Explicit computation of certain Arakelov–Green functions

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Abstract Arakelov–Green functions defined on metrized graphs have an important role in relating arithmetical problems on algebraic curves to graph-theoretical problems. In this paper, we clarify the combinatorial interpretation of certain Arakelov–Green functions by using electric circuit theory. The formulas we give clearly show that such functions are piecewisely defined, and each piece is a linear or quadratic function on each pair of edges of metrized graphs. These formulas lead to an efficient algorithm for explicit computation of Arakelov–Green functions.

1. Introduction

Algebraic geometers have powerful tools due to intersection theory over complex numbers to study curves and varieties in general. Given the success of algebraic geometers, it is the desire of number theorists and arithmetic geometers to utilize intersection theory for studying arithmetic properties of algebraic curves. However, if one works over fields other than complex numbers, many difficulties arise, because various nice properties of complex numbers are no longer in use. Additional new tools should be used to overcome these difficulties. This is what Arakelov [1] did over archimedean fields in his studies, which we now know as Arakelov theory. Arakelov introduced an intersection pairing on arithmetic surfaces. The key part was to consider the contribution to the intersection number that comes from the infinite places. This contribution is defined by using Arakelov–Green functions for the Riemann surfaces associated to the arithmetic surfaces. He used analysis and studied the Laplace operator on those associated Riemann surfaces to derive global results on arithmetic surfaces. We note that the use of admissible metrized line bundles, metrized line bundles satisfying certain analytic criteria, on arithmetic surfaces is another important tool considered in Arakelov theory. Faltings's arithmetic analogues of the Riemann–Roch theorem and adjunction formula from classical intersection theory on surfaces are

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two striking examples of the successes of Arakelov theory. These kinds of successes enabled Faltings [11] to prove the Mordell conjecture among other results in arithmetic geometry.

We have a similar story for nonarchimedean fields. In this case, we have metrized graphs as nonarchimedean analogues of Riemann surfaces. Again we have Arakelov–Green functions and Laplacian operators on metrized graphs. Reduction graphs, the dual graphs associated to the special fiber curve, are examples of metrized graphs. Rumely [12], who introduced metrized graphs to study arithmetic properties of algebraic curves and developed capacity theory, contributed to the development of local intersection theory for algebraic curves defined over nonarchimedean fields. Metrized graphs were further developed by Chinburg and Rumely [4] and by Zhang [13]. Chinburg and Rumely[4] introduced "capacity pairing" and used metrized graphs in their work. Later, Zhang [13] introduced another intersection pairing as a nonarchimedean analogue of Arakelov's pairing on a Riemann surface, and he showed that the analogous Riemann–Roch theorem and adjunction formula hold for this admissible pairing. Baker and Rumely [3] used harmonic analysis on metrized graphs to study Arakelov–Green functions and related continuous Laplacian operators. Various arithmetic results have been obtained after these studies, for example, the proof of the effective Bogomolov conjecture over function fields of characteristic zero (see [9], [14]).

Metrized graphs and Arakelov–Green functions on metrized graphs have important roles in the articles [3], [4], [9], [12], [13], and [14]. The basic interest in Arakelov–Green functions is to find their values on any given points of metrized graphs. Our aim in this article is to address this issue by giving formulas that explain how Arakelov–Green functions behave on any pairs of edges of metrized graphs. This leads to an efficient algorithm that can be used for both symbolic and numerical computations of Arakelov–Green functions. We think that the formulas we give for Arakelov–Green functions can be used to study spectral properties of the Laplacian operator given in [3].

In Section 2, we give a short description of metrized graphs and their discrete Laplacian matrix. In Section 3, we first review basic facts about the resistance function $r(x, y)$ on a metrized graph Γ. Then we obtain formulas that express $r(x, y)$ in terms of the endpoints of the edges that contain x and y (see Lemma 3.1, Lemma 3.2, and Theorem 3.3). This means that one needs basically the effective resistance values between any two vertices in Γ to obtain the values of $r(x, y)$.

In Section 4, we first describe Arakelov–Green functions on a metrized graph Γ. Baker and Rumely showed that the Arakelov–Green function $g_{\mu_{\mathrm{can}}}(x, y)$ can be expressed in terms of the tau constant $\tau(\Gamma)$ of the metrized graph Γ and the resistance function $r(x, y)$ (see Theorem 4.2). Combining this fact and our results from Section 3 about the resistance function, we obtain our main result in Theorem 4.3. In this way, we show that $g_{\mu_{\mathrm{can}}}(x, y)$ on Γ is a piecewisely defined quadratic or linear function in both x and y by explicitly giving the coefficients of each piece in terms of the effective resistance values between the related vertices of Γ. If x (or y) belongs to an edge whose removal disconnects Γ, then $g_{\mu_{\text{can}}}(x, y)$ is linear in x (or y). Otherwise it will be quadratic. We suggest that a matrix Z of size $e \times e$ can be used to describe $g_{\mu_{\text{can}}}(x, y)$, where e is the number of edges in Γ.

In Section 6, we give several examples of computations of $g_{\mu_{\text{can}}}(x, y)$ by finding the matrix Z, which we call the *value matrix*. We know that the tau constant can be computed symbolically and numerically by using either theoretical work in various cases (see [8] and [7]) or computer algorithms in all cases (see [6]). Therefore, we conclude the same things for computation of $g_{\mu_{\text{can}}}(x, y)$ by both using the results of Section 4 and our previous results on the tau constant.

2. Metrized graphs

In this section, we give a brief review of metrized graphs and their discrete Laplacian matrix.

A metrized graph Γ is a finite connected graph equipped with a distinguished parameterization of each of its edges. A metrized graph Γ can have multiple edges and self-loops. For any given $p \in \Gamma$, the number $\upsilon(p)$ of directions emanating from p will be called the *valence* of p. By definition, there can be only finitely many $p \in \Gamma$ with $\upsilon(p) \neq 2$.

