Multiplicity of solutions for Neumann problems resonant at any eigenvalue

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Abstract   We consider a semilinear Neumann problem with a reaction which is resonant both at $\pm\infty$ and at zero with respect to any eigenvalue, possibly the same one. Using the reduction method and Morse theory, we show that the problem has at least two nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Neumann problem:

$$(1.1) \qquad \begin{cases} -\Delta u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Our goal is to prove a multiplicity theorem when resonance can occur with respect to any eigenvalue of the negative Neumann Laplacian (hereafter denoted by $-\Delta^N$) both at zero and at $\pm\infty$. Recently this situation was investigated by Gasiński–Papageorgiou [6], who assumed that

$$\lim_{|\zeta| \to +\infty} \frac{f(z, \zeta)}{\zeta}$$

exists and belongs to the spectrum of $-\Delta^N$ and that the eigenvalues with respect to which resonance occurs at infinity and at zero are different. No such restrictions are imposed here. Other multiplicity results for resonant Neumann problems can be found in work from Filippakis–Papageorgiou [5] (resonance at infinity with respect to the principal eigenvalue $\lambda_0 = 0$) and Tang [12] and Tang–Wu [13] (resonance occurring only at zero).

Our approach uses the reduction method due to Amann [2] and Morse theory.

2. Mathematical background

Let X be a Banach space, and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). For a given $\varphi \in C^1(X)$, we say that φ satisfies the Palais–Smale condition if every sequence $\{x_n\}_{n \geq 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(x_n) \longrightarrow 0$ in X^*, admits a strongly convergent subsequence. For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \big\{ x \in X : \varphi(x) \leq c \big\},$$
$$K_\varphi = \big\{ x \in X : \varphi'(x) = 0 \big\},$$
$$K_\varphi^c = \big\{ x \in K_\varphi : \varphi(x) = c \big\}.$$

If (Y_1, Y_2) is a topological pair with $Y_2 \subseteq Y_1 \subseteq X$, then for every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group with integer coefficients. The critical groups of φ at an isolated $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\} \big) \quad \forall k \geq 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that this definition is independent of the particular choice of the neighborhood U. Suppose that $\varphi \in C^1(X)$ satisfies the Palais–Smale condition, and assume that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geq 0.$$

The second deformation theorem (see, e.g., Papageorgiou–Kyritsi [11, p. 349]) implies that the above definition is independent of the particular level $c < \inf \varphi(K_\varphi)$. Suppose that K_φ is finite. We set

$$M(t, x) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \forall t \in \mathbb{R}, x \in K_\varphi,$$

$$P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \forall t \in \mathbb{R}.$$

We have the Morse relation, which says that

(2.1) $$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t),$$

where

$$Q(t) = \sum_{k \geq 1} \beta_k t^k$$

is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k (see Chang [3]).

3. Multiplicity theorem

In what follows, by $\{\widehat{\lambda}_k\}_{k\geq 0}$ we denote the distinct eigenvalues of $-\Delta^N$. Recall that $\widehat{\lambda}_0 = 0$ is the principal eigenvalue. By $E(\widehat{\lambda}_k)$ we denote the corresponding eigenspace.

The hypotheses on the reaction $f(z,\zeta)$ are the following:

<u>H</u>: $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that $f(z,0)=0$ and $f(z,\cdot)\in C^1(\mathbb{R})$ for almost all $z\in\Omega$ and

(i)   there exists $\eta\in L^\infty(\Omega)_+$, $\eta(z)\leq\widehat{\lambda}_{m+1}$ for almost all $z\in\Omega$, $\eta\neq\widehat{\lambda}_{m+1}$ with $m\geq 0$, such that

$$\left|f'_\zeta(z,\zeta)\right|\leq\eta(z)\quad\text{for almost all }z\in\Omega,\text{ all }\zeta\in\mathbb{R};$$

