Duality theorems and topological structures of groups

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Abstract We introduce four different notions of weak Tannaka-type duality theorems, and we define three categories of topological groups, called *T-type groups*, *strongly T-type groups*, and *NOS-groups*.

We call a *one-parameter subgroup* a nontrivial homomorphic image of the additive group $\mathbf{R}$ of real numbers into a topological group G. When G does not contain any one-parameter subgroup, we call G a NOS-group.

The aim of this paper is to show the following relations. In the table below, the symbol $\Longleftrightarrow$ means that for a given topological group G the duality theorem on the left-hand side holds if and only if G is of type cited on the right-hand side:

(1) u-duality $\Longleftrightarrow$ T-type,
(2) i-duality $\Longleftrightarrow$ strongly T-type,
(3) b-duality $\Longleftrightarrow$ locally compact,
(4) c-duality $\Longleftrightarrow$ locally compact NOS.

We give in the last section some examples which show the actual differences among (1)–(4).

0. Introduction

Four types of weak Tannaka-type duality theorems for topological groups are stated as follows.

We take the set $\Omega \equiv \{D = (\mathcal{H}^D, T_g^D)\}$ of all unitary representations of given topological group G, dimensions of which are bounded by $\max(\aleph_0, \#G)$. Then there exist relations between elements of Ω as

(1) unitary equivalence: $D_1 \sim_W D_2$ (W: intertwining operator),
(2) direct sum: $D_1 \oplus D_2$,
(3) tensor product: $D_1 \otimes D_2$,
(4) contragradient: $D \to \overline{D}$.

We consider an operator field $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$ on Ω satisfying the following:

(B-0) for each $D \in \Omega$, A^D is an operator in a *certain category* on the representation Hilbert space $\mathcal{H}^D$:

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(B-1) $D_1 \sim_W D_2 \Longrightarrow W A^{D_1} W^{-1} = A^{D_2}$,
(B-2) $A^{D_1} \oplus A^{D_2} = A^{D_1 \oplus D_2}$,
(B-3) $A^{D_1} \otimes A^{D_2} = A^{D_1 \otimes D_2}$,
(B-4) $\overline{A^D} = A^{\overline{D}}$.

In the condition (B-0), for the terminology *certain category* we consider four cases as follows.

At first, we take the *unitarity* property, that is,

(B-01) for each $D \in \Omega$, A^D is a *unitary* operator on the representation Hilbert space $\mathcal{H}^D$.

We shall call such an operator field which satisfies (B-01)–(B-4) simply a *birepresentation* of G and write $\mathcal{U}$ for the set of all birepresentations.

Secondly, we take *isometricity*, that is,

(B-02) for each $D \in \Omega$, A^D is an *isometric* operator on the representation Hilbert space $\mathcal{H}^D$.

We shall call such an operator field which satisfies (B-02), (B-1), (B-2), (B-3), and (B-4) an *isobirepresentation* of G and write $\mathcal{J}$ for the set of all isobirepresentations.

Thirdly, we take nonzero and *uniform boundedness*, that is,

(B-03) for each $D \in \Omega$, A^D is a *nonzero bounded* operator such that $\|A^D\| \leq 1$ on the representation Hilbert space $\mathcal{H}^D$.

We shall call such an operator field which satisfies (B-03), (B-1), (B-2), (B-3), (B-4) a *bd-birepresentation* of G and write $\mathcal{B}$ the set of all bd-birepresentations.

Lastly, we take the property *nonzero closed with a common fixed domain* $\mathcal{D}^D$ *dense in* $\mathcal{H}^D$ *and range in the same subspace* for each D.

(B-04) For each $D \in \Omega$, A^D is a *nonzero closed operator on* $\mathcal{H}^D$ *with domain and range a fixed dense subspace* $\mathcal{D}^D$ *in common.*

But in this case, we must assume relations for subspaces $\mathcal{D}^D$ as

(B-041) $\mathcal{D}^{D_1} \sim_W \mathcal{D}^{D_2} \Longrightarrow W\mathcal{D}^{D_1} = \mathcal{D}^{D_2}$,
(B-042) $\mathcal{D}^{D_1} \oplus \mathcal{D}^{D_2} \subset \mathcal{D}^{D_1 \oplus D_2}$,
(B-043) $\mathcal{D}^{D_1} \otimes \mathcal{D}^{D_2} \subset \mathcal{D}^{D_1 \otimes D_2}$,
(B-044) $\overline{\mathcal{D}}^D = \mathcal{D}^{\overline{D}}$.

We shall call such an operator field that satisfies (B-04), (B-1), (B-2), (B-3), (B-4) a *cl-birepresentation* of G and write $\mathcal{C}$ for the set of all cl-birepresentations.

Of course, a birepresentation is an isobirepresentation, an isobirepresentation is a bd-birepresentation, and a bd-birepresentation is a cl-birepresentation.

PROPOSITION 0.1

We have

$$\mathcal{U} \subset \mathcal{J} \subset \mathcal{B} \subset \mathcal{C}.$$

On the space $\mathcal{B}$, we give a topology which is the product of weak topologies τ^D on each spaces $\mathcal{B}^D$ of bounded operators on Hilbert spaces $\mathcal{H}^D$. We write this topology on $\mathcal{B}$ as τ. The topologies on $\mathcal{U}, \mathcal{J}$ are the restriction of τ onto each of the spaces.

In the case of cl-birepresentations, we must consider for $\mathcal{C}^D$ the space of closed operators with domain $\mathcal{D}^D$ at each component D, the topology which is the weakest topology $\tau_{\mathcal{C}}^D$ making continuous all matrix elements $\langle A^D v^D, u^D \rangle$ ($v^D, u^D \in \mathcal{D}^D$), and give the product topology τ_C on the space $\mathcal{C}$.

PROPOSITION 0.2

Topology τ is the restriction of $\tau_{\mathcal{C}}$ to the space $\mathcal{B}$ from $\mathcal{C}$.

Proof

For any fixed D and v^D, u^D in $\mathcal{D}^D$, $\langle A^D v^D, u^D \rangle$ is a continuous function of A^D with respect to the weak topology τ^D on the space $\mathcal{B}^D$.

Therefore it is sufficient to show for any v^D, u^D ($\|v^D\| = \|u^D\| = 1$) in $\mathcal{H}^D$, that the function $\langle A^D v^D, u^D \rangle$ is continuous with respect to $\tau_{\mathcal{C}}^D|_{\mathcal{B}^D}$.

The common fixed domain $\mathcal{D}^D$ is dense in $\mathcal{H}^D$ for each D, and we assumed by (B-03) that the components of $\mathcal{B}^D$ are bounded by 1. So for any $\varepsilon > 0$ as in (B-03), there exist v_0^D, u_0^D ($\|v_0^D\| = \|u_0^D\| = 1$) in $\mathcal{D}^D$ such that

$$\|v^D - v_0^D\| < \varepsilon, \qquad \|u^D - u_0^D\| < \varepsilon. \tag{0.1}$$

This shows for any A^D bounded as $\|A^D\| \leq 1$,

$$\begin{aligned} &\left|\langle A^D v^D, u^D \rangle - \langle A^D v_0^D, u_0^D \rangle\right| \\ &\leq \left|\langle A^D v^D - A^D v_0^D, u^D \rangle\right| + \left|\langle A^D v_0^D, u^D - u_0^D \rangle\right| \\ &\leq \|A^D v^D - A^D v_0^D\| \times \|u^D\| + \|A^D v_0^D\| \times \|u^D - u_0^D\| \\ &\leq \|A^D\| \times \|v^D - v_0^D\| + \|A^D\| \times \|u^D - u_0^D\| \leq \varepsilon. \end{aligned} \tag{0.2}$$

Therefore on $\mathcal{B}^D$ the function $\langle A^D v^D, u^D \rangle$ is continuous with respect to $\tau_{\mathcal{C}}^D$ as a limit of uniform convergence of continuous functions $\langle A^D v_0^D, u_0^D \rangle$. □

For any $g \in G$ the operator field $\mathbf{T}_g \equiv \{T_g^D\}_{D \in \Omega}$ gives a birepresentation.

Our weak Tannaka-type duality theorems assert the converses, which are separated into the set theoretical part and the topological part.

SET PART OF U-DUALITY'S ASSERTION

For any birepresentation $\mathbf{U} \equiv \{U^D\}_{D \in \Omega}$, there exists a unique $g \in G$ such that $U^D = T_g^D$ ($\forall D \in \Omega$).

SET PART OF I-DUALITY'S ASSERTION

For any isobirepresentation $\mathbf{J} \equiv \{J^D\}_{D\in\Omega}$, *there exists a unique* $g \in G$ *such that* $J^D = T_g^D \ (\forall D \in \Omega)$.

SET PART OF B-DUALITY'S ASSERTION

For any bd-birepresentation $\mathbf{B} \equiv \{B^D\}_{D\in\Omega}$, *there exists a unique* $g \in G$ *such that* $B^D = T_g^D \ (\forall D \in \Omega)$.

SET PART OF C-DUALITY'S ASSERTION

For any cl-birepresentation $\mathbf{C} \equiv \{C^D\}_{D\in\Omega}$, *there exists a unique* $g \in G$ *such that* $C^D = T_g^D \ (\forall D \in \Omega)$.

