{(x, np, x) \in G_A \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$, where $p$ is the period of $x$.

(c) When $x$ is eventually periodic and has period $p$, $(x, np, x)$ is attracting if and only if $n$ is positive.

**Proof**

Both (a) and (b) are obvious. We prove (c). Suppose that $x$ is an eventually periodic point whose period is $p$. Let $(x, np, x) \in (G_A)_x$. Assume that $n$ is positive. Choose $k, l \in \mathbb{Z}_+$ so that $\sigma_A^k(x) = \sigma_A^l(x)$ and $pn = k - l$. Define a clopen neighborhood $V$ and $W$ of $x$ by

$$V = \{(y_n)_n \in X_A \mid y_i = x_i \forall i = 1, 2, \ldots, k + 1\}$$

and

$$W = \{(y_n)_n \in X_A \mid y_i = x_i \forall i = 1, 2, \ldots, l + 1\}.$$ 

We have that $V \subset W$ and $\sigma_A^k(V) = \sigma_A^l(W)$. Then

$$U = \{(y, np, z) \in G_A \mid y \in V, z \in W, \sigma_A^k(y) = \sigma_A^l(z)\}$$

is a compact open $G_A$-set such that $(x, np, x) \in U$, $r(U) = V$, $s(U) = W$, and $\pi_U = (\sigma_A^l | V)^{-1} \circ (\sigma_A^k | W)$. It is easy to see that

$$\lim_{m \to +\infty} (\pi_U)^m(z) = x$$

holds for any $z \in s(U)$. Thus $(x, np, x)$ is attracting.

Suppose that $U \subset G_A$ is a compact open $G_A$-set containing $(x, 0, x)$. Then $\pi_U(y) = y$ for any $y$ sufficiently close to $x$, and so $(x, 0, x)$ is not attracting.

Assume that $n$ is negative. Let $U \subset G_A$ be a compact open $G_A$-set containing $(x, np, x)$. By the argument above, $(x, -np, x)$ is attracting. Hence there exists
a clopen neighborhood $V$ of $x$ such that $V \subset s(U)$ and $V \subset \pi_U(V)$. This means that $(x, np, x)$ cannot be an attracting element. □

**Proposition 3.4**

There exists an isomorphism $\Phi : H^1(G_A) \rightarrow H^A$ such that $\Phi([\omega])$ is in $H^A_+$ if and only if $\omega(g) \geq 0$ for every attracting $g \in G_A$.

**Proof**

Let $\omega \in \text{Hom}(G_A, \mathbb{Z})$. Define $\xi \in C(X_A, \mathbb{Z})$ by

$$\xi(x) = \omega((x, 1, \sigma_A(x))).$$

Let us verify that the map $\omega \mapsto \xi$ is surjective. For a given $\xi \in C(X_A, \mathbb{Z})$, we can define $\omega \in \text{Hom}(G_A, \mathbb{Z})$ as follows. Take $(x, n, y) \in G_A$. There exists $k, l \in \mathbb{Z}_+$ such that $k - l = n$ and $\sigma^k_A(x) = \sigma^l_A(y)$. Put

$$\omega((x, n, y)) = \sum_{i=0}^{k-1} \xi(\sigma^i_A(x)) - \sum_{j=0}^{l-1} \xi(\sigma^j_A(y)).$$

Clearly this gives a well-defined continuous homomorphism from $G_A$ to $\mathbb{Z}$. If there exists $\eta \in C(X_A, \mathbb{Z})$ such that $\omega = \partial(\eta)$, then $\xi = \eta - \eta \circ \sigma_A$, that is, $[\xi] = 0$ in $H^A$. It is also easy to see that the converse holds. Therefore $\Phi : [\omega] \mapsto [\xi]$ is an isomorphism from $H^1(G_A)$ to $H^A$.