(ii)   we have

$$\widehat{\lambda}_m\leq\liminf_{\zeta\to\pm\infty}\frac{f(z,\zeta)}{\zeta}\quad\text{uniformly for almost all }z\in\Omega$$

and if

$$F(z,\zeta)=\int_0^\zeta f(z,s)\,ds,$$

then

$$f(z,\zeta)\zeta-2F(z,\zeta)\longrightarrow-\infty\quad\text{as }\zeta\to\pm\infty$$

uniformly for almost all $z\in\Omega$;

(iii) there exist integer $l\geq 0$ and $\delta>0$ such that

$$\widehat{\lambda}_l\leq\frac{f(z,\zeta)}{\zeta}\leq\widehat{\lambda}_{l+1}\quad\text{for almost all }z\in\Omega,\text{ all }0<|\zeta|\leq\delta.$$

REMARK 3.1

From hypotheses $H(\mathrm{i})$ and $H(\mathrm{ii})$ and the mean value theorem, we have

$$\widehat{\lambda}_m\leq\liminf_{\zeta\to\pm\infty}\frac{f(z,\zeta)}{\zeta}\leq\limsup_{\zeta\to\pm\infty}\frac{f(z,\zeta)}{\zeta}\leq\eta(z)$$

uniformly for almost all $z\in\Omega$. So resonance can occur at $\pm\infty$ with respect to any eigenvalue $\widehat{\lambda}_m$, $m\geq 0$. However, note that we do not require that the quotient $f(z,\zeta)/\zeta$ necessarily has a limit as $\zeta\longrightarrow\pm\infty$.

Similarly, hypothesis $H(\mathrm{iii})$ permits resonance at zero to any eigenvalue $\widehat{\lambda}_l$, $l\geq 0$, which need not be distinct from $\widehat{\lambda}_m$.

Let $\varphi\colon H^1(\Omega)\longrightarrow\mathbb{R}$ be the energy functional for problem (1.1), defined by

$$\varphi(u)=\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega F\big(z,u(z)\big)\,dz\quad\forall u\in H^1(\Omega).$$

Evidently $\varphi \in C^2(H^1(\Omega))$. Also, we consider the following orthogonal direct sum decomposition of $H^1(\Omega)$:

$$H^1(\Omega) = Y \oplus \widehat{H} \quad \text{with } Y = \bigoplus_{i=0}^{m} E(\widehat{\lambda}_i), \widehat{H} = Y^\perp.$$

PROPOSITION 3.2

If hypotheses H hold, then there exists a C^1-map $\vartheta \colon Y \longrightarrow \widehat{H}$ such that

$$\inf_{\widehat{u} \in \widehat{H}} \varphi(y + \widehat{u}) = \varphi(y + \vartheta(y)) \quad \forall y \in Y.$$

Proof
We fix $y \in Y$, and let $\varphi_y \colon H^1(\Omega) \longrightarrow \mathbb{R}$ be the C^2-functional defined by

$$\varphi_y(u) = \varphi(y + u) \quad \forall u \in H^1(\Omega).$$

Let $i \colon \widehat{H} \longrightarrow H^1(\Omega)$ be the inclusion map, and let $\widehat{\varphi}_y \colon \widehat{H} \longrightarrow \mathbb{R}$ be defined by

$$\widehat{\varphi}_y = \varphi_y \circ i.$$

Recall that $i^* = p_{\widehat{H}^*}$ (the orthogonal projection on $\widehat{H}^*$). From the chain rule, we have

$$\widehat{\varphi}_y'(\widehat{u}) = p_{\widehat{H}^*} \varphi_y'(\widehat{u}) = p_{\widehat{H}^*}\big(A(y + \widehat{u}) - N_f(y + \widehat{u})\big),$$

where $A \in \mathcal{L}(H^1(\Omega); H^1(\Omega)^*)$ is defined by

$$\big\langle A(u), v \big\rangle = \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^N} \, dz \quad \forall u, v \in H^1(\Omega)$$

and

$$N_f(u)(\cdot) = f\big(\cdot, u(\cdot)\big) \quad \forall u \in H^1(\Omega).$$

Let $\langle \cdot, \cdot \rangle_{\widehat{H}}$ denote the duality brackets for the pair $(\widehat{H}^*, \widehat{H})$, and let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(H^1(\Omega)^*, H^1(\Omega))$. For $\widehat{u}_1, \widehat{u}_2 \in \widehat{H}$, we have

$$\big\langle \widehat{\varphi}_y'(\widehat{u}_1) - \widehat{\varphi}_y'(\widehat{u}_2), \widehat{u}_1 - \widehat{u}_2 \big\rangle_{\widehat{H}}$$