TOPOLOGICAL ASSERTION OF THESE DUALITIES

Moreover, the topologies given above coincide with the original topology of G *under the correspondence* $g \to \mathbf{T}_g$.

PROPOSITION 0.3

$$\textit{c-duality} \implies \textit{b-duality} \implies \textit{i-duality} \implies \textit{u-duality}.$$

Proof

From Proposition 0.1, the set part of this implication is valid. Proposition 0.2 shows that the topological part is also satisfied. □

NOTATION

For a representation $D \equiv \{\mathcal{H}^D, T_g^D\}$ of G, we take its cyclic subrepresentation on the closed subspace $(\mathcal{H}^D)$ of $\mathcal{H}^D$ spanned by $\{T_g^D v^D\}_{g\in G}$ for a fixed $v^D \in \mathcal{H}^D, \|v^D\| = 1$. We express it as $(D) = \{(\mathcal{H}^D), T_g^D, v^D\}$.

Hereafter, for two cyclic representations $D_j = \{\mathcal{H}^j, T_g^j, v^j\}, \ j = 1, 2,$ we denote cyclic subrepresentations contained in $D_1 \oplus D_2$ and $D_1 \otimes D_2$, respectively, as

$$(D_1 \oplus D_2) \equiv \big\{(\mathcal{H}^1 \oplus \mathcal{H}^2), T_g^1 \oplus T_g^2, v^1 \oplus v^2\big\},$$
$$(D_1 \otimes D_2) \equiv \big\{(\mathcal{H}^1 \otimes \mathcal{H}^2), T_g^1 \otimes T_g^2, v^1 \otimes v^2\big\}.$$

In Section 1 of this paper, we give the notion of an SSUR of G, and "complete" and "b-complete" properties for G, after [8, Section 2]. Using these concepts, we define categories of T-type and strongly T-type groups for topological groups.

Section 2 is devoted to proving the condition (W-3′) in [8, Section 8], for topological groups G to have an SSUR. This shows that a T-type group is a well-behaved group, and a strongly T-type group is strongly well behaved as is stated in [8, Sections 7, 8].

We show in Section 3 that a T-type group satisfies weak, Tannaka-type i-duality.

Since strongly T-type deduces T-type, this gives also the following.

(1) A T-type group satisfies u-duality.
(2) A strongly-T-type group satisfies i-duality.

The converse problems for T-type groups and strongly T-type groups are solved in Section 4. And we obtain the following:

(3) if a topological group G satisfies u-duality, then G is T-type;
(4) if a topological group G satisfies i-duality, then G is strongly T-type.

We discuss in Section 5 the similar problems for locally compact groups and locally compact NOS groups.

Summarizing the above results, we state the main theorem in Section 6.

We give in the same section several examples of groups of each type.

1. Separating systems of unitary representations: T-type groups and strongly T-type groups

We consider a Hausdorff (i.e., T_2-) topological group G.

DEFINITION 1.1

We say that a set $\Omega_0 \equiv \{D_\alpha \equiv \{\mathcal{H}^{D_\alpha}, T_g^{D_\alpha}, v^{D_\alpha}\}\}_{\alpha \in A}$ of cyclic unitary representations of G gives a *separating system of unitary representations (SSUR)* if for any neighborhood V of the unit e in G, there exist a positive definite function $\eta^D(g) \equiv \langle T_g^D v^D, v^D \rangle$ ($D \in \Omega_0$, $\|v^D\| = 1$) and $\varepsilon > 0$ such that

$$F(D,\varepsilon) \equiv \{g \in G \mid |1 - \eta^D(g)| < \varepsilon\} \subset V. \tag{1.1}$$

For any given cyclic unitary representation $D \equiv \{\mathcal{H}^D, T_g^D, v^D\}$ ($\|v^D\| = 1$), and the trivial representation $I = \{\mathbf{C}, I_g, v_0\}$ of G, we define a unitary representation $D_p \equiv I \oplus D \oplus \overline{D}$ and its cyclic part (D_p), whose representation space is spanned by the vector

$$v_p \equiv (2^{-1/2})v_0 \oplus (1/2)(v^D \oplus v^{\overline{D}}).$$

Here $\overline{D} \equiv \{\mathcal{H}^{\overline{D}}, T_g^{\overline{D}}, v^{\overline{D}}\}$ is the contragradient representation of D and $v^{\overline{D}}$ is the vector in the space $\mathcal{H}^{\overline{D}}$ corresponding to v^D. We showed in our previous paper (see [8, Corollaries 1-2-1, 1-2-2]) that

$$1 \geq \langle T_g^{D_p} v_p, v_p \rangle \geq 0. \tag{1.2}$$

For any D in Ω we write $\eta^D(g) \equiv \langle T_g^D v^D, v^D \rangle$ and

$$F(D,\varepsilon) \equiv \{g \in G \mid |1 - \eta^D(g)| < \varepsilon\};$$

then

$$1 > \forall \varepsilon \geq 0, \exists \delta > 0, \quad F((D_p),\delta) \subset F(D,\varepsilon). \tag{1.3}$$

Using (1.3), if a topological group G has an SSUR $\Omega_0 \equiv \{D_\alpha\}_{\alpha\in A}$, then we can select a new SSUR $\Omega_1 \equiv \{(D_{\alpha,p})_{\alpha\in A}\}$, for which any $(D_{\alpha,p}) = \{(\mathcal{H}^{D_{\alpha,p}}), T_g^{D_{\alpha,p}}, v^{D_{\alpha,p}}\} \in \Omega_1$ has a nonnegative-valued positive definite function $\eta^{D_{\alpha,p}}(g) \equiv \langle T_g^{D_{\alpha,p}} v^{D_{\alpha,p}}, v^{D_{\alpha,p}}\rangle$.

For a given Hilbert space $\mathcal{H}$, we denote by $B(\mathcal{H})$ the space of all bounded operators on $\mathcal{H}$, by $U(\mathcal{H})$ the space of all unitary operators on $\mathcal{H}$, and by $J(\mathcal{H})$ the space of all isometric operators on $\mathcal{H}$. Put the weak topologies on each of the spaces.

For any isometric operator J on a Hilbert space $\mathcal{H}$, we get

$$\begin{aligned} \|Jv - v\|^2 &= \|Jv\|^2 + \|v\|^2 - 2\Re\langle Jv, v\rangle \\ &= 2\big(\langle v, v\rangle - \Re(\langle Jv, v\rangle)\big) = 2\Re\big(\langle v - Jv, v\rangle\big). \end{aligned} \tag{1.4}$$

This shows that on $J(\mathcal{H})$, and so on $U(\mathcal{H})$ too, the weak topology coincides with the strong topology.

Moreover, $U(\mathcal{H})$ becomes a topological group with the multiplication of operators and this topology. As a group topology, this topology gives a uniform structure on $U(\mathcal{H})$.

For a topological group G, let $D \equiv \{\mathcal{H}^D, T_g^D\}$ be any unitary representation. Then the map $G \ni g \to T_g^D \in U(\mathcal{H}^D) \subset J(\mathcal{H}^D)$ is continuous for each D.

Construct $\mathbf{U}(\Omega) \equiv \prod_{D\in\Omega} U(\mathcal{H}^D) \subset \mathbf{J}(\Omega) \equiv \prod_{D\in\Omega} J(\mathcal{H}^D)$ with natural product topologies. The maps

$$G \ni g \longmapsto (T_g^D)_{D\in\Omega} \in \mathcal{U} \quad \Big(\subset \prod_{D\in\Omega} U(\mathcal{H}^D) = \mathbf{U}(\Omega)\Big), \tag{1.5}$$

$$G \ni g \longmapsto (T_g^D)_{D\in\Omega} \in \mathcal{U} \subset \mathcal{J} \subset \prod_{D\in\Omega} J(\mathcal{H}^D) = \mathbf{J}(\Omega) \tag{1.6}$$

are into-homomorphisms as topological groups.

When G is a T_2-topological group with an SSUR [7, Lemma 1.4] shows that the map (1.5) is an into-isomorphism, so by this map, G is embedded as a topological group in $\mathbf{J}(\Omega)$.

We denote by G_J the image of G under the map (1.6) into $\mathbf{J}(\Omega)$.

LEMMA 1.1

Let G be a T_2-topological group with an SSUR.

(1) *The weak Tannaka-type u-duality theorem is valid for G, if and only if $G_J = \mathcal{U}$ and the map (1.5) is an isomorphism between G and its image G_J in $\mathbf{U}(\Omega)$ as topological groups.*

(2) *The weak Tannaka-type i-duality theorem is valid for G, if and only if $G_J = \mathcal{J}$ and the map (1.6) is an isomorphism between G and its image G_J in $\mathbf{J}(\Omega)$ as topological spaces. In this case, $G_J = \mathcal{U} = \mathcal{J}$.*

Proof

It is obvious from the definitions. □

In [8, Section 2, Definition 2.1], we defined the notions of *l-Cauchy* and *b-Cauchy* properties for a filter base as follows.