We would like to show that $[\xi]$ is in $H^A_+$ if and only if $\omega(g) \geq 0$ for every attracting $g \in G_A$. Let $x \in X_A$ be an eventually periodic point whose period is $p$, and let $g = (x, np, x)$ be an attracting element. By the lemma above, $n$ is positive. There exists $k, l \in \mathbb{Z}_+$ such that $k - l = np$ and $\sigma^k_A(x) = \sigma^l_A(x)$. Then one has

$$\omega(g) = \sum_{i=0}^{k-1} \xi(\sigma^i_A(x)) - \sum_{j=0}^{l-1} \xi(\sigma^j_A(x))$$

$$= \sum_{i=l}^{k-1} \xi(\sigma^i_A(x))$$

$$= n \sum_{i=1}^{l+p-1} \xi(\sigma^i_A(x)).$$

Notice that $O = \{\sigma_A^l(x), \sigma_A^{l+1}(x), \ldots, \sigma_A^{l+p-1}(x)\}$ is a finite $\sigma_A$-invariant set. By Lemma 3.2, $[\xi]$ belongs to $H^A_+$ if and only if

$$\sum_{y \in O} \xi(y) \geq 0$$

for any finite $\sigma_A$-invariant set $O \subset X_A$, thereby completing the proof. □

Consequently we have the following.
THEOREM 3.5
Let \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) be two irreducible one-sided topological Markov shifts. If \((X_A, \sigma_A)\) is continuously orbit equivalent to \((X_B, \sigma_B)\), then there exists an isomorphism \(\Phi : H_A \to H_B\) such that \(\Phi(H_A^+) = H_B^+\). In particular, \((\bar{X}_A, \bar{\sigma}_A)\) is flow equivalent to \((\bar{X}_B, \bar{\sigma}_B)\).

Proof
Consider the étale groupoids \(G_A\) and \(G_B\). By Theorem 2.3, \(G_A\) and \(G_B\) are isomorphic. Let \(\varphi : G_A \to G_B\) be an isomorphism. For \(g \in G_A\), \(g\) is attracting in \(G_A\) if and only if \(\varphi(g)\) is attracting in \(G_B\). It follows from Proposition 3.4 above that \((H_A, H_A^+)\) is isomorphic to \((H_B, H_B^+)\). Then, Lemma 3.1 implies that \((\bar{H}_A, \bar{H}_A^+)\) is isomorphic to \((\bar{H}_B, \bar{H}_B^+)\). By Theorem 2.4, \((\bar{X}_A, \bar{\sigma}_A)\) is flow equivalent to \((\bar{X}_B, \bar{\sigma}_B)\). \(\square\)

THEOREM 3.6
Let \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) be two irreducible one-sided topological Markov shifts. The following conditions are equivalent.

(a) \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent.
(b) The étale groupoids \(G_A\) and \(G_B\) are isomorphic.
(c) There exists an isomorphism \(\Psi : O_A \to O_B\) such that \(\Psi(D_A) = D_B\).
(d) \(O_A\) is isomorphic to \(O_B\) and \(\text{sgn}(|\det(id - A)|) = \text{sgn}(|\det(id - B)|)\).
(e) There exists an isomorphism \(\Phi : BF(A^t) \to BF(B^t)\) such that \(\Phi(u_A) = u_B\) and \(\text{sgn}(|\det(id - A)|) = \text{sgn}(|\det(id - B)|)\).

Proof
The equivalence between (a), (b), and (c) is already known (see Theorem 2.3). As mentioned in Section 2.5, \((K_0(O_A), [1])\) is isomorphic to \((BF(A^t), u_A)\), and so (d) \(\Rightarrow\) (e) holds. The implication (e) \(\Rightarrow\) (a) follows from [12, Theorem 1.1].