(3.1)
$$= \big\langle A(\widehat{u}_1) - A(\widehat{u}_2), \widehat{u}_1 - \widehat{u}_2 \big\rangle$$

$$- \int_\Omega \big(f(z, y + \widehat{u}_1) - f(z, y + \widehat{u}_2)\big)(\widehat{u}_1 - \widehat{u}_2) \, dz.$$

Hypothesis $H_1(\mathrm{i})$ and the mean value theorem imply that

$$\big| f\big(z, (y + \widehat{u}_1)(z)\big) - f\big(z, (y + \widehat{u}_2)(z)\big) \big| \leq \eta(z) \big| \widehat{u}_1(z) - \widehat{u}_2(z) \big|$$

for almost all $z \in \Omega$. Using this in (3.1), we obtain

(3.2)
$$\big\langle \widehat{\varphi}_y'(\widehat{u}_1) - \widehat{\varphi}_y'(\widehat{u}_2), \widehat{u}_1 - \widehat{u}_2 \big\rangle_{\widehat{H}} \geq \big\| \nabla(\widehat{u}_1 - \widehat{u}_2) \big\|_2^2 - \int_\Omega \eta(z)|\widehat{u}_1 - \widehat{u}_2|^2 \, dz$$

$$\geq \xi_0 \|\widehat{u}_1 - \widehat{u}_2\|^2$$

for some $\xi_0 > 0$ (see Gasiński–Papageorgiou [6, Proposition 2.5]), so $\widehat{\varphi}_y$ is strictly convex.

Also, we have

$$(3.3) \qquad \langle \widehat{\varphi}'_y(\widehat{u}), \widehat{u} \rangle = \langle \widehat{\varphi}'_y(\widehat{u}) - \widehat{\varphi}'_y(0), \widehat{u} \rangle + \langle \widehat{\varphi}'_y(0), \widehat{u} \rangle \geq \xi_0 \|\widehat{u}\|^2 - c_1 \|\widehat{u}\|,$$

for some $c_1 > 0$ (see (3.1)), so

$$(3.4) \qquad \qquad \text{the map } \widehat{u} \longmapsto \widehat{\varphi}'_y(\widehat{u}) \text{ is coercive on } \widehat{H}.$$

But $\widehat{\varphi}'_y$ is maximal monotone (see Filippakis–Papageorgiou [5]). Hence, it is surjective (see Papageorgiou–Kyritsi [11, Corollary 3.2.28, p. 172]). So, we can find $\widehat{u}_0 \in \widehat{H}$ such that

$$\widehat{\varphi}'_y(\widehat{u}_0) = 0.$$

From the strict convexity of $\widehat{\varphi}_y$, it follows that $\widehat{u}_0$ is the unique global minimizer of $\widehat{\varphi}_y$. Let $\vartheta \colon Y \longrightarrow \widehat{H}$ be the map that assigns to each $y \in Y$ the unique minimizer $\widehat{u}_0 = \widehat{u}_0(y)$ of $\widehat{\varphi}_y$. We have

$$(3.5) \quad \widehat{\varphi}'_y\big(\vartheta(y)\big) = p_{\widehat{H}^*} \varphi'\big(y + \vartheta(y)\big) \qquad \text{and} \qquad \varphi\big(y + \vartheta(y)\big) = \inf_{\widehat{u} \in \widehat{H}} \varphi(y + \widehat{u}).$$

Let $\sigma \colon Y \times \widehat{H} \longrightarrow \widehat{H}^*$ be defined by

$$(3.6) \qquad \qquad \sigma(y, \widehat{u}) = p_{\widehat{H}^*} \varphi'(y + \widehat{u}) \quad \forall (y, \widehat{u}) \in Y \times \widehat{H}.$$