DEFINITION 1.2

A filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ on G, where Γ is a partially ordered set, is called *l-Cauchy* (hereafter simply *Cauchy*), if for any neighborhood V of $e \in G$, there exists an $\alpha \in \Gamma$ such that

$$\forall\beta,\gamma \succ \alpha\ (\beta,\gamma \in \Gamma), \quad F_\beta^{-1}F_\gamma \subset V.$$

We say that $\mathcal{F}$ is *b-Cauchy* (both Cauchy) when both of $\mathcal{F}$ and $\mathcal{F}^{-1} \equiv \{F_\alpha^{-1}\}_{\alpha\in\Gamma}$ are Cauchy at the same time.

If any Cauchy (resp., b-Cauchy) filter base has limit points in G, we say that G is *complete* (resp., *b-complete*).

Evidently a b-Cauchy filter base is also Cauchy, so a complete group is b-complete too.

If a filter base $\mathcal{F}$ converges to a point g_0 in G, then $\mathcal{F}^{-1}$ converges to g_0^{-1}.

We consider the topological group $\mathbf{G} \equiv \mathbf{U}(\Omega) = \prod_{D\in\Omega} U(\mathcal{H}^D)$ and a Cauchy (resp., b-Cauchy) filter base $\mathcal{F} \equiv \{F_\alpha\}$ on $\mathbf{G}$. Projection image $\mathcal{F}^D \equiv \{F_\alpha^D \equiv \mathrm{Proj}_{\mathcal{H}^D} F_\alpha\}$ for any $D \in \Omega$ gives a Cauchy (resp., b-Cauchy) filter base on $U(\mathcal{H}^D)$.

Conversely, for a filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ on $\mathbf{U}(\Omega)$ to be Cauchy (resp., b-Cauchy), it is enough that, for any D in Ω, $\mathcal{F}^D$ are Cauchy (resp., b-Cauchy).

Since on $\mathbf{U}(\Omega)$ the weak topology is equivalent to the strong topology, we can consider these Cauchy or b-Cauchy properties in the sense of strong topology on $\mathbf{U}(\Omega)$.

Let $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ be a Cauchy filter base on $\mathbf{U}(\Omega)$.

For any $v \in \mathcal{H}^D$ for a fixed D, a Cauchy filter base $\{F_\alpha^D v\}_{\alpha\in\Gamma}$ converges to a vector $u(v)$ in the Hilbert space $\mathcal{H}^D$, that is, for any $U_\alpha^D \in F_\alpha^D$ and any $v \in \mathcal{H}^D$, strong-$\lim_\alpha U_\alpha^D v = u(v)$. Then, for any $a, b \in \mathbf{C}$,

$$\lim_\alpha U_\alpha^D(av_1 + bv_2) = au(v_1) + bu(v_2), \qquad \|u(v)\| = \lim_\alpha \|U_\alpha^D v\| = \|v\|. \tag{1.7}$$

Therefore the map $\mathcal{H}^D \ni v \to u(v) \in \mathcal{H}^D$ is linear and isometric. Thus there exists an isometric operator B^D such that $u(v) = B^D v$.

LEMMA 1.2

Any Cauchy filter base on $\mathbf{U}(\Omega) = \prod_{D\in\Omega} U(\mathcal{H}^D)$ converges to a $\mathbf{B} \equiv (B^D)_{D\in\Omega} \in \mathbf{J}(\Omega) = \prod_{D\in\Omega} J(\mathcal{H}^D)$, where B^D are isometric operators.

For a topological group G, any filter base $\mathcal{F}$ on it is mapped to a filter base $\mathcal{F}_J$ in G_J. And if $\mathcal{F}$ is Cauchy (resp., b-Cauchy), then $\mathcal{F}_J$ in $\mathbf{U}(\Omega)$ is also Cauchy (resp., b-Cauchy).

LEMMA 1.3

(1) *A Cauchy filter base $\mathcal{F}_J$ on a group G_J converges to an element $\mathbf{B} \equiv (B^D)_{D\in\Omega}$ in $\mathbf{J}(\Omega)$.*

(2) *If $\mathcal{F}_J$ is b-Cauchy, then $\mathcal{F}_J$ converges to an element in* $\mathbf{U}(\Omega)$.

Proof

(1) The result is deduced directly from Lemma 1.2.

(2) If $\mathcal{F}_J$ is b-Cauchy, then $\mathcal{F}_J$ and $\mathcal{F}_J^{-1}$ are both Cauchy. So by Lemma 1.2, $\mathcal{F}_J$ converges to a $\mathbf{B} \equiv (B^D)_{D\in\Omega} \in \mathbf{J}(\Omega)$ and $\mathcal{F}_J^{-1}$ converges to a $\mathbf{C} \equiv (C^D)_{D\in\Omega} \in \mathbf{J}(\Omega)$, where B^D and C^D are isometric operators. This means that for each $D \in \Omega$, the component $\mathcal{F}_J^D \equiv \{F_\alpha^D\}$ of $\mathcal{F}_J$ converges to B^D and $(\mathcal{F}^{-1})_J^D \equiv \{(F_\alpha^D)^{-1}\}$ of $\mathcal{F}_J^{-1}$ converges to C^D under the strong topology of $J(\mathcal{H}^D)$, so that, for any neighborhood V^D of I^D (the identity operator in $J(\mathcal{H}^D)$), $B^D \in F_\alpha^D V^D$ and $C^D \in V^D (F_\alpha^D)^{-1}$.

For given neighborhood W^D of I^D in $J(\mathcal{H}^D)$, take V^D as $(V^D)^3 \subset W^D$, and take α as $F_\alpha^{-1} F_\alpha \subset V^D$. Then $C^D B^D \in V^D F_\alpha^{-1} F_\alpha V^D \subset (V^D)^3 \subset W^D$, that is, $C^D B^D \in \bigcap W^D$; here W^D runs any neighborhood of I^D.

We get $C^D B^D = I^D$.

But C^D is an isometric operator having the range the full space $\mathcal{H}^D$ and therefore is a unitary operator on $\mathcal{H}^D$.

Therefore we have

$$\forall D \in \Omega, \quad B^D = (C^D)^{-1}. \tag{1.8}$$

Thus B^D must be a unitary operator, and $\mathbf{B} \equiv (B^D)_{D\in\Omega} \in \mathbf{U}(\Omega)$. □

DEFINITION 1.3

(1) We say that a topological group G is *T-type* if

(T-1) G has an SSUR (separating condition), and

(T-2) G is b-complete.

(2) We say that a topological group G is *strongly T-type* if

(T-1) G has an SSUR (separating condition), and

(T-2′) G is complete.

Since a complete group is b-complete, so a strongly T-type group is a T-type group.

REMARK

The conditions (T-1), (T-2), and (T-2′) above are just the same as the conditions (W-1), (W-2), and (W-2′) defined in [8, Sections 7, 8], respectively.

In the succeeding sections, we shall discuss relations between groups of this type and two types of weak Tannaka-type dualities.

We showed in [8, Lemma 7.1] that any locally compact groups and any closed-type inductive limits of such groups are strongly T-type, therefore of T-type.

2. Birepresentation and isobirepresentation of G

Now we discuss some elementary properties of isobirepresentations on G with an SSUR. Since any birepresentation is also an isobirepresentation, these properties are valid for it too.

The following arguments in Lemma 2.1 and its corollaries are just similar to those of [7, Section 6]. The only difference is to change the letter U to J; we will repeat it to confirm.

The condition (B-4) assures that for any isobirepresentation $\mathbf{J} \equiv \{J^D\}$,

$$J^{\overline{D}} = \overline{(J^D)}. \tag{2.1}$$

LEMMA 2.1

For $D^0 \equiv D \oplus \overline{D}$, the matrix element $\langle J^{D^0}(u \oplus \overline{u}), v \oplus \overline{v}\rangle$ is real valued.

Proof

$\langle J^{D^0}(u \oplus \overline{u}), v \oplus \overline{v}\rangle = \langle J^D u, v\rangle + \langle J^{\overline{D}}\overline{u}, \overline{v}\rangle = \langle J^D u, v\rangle + \langle \overline{(J^D u)}, \overline{v}\rangle = \langle J^D u, v\rangle + \overline{\langle J^D u, v\rangle} \in \mathbf{R}$. □

COROLLARY 2.1.1

Put $D_p \equiv I \oplus D \oplus \overline{D}$. Take vectors $w_0 \in \mathcal{H}^I, w \in \mathcal{H}^D$ such that $2^{1/2}\|w_0\| = 2\|w\| = 1$, and put $v_p \equiv w_0 \oplus w \oplus \overline{w}$. Then the matrix element

$$\langle J^{D_p} v_p, v_p\rangle = \left\langle J^{D_p}(w_0 \oplus w \oplus \overline{w}), w_0 \oplus w \oplus \overline{w}\right\rangle \geq 0. \tag{2.2}$$

Proof

We have

$$\begin{aligned}\langle J^{D_p} v_p, v_p\rangle &= \left\langle (I w_0) \oplus (J^D w) \oplus (J^{\overline{D}}\overline{w}), w_0 \oplus w \oplus \overline{w}\right\rangle \\ &= \langle w_0, w_0\rangle + \langle J^D w, w\rangle + \langle J^{\overline{D}}\overline{w}, \overline{w}\rangle \\ &= \|w_0\|^2 + 2\Re\langle J^D w, w\rangle \\ &= 2^{-1} + 2\Re\langle J^D w, w\rangle.\end{aligned}$$