For a metrized graph Γ, we will denote a vertex set for Γ by $V(\Gamma)$. We require that $V(\Gamma)$ be finite and nonempty and that $p \in V(\Gamma)$ for each $p \in \Gamma$ if $\upsilon(p) \neq 2$. For a given metrized graph Γ, it is possible to enlarge the vertex set $V(\Gamma)$ by considering additional valence 2 points as vertices.

For a given metrized graph Γ with vertex set $V(\Gamma)$, the set of edges of Γ is the set of closed line segments with endpoints in $V(\Gamma)$. The endpoints of an edge can possibly be identical, in which case the edge is a self-loop. We will denote the set of edges of Γ by $E(\Gamma)$. However, if e_i is an edge, then by $\Gamma - e_i$ we mean the graph obtained by deleting the *interior* of e_i.

We denote the length of an edge $e_i \in E(\Gamma)$ by L_i, which represents a positive real number. The total length of Γ, which is denoted by $\ell(\Gamma)$, is given by $\ell(\Gamma) = \sum_{i=1}^{e} L_i$. Here, e denotes the number of edges in $E(\Gamma)$.

To have a well-defined discrete Laplacian matrix L for a metrized graph Γ, we first choose a vertex set $V(\Gamma)$ for Γ in such a way that there are no self-loops, and no multiple edges connecting any two vertices. This can be done by enlarging the vertex set by considering additional valence 2 points as vertices whenever needed. We call such a vertex set $V(\Gamma)$ *adequate*. If distinct vertices p and q are the endpoints of an edge, then we call them *adjacent* vertices.

Let Γ be a metrized graph with e edges and an adequate vertex set $V(\Gamma)$ containing v vertices. Fix an ordering of the vertices in $V(\Gamma)$. Let $\{L_1, L_2, \dots, L_e\}$ be a labeling of the edge lengths. The matrix $\mathrm{A} = (a_{pq})_{v \times v}$ given by

$$a_{pq} = \begin{cases} 0 & \text{if } p = q \text{ or if } p \text{ and } q \text{ are not adjacent,} \\ \frac{1}{L_k} & \text{if } p \neq q \text{ and an edge of length } L_k \text{ connects } p \text{ and } q \end{cases}$$

is called the *adjacency matrix* of Γ. Let $\mathrm{D} = \mathrm{diag}(d_{pp})$ be the $v \times v$ diagonal matrix given by $d_{pp} = \sum_{s \in V(\Gamma)} a_{ps}$. Then $\mathrm{L} := \mathrm{D} - \mathrm{A}$ is called the *discrete Laplacian matrix* of Γ. That is, $\mathrm{L} = (l_{pq})_{v \times v}$ where

$$l_{pq} = \begin{cases} 0 & \text{if } p \neq q \text{ and } p \text{ and } q \text{ are not adjacent,} \\ -\frac{1}{L_k} & \text{if } p \neq q \text{ and } p \text{ and } q \text{ are connected by} \\ & \quad \text{an edge of length } L_k, \\ -\sum_{s \in V(\Gamma) - \{p\}} l_{ps} & \text{if } p = q. \end{cases}$$

Since a metrized graph Γ is represented by connected graphs, one of the eigenvalues of L is 0 and the others are positive. Thus, L is not invertible. However, it has generalized inverses. In particular, it has the pseudoinverse L^+, also known as the *Moore–Penrose generalized inverse*, which is uniquely determined by the following properties:

$$\begin{array}{llll} \text{(a)} & \mathrm{L}\mathrm{L}^+\mathrm{L} = \mathrm{L}, & \text{(c)} & (\mathrm{L}\mathrm{L}^+)^T = \mathrm{L}\mathrm{L}^+, \\ \text{(b)} & \mathrm{L}^+\mathrm{L}\mathrm{L}^+ = \mathrm{L}^+, & \text{(d)} & (\mathrm{L}^+\mathrm{L})^T = \mathrm{L}^+\mathrm{L}. \end{array}$$

One can find more information about L and L^+ in [6, Section 3] and the references therein.

3. Resistance function $r(x, y)$

In this section, we study the resistance and voltage functions on a metrized graph Γ. After reviewing the facts that we will use about these functions, we consider the following problem. If one considers these functions on a graph having only combinatorial nature, consisting of vertices and edges between these vertices, one can compute the resistance and voltage functions by using the discrete Laplacian matrix of the graph. However, these functions are continuous functions on a metrized graph Γ. A metrized graph being more than a combinatorial graph has additional structures, but still has the combinatorial properties of a graph. Therefore, there should be a way to relate the values of continuous resistance and voltage functions on Γ with the values of discrete resistance and voltage functions on the vertices of a combinatorial graph. Our goal is to clarify this relation in this section. The results we obtain in this section will be used in the next section.

For any x, y, z in Γ, the voltage function $j_z(x, y)$ on a metrized graph Γ is a symmetric function in x and y which satisfies $j_x(x, y) = 0$ and $j_z(x, y) \geq 0$ for all x, y, z in Γ. For each vertex set $V(\Gamma)$, $j_z(x, y)$ is continuous on Γ as a function of all three variables. For fixed z and y it has the following physical interpretation: if Γ is viewed as a resistive electric circuit with terminals at z and y, with the resistance in each edge given by its length, then $j_z(x, y)$ is the voltage difference between x and z, when unit current enters at y and exits at z (with reference voltage 0 at z).