If $\sigma(y_0, \widehat{u}_0) = 0$, then from (3.5), we have that $\widehat{u}_0 = \vartheta(y_0)$. Recall that $\varphi \in C^2(H^1(\Omega))$. So, from (3.6), it is clear that $\sigma \in C^1(Y \times \widehat{H}, \widehat{H}^*)$, and for all $\widehat{u} \in \widehat{H}$, we have

$$\begin{aligned}
\big\langle \sigma'_{\widehat{u}}(y_0, \widehat{u}_0)\widehat{u}, \widehat{u} \big\rangle_{\widehat{H}} &= \big\langle p_{\widehat{H}^*} \varphi''(y_0 + \widehat{u}_0)\widehat{u}, \widehat{u} \big\rangle_{\widehat{H}} \\
&= \big\langle \varphi''(y_0 + \widehat{u}_0)\widehat{u}, \widehat{u} \big\rangle \\
&= \|\nabla \widehat{u}\|_2^2 - \int_\Omega f'_\zeta(z, y_0 + \widehat{u}_0)\widehat{u}^2 \, dz \\
&\geq \|\nabla \widehat{u}\|_2^2 - \int_\Omega \eta \widehat{u}^2 \, dz \\
&\geq \xi_0 \|\widehat{u}\|^2,
\end{aligned}$$

where we used the orthogonality of Y and $\widehat{H}$, hypothesis $H(\mathrm{i})$, and [6, Proposition 2.5]. So $\sigma'_{\widehat{u}}(y_0, \widehat{u}_0) \in \mathcal{L}(\widehat{H}, \widehat{H}^*)$ is an isomorphism.

Invoking the implicit function theorem, we conclude that $\vartheta \in C^1(Y; \widehat{H})$. $\qquad \square$

Let $\overline{\varphi} \colon Y \longrightarrow \mathbb{R}$ be defined by

$$(3.7) \qquad \qquad \overline{\varphi}(y) = \varphi\big(y + \vartheta(y)\big) \quad \forall y \in Y.$$

PROPOSITION 3.3

If hypotheses H hold, then $\overline{\varphi}$ is anticoercive; that is, if $\|y\| \longrightarrow \infty$, then $\overline{\varphi}(y) \longrightarrow -\infty$.

Proof

We argue by contradiction. So, suppose that the result is not true. Then we can find $M > 0$ and a sequence $\{y_n\}_{n \geq 1} \subseteq Y$ such that

$$\|y_n\| \longrightarrow +\infty \qquad \text{and} \qquad -M \leq \overline{\varphi}(y_n) \quad \forall n \geq 1.$$

From the definition of $\overline{\varphi}$ (see (3.7)), we have

(3.8)
$$-M \leq \overline{\varphi}(y_n) \leq \varphi(y_n) = \frac{1}{2}\|\nabla y_n\|_2^2 - \int_\Omega F(z, y_n)\, dz$$

$$\leq \int_\Omega \Big(\frac{1}{2}\lambda_m y_n^2 - F(z, y_n)\Big)\, dz$$

since $y_n \in Y$. Let

$$\widehat{f}(z, \zeta) = f(z, \zeta) - \widehat{\lambda}_m \zeta,$$

and let

$$\widehat{F}(z, \zeta) = \int_0^\zeta \widehat{f}(z, s)\, ds.$$

Then

$$0 \leq \liminf_{\zeta \to \pm\infty} \frac{\widehat{f}(z, \zeta)}{\zeta} \leq \limsup_{\zeta \to \pm\infty} \frac{\widehat{f}(z, \zeta)}{\zeta} \leq \eta(z) - \widehat{\lambda}_m$$

uniformly for almost all $z \in \Omega$, so

(3.9)
$$0 \leq \liminf_{\zeta \to \pm\infty} \frac{2\widehat{F}(z, \zeta)}{\zeta^2} \leq \limsup_{\zeta \to \pm\infty} \frac{2\widehat{F}(z, \zeta)}{\zeta^2} \leq \eta(z) - \widehat{\lambda}_m$$

uniformly for almost all $z \in \Omega$ (see [1, Remark 26]).