But $|\langle J^D w, w\rangle| \leq \|w\|^2 = 2^{-2}$. So $-2^{-1} \leq 2\Re\langle J^D w, w\rangle \leq 2^{-1}$, whence $\langle J^{D_p} w_p, w_p\rangle \geq 0$. □

COROLLARY 2.1.2

As in the case of Corollary 2.1.1, for $D_p \equiv I \oplus D \oplus \overline{D}$,

$$\forall g \in G, \quad \langle T_g^{D_p} J^{D_p} v_p, v_p\rangle \geq 0 \quad (v_p = w_0 \oplus w \oplus \overline{w}). \tag{2.3}$$

Proof

For any isobirepresentations $\mathbf{J} \equiv \{J^D\}$ and $\mathbf{T}_g \equiv \{T_g^D\}, \mathbf{T}_g\mathbf{J} \equiv \{T_g^D J^D\}$ is also an isobirepresentation. So we can apply the result of Corollary 2.1.1. □

Let $D = \{\mathcal{H}^D, T_g^D, v^D\}$ be a cyclic unitary representation of G. Denote by $\eta^D(g) \equiv \langle T_g^D v^D, v^D\rangle$ the positive definite function to which D belongs, and put

$$K^D(g) \equiv \langle T_g^D J^D v^D, v^D\rangle.$$

LEMMA 2.2

Let $\mathbf{J}$ *be an isobirepresentation. Then, for any* $D \equiv \{\mathcal{H}^D, T_g^D, v^D\}$ $(\|v^D\| = 1)$ *in* Ω,

$$\sup_{g\in G} \left|K^D(g)\right| = 1. \tag{2.4}$$

Proof

Since $\|v^D\| = 1$ and J^D, T_g^D are isometric, $|K^D(g)| \leq 1$.

Now consider the family $\mathfrak{F} \equiv \{\zeta^D(g) = \langle T_g^D u, w\rangle\}$ of matrix elements. Here D runs over Ω, and u, w are any unit vectors in $\mathcal{H}^D$. For two $\zeta^{D_1}(g), \zeta^{D_2}(g)$,

$$\overline{\zeta^D(g)} = \zeta^{\overline{D}}(g), \tag{2.5}$$

$$\zeta^{D_1}(g) + \zeta^{D_2}(g) = \zeta^{D_1 \oplus D_2}(g), \tag{2.6}$$

$$\zeta^{D_1}(g) \times \zeta^{D_2}(g) = \zeta^{D_1 \otimes D_2}(g). \tag{2.7}$$

Therefore $\mathfrak{F}$ is a $*$-algebra contained in the $*$-algebra $\mathcal{C}^b(G)$ of all bounded continuous functions on G.

Define norm $\|\zeta^D\| \equiv \sup_{g\in G} |\zeta^D(g)|$ on $\mathfrak{F}$. Consider the completion $\mathfrak{F}^C$ of $\mathfrak{F}$ with respect to this norm. Then as a set of uniform limits of continuous functions of $\mathfrak{F}$, $\mathfrak{F}^C$ becomes a C^*-algebra of continuous functions on G.

Applying the Gelfand representation theorem, there exists a locally compact space X, and $\mathfrak{F}^C$ is isomorphic to the space $\mathcal{C}^b(X)$ of all bounded continuous functions on X under the correspondence $\mathfrak{F}^C \ni f \longrightarrow f^\sim \in \mathcal{C}^b(X)$. A point x of X is considered as a homomorphic map such that

$$\psi^x : \mathcal{C}^b(X) \longrightarrow \mathbf{C}, \tag{2.8}$$

$$\psi^x(\varphi) \equiv \varphi(x) \quad \big(\varphi \in \mathcal{C}^b(X)\big). \tag{2.9}$$

For any element g in G and f in $\mathfrak{F}^C$,

$$f \longmapsto f(g) \tag{2.10}$$

gives a homomorphic map from $\mathfrak{F}^C$ to $\mathbf{C}$. So there exits a unique element x_g in X as

$$f(g) = f^\sim(x_g). \tag{2.11}$$

The existence of an SSUR assures us that the map $g \longmapsto x_g$ is one-to-one. So by this map, G is embedded into X. But $\mathcal{C}^b(X)$ is given as the space of $\{f^\sim \mid f \in \mathfrak{F}^C\}$ and $\mathfrak{F}^C \subset \mathcal{C}^b(G)$. From this we conclude that the image of G is dense in X. That is, for any $x \in X$, $\delta > 0$, and $f^\sim(x) \in \mathfrak{F}^C$, there exists $g_0 \in G$ such that

$$\left|f^\sim(g_0) - f^\sim(x)\right| < \delta. \tag{2.12}$$

For a given isobirepresentation $\mathbf{J} \equiv \{J^D\}_{D\in\Omega}$, consider the map

$$\zeta^D(g) = \langle T_g^D v^D, u^D\rangle \longrightarrow \langle T_g^D J^D v^D, u^D\rangle \equiv \theta_{\mathbf{J}}(\zeta^D)(g). \tag{2.13}$$

By analogous consideration as in (2.5)–(2.7), we get

$$\begin{aligned}
\overline{\theta_{\mathbf{J}}\zeta^D(g)} &= \overline{\langle T_g^D J^D v^D, u^D\rangle} = \langle \overline{T_g^D J^D} v^{\overline{D}}, u^{\overline{D}}\rangle = \langle T_g^{\overline{D}} J^{\overline{D}} v^{\overline{D}}, u^{\overline{D}}\rangle \\
&= \theta_{\mathbf{J}}(\zeta^{\overline{D}})(g), \\
\theta_{\mathbf{J}}(\zeta^{D_1})(g) + \theta_{\mathbf{J}}(\zeta^{D_2})(g) &= \langle T_g^{D_1} J^{D_1} v^{D_1}, u^{D_1}\rangle + \langle T_g^{D_2} J^{D_2} v^{D_2}, u^{D_2}\rangle \\
&= \big\langle (T_g^{D_1} J^{D_1} v^{D_1} \oplus T_g^{D_2} J^{D_2} v^{D_2}), (u^{D_1} \oplus u^{D_2})\big\rangle \\
&= \theta_{\mathbf{J}}(\zeta^{D_1\oplus D_2})(g), \\
\theta_{\mathbf{J}}(\zeta^{D_1})(g) \times \theta_{\mathbf{J}}(\zeta^{D_2})(g) &= \langle T_g^{D_1} J^{D_1} v^{D_1}, u^{D_1}\rangle \times \langle T_g^{D_2} J^{D_2} v^{D_2}, u^{D_2}\rangle \\
&= \big\langle (T_g^{D_1} J^{D_1} v^{D_1} \otimes T_g^{D_2} J^{D_2} v^{D_2}), (u^{D_1} \otimes u^{D_2})\big\rangle \\
&= \theta_{\mathbf{J}}(\zeta^{D_1\otimes D_2})(g).
\end{aligned}$$

Consider the case $\Sigma_j \langle T_g^{D_j} v^{D_j}, u^{D_j}\rangle \equiv 0$ as a function on G for some countable set $\{D_j\} \subset \Omega$ and $\{v^{D_j}, u^{D_j} \in \mathcal{H}^{D_j}\}$ such that $\Sigma_j \|v^{D_j}\|^2, \Sigma_j \|u^{D_j}\|^2 < \infty$.

Put $D \equiv \Sigma_j^{\oplus} D_j, v \equiv \Sigma_j^{\oplus} v^{D_j}, u \equiv \Sigma_j^{\oplus} u^{D_j}$; then

$$\text{for any } g, h \in G, \quad \langle T_g^D v, T_h^D u\rangle \equiv 0,$$

that is, $[\{T_g^D v \mid g \in G\}] \perp [\{T_h^D u \mid h \in G\}]$.

The condition (B-2) shows that the operator J^D of isobirepresentation $\mathbf{J} \equiv \{J^D\}_{D\in\Omega}$ keeps invariant subspaces; therefore $[\{T_g^D v \mid g \in G\}] \perp J^D u$. This concludes

$$0 = \langle T_g^D v, J^D u\rangle = \Sigma_j \langle T_g^{D_j} v^{D_j}, J^{D_j} u^{D_j}\rangle. \tag{2.14}$$

Therefore the map (2.13) generates a *-algebra homomorphism

$$f^{\sim}(g) \longrightarrow \theta_{\mathbf{J}}(f^{\sim})(e) \equiv f^{\sim}(x_J) \tag{2.15}$$

of the space $\mathfrak{F}$, and of $\mathfrak{F}^C$ to C; that is, it gives an element $x_J \in X$.