The effective resistance between two points x, y of a metrized graph Γ is given by $r(x, y) = j_y(x, x)$, where $r(x, y)$ is the resistance function on Γ. The resistance

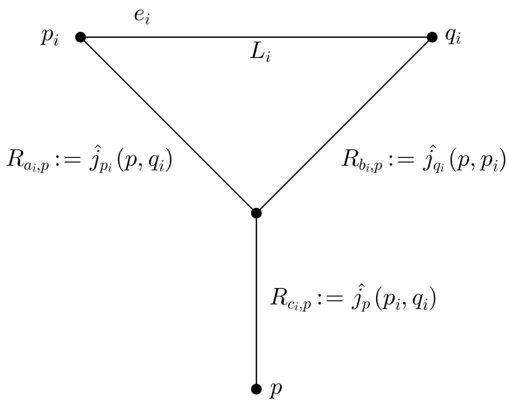

Figure 1. Circuit reduction with reference to an edge and a point.

function inherits certain properties of the voltage function. For any x, y in Γ, $r(x,y)$ on Γ is a symmetric function in x and y, and it satisfies $r(x,x)=0$. For each vertex set $V(\Gamma)$, $r(x,y)$ is continuous on Γ as a function of two variables and $r(x,y)\geq 0$ for all x, y in Γ. If a metrized graph Γ is viewed as a resistive electric circuit with terminals at x and y, with the resistance in each edge given by its length, then $r(x,y)$ is the effective resistance between x and y.

The proofs of the facts mentioned above can be found in [4], [3, Sections 1.5 and 6], and [13, Appendix]. The voltage function $j_z(x,y)$ and the resistance function $r(x,y)$ are also studied in [2] and [10].

We will denote by R_i the resistance between the endpoints of an edge e_i of a graph Γ when the interior of the edge e_i is deleted from Γ.

Let Γ be a metrized graph with $p\in V(\Gamma)$, and let $e_i\in E(\Gamma)$ having endpoints p_i and q_i. If $\Gamma - e_i$ is connected, then Γ can be transformed into the graph in Figure 1 by circuit reductions. More details on this fact can be found in [4] and [5, Section 2]. Note that, in Figure 1, we have $R_{a_i,p}=\hat{j}_{p_i}(p,q_i)$, $R_{b_i,p}=\hat{j}_{q_i}(p,p_i)$, and $R_{c_i,p}=\hat{j}_p(p_i,q_i)$, where $\hat{j}_x(y,z)$ is the voltage function in $\Gamma - e_i$. We have $R_{a_i,p}+R_{b_i,p}=R_i$ for each $p\in\Gamma$.

If $\Gamma - e_i$ is not connected, then we set $R_{b_i,p}=R_i=\infty$ and $R_{a_i,p}=0$ if p belongs to the component of $\Gamma - e_i$ containing p_i, and we set $R_{a_i,p}=R_i=\infty$ and $R_{b_i,p}=0$ if p belongs to the component of $\Gamma - e_i$ containing q_i. We will use this notation for the rest of the paper.

Recall that the function $r(x,y)$ is defined on Γ and has nonnegative real number values. Therefore, when we write an equality as in Lemma 3.1 below, we mean that $x,y\in\Gamma$ on the left-hand side of the equality and that x,y are the corresponding real numbers via the parameterization. For example, if x is on edge e_i of length L_i with endpoints p_i and q_i, then we consider a parameterization identifying e_i by the interval $[0,L_i]$ so that the points p_i and q_i correspond to 0 and L_i, respectively, and that $x\in[0,L_i]$. We follow this approach in the rest of the paper. One should note that the direction of parameterization presents no

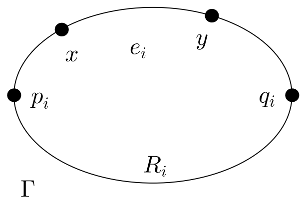

Figure 2. Circuit reduction with reference to an edge e_i having endpoints p_i and q_i.

problem in our computations as long as one is careful about the adjustment of the relevant formulas.

LEMMA 3.1

Let $e_i \in E(\Gamma)$ be an edge of length L_i with endpoints p_i and q_i. If both x and y belong to the same edge e_i, then

$$r(x,y) = |x-y| - (x-y)^2 \frac{L_i - r(p_i,q_i)}{L_i^2}.$$

Proof

Through circuit reductions, this case can be illustrated as in Figure 2. With abuse of notation, x and y denote points on e_i and also their distances to the vertex p_i. The result follows from the fact that $r(p_i,q_i) = \frac{L_i R_i}{L_i + R_i}$ and that x and y are connected by two parallel edges with edge lengths $|x-y|$ and $L_i + R_i - |x-y|$. $\square$

Note that L_i and $r(p_i,q_i)$ can be expressed in terms of the entries of the discrete Laplacian matrix L and its pseudoinverse L^+, respectively. In this way, whenever x and y are chosen from the same edge, we can express the continuous function $r(x,y)$ as a piecewise linear or quadratic function with coefficients obtained by using the discrete graph representation of metrized graphs. The condition that both x and y are on the same edge is an essential hypothesis in Lemma 3.1. A relevant question is: what would be the corresponding formula of $r(x,y)$ if x and y are chosen from different edges of Γ? In the rest of this section, we provide an answer to this question. First, we need the following technical lemma.

LEMMA 3.2

Let $e_i \in E(\Gamma)$ be an edge of length L_i with endpoints p_i and q_i. If x belongs to the edge e_i, then for any vertex $p \in V(\Gamma)$ we have

$$r(p,x) = -x^2 \frac{L_i - r(p_i,q_i)}{L_i^2} + x \frac{L_i - r(p_i,q_i) + r(p,q_i) - r(p,p_i)}{L_i} + r(p,p_i).$$

Proof

Using circuit reductions, this case can be illustrated as in Figure 3. Applying circuit reductions on the electric circuit given in Figure 3, we obtain

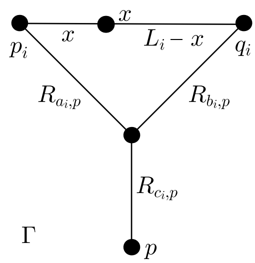

Figure 3. Circuit reduction with reference to an edge and a vertex.