Also, we have (see hypothesis $H(\mathrm{ii})$)

(3.10) $\qquad \lim_{\zeta \to \pm\infty} \big(\widehat{f}(z, \zeta)\zeta - 2\widehat{F}(z, \zeta)\big) = -\infty \quad$ uniformly for almost all $z \in \Omega$.

Let

$$v_n = \frac{y_n}{\|y_n\|} \quad \forall n \geq 1.$$

Then

$$\|v_n\| = 1 \quad \forall n \geq 1,$$

and by virtue of the finite dimensionality of Y, passing to a subsequence if necessary, we may assume that

$$v_n \longrightarrow v \quad \text{in } H^1(\Omega),$$

with $\|v\| = 1$. Hence,

$$\big|\{v \neq 0\}\big|_N > 0,$$

where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$, and

$$\big|y_n(z)\big| \longrightarrow +\infty \quad \text{for almost all } z \in \{v \neq 0\}.$$

By virtue of (3.10), for a given $\beta > 0$, we can find $\widehat{M} = \widehat{M}(\beta) > 0$ such that

$$(3.11) \qquad \widehat{f}(z,\zeta)\zeta - 2\widehat{F}(z,\zeta) \leq -\beta \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \geq \widehat{M}.$$

For almost all $z \in \Omega$ and all $\zeta \geq \widehat{M}$, we have (see (3.20))

$$(3.12) \qquad \frac{d}{d\zeta}\frac{\widehat{F}(z,\zeta)}{\zeta^2} = \frac{\widehat{f}(z,\zeta)\zeta - 2\widehat{F}(z,\zeta)}{\zeta^3} \leq -\frac{\beta}{\zeta^3}.$$

Integrating (3.12) on $[\zeta, y]$ (with $\widehat{M} \leq \zeta \leq y$), we have

$$\frac{\widehat{F}(z,y)}{y^2} - \frac{\widehat{F}(z,\zeta)}{\zeta^2} \leq \frac{\beta}{2}\Big(\frac{1}{y^2} - \frac{1}{\zeta^2}\Big) \quad \text{for almost all } z \in \Omega.$$

We let $y \longrightarrow +\infty$ and obtain (see (3.9))

$$-\frac{\widehat{F}(z,\zeta)}{\zeta^2} \leq -\frac{\beta}{2\zeta^2} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geq \widehat{M},$$

so

$$F(z,\zeta) \geq \frac{\beta}{2} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geq \widehat{M}.$$

Since $\beta > 0$ is arbitrary, we conclude that $F(z,\zeta) \longrightarrow +\infty$ as $\zeta \to +\infty$ uniformly for almost all $z \in \Omega$.

Similarly, we show that $\widehat{F}(z,\zeta) \longrightarrow +\infty$ as $\zeta \to -\infty$. Then, from (3.8), we have

$$-M \leq \overline{\varphi}(y_n) \leq -\int_\Omega \widehat{F}(z, y_n)\, dz \longrightarrow -\infty$$

(recall that $|y_n(z)| \to +\infty$ for almost all $z \in \{v \neq 0\}$), a contradiction. $\qquad \square$

We assume that $K_{\overline{\varphi}}$ is finite. (Otherwise we already have a whole sequence of nontrivial solutions of (1.1) and so we are done.) From Proposition 3.3, we know that $-\overline{\varphi}$ is coercive; hence, it satisfies the Palais–Smale condition. Therefore, $\overline{\varphi}$ satisfies the Palais–Smale condition, and we can consider the critical groups of $\overline{\varphi}$ at infinity. In fact, using ideas from Liu [9], we can compute exactly these critical groups of $\overline{\varphi}$.

PROPOSITION 3.4
If hypotheses H hold, then

$$C_k(\overline{\varphi}, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geq 0,$$

with $d_m = \dim Y \geq 1$.