Put $f^{\sim}(g) \equiv \langle T_g^D v^D, J^D v^D\rangle$, and apply (2.12). We obtain

$$\begin{aligned}
\big|f^{\sim}(g_0) - f^{\sim}(x_J)\big| &= \big|\langle T_{g_0}^D v^D, J^D v^D\rangle - \langle J^D v^D, J^D v^D\rangle\big| \\
&= \big|\langle T_{g_0}^D v^D, J^D v^D\rangle - 1\big| = \big|1 - K(g_0^{-1})\big| < \delta.
\end{aligned} \tag{2.16}$$

This proves (2.4). □

In [7, Sections 7, 8], we give the following conditions to a topological group G:

(W-3) For any cyclic unitary representation

$$D \equiv \{\mathcal{H}^D, T_g^D, v^D\} \quad \big(\|v^D\| = 1\big),$$

and any birepresentation $\mathbf{U} \equiv \{U^D\}_D$, there holds

$$\sup_{g\in G} \big|\langle T_g^D U^D v^D, v^D\rangle\big| = 1;$$

(W-3′) For any cyclic unitary representation

$$D \equiv \{\mathcal{H}^D, T_g^D, v^D\} \quad (\|v^D\| = 1),$$

and any isobirepresentation $\mathbf{J} \equiv \{J^D\}_D$, there holds

$$\sup_{g\in G} \left|\langle T_g^D J^D v^D, v^D\rangle\right| = 1.$$

After the above argument, we obtain the following lemma.

LEMMA 2.3

If a topological group G has an SSUR, then (W-3′) holds for G, and so does (W-3).

We defined *well-behaved group* as a topological group which satisfies the conditions (T-1), (T-2) ((W-1), (W-2) in [8]), and (W-3), and we defined *strongly well-behaved group* as a topological group which satisfies the conditions (T-1), (T-2′) ((W-1), (W-2′) in [7]), and (W-3′). Therefore we ge the following.

COROLLARY 2.3.1

(1) *A T-type group is a well-behaved group.*

(2) *A strongly T-type group is a strongly well-behaved group.*

As in Section 2, for representations of type (D_p), the set $\{(D_p)\}_D$ gives an SSUR of G, and also

$$\inf_{g\in G}\left(1 - K^{D_p}(g)\right) = 0. \tag{2.17}$$

We denote by Ω_+ the set of all cyclic representations $D = (\mathcal{H}^D, T_g^D, v^D)$ ($\|v^D\| = 1$) satisfying

$$K^D(g) = \langle T_g^D J^D v^D, v^D\rangle \geq 0 \quad (g \in G).$$

As was shown, Ω_+ contains cyclic representations of type (D_p).

Introduce a subset of G,

$$F(D,\varepsilon) \equiv \left\{g \,\middle|\, 1 - K^D(g^{-1}) < \varepsilon\right\} \tag{2.18}$$

for $\varepsilon > 0, D \in \Omega_+$, and consider the family of subsets

$$\mathrm{Z} \equiv \left\{F(D,\varepsilon)\right\}_{D\in\Omega_+,\varepsilon>0}. \tag{2.19}$$

LEMMA 2.4

Let G be a topological group with SSUR, and let $\mathbf{J}$ be an isobirepresentation of G. Then Z gives a Cauchy filter base on G by the order of sets inclusion.

If a given $\mathbf{J}$ is a birepresentation $\mathbf{U} \equiv \{U^D\}_D$, that is, for any $D, J^D (= U^D)$ is a unitary operator, then Z is b-Cauchy.

Proof

The arguments are almost the same as in [8, Sections 7, 8]. So we will trace the argument there shortly.

Lemma 2.2 shows that for any $D \in \Omega_+$ and $\varepsilon > 0$, $F(D,\varepsilon)$ is not empty:

$$\varepsilon_1 > \varepsilon_2 \quad \Longrightarrow \quad F(D,\varepsilon_1) \supseteq F(D,\varepsilon_2). \tag{2.20}$$

And for $D^0 \equiv (D^1 \otimes D^2)$, we obtain

$$1 - K^{D^0}(g^{-1}) \geq 1 - K^{D^1}(g^{-1}),\ 1 - K^{D^2}(g^{-1}). \tag{2.21}$$

Therefore

$$F(D^1,\varepsilon) \cap F(D^2,\varepsilon) \supseteq F(D^0,\varepsilon) \neq \phi. \tag{2.22}$$

This shows that Z is a filter base.

To show that Z is l-Cauchy, we showed in [8, (7.8)] that the condition $1 - K^D(g^{-1}) < \varepsilon$ leads to

$$\|U^D v^D - T_g^D v^D\| \leq (2\varepsilon)^{1/2}. \tag{2.23}$$

Using this relation we get for any $g, h \in F(D,\varepsilon)$,

$$\|T_g^D v^D - T_h^D v^D\| = \|T_{h^{-1}g}^D v^D - v^D\| \leq 2(2\varepsilon)^{1/2}. \tag{2.24}$$

For an arbitrary given neighborhood V of e in G, if we take the above D in Ω_+ as

$$\{g \in G \mid |\langle T_g^D v^D - v^D, v^D\rangle| < \delta\} \subset V,$$

then $4\varepsilon < \delta$ leads to

$$F(D,\varepsilon)^{-1} F(D,\varepsilon) \subset V, \tag{2.25}$$

that is, Z is l-Cauchy.

Next we consider the case where a given isobirepresentation $\mathbf{J}$ is a birepresentation $\mathbf{U} = \{U^D\}_D$.

We will show that the family of subsets

$$\mathrm{Z}^{-1} \equiv \{F(D,\varepsilon)^{-1}\}_{D\in\Omega_+,\varepsilon>0} \tag{2.26}$$

gives a Cauchy filter base. In the above arguments, we proved that $\mathrm{Z} \equiv \{F(D,\varepsilon)\}$ gives a filter base, so $\mathrm{Z}^{-1} \equiv \{F(D,\varepsilon)^{-1}\}$ is also a filter base. Hence it is sufficient to see that Z^{-1} is Cauchy.

For an arbitrary given neighborhood V_0 of e in G and $\delta > 0$, take $D \in \Omega_+$ and a normalized vector w in $\mathcal{H}^D$ as

$$\{g \in G \mid |\langle T_g^D w - w, w\rangle| < \delta\} \subset V_0.$$

Since U^D is unitary, we can take $v^D \equiv (U^D)^{-1} w$ and $\varepsilon > 0$ as $\varepsilon < \delta$.

Consider $K^D(g) = \langle T_g^D U^D v^D, v^D\rangle \geq 0$ $(g \in G)$, and consider $F(D,\varepsilon)$ as above.

As in (2.23), the relation $1 - K^D(g^{-1}) < \varepsilon$ gives that for any $g \in F(D,\varepsilon)$,

$$\begin{aligned} \|T^D_{g^{-1}}w - (U^D)^{-1}w\| &= \|w - T^D_g(U^D)^{-1}w\| \\ &= \|U^D v^D - T_g v^D\| \le (2\varepsilon)^{1/2}. \end{aligned} \tag{2.27}$$

Consequently we see that for any $g, h \in F(D,\varepsilon)$,

$$\begin{aligned} \|T^D_{hg^{-1}}w - w\| &= \|T^D_{g^{-1}}w - T^D_{h^{-1}}w\| \\ &\le \|T^D_{g^{-1}}w - (U^D)^{-1}w\| + \|(U^D)^{-1}w - T^D_{h^{-1}}w\| \le 2(2\varepsilon)^{1/2}. \end{aligned}$$

By analogous arguments as after (2.24) or in the proof of [8, Section 7, Lemma 7.4], this leads to $hg^{-1} \in V_0$. Therefore

$$F(D,\varepsilon)F(D,\varepsilon)^{-1} \subset V_0.$$

This shows that $\{F(D,\varepsilon)^{-1}\}$ gives a Cauchy filter, and Z is b-Cauchy. □

3. Proof of a Tannaka-type weak duality theorem for T-type groups and strongly T-type groups

PROPOSITION 3.1

(1) *For a T-type group G, a weak Tannaka-type u-duality theorem is valid.*

(2) *For a strongly-T-type group G, a weak Tannaka-type i-duality theorem is valid.*

Proof

We fix an isobirepresentation $\mathbf{J} \equiv \{J^D\}$ and show that there exists a unique g in G such that

$$\{J^D\} = \{T^D_g\}. \tag{3.1}$$

Lemma 2.4 shows that for T-type (also for strongly T-type) group G and $\mathbf{J}$ as above, $\mathrm{Z} \equiv \{F(D,\varepsilon)\}_{D\in\Omega_+,\varepsilon>0}$ gives a Cauchy filter base.

Especially in the case that $\mathbf{J}$ is a birepresentation, Z give a b-Cauchy filter base.

Therefore if G is T-type and $\mathbf{J}$ is a birepresentation, or G is strongly-T-type and $\mathbf{J}$ is an isobirepresentation, Z converges to a point in G.

In both cases we write this limit point as $(g_{\mathbf{J}})^{-1}$. Then,

$$\bigcap_{(D,\varepsilon)} \overline{F(D,\varepsilon)} = \{(g_{\mathbf{J}})^{-1}\}. \tag{3.2}$$

So $1 = \langle T^D_{(g_{\mathbf{J}})^{-1}} J^D v^D, v^D\rangle$, and

$$\forall D \in \Omega_+, \quad J^D v^D = T^D_{g_{\mathbf{J}}} v^D. \tag{3.3}$$

For a general cyclic representation D, consider $(D_p) \in \Omega_+$ as in Section 1; then we obtain from $J^{D_p}v_p = T^{D_p}_{g_{\mathbf{J}}} v_p$,

$$Iw_0 \oplus J^D w \oplus J^{\overline{D}}\overline{w} = Iw_0 \oplus T^D_{g_{\mathbf{J}}} w \oplus T^{\overline{D}}_{g_{\mathbf{J}}}\overline{w}. \tag{3.4}$$

So for any D in Ω, we get $J^D w = T^D_{gJ} w$.