$$(1)\qquad r(p,x) = \frac{(x+R_{a_i,p})(L_i - x + R_{b_i,p})}{L_i + R_i} + R_{c_i,p}, \qquad r(p_i,q_i) = \frac{L_i R_i}{L_i + R_i},$$

$$(2)\quad r(p,p_i) = \frac{R_{a_i,p}(L_i + R_{b_i,p})}{L_i + R_i} + R_{c_i,p}, \qquad r(p,q_i) = \frac{R_{b_i,p}(L_i + R_{a_i,p})}{L_i + R_i} + R_{c_i,p}.$$

Then the result follows from these equations. □

THEOREM 3.3

Let $e_i \in E(\Gamma)$ be an edge of length L_i with endpoints p_i and q_i, and let $e_j \in E(\Gamma)$ be an edge of length L_j with endpoints p_j and q_j. Suppose that the edges e_i and e_j are distinct, but their endpoints are not necessarily distinct. If x belongs to the edge e_i and y belongs to the edge e_j, then we have

$$\begin{aligned} r(x,y) = {} & -x^2\frac{L_i - r(p_i,q_i)}{L_i^2} - y^2\frac{L_j - r(p_j,q_j)}{L_j^2} + \frac{2xy}{L_iL_j}\big(j_{p_j}(p_i,q_j) - j_{p_j}(q_i,q_j)\big) \\ & + \frac{x}{L_i}\big(L_i - 2j_{p_i}(q_i,p_j)\big) + \frac{y}{L_j}\big(L_j - 2j_{p_j}(p_i,q_j)\big) + r(p_i,p_j). \end{aligned}$$

Proof

Applying Lemma 3.2 with edge e_j containing y and vertex p_i, we obtain

$$(3)\qquad \begin{aligned} r(p_i,y) = {} & -y^2\frac{L_j - r(p_j,q_j)}{L_j^2} \\ & + y\frac{L_j - r(p_j,q_j) + r(p_i,q_j) - r(p_i,p_j)}{L_j} + r(p_i,p_j). \end{aligned}$$

Similarly, applying Lemma 3.2 with edge e_j containing y and vertex q_i gives

$$(4)\qquad \begin{aligned} r(q_i,y) = {} & -y^2\frac{L_j - r(p_j,q_j)}{L_j^2} \\ & + y\frac{L_j - r(p_j,q_j) + r(q_i,q_j) - r(q_i,p_j)}{L_j} + r(q_i,p_j). \end{aligned}$$

Now, we fix a point $y \in E(e_j)$, consider it as a vertex, and apply Lemma 3.2 with edge e_i containing x and vertex y. In this way, we obtain

$$(5)\qquad r(x,y) = -x^2\frac{L_i - r(p_i,q_i)}{L_i^2} + x\frac{L_i - r(p_i,q_i) + r(y,q_i) - r(y,p_i)}{L_i} + r(y,p_i).$$

Using the fact that the resistance function is symmetric, we substitute (3) and (4) into (5) to obtain

$$
\begin{aligned}
r(x,y) = & -x^2\frac{L_i - r(p_i,q_i)}{L_i^2} - y^2\frac{L_j - r(p_j,q_j)}{L_j^2} \\
& + \frac{xy}{L_iL_j}\big(r(p_i,p_j) - r(p_i,q_j) - r(q_i,p_j) + r(q_i,q_j)\big) \\
& + \frac{x}{L_i}\big(L_i - r(p_i,q_i) + r(q_i,p_j) - r(p_i,p_j)\big) \\
& + \frac{y}{L_j}\big(L_j - r(p_j,q_j) + r(p_i,q_j) - r(p_i,p_j)\big) \\
& + r(p_i,p_j).
\end{aligned} \tag{6}
$$

Then the result follows using the fact that $2j_x(y,z) = r(x,y) + r(x,z) - r(y,z)$ for any $x,y,z \in \Gamma$. □

REMARK 3.4

Whenever the edges e_i and e_j that x and y belong to are bridges, that is, $\Gamma - e_i$ or $\Gamma - e_j$ is disconnected, we obtain the following results by letting $R_i \to \infty$ or $R_j \to \infty$ in the formulas given in Lemma 3.1 and Theorem 3.3.

(a) $r(x,y) = |x-y|$ if both x and y are on the same edge that is a bridge.

(b)

$$
r(p,x) = \begin{cases} x + r(p,p_i) & \text{if } p \text{ is on the side of } p_i, \\ L_i - x + r(p,q_i) & \text{if } p \text{ is on the side of } q_i. \end{cases}
$$

(c) If both e_i and e_j are bridges that are distinct edges, then we have that

$$
r(x,y) = \begin{cases} x + y + r(p_i,p_j) & \text{if } p_i \text{ and } p_j \text{ are between } x \text{ and } y, \\ x + L_j - y + r(p_i,q_j) & \text{if } p_i \text{ and } q_j \text{ are between } x \text{ and } y, \\ L_i - x + y + r(q_i,p_j) & \text{if } q_i \text{ and } p_j \text{ are between } x \text{ and } y, \\ L_i - x + L_j - y + r(q_i,q_j) & \text{if } q_i \text{ and } q_j \text{ are between } x \text{ and } y. \end{cases}
$$

(d) Suppose that only e_i is a bridge. (The case that only e_j is a bridge can be done by imitating this case.) Then we have two cases. If y is on the side of p_i, we have

$$
r(x,y) = x - y^2\frac{L_j - r(p_j,q_j)}{L_j^2} + y\frac{L_j - r(p_j,q_j) + r(p_i,q_j) - r(p_i,p_j)}{L_j} + r(p_i,p_j).
$$

If y is on the side of q_i, we have

$$
\begin{aligned}
r(x,y) = & L_i - x - y^2\frac{L_j - r(p_j,q_j)}{L_j^2} \\
& + y\frac{L_j - r(p_j,q_j) + r(q_i,q_j) - r(q_i,p_j)}{L_j} + r(q_i,p_j).
\end{aligned}
$$

REMARK 3.5
Since $2j_x(y,z) = r(x,y) + r(x,z) - r(y,z)$ for any $x, y, z \in \Gamma$, one can use Theorem 3.3 for each of $r(x,y)$, $r(x,z)$, and $r(y,z)$ to express the voltage function $j_x(y,z)$ in terms of its values on vertices of Γ.