Suppose that \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent. It follows from the theorem above that the two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent. Therefore, by [17], we have that \(\det(id - A) = \det(id - B)\) (see Theorem 2.6). Since (a) \(\Rightarrow\) (c) is already known, \(O_A\) is isomorphic to \(O_B\). Thus we have obtained (d). This completes the proof. \(\square\)

As mentioned in Section 2.5, \(\det(id - A) = 0\) when \(BF(A^t)\) is infinite, and \(\det(id - A)\) equals the cardinality of \(BF(A^t)\) when \(BF(A^t)\) is finite. Hence, our invariant of the continuous orbit equivalence consists of a finitely generated abelian group \(F\), an element \(u \in F\), and \(s \in \{-1, 0, 1\}\) such that \(F\) is an infinite group if and only if \(s = 0\). Conversely, for any such triplet \((F, u, s)\), there exists an irreducible one-sided topological Markov shift whose invariant is equal to \((F, u, s)\). This is probably known to experts, but the authors are not aware of a specific reference and thus include a proof for completeness.
LEMMA 3.7
Let $F$ be a finitely generated abelian group, and let $u \in F$. Let $s = 0$ when $F$ is infinite, and let $s$ be either $-1$ or $1$ when $F$ is finite. There exists an irreducible one-sided topological Markov shift $(X_A, \sigma_A)$ such that $(F, u) \cong (\text{BF}(A^t), u_A)$ and the sign of $\det(\text{id} - A)$ equals $s \in \{-1, 0, 1\}$.

Proof
Suppose that we are given $(F, u, s)$. It suffices to find a square irreducible matrix $A$ with entries in $\mathbb{Z}_+$ satisfying the desired properties (see [10, Section 2.3] or [5, Remark 2.16]). Let $A = [A(i, j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\mathbb{Z}_+$ such that $A(1, 1) = 2$, $A(i, i) \geq 2$, and $A(i, j) = 1$ for all $i, j$ with $i \neq j$. Let $d_i = A(i, i) - 2$, and let $r = |\{i \mid d_i = 0\}| - 1$. Then it is straightforward to see that $\text{BF}(A^t) \cong \mathbb{Z}^r \oplus \bigoplus_{d_i \geq 2} \mathbb{Z}/d_i \mathbb{Z}$ and $\det(\text{id} - A) = (-1)^N \prod_{i=2}^N d_i$.

Therefore we can construct such $A$ so that $\text{BF}(A^t) \cong F$ and the sign of $\det(\text{id} - A)$ equals $s$. In what follows, we identify $\text{BF}(A^t)$ with $F$. Note that $u_A \in \text{BF}(A^t)$ is zero. Choose $(c_1, c_2, \ldots, c_N) \in \mathbb{Z}^N$ whose equivalence class in $\text{BF}(A^t)$ equals $u$. Since $u_A$ is zero, we may assume that $c_i \in \mathbb{Z}_+$ for all $i$. We now construct a new matrix $B$ as follows. Set

$$\Sigma = \{ (i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid 1 \leq i \leq N, 0 \leq j \leq c_i \}.$$

Define $B = [B((i, j), (k, l))]|_{(i, j), (k, l) \in \Sigma}$ by

$$B((i, j), (k, l)) = \begin{cases} A(i, k), & j = c_i, l = 0, \\ 1, & i = k, j + 1 = l, \\ 0, & \text{otherwise}. \end{cases}$$

The group $\text{BF}(A^t)$ is the abelian group with generators $e_1, \ldots, e_N$ and relations

$$e_i = \sum_{j=1}^N A(i, j) e_j,$$

and $u$ equals $\sum c_i e_i$. The group $\text{BF}(B^t)$ is the abelian group with generators $\{f_{i, j} \mid (i, j) \in \Sigma\}$ and relations

$$f_{i, j} = f_{i, j'}, \quad f_{i, c_i} = \sum_{k=1}^N A(i, k) f_{k, 0},$$

and $u_B$ equals $\sum f_{i, j}$. Hence $(\text{BF}(A^t), u)$ is isomorphic to $(\text{BF}(B^t), u_B)$. It is also easy to see that $\det(\text{id} - A) = \det(\text{id} - B)$. The proof is completed.

□

For $i = 1, 2$, let $G_i$ be a minimal essentially principal étale groupoid whose unit space is a Cantor set. It has been shown that the following conditions are mutually
equivalent (see [16, Theorem 3.10]). For a group \( \Gamma \), we let \( D(\Gamma) \) denote the commutator subgroup.

- \( G_1 \) and \( G_2 \) are isomorphic as étale groupoids.
- \( [[G_1]] \) and \( [[G_2]] \) are isomorphic as discrete groups.
- \( D([[G_1]]) \) and \( D([[G_2]]) \) are isomorphic as discrete groups.