This concludes the proof of the assertion. □

4. Converse of duality theorems for T-type or strongly T-type groups

PROPOSITION 4.1

For a T_2-topological group G, if a weak Tannaka-type u-duality theorem holds, then G has an SSUR; that is, the condition (T-1) in Definition 1.3 is satisfied.

Proof

By Lemma 1.1, the inverse map of (1.5) must be continuous. A fundamental system of neighborhoods V of e in the image G_J of G in $\mathcal{U}$ is given as the collection of

$$V_1 \equiv \bigcap_{1\le j\le n} \{\mathbf{T}_g = (T^D_g)_{D\in\Omega} \mid \|v_j - T^{D_j}_g v_j\|^2 < \varepsilon_j\} \tag{4.1}$$

for a finite set $\{(D_j, v_j, \varepsilon_j)\}$, where $D_j \in \Omega$ and $v_j \in \mathcal{H}^{D_j}$ ($\|v_j\| = 1$), $\varepsilon_j > 0$ ($j = 1, 2, \ldots, n$).

Consider the representations $D_0 \equiv \Sigma^{\oplus}_j D_j$ and $v_0 = n^{-(1/2)} \Sigma^{\oplus}_j v_j$, $\varepsilon_0 = \mathrm{Min}_j\, \varepsilon_j$; then

$$V_1 \supseteq V_2(\varepsilon_0) \equiv \{\mathbf{T}_g = (T^D_g)_{D\in\Omega} \mid \|v_0 - T^{D_0}_g v_0\|^2 < \varepsilon_0\}. \tag{4.2}$$

The evaluation

$$\begin{aligned} \|v_0 - T^{D_0}_g v_0\|^2 &= 2\big(1 - \Re(\langle T^{D_0}_g v_0, v_0\rangle)\big) \\ &\le 2\big|1 - \langle T^{D_0}_g v_0, v_0\rangle\big| \end{aligned} \tag{4.3}$$

shows that, if we take $\delta < 2^{-1}\varepsilon_0$, then

$$V_2(\varepsilon_0) \supset V_\delta \equiv \{\mathbf{T}_g = (T^D_g)_{D\in\Omega} \mid |1 - \langle T^{D_0}_g v_0, v_0\rangle| < \delta\}. \tag{4.4}$$

Since the inverse map of (1.5) is continuous, for any neighborhood V of e in G, there exist $V, V_1, V_2(\varepsilon_0)$, and V_δ such that

$$V \supseteq V_1 \supseteq V_2(\varepsilon_0) \supseteq V_\delta. \tag{4.5}$$

This shows the separating condition (T-1) in Definition 1.3. □

As is shown in Proposition 0.3, u-duality follows from i-duality. Then we have the following.

COROLLARY 4.1.1

For a topological group G, if the weak Tannaka-type i-duality theorem holds, then the condition (T-1) is satisfied.

Now we discuss about the conditions (T-2) and (T-2′).

PROPOSITION 4.2

Let G be a T_2-topological group.

(1) *For the weak Tannaka-type i-duality theorem to hold, G must be complete; that is, the condition (T-2′) should hold.*

(2) *For the weak Tannaka-type u-duality theorem to hold, G must be b-complete; that is, the condition (T-2) should hold.*

Proof

The map (1.6) is isomorphic. We consider the image G_J in $\mathbf{U}(\Omega) \subset \mathbf{J}(\Omega)$.

Lemma 1.3 asserts that a Cauchy filter base $\mathcal{F}_J$ on G_J has a limit $\mathbf{B} \equiv (B^D)_D$ in $\mathbf{J}(\Omega)$. And if $\mathcal{F}$ is a b-Cauchy filter base, then $\mathbf{B}$ is in $\mathbf{U}(\Omega)$.

Obviously any element of G_J satisfies the conditions (B-1)–(B-4) of birepresentation and isobirepresentation in Section 0. And these conditions are valid for its limit point $\mathbf{B} \equiv (B^D)_D$ of these elements too.

Consequently, we see that $\mathbf{B}$ is an isobirepresentation. If $\mathbf{B}$ is in $\mathbf{U}(\Omega)$, it is a birepresentation too.

If the weak Tannaka-type i-duality theorem holds for G, then $\mathbf{B} \in G_J$. This shows that any Cauchy filter base $\mathcal{F}$ on G_J converges to a point in G_J; that is, G_J is complete. Equivalently G must be complete; that is, G is strongly T-type.

In the case where the weak Tannaka-type u-duality theorem holds for G, a b-Cauchy filter base $\mathcal{F}$ must converge to a point $\mathbf{B}$ in $\mathcal{U}(\Omega)$. That is, $\mathbf{B}$ turns to be a birepresentation. So the assumption that the weak Tannaka-type u-duality is valid for G induces the assertion that G is b-complete. Hence, in this case, G is a T-type group. □

5. Case of locally compact groups and NOS-groups

We consider the case of locally compact groups. Here we quote the papers [3]–[5], and [8].

In [3] and [4], we have shown that, for locally compact groups, the b-duality theorem is valid. And in [5] and [8], it is shown that b-duality is false if the group is not locally compact.

Altogether we showed

$$\text{b-duality} \quad \Longleftrightarrow \quad \text{locally compact.}$$

And it is remarkable that for locally compact groups to prove the weak duality theorem we do not need the condition (B-4) in the definition of birepresentations (cf. [3], [4]).

Next we consider the case where c-duality is false for a locally compact group G.

In [4], we introduced so-called Katz–Takesaki operator $W_{\mathfrak{R}}$ on the regular representation $\mathfrak{R} \equiv (\mathfrak{H}, R_g), \mathfrak{H} = L^2(G)$ of a given locally compact group G. $W_{\mathfrak{R}}$ is a unitary operator defined on the space $\mathfrak{H} \otimes \mathfrak{H}$. And $W_{\mathfrak{R}}$ gives an intertwining operator from $\mathfrak{R} \otimes \mathfrak{R}$ to $\Sigma^{\oplus}\mathfrak{R}$ (a multiple of $\mathfrak{R}$). $W_{\mathfrak{R}}$ is written for any fixed

CONS $\{\Phi_\alpha\}_\alpha$ in $\mathfrak{H}$ as

$$\big(W_{\mathfrak{R}}(f_1 \otimes f_2)\big)(g) = \big\{\langle R_g f_2, \Phi_\alpha\rangle f_1(g)\big\}_\alpha \in \Sigma_\alpha^\oplus \mathfrak{H} \quad (f_1 \otimes f_2 \in \mathfrak{H} \otimes \mathfrak{H})$$

(cf. [4]).

The group G satisfies the b-duality theorem, so G is identified as

$$G \approx G_R \equiv \big\{U \,\big|\, W_{\mathfrak{R}}(U \otimes U) = (I \otimes U)W_{\mathfrak{R}} \neq 0\big\} \subset U(\mathfrak{H}). \tag{5.1}$$

To extend the discussion to the case of c-duality, we must extend G_R to the space

$$G_C^R \equiv \big\{C \,\big|\, W_{\mathfrak{R}}(C \otimes C) = (I \otimes C)W_{\mathfrak{R}} \neq 0\big\} \subset C(\mathfrak{H}). \tag{5.2}$$

Here $C(\mathfrak{H})$ is the space of closed operators C on $\mathfrak{H}$ with a common domain $\mathcal{D}_R$ satisfying (B-041)–(B-044) for the case

$$D_1 = D_2 = \mathfrak{R}.$$

An element C of G_C^R is called an *admissible operator*. It is a closed operator with domain $\mathcal{D}_R$ and range in the same $\mathcal{D}_R$ satisfying

$$W_{\mathfrak{R}}(C \otimes C) = (I \otimes C)W_{\mathfrak{R}}. \tag{5.3}$$

We must show the following proposition which is an extension of [3, Lemma 2.4].

PROPOSITION 5.1

For any cl-birepresentation $\mathbf{C} = \{C^D\}_{D\in\Omega}$, its component $C^{\mathfrak{R}}$ on the regular representation $\mathfrak{R}$ is an admissible operator. For two cl-birepresentations $\mathbf{C}_1$ and $\mathbf{C}_2$, if

$$C_1^{\mathfrak{R}} = C_2^{\mathfrak{R}}, \tag{5.4}$$

then $\mathbf{C}_1 = \mathbf{C}_2$, that is, the same $C_1^D = C_2^D$ for any D in Ω.

Proof

The proof is done similarly to the argument in [3]. It was shown that $D \otimes \mathfrak{R}$ is unitary equivalent to $\Sigma_{\mathbf{dim}D}^\oplus \mathfrak{R}$ with an intertwining operator W_D. So, for a cl-birepresentation $\mathbf{C} = \{C^D\}$,

$$W_D(C^D \otimes C^{\mathfrak{R}}) = (I \otimes C^{\mathfrak{R}})W_D. \tag{5.5}$$

When $D = \mathfrak{R}$, this shows that the operator $C^{\mathfrak{R}}$ is an admissible operator.