4. Arakelov–Green function $g_{\mu_{\mathrm{can}}}(x,y)$

In this section, we first give the definition of Arakelov–Green functions $g_\mu(x,y)$ on a metrized graph Γ. Then we study the Arakelov–Green function $g_{\mu_{\mathrm{can}}}(x,y)$ defined with respect to a canonical measure μ_{can} on Γ. Our goal is to clarify the combinatorial interpretation of $g_{\mu_{\mathrm{can}}}(x,y)$.

For any real-valued, signed Borel measure μ on Γ with $\mu(\Gamma) = 1$ and $|\mu|(\Gamma) < \infty$, define the function $j_\mu(x,y) = \int_\Gamma j_z(x,y)\,d\mu(z)$. Clearly $j_\mu(x,y)$ is symmetric, and is jointly continuous in x and y. Chinburg and Rumely [4] discovered that there is a unique real-valued, signed Borel measure $\mu = \mu_{\mathrm{can}}$ such that $j_\mu(x,x)$ is constant on Γ. The measure μ_{can} is called the *canonical measure*. One can find several interpretations of μ_{can} in [3] and [5]. Baker and Rumely [3, Section 14] called the constant $\frac{1}{2}j_\mu(x,x)$ the *tau constant* of Γ and denoted it by $\tau(\Gamma)$. The following lemma gives a description of the tau constant.

LEMMA 4.1 ([3, LEMMA 14.4])
For any fixed y in Γ, $\tau(\Gamma) = \frac{1}{4}\int_\Gamma (\frac{\partial}{\partial x} r(x,y))^2\,dx$.

One can find more detailed information on $\tau(\Gamma)$ in [10], [6], [8], and [7].

Let μ be a real-valued, signed Borel measure of total mass 1 on Γ. In [3], the *Arakelov–Green function* $g_\mu(x,y)$ associated to μ is defined to be

$$g_\mu(x,y) = \int_\Gamma j_z(x,y)\,d\mu(z) - \int_{\Gamma^3} j_z(x,y)\,d\mu(z)\,d\mu(x)\,d\mu(y),$$

where the latter integral is a constant that depends on Γ and μ.

As shown in [3], $g_\mu(x,y)$ is continuous, symmetric (i.e., $g_\mu(x,y) = g_\mu(y,x)$, for each x and y), and for each y, $\int_\Gamma g_\mu(x,y)\,d\mu(x) = 0$. More precisely, as shown in [3], one can characterize $g_\mu(x,y)$ as the unique function on $\Gamma \times \Gamma$ such that the following hold.

(a) $g_\mu(x,y)$ is jointly continuous in x, y and belongs to $\mathrm{BDV}_\mu(\Gamma)$ as a function of x, for each fixed y, where $\mathrm{BDV}_\mu(\Gamma) := \{f \in \mathrm{BDV}(\Gamma) : \int_\Gamma f\,d\mu = 0\}$ and $\mathrm{BDV}(\Gamma)$ is a space of continuous functions of bounded differential variation Γ.
(b) For fixed y, g_μ satisfies the identity $\Delta_x g_\mu(x,y) = \delta_y(x) - \mu(x)$.
(c) $\iint_{\Gamma\times\Gamma} g_\mu(x,y)\,d\mu(x)\,d\mu(y) = 0$.

Precise definitions of $\mathrm{BDV}(\Gamma)$ and of Δf for $f \in \mathrm{BDV}(\Gamma)$ can be found in [3].

The Arakelov–Green function $g_\mu(x,y)$ satisfies the following properties. (A detailed proof can be found in [4, Theorem 2.11] and [10, p. 34].)

THEOREM 4.2 ([3, THEOREM 14.1])

(a) *The probability measure* $\mu_{\mathrm{can}} = \Delta_x(\frac{1}{2}r(x,y)) + \delta_y(x)$ *is independent of* $y \in \Gamma$.

(b) μ_{can} *is the unique measure* μ *of total mass* 1 *on* Γ *for which* $g_\mu(x,x)$ *is a constant independent of* x.

(c) *There is a constant* $\tau(\Gamma) \in \mathbb{R}$ *such that* $g_{\mu_{\mathrm{can}}}(x,y) = -\frac{1}{2}r(x,y) + \tau(\Gamma)$.

Since $r(x,x) = 0$ for every $x \in \Gamma$, the diagonal values $g_{\mu_{\mathrm{can}}}(x,x)$ are constant on Γ, and are equal to the tau constant $\tau(\Gamma)$.

Next, we state the main result of this paper.