Proof
Let $m_0 < \inf \overline{\varphi}(K_{\overline{\varphi}})$. Since $\overline{\varphi}$ is anticoercive (see Proposition 3.3), we can find $\gamma < \beta < m_0$ and $0 < \varrho < R$ such that

$$C_r \subseteq \overline{\varphi}^\gamma \subseteq C_\varrho \subseteq \overline{\varphi}^\beta,$$

where, for every $r > 0$, $C_r = \{y \in Y : \|y\| \geq r\}$.

For the triples (C_R, C_ϱ, Y) and $(\overline{\varphi}^\gamma, \overline{\varphi}^\beta, Y)$ we consider the corresponding long exact sequences of homology groups. So, we have

$$\vdots \qquad\qquad \vdots$$

$$H_k(C_\varrho, C_R) \xrightarrow{\ h_*|_{C_\varrho}\ } H_k(\overline{\varphi}^\beta, \overline{\varphi}^\gamma)$$

$$\downarrow i^* \qquad\qquad \downarrow \widehat{i}_*$$

$$H_k(Y, C_R) \xrightarrow{\ h_*\ } H_k(Y, \overline{\varphi}^\gamma)$$

(3.13)
$$\downarrow j_* \qquad\qquad \downarrow \widehat{j}_*$$

$$H_k(Y, C_\varrho) \xrightarrow{\ h_*\ } H_k(Y, \overline{\varphi}^\beta)$$

$$\downarrow \partial_* \qquad\qquad \downarrow \widehat{\partial}_*$$

$$H_{k-1}(C_\varrho, C_R) \xrightarrow{\ h_*|_{C_\varrho}\ } H_{k-1}(\overline{\varphi}^\beta, \overline{\varphi}^\gamma)$$

$$\vdots \qquad\qquad \vdots$$

In (3.13) all squares are commutative (see Granas–Dugundji [7, p. 377]) and the maps $i_*, j_*, \widehat{i}_*, \widehat{j}_*, h_*$ are the group homomorphisms induced by the corresponding inclusion maps. Finally, ∂_* and $\widehat{\partial}_*$ are the corresponding boundary homomorphisms.

Since $\gamma < \beta < m_0 < \inf \overline{\varphi}(K_{\overline{\varphi}})$, we have

(3.14)
$$H_k(\overline{\varphi}^\beta, \overline{\varphi}^\gamma) = 0 \quad \forall k \geq 0$$

by the second deformation theorem (see Papageorgiou–Kyritsi [11, Theorem 4.6.1, p. 349]). We consider the map $\xi \colon C_\varrho \longrightarrow C_R$, defined by

$$\xi(u) = \begin{cases} R\dfrac{u}{\|u\|} & \text{if } \varrho \leq \|u\| \leq R, \\ u & \text{if } R < \|u\|. \end{cases}$$

Evidently ξ is continuous and $\xi|_{C_R} = id|_{C_R}$. Therefore, C_R is a retract of C_ϱ. Moreover, if $h \colon [0,1] \times C_\varrho \longrightarrow Y$ is defined by

$$h(t, u) = (1 - t)u + tR\frac{u}{\|u\|} \quad \forall (t, u) \in [0, 1] \times C_\varrho,$$

then we see that C_ϱ is deformable into C_R over Y. So, invoking [4, Theorem 6.5, p. 325], we have that C_R is a deformation retract of C_ϱ. Hence (see Granas–Dugundji [7, p. 387]),

(3.15)
$$H_k(C_\varrho, C_R) = 0 \quad \forall k \geq 0.$$

From the exactness of the long homology sequence in (3.13), we have (see (3.15))

$$0 = \operatorname{im} i_* = \ker j_* \qquad \text{and} \qquad \operatorname{im} j_* = \ker \partial_* = H_k(Y, C_\varrho)$$

and (see (3.14))

$$0 = \operatorname{im} \widehat{i}_* = \ker \widehat{j}_* \qquad \text{and} \qquad \operatorname{im} \widehat{j}_* = \ker \widehat{\partial}_* = H_k(Y, \overline{\varphi}^\beta).$$

From these equalities, it follows that both j_* and $\widehat{j}_*$ are group homomorphisms. Then invoking [7, Lemma D.1, p. 610], we infer that h_* is an isomorphism. So, we have

$$H_k(Y, C_\varrho) = H_k(Y, \overline{\varphi}^\beta);$$

hence,

$$(3.16) \qquad H_k(Y, C_\varrho) = C_k(\overline{\varphi}, \infty) \quad \forall k \geq 0.$$

Recall that $\beta < \inf \overline{\varphi}(K_{\overline{\varphi}})$.