For two $\mathbf{C}_i \equiv (C_i^D)_{D\in\Omega}$ $(i = 1, 2)$,

$$W_D(C_1^D \otimes C^{\mathfrak{R}}) = (I \otimes C^{\mathfrak{R}})W_D = W_D(C_2^D \otimes C^{\mathfrak{R}}); \tag{5.6}$$

that is,

$$C_1^D u \otimes C^{\mathfrak{R}} v = C_2^D u \otimes C^{\mathfrak{R}} v, \tag{5.7}$$

for any u in $\mathcal{D}^D$ and v in $\mathcal{D}^{\mathfrak{R}}$. The nonzero assumption of cl-birepresentation leads $C_1^D u = C_2^D u$ for any u in $\mathcal{D}^D$, that is,

$$C_1^D = C_2^D \tag{5.8}$$

for any D in Ω. □

After Proposition 5.1, to show c-duality, it is sufficient to show that for any admissible operator C, there exists an element g in G such that $C = R_g$.

PROPOSITION 5.2

Let C_1 and C_2 be two admissible operators; then their product C_1C_2 is also an admissible operator.

Proof

$$\begin{aligned} W_R(C_1C_2 \otimes C_1C_2) &= W_{\mathfrak{R}}(C_1 \otimes C_1)(C_2 \otimes C_2) = (I \otimes C_1)W_{\mathfrak{R}}(C_2 \otimes C_2) \\ &= (I \otimes C_1)(I \otimes C_2)W_{\mathfrak{R}} = (I \otimes C_1C_2)W_{\mathfrak{R}}. \end{aligned}$$ □

PROPOSITION 5.3

For an admissible operator C, its conjugate operator $\overline{C}$ is an admissible operator too.

Proof

We have

$$\begin{aligned} W_{\mathfrak{R}}(\overline{C} \otimes \overline{C}) &= \overline{(C \otimes C)\overline{W_{\mathfrak{R}}}} = \overline{(C \otimes C)W_{\mathfrak{R}}^{-1}} \\ &= \overline{W_{\mathfrak{R}}^{-1}(I \otimes C)} = \overline{\overline{W_{\mathfrak{R}}}(I \otimes C)} = (I \otimes \overline{C})W_{\mathfrak{R}}. \end{aligned} \tag{5.9}$$ □

COROLLARY 5.3.1

For an admissible operator C, the positive definite operator $\overline{C}C$ (i.e., C_P) is an admissible operator too; that is,

$$W_{\mathfrak{R}}(C_P \otimes C_P) = (I \otimes C_P)W_{\mathfrak{R}}. \tag{5.10}$$

Proof

From Propositions 5.2 and 5.3, the result is direct. □

COROLLARY 5.3.2

For an admissible operator C, the positive definite operator $(C_P)^t \equiv (\overline{C}C)^t$ ($t \in$ $\mathbf{R}$) is an admissible operator too.

Proof

For positive definite operators C_P, we can define uniquely t-power $(C_P)^t$:

$$\begin{aligned} (C_P)^t \otimes (C_P)^t &= (C_P \otimes C_P)^t = \big(W_{\mathfrak{R}}^{-1}(I \otimes C_P)W_{\mathfrak{R}}\big)^t \\ &= W_{\mathfrak{R}}^{-1}(I \otimes C_P)^t W_{\mathfrak{R}} = W_{\mathfrak{R}}^{-1}\big(I \otimes (C_P)^t\big)W_{\mathfrak{R}}. \end{aligned} \tag{5.11}$$

This shows that $(C_P)^t$ is an admissible operator. □

PROPOSITION 5.4

Take the spectral decomposition

$$(C_P)^t = \int_0^\infty \lambda^t \, dE(\lambda)$$

in Corollary 5.3.2, and consider the self-adjoint operator

$$L \ (\equiv \log C_P) \equiv \int_0^\infty (\log \lambda) \, dE(\lambda);$$

then

$$W_{\mathfrak{R}}(L \otimes I + I \otimes L) = (I \otimes L) W_{\mathfrak{R}}. \tag{5.12}$$

Proof

For v and u in $\mathcal{D}_R$, take derivatives with respect to t of the both sides of (5.11) in the strong sense, and put $t = 0$ as shown below:

$$\begin{aligned}
\Big(\int_0^\infty \lambda^t \, dE(\lambda) \otimes \int_0^\infty \lambda^t \, dE(\lambda)\Big)(u \otimes v) &= (C_P)^t \otimes (C_P)^t (u \otimes v) \\
&= W_{\mathfrak{R}}^{-1}\big(I \otimes (C_P)^t\big) W_{\mathfrak{R}}(u \otimes v) \\
&= W_{\mathfrak{R}}^{-1}\Big(I \otimes \int_0^\infty \lambda^t \, dE(\lambda)\Big) W_{\mathfrak{R}}(u \otimes v).
\end{aligned}$$

Then

$$(d/dt)(C_P)^t u|_{t=0} = \int_0^\infty \log(\lambda) \, dE(\lambda) u \equiv Lu, \tag{5.13}$$

$$\begin{aligned}
(d/dt)(C_P)^t (u \otimes v)|_{t=0} &= (Lu) \otimes v + u \otimes (Lv) \\
&= (L \otimes I + I \otimes L)(u \otimes v),
\end{aligned} \tag{5.14}$$

and combining these results, we get

$$W_{\mathfrak{R}}(L \otimes I + I \otimes L)(u \otimes v) = (I \otimes L) W_{\mathfrak{R}}(u \otimes v). \tag{5.15}$$

□

PROPOSITION 5.5

For a nonunitary admissible operator C, there exists a one-parameter subgroup $\{(C_P)^{it} \ (t \in \mathbf{R})\}$ of admissible unitary operators.

Proof

Since the operator C is not unitary, the above C_P is not I, so L is not zero, $\{(C_P)^{it} = \exp(itL) \ (t \in \mathbf{R})\}$ is a group of unitary operators, and

$$\begin{aligned}
(C_P)^{it} \otimes (C_P)^{it} &= \exp(itL) \otimes \exp(itL) \\
&= \exp\big(it(L \otimes I + I \otimes L)\big) \\
&= W_{\mathfrak{R}}^{-1} \exp\big(it(I \otimes L)\big) W_{\mathfrak{R}} \\
&= W_{\mathfrak{R}}^{-1}\big(I \otimes (C_P)^{it}\big) W_{\mathfrak{R}}.
\end{aligned}$$

Therefore $(C_P)^{it}$ is an admissible operator. □

PROPOSITION 5.6

A locally compact NOS-group G satisfies c-duality.

Proof

If c-duality is not valid for G, there exists a nontrivial cl-birepresentation $\mathbf{C} \equiv (C^D)_{D\in\Omega}$ of G, which is not a birepresentation. Let $C^{\mathfrak{R}}$ be its component on the regular representation $\mathfrak{R}$.

By Proposition 5.1, $C^{\mathfrak{R}}$ is nonzero. If $C^{\mathfrak{R}}$ is unitary, the b-duality theorem for locally compact groups asserts that $C^{\mathfrak{R}}$ must be an element of G_R. So by Proposition 5.1, $\mathbf{C}$ must be a birepresentation.

Consequently, $C^{\mathfrak{R}}$ is nonzero and nonunitary.

Thus, using Proposition 5.5, we get a one-parameter subgroup of admissible unitary operators. Again by the b-duality theorem, G must have a one-parameter subgroup. This contradicts our assumption that G is an NOS-group. □

Next we consider the case where the c-duality theorem is valid for a topological group G.

PROPOSITION 5.7

If G has a one-parameter subgroup, then the c-duality theorem is false for G.

Proof

Let $K = \{g_t\}_t$ be a one-parameter subgroup of G. Then $\{T^D_{g_t}\}_t$ is a one-parameter subgroup in $\mathcal{U}(\mathcal{H}^D)$ for any D in Ω. Especially for the regular representation, $\{R_{g_t}\}_t$ is a one-parameter subgroup in $\mathcal{U}(L^2(G))$.

For a C^∞-function f with compact support on $\mathbf{R}$, we define

$$T^D_f \equiv \int f(t) T^D_{g_t}\, dt. \tag{5.16}$$

Then, for any v^D in $\mathcal{H}^D$, $T^D_f v^D$ is a C^∞-vector with respect to the parameter t.

By Stone's theorem, there exists a self-adjoint operator A^D on the space $\mathcal{H}^D$, and $\{T^D_f v^D \mid v^D \in \mathcal{H}^D, f \in C^\infty(\mathbf{R})\}$ spans the domain of A^D and

$$T^D_{g_t} = \exp(iA^D t) \quad (\forall t \in \mathbf{R}) \quad = \int e^{i\lambda t}\, dE^D(\lambda). \tag{5.17}$$

Here

$$A^D = \int \lambda\, dE^D(\lambda) \tag{5.18}$$

is the spectral decomposition of A.