THEOREM 4.3

Suppose that $e_i \in E(\Gamma)$ *is an edge of length* L_i *with endpoints* p_i *and* q_i, *and suppose that* $e_j \in E(\Gamma)$ *is an edge of length* L_j *with endpoints* p_j *and* q_j. *Assume that the edges* e_i *and* e_j *are not bridges and distinct edges, but their endpoints are not necessarily distinct. If* x *belongs to the edge* e_i *and* y *belongs to the edge* e_j, *we have*

$$\begin{aligned} g_{\mu_{\mathrm{can}}}(x,y) = \tau(\Gamma) + x^2\frac{L_i - r(p_i,q_i)}{2L_i^2} + y^2\frac{L_j - r(p_j,q_j)}{2L_j^2} \\ - \frac{xy}{L_iL_j}\big(j_{p_j}(p_i,q_j) - j_{p_j}(q_i,q_j)\big) - \frac{x}{2L_i}\big(L_i - 2j_{p_i}(q_i,p_j)\big) \\ - \frac{y}{2L_j}\big(L_j - 2j_{p_j}(p_i,q_j)\big) - \frac{1}{2}r(p_i,p_j). \end{aligned}$$

If both x *and* y *belong to the same edge* e_i *of length* L_i *with endpoints* p_i *and* q_i, *then we have*

$$g_{\mu_{\mathrm{can}}}(x,y) = \tau(\Gamma) - \frac{1}{2}|x-y| + (x-y)^2\frac{L_i - r(p_i,q_i)}{2L_i^2}.$$

Proof
The result follows from Theorem 4.2 along with Lemma 3.1 and Theorem 3.3. □

If any of the involved edges in Theorem 4.3 is a bridge, then we interpret the given formulas by using Remark 3.4. In such cases, we obtain the following modified version of Theorem 4.3 by applying Theorem 4.2 and Remark 3.4.

THEOREM 4.4

Suppose that $e_i \in E(\Gamma)$ *is an edge of length* L_i *with endpoints* p_i *and* q_i, *and suppose that* $e_j \in E(\Gamma)$ *is an edge of length* L_j *with endpoints* p_j *and* q_j. *Let* x *and* y *belong to* e_i *and* e_j, *respectively.*

(a) *If both* x *and* y *are on the same edge that is a bridge, then we have*

$$g_{\mu_{\mathrm{can}}}(x,y) = \tau(\Gamma) - \frac{1}{2}|x-y|.$$

(b) *If both e_i and e_j are bridges that are distinct edges, then we have that*

$$g_{\mu_{\mathrm{can}}}(x,y)=\begin{cases}\tau(\Gamma)-\frac{1}{2}(x+y+r(p_i,p_j)) \\ \quad \textit{if } p_i \textit{ and } p_j \textit{ are between } x \textit{ and } y, \\ \tau(\Gamma)-\frac{1}{2}(x+L_j-y+r(p_i,q_j)) \\ \quad \textit{if } p_i \textit{ and } q_j \textit{ are between } x \textit{ and } y, \\ \tau(\Gamma)-\frac{1}{2}(L_i-x+y+r(p_j,q_i)) \\ \quad \textit{if } p_j \textit{ and } q_i \textit{ are between } x \textit{ and } y, \\ \tau(\Gamma)-\frac{1}{2}(L_i-x+L_j-y+r(q_i,q_j)) \\ \quad \textit{if } q_i \textit{ and } q_j \textit{ are between } x \textit{ and } y.\end{cases}$$

(c) *Suppose that only e_i is a bridge. (The case in which only e_j is a bridge can be proved similarly.) Then we have two cases. If y is on the side of p_i, then we have*

$$g_{\mu_{\mathrm{can}}}(x,y)=\tau(\Gamma)+y^2\frac{L_j-r(p_j,q_j)}{2L_j^2}-y\frac{L_j-r(p_j,q_j)+r(p_i,q_j)-r(p_i,p_j)}{2L_j}$$
$$-\frac{1}{2}\big(x+r(p_i,p_j)\big).$$

If y is on the side of q_i, then we have

$$g_{\mu_{\mathrm{can}}}(x,y)=\tau(\Gamma)+y^2\frac{L_j-r(p_j,q_j)}{2L_j^2}-y\frac{L_j-r(p_j,q_j)+r(q_i,q_j)-r(q_i,p_j)}{2L_j}$$
$$-\frac{1}{2}\big(L_i-x+r(q_i,p_j)\big).$$

Recall that $g_{\mu_{\mathrm{can}}}(x,y)$ is a symmetric function, that is, $g_{\mu_{\mathrm{can}}}(x,y)=g_{\mu_{\mathrm{can}}}(y,x)$, and recall that it is continuous in x and y. It is clear from Theorem 4.3 that $g_{\mu_{\mathrm{can}}}(x,y)$ is a piecewisely defined function on each pair of edges (e_i,e_j). Based on this information about $g_{\mu_{\mathrm{can}}}(x,y)$ and Theorem 4.3, we suggest that a matrix Z defined below can be used to describe $g_{\mu_{\mathrm{can}}}(x,y)$. We call Z the *value matrix* of $g_{\mu_{\mathrm{can}}}(x,y)$.

We define $\mathrm{Z}=(z_{ij})$ as a matrix of size $e\times e$, where e is the number of edges of Γ, such that z_{ij} is equal to $g_{\mu_{\mathrm{can}}}(x,y)$ when $x\in e_i$ and $y\in e_j$. We note that Z is a symmetric matrix. The diagonal values of Z are of the form $a_2(x-y)^2+a_1(x-y)+a_0$ for some constants a_0, a_1, and a_2, where $a_2=0$ if and only if the edge that x and y belong to is a bridge. Other entries of Z are of the form $ax^2+by^2+cxy+dx+ey+f$ for some constants a, b, c, d, e, and f, where e_i is a bridge if and only if $a=0$ and where e_j is a bridge if and only if $b=0$. We provide various examples in Section 6.