As before, using the radial retraction and [4, Theorem 6.5, p. 325], we show that $\partial B_\varrho = \{y \in Y : \|y\| = \varrho\}$ is a deformation retract of C_ϱ. Therefore,

$$H_k(Y, C_\varrho) = H_k(Y, \partial B_\varrho) \quad \forall k \geq 0,$$

so (see Maunder [10, Example 4.3.12, pp. 120–121])

$$H_k(Y, C_\varrho) = \delta_{k, d_m} \mathbb{Z} \quad \forall k \geq 0,$$

and thus (see (3.16))

$$C_k(\overline{\varphi}, \infty) = \delta_{k, d_m} \mathbb{Z} \quad \forall k \geq 0. \qquad \square$$

Now, we are ready for the multiplicity theorem.

THEOREM 3.5

If hypotheses H hold, then problem (1.1) has at least two nontrivial smooth solutions $u_0, \widehat{u} \in C^1(\overline{\Omega})$.

Proof

Since $\varphi \in C^2(H^1(\Omega))$ and $\vartheta \in C^1(Y; \widehat{H})$ (see Proposition 3.2), it follows that $\overline{\varphi} \in C^1(Y)$ (see (3.7)). Also, from Proposition 3.3, we know that $\overline{\varphi}$ is anticoercive. Since Y is finite-dimensional, we can find $y_0 \in Y$ such that

$$\overline{\varphi}(y_0) = \max\{\overline{\varphi}(y) : y \in Y\}.$$

Then from Chang [3, Example 1, p. 33], we have

$$(3.17) \qquad C_k(\overline{\varphi}, y_0) = \delta_{k, d_m} \mathbb{Z} \quad \forall k \geq 0.$$

Hypothesis $H(\text{iii})$ implies that

$$(3.18) \qquad C_k(\varphi, 0) = \delta_{k, d_l} \mathbb{Z} \quad \forall k \geq 0,$$

where

$$d_l = \dim \bigoplus_{i=0}^{l} E(\widehat{\lambda}_i).$$

This can be established as in the work of Li–Perera–Su [8, Proposition 1.1], who deal with Dirichlet spaces. Since their proof uses only the homotopy invariance of critical groups and the orthogonal direct sum decomposition of the ambient space, it carries also to the Neumann case. From (3.18) and since $\vartheta(0) = 0$ (see (3.3) and (3.5)), we have (see Liu [9, Lemma 2.3, p. 501])

$$(3.19) \qquad C_k(\overline{\varphi}, 0) = \delta_{k, d_\varrho} \mathbb{Z} \quad \forall k \geq 0.$$

Finally, from Proposition 3.4, we have

$$(3.20) \qquad C_k(\overline{\varphi}, \infty) = \delta_{k, d_m} \mathbb{Z} \quad \forall k \geq 0.$$

If $K_{\overline{\varphi}} = \{0, y_0\}$, then from (3.17), (3.19), and (3.20) and the Morse relation with $t = -1$ (see (2.1)), we have

$$(-1)^{d_l} = 0,$$

a contradiction. So, $\overline{\varphi}$ has at least one more critical point $\widehat{y} \notin \{0, y_0\}$. Let $u_0 = y_0 + \vartheta(y_0)$, and let $\widehat{u} = \widehat{y} + \vartheta(\widehat{y})$. Then $u_0, \widehat{u}$ are solutions of (1.1), and by standard regularity theory, we have $u_0, \widehat{u} \in C^1(\overline{\Omega})$. $\qquad \square$

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