Since $T^D_{g_t}$ satisfies (B-1)–(B-3),

$$\exp(iA^{D_1} t) \otimes \exp(iA^{D_2} t) = \exp(iA^{D_1 \otimes D_2} t). \tag{5.19}$$

For the case where D_2 is a regular representation,

$$\exp(iA^{D_1}t)\otimes\exp(iA^{\mathfrak{R}}t)=W_{D_1}^{-1}\big(I\otimes\exp(iA^{\mathfrak{R}}t)\big)W_{D_1}. \tag{5.20}$$

Here W_{D_1} is the intertwining operator corresponding to the equivalence relation of

$$D_1\otimes\mathfrak{R}\sim_{W_{D_1}}\Sigma^{\oplus}_{\dim(D_1)}\mathfrak{R}. \tag{5.21}$$

Moreover, when D_1 is $\mathfrak{R}$,

$$\exp(iA^{\mathfrak{R}}t)\otimes\exp(iA^{\mathfrak{R}}t)=W_{\mathfrak{R}}^{-1}\big(I\otimes\exp(iA^{\mathfrak{R}}t)\big)W_{\mathfrak{R}}. \tag{5.22}$$

In the cases (5.19) and (5.20), we differentiate with respect to t both sides and put $t=0$; then we obtain

$$A^{D_1}\otimes I_{\mathfrak{R}}+I_{D_1}\otimes A^{\mathfrak{R}}=W_{D_1}^{-1}(I_D\otimes A^{\mathfrak{R}})W_{D_1},$$
$$A^{\mathfrak{R}}\otimes I_{\mathfrak{R}}+I_{\mathfrak{R}}\otimes A^{\mathfrak{R}}=W_{\mathfrak{R}}^{-1}(I_{\mathfrak{R}}\otimes A^{\mathfrak{R}})W_{\mathfrak{R}}.$$

For the spectral decomposition (5.18) of A^D and for any $a,b\in\mathbf{R}$, take the projection operator

$$P^D_{(a,b)}\equiv\int_a^b dE^D(\lambda).$$

The subspace $\mathcal{D}^D_0\equiv\bigcup_{(a,b)}P_{(a,b)}\mathcal{H}^D$ is dense in $\mathcal{H}^D$ and contained in the domain of A^D.

Put for any $v\in\mathcal{D}^D_0$,

$$C^D_0v\equiv\exp(A^D)v=\sum_{k=0}^{\infty}(k!)^{-1}(A^D)^kv. \tag{5.23}$$

The right-hand side converges in the norm sense. C^D_0 is closable, and its closure C^D is a positive definite operator.

Moreover, for any $v^D\in\mathcal{D}^D_0$ and $v^{\mathfrak{R}}\in\mathcal{D}^{\mathfrak{R}}_0$,

$$\begin{aligned}
&(C^D_0\otimes C^{\mathfrak{R}})(v^D\otimes v^{\mathfrak{R}})\\
&\quad=(C^D_0v^D\otimes C^{\mathfrak{R}}v^{\mathfrak{R}})\\
&\quad=\Big(\sum_{k=0}^{\infty}(k!)^{-1}(A^D)^k\Big)v^D\otimes\Big(\sum_{k=0}^{\infty}(k!)^{-1}(A^{\mathfrak{R}})^k\Big)v^{\mathfrak{R}}\\
&\quad=\sum_{k=0}^{\infty}(k!)^{-1}(A^D\otimes I_{\mathfrak{R}}+I_D\otimes A^{\mathfrak{R}})^k(v^D\otimes v^{\mathfrak{R}}) \qquad (5.24)\\
&\quad=W_D^{-1}\Big(\sum_{k=0}^{\infty}(k!)^{-1}(I_D\otimes A^{\mathfrak{R}})^k\Big)W_D(v^D\otimes v^{\mathfrak{R}})\\
&\quad=W_D^{-1}\Big(I_D\otimes\Big(\sum_{k=0}^{\infty}(k!)^{-1}(A^{\mathfrak{R}})^k\Big)\Big)W_D(v^D\otimes v^{\mathfrak{R}})\\
&\quad=W_D^{-1}(I_D\otimes C^{\mathfrak{R}})W_D(v^D\otimes v^{\mathfrak{R}}),
\end{aligned}$$

$$
\begin{aligned}
&(C^{\mathfrak{R}} \otimes C^{\mathfrak{R}})(v_1^{\mathfrak{R}} \otimes v_2^{\mathfrak{R}}) \\
&\quad = (C^{\mathfrak{R}} v_1^{\mathfrak{R}} \otimes C^{\mathfrak{R}} v_2^{\mathfrak{R}}) \\
&\quad = \Big(\sum_{k=0}^{\infty} (k!)^{-1} (A^{\mathfrak{R}})^k\Big) \otimes \Big(\sum_{k=0}^{\infty} (k!)^{-1} (A^{\mathfrak{R}})^k\Big) (v_1^{\mathfrak{R}} \otimes v_2^{\mathfrak{R}}) \\
&\quad = \Big(\sum_{k=0}^{\infty} (k!)^{-1} (A^{\mathfrak{R}} \otimes I_{\mathfrak{R}} + I_R \otimes A^{\mathfrak{R}})^k\Big) (v_1^{\mathfrak{R}} \otimes v_2^{\mathfrak{R}}) \\
&\quad = W_{\mathfrak{R}}^{-1} \Big(I_R \otimes \Big(\sum_{k=0}^{\infty} (k!)^{-1} (A^{\mathfrak{R}})^k\Big)\Big) W_{\mathfrak{R}} (v_1^{\mathfrak{R}} \otimes v_2^{\mathfrak{R}}) \\
&\quad = W_{\mathfrak{R}}^{-1} (I_R \otimes C^{\mathfrak{R}}) W_{\mathfrak{R}} (v_1^{\mathfrak{R}} \otimes v_2^{\mathfrak{R}}).
\end{aligned} \tag{5.25}
$$

This equality is valid on dense sets of domains of operators in both sides, so taking the closure of operators, we get that $C^{\mathfrak{R}}$ is a nonzero, nonunitary admissible operator, and $\mathbf{C} \equiv \{C^D\}$ gives a cl-birepresentation not belonging to G_J.

This shows that c-duality is false for G. □

6. Main theorem and examples

Summarizing the results in Sections 3–5, we obtain the following.

MAIN THEOREM

Let G be a T_2-topological group.

(1) *For G, the weak Tannaka-type u-duality theorem holds if and only if G is a T-type group.*

(2) *For G, the weak Tannaka-type i-duality theorem holds if and only if G is a strongly T-type group.*

(3) *For G, the weak Tannaka-type b-duality theorem holds if and only if G is a locally compact group.*

(4) *For G, the weak Tannaka-type c-duality theorem holds if and only if G is a locally compact NOS-group.*

EXAMPLE 1

Let $\mathcal{H}$ be a Hilbert space of infinite dimension, and let $G \equiv U(\mathcal{H})$ be the group of all unitary operators on $\mathcal{H}$ with the weak (resp., strong) topology of operator space.

LEMMA 6.1

The group $G = U(\mathcal{H})$ is a topological group and has an SSUR.

Proof
G is a topological group by the strong topology, and this topology is equivalent to the weak one. Consider the identical representation $D_0 \equiv \{\mathcal{H}, T_U\}$,

$$G \ni U \longmapsto T_U \ (\equiv U) \in U(\mathcal{H}). \tag{6.1}$$

Evidently this representation is cyclic with cyclic vector v, which is any normalized vector in $\mathcal{H}$. We denote this cyclic representation as $D_v \equiv \{\mathcal{H}, T_U, v\}$ and take the family $\Omega_0 \equiv \{D_v\}_v$. We show that this family gives an SSUR of G.

A fundamental system of neighborhoods of $e\ (= I)$ is given by the family of sets

$$\mathcal{V} \equiv \big\{V(v,\varepsilon) \equiv \big\{U \in G \,\big|\, \|Uv - v\| < \varepsilon\big\} \,\big|\, v \in \mathcal{H}, \|v\| = 1, \varepsilon > 0\big\}.$$

For any D_v in Ω_0, $\|Uv - v\|^2 = 2(1 - \Re\langle Uv, v\rangle)$ and $|1 - \langle T_U v, v\rangle|^2 = |1 - \Re(\langle T_U v, v\rangle)|^2 + |\Im(\langle T_U v, v\rangle)|^2$, so

$$\big|1 - \langle T_U v, v\rangle\big| \geq \big|1 - \Re\langle T_U v, v\rangle\big| = (1/2)\|Uv - v\|^2. \tag{6.2}$$

As in Section 1, we put $\eta(g) \equiv \langle T_U v, v\rangle$ and $F(D,\varepsilon) \equiv \{g \in G \mid |1 - \eta(g)| < \delta\}$.

This means that for $\delta \leq 2^{-1}\varepsilon^2$,

$$F(D,\delta) \subset V(v,\varepsilon); \tag{6.3}$$

that is, Ω_0 gives an SSUR for G. □

LEMMA 6.2
The group $G \equiv U(\mathcal{H})$ is b-complete but not complete.

Proof
Lemma 1.3(1) shows that any Cauchy filter base on G_J converges to an element in $\mathcal{J}(\Omega)$, and Lemma 1.3(2) shows that any b-Cauchy filter base converges to an element in $\mathcal