5. Arakelov–Green function $g_{\mu_D}(x,y)$

In this section, we consider another important Arakelov–Green function $g_{\mu_D}(x,y)$ defined by Zhang [13, Section 3] as the generalization of $g_{\mu_{\mathrm{can}}}(x,y)$. Here, $g_{\mu_D}(x,y)$ is defined with respect to the measure $\mu_D(x)$, where D is a divisor on Γ. More precisely, for any divisor $D=\sum_{q\in V(\Gamma)} a_q \cdot q$ on Γ with $\deg(D)\neq -2$ and for the corresponding measure (called the admissible metric on Γ with respect to D)

$$\mu_D(x)=\frac{1}{\deg(D)+2}\Big(\sum_{q\in V(\Gamma)} a_q\delta_q(x)+2\mu_{\mathrm{can}}(x)\Big),$$

$g_{\mu_D}(x,y)$ can be given as follows (see [10, Section 4.4]):

$$(7)\qquad g_{\mu_D}(x,y)=\frac{1}{\deg(D)+2}\Big(\sum_{s\in V(\Gamma)} a_s\cdot j_s(x,y)+4\tau(\Gamma)-r(x,y)\Big)-c_{\mu_D},$$

where

$$c_{\mu_D}=\frac{1}{2(\deg(D)+2)^2}\Big(8\tau(\Gamma)\big(\deg(D)+1\big)+\sum_{q,s\in V(\Gamma)} a_q\cdot a_s\cdot r(q,s)\Big).$$

Note that $g_{\mu_D}(x,y)=g_{\mu_{\mathrm{can}}}(x,y)$ and $\mu_D(x)=\mu_{\mathrm{can}}$ if $D=0$.

Using Theorem 3.3, Lemma 3.1, Remark 3.5, and (7), one can extend Theorems 4.3 and 4.4 to a formula for $g_{\mu_D}(x,y)$.

6. Computational examples for $g_{\mu_{\mathrm{can}}}(x,y)$

We first give two examples for symbolic computations, and then an example with numerical computations. Given a metrized graph Γ with discrete Laplacian L, we first compute the pseudoinverse L^+ of L. We can compute the tau constant $\tau(\Gamma)$ symbolically for certain graphs or numerically for all graphs by using L and L^+ as shown in [6, Theorem 1.1]. Then we compute the resistance matrix R using the matrix L^+ along with Lemma 6.1 given below. Finally, we compute the value matrix Z using either Theorem 4.3 or Theorem 4.4.

We first recall that both voltage and resistance values on vertices can be expressed in terms of the entries of the pseudoinverse L^+ of L (see [5, Lemmas 3.4 and 3.5] and the related references therein).

LEMMA 6.1

For any p, q, s in $V(\Gamma)$, we have

$$r(p,q)=l^+_{pp}-2l^+_{pq}+l^+_{qq} \qquad \textit{and} \qquad j_p(q,s)=l^+_{pp}-l^+_{pq}-l^+_{ps}+l^+_{qs}.$$

Suppose that J_e denotes an $e\times e$ matrix where each entry is equal to 1.

EXAMPLE 6.1

Let Γ be the circle graph with three vertices as illustrated in Figure 4. The total length of Γ is $\ell(\Gamma)=a+b+c$. We have $\tau(\Gamma)=\ell(\Gamma)/12$ and the following discrete Laplacian matrix L, its pseudoinverse L^+, the resistance matrix R, and the value matrix Z with respect to the ordered endpoints of edges $\{(v_1,v_2),(v_1,v_3),(v_2,v_3)\}$:

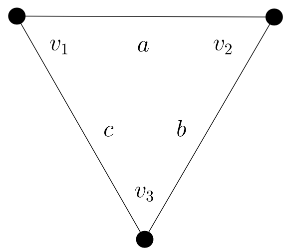

Figure 4. A circle graph with vertices $\{v_1, v_2, v_3\}$ and edge lengths $\{a, b, c\}$.

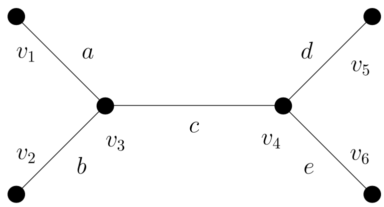

Figure 5. A tree graph with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge lengths $\{a, b, c, d, e\}$.

$$\mathrm{L} = \begin{bmatrix} 1/a+1/b & -1/a & -1/b \\ -1/a & 1/a+1/c & -1/c \\ -1/b & -1/c & 1/b+1/c \end{bmatrix},$$

$$\mathrm{R} = \frac{1}{\ell(\Gamma)} \begin{bmatrix} 0 & ab+ac & ab+bc \\ ab+ac & 0 & ac+bc \\ ab+bc & ac+bc & 0 \end{bmatrix},$$

$$\mathrm{L}^{+} = \frac{1}{9\ell(\Gamma)} \begin{bmatrix} bc+a(4b+c) & bc-2a(b+c) & -2bc+a(-2b+c) \\ bc-2a(b+c) & bc+a(b+4c) & a(b-2c)-2bc \\ -2bc+a(-2b+c) & a(b-2c)-2bc & 4bc+a(b+c) \end{bmatrix},$$

$$\begin{aligned} \mathrm{Z} &= \tau(\Gamma) J_3 \\ &\quad - \frac{1}{2\ell(\Gamma)} \begin{bmatrix} -(x-y)^2+(a+b+c)|x-y| & (a+b+c-x-y)(x+y) & (b+c+x-y)(a-x+y) \\ (a+b+c-x-y)(x+y) & -(x-y)^2+(a+b+c)|x-y| & (b+c-x-y)(a+x+y) \\ (b+c+x-y)(a-x+y) & (b+c-x-y)(a+x+y) & -(x-y)^2+(a+b+c)|x-y| \end{bmatrix}. \end{aligned}$$

Since Γ has no bridge, each entry of Z is a quadratic function in both x and y.