The étale groupoid \( G_A \) arising from \( (X_A, \sigma_A) \) is minimal, essentially principal, and purely infinite (see [16, Lemma 6.1]). Hence \( D([[G_A]]) \) is simple by [16, Theorem 4.16]. Moreover, \( D([[G_A]]) \) is finitely generated (see [16, Corollary 6.25]), \( [[G_A]] \) is of type \( \mathsf{F}_\infty \) (see [16, Theorem 6.21]), and \( [[G_A]]/D([[G_A]]) \) is isomorphic to \( (H_0(G_A) \otimes \mathbb{Z}_2) \oplus H_1(G_A) \) (see [16, Corollary 6.24]). Theorem 3.6 tells us that the isomorphism class of \( [[G_A]] \) and \( D([[G_A]]) \) is determined by \( (H_0(G_A), [1_{X_A}], \det(id-A)) \) (see also Theorem 2.7). By Lemma 3.7, for each triplet \( (F, u, s) \) there exists \( (X_A, \sigma_A) \) whose invariant agrees with it. In particular, the simple finitely generated groups \( D([[G_A]]) \) are parameterized by such triplets \( (F, u, s) \).

We conclude this article by giving a corollary. We denote by \( \mathbb{K} \) the \( C^* \)-algebra of all compact operators on \( \ell^2(\mathbb{Z}) \). Let \( \mathcal{C} \cong \mathcal{C}_0(\mathbb{Z}) \) be the maximal abelian subalgebra of \( \mathbb{K} \) consisting of diagonal operators.

**Corollary 3.8**

Let \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) be two irreducible two-sided topological Markov shifts. The following conditions are equivalent.

(a) \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are flow equivalent.

(b) There exists an isomorphism \( \Psi : \mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_B \otimes \mathbb{K} \) such that \( \Psi(D_A \otimes \mathcal{C}) = D_B \otimes \mathcal{C} \).

**Proof**

From [5, Theorem 4.1], we know that (a) \( \Rightarrow \) (b). Let us assume (b). In what follows, we identify the Bowen–Franks group with the \( K_0 \)-group of the Cuntz–Krieger algebra. We have the isomorphism \( K_0(\Psi) : \mathsf{BF}(A^t) \to \mathsf{BF}(B^t) \). By Lemma 3.7, there exists an irreducible one-sided topological Markov shift \( (X_C, \sigma_C) \) such that \( (\mathsf{BF}(B^t), K_0(\Psi)(u_A)) \cong (\mathsf{BF}(C^t), u_C) \) and \( \det(id-B) = \det(id-C) \). It follows from Theorem 2.6 that \( (X_B, \sigma_B) \) is flow equivalent to \( (X_C, \sigma_C) \). Moreover, by Huang’s theorem (see [8, Theorem 2.15]) and its proof, there exists an isomorphism \( \Phi : \mathcal{O}_B \otimes \mathbb{K} \to \mathcal{O}_C \otimes \mathbb{K} \) such that \( \Phi(D_B \otimes \mathcal{C}) = D_C \otimes \mathcal{C} \) and \( K_0(\Phi)(K_0(\Psi)(u_A)) = u_C \). Then \( \Phi \circ \Psi \) is an isomorphism from \( \mathcal{O}_A \otimes \mathbb{K} \) to \( \mathcal{O}_C \otimes \mathbb{K} \) such that \( \phi = \Phi \circ \Psi \circ D_A \otimes \mathcal{C} = D_C \otimes \mathcal{C} \) and \( K_0(\Phi \circ \Psi)(u_A) = u_C \). In the same way as the proof of [12, Theorem 4.1], we can conclude that \( (\mathcal{O}_A, D_A) \) is isomorphic to \( (\mathcal{O}_C, D_C) \). By virtue of Theorem 3.6, we get that \( \det(id-A) = \det(id-C) \). Therefore \( \det(id-A) = \det(id-B) \). Hence, by Theorem 2.6, \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are flow equivalent. \( \square \)
References


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