EXAMPLE 6.2

Let Γ be the tree graph as given in Figure 5. The list of the ordered end-points of the edges is $\{(v_1, v_3), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_4, v_6)\}$, and the list of the corresponding edge lengths in order is given by $\{a,b,c,d,e\}$. In this case, $\tau(\Gamma)=\frac{1}{4}(a+b+c+d+e)$, and the Laplacian matrix L and the resistance matrix R are given as follows:

$$\mathrm{L}=\begin{bmatrix} 1/a & 0 & -1a & 0 & 0 & 0 \\ 0 & 1/b & -1/b & 0 & 0 & 0 \\ -1/a & -1/b & 1/a+1/b+1/c & -1/c & 0 & 0 \\ 0 & 0 & -1/c & 1/c+1/d+1/e & -1/d & -1/e \\ 0 & 0 & 0 & -1/d & 1/d & 0 \\ 0 & 0 & 0 & -1/e & 0 & 1/e \end{bmatrix},$$

$$\mathrm{R}=\begin{bmatrix} 0 & a+b & a & a+c & a+c+d & a+c+e \\ a+b & 0 & b & b+c & b+c+d & b+c+e \\ a & b & 0 & c & c+d & c+e \\ a+c & b+c & c & 0 & d & e \\ a+c+d & b+c+d & c+d & d & 0 & d+e \\ a+c+e & b+c+e & c+e & e & d+e & 0 \end{bmatrix},$$

$$\mathrm{Z}=\tau(\Gamma)J_5 - \frac{1}{2}\begin{bmatrix} |x-y| & a-x+b-y & a-x+y & a-x+c+y & a-x+c+y \\ a-x+b-y & |x-y| & b-x+y & b-x+c+y & b-x+c+y \\ a-x+y & b-x+y & |x-y| & c-x+y & c-x+y \\ a-x+c+y & b-x+c+y & c-x+y & |x-y| & x+y \\ a-x+c+y & b-x+c+y & c-x+y & x+y & |x-y| \end{bmatrix}.$$

We consider the following cases to clarify how we use the value matrix Z.

If $x,y\in e_1$, then $g_{\mu_{\mathrm{can}}}(x,y)=\tau(\Gamma)-\frac{1}{2}|x-y|$, where $0\le x\le a$, $0\le y\le a$, and v_1 corresponds to 0.

If $x,y\in e_3$, then $g_{\mu_{\mathrm{can}}}(x,y)=\tau(\Gamma)-\frac{1}{2}|x-y|$, where $0\le x\le c$, $0\le y\le c$, and v_3 corresponds to 0.

If $x\in e_2$ and $y\in e_4$, then $g_{\mu_{\mathrm{can}}}(x,y)=\tau(\Gamma)-\frac{1}{2}(b-x+c+y)$, where $0\le x\le b$, $0\le y\le d$, and both v_2 and v_4 correspond to 0.

Note that each entry of Z is a linear function in both x and y because of the fact that Γ is a tree, that is, has no bridges.

EXAMPLE 6.3

In this example, we consider the tetrahedral graph with edge lengths given as in Figure 6.

In this case, we have $\tau(\Gamma)=5/16$. The discrete Laplacian matrix L, its pseudoinverse of L^+, and the resistance matrix R are as follows:

$$\mathrm{L}=\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \qquad \mathrm{L}^+=\frac{1}{48}\begin{bmatrix} 19 & 7 & 7 & 7 \\ 7 & 19 & 7 & 7 \\ 7 & 7 & 19 & 7 \\ 7 & 7 & 7 & 19 \end{bmatrix},$$

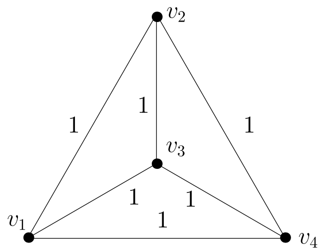

Figure 6. Tetrahedral graph with vertices $\{v_1, v_2, v_3, v_4\}$.

$$\mathrm{R} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

If the ordered endpoints of edges are $\{(v_1,v_2),(v_1,v_3),(v_1,v_4),(v_2,v_3),(v_2,v_4),(v_3,v_4)\}$, then we have the value matrix $\mathrm{Z} = \frac{5}{16}J_6 - \frac{1}{4}[C_1,C_2,C_3,C_4,C_5,C_6]$, where the C_i's with $i \in \{1,2,3,4,5,6\}$ are columns as given below:

$$[C_1,C_2,C_3] = \begin{bmatrix} -x^2+2xy-y^2+2|x-y| & 2x-x^2+2y-xy-y^2 & 2x-x^2+2y-xy-y^2 \\ 2x-x^2+2y-xy-y^2 & -x^2+2xy-y^2+2|x-y| & 2x-x^2+2y-xy-y^2 \\ 2x-x^2+2y-xy-y^2 & 2x-x^2+2y-xy-y^2 & -x^2+2xy-y^2+2|x-y| \\ 1-x^2+y+xy-y^2 & 1+x-x^2+y-xy-y^2 & 1+x-x^2+y-y^2 \\ 1-x^2+y+xy-y^2 & 1+x-x^2+y-y^2 & 1+x-x^2+y-xy-y^2 \\ 1+x-x^2+y-y^2 & 1-x^2+y+xy-y^2 & 1+x-x^2+y-xy-y^2 \end{bmatrix},$$

$$[C_4,C_5,C_6] = \begin{bmatrix} 1-x^2+y+xy-y^2 & 1-x^2+y+xy-y^2 & 1+x-x^2+y-y^2 \\ 1+x-x^2+y-xy-y^2 & 1+x-x^2+y-y^2 & 1-x^2+y+xy-y^2 \\ 1+x-x^2+y-y^2 & 1+x-x^2+y-xy-y^2 & 1+x-x^2+y-xy-y^2 \\ -x^2+2xy-y^2+2|x-y| & 2x-x^2+2y-xy-y^2 & 1-x^2+y+xy-y^2 \\ 2x-x^2+2y-xy-y^2 & -x^2+2xy-y^2+2|x-y| & 1+x-x^2+y-xy-y^2 \\ 1-x^2+y+xy-y^2 & 1+x-x^2+y-xy-y^2 & -x^2+2xy-y^2+2|x-y| \end{bmatrix}.$$

Note that each entry of Z is a quadratic function in both x and y as expected, because Γ has no bridge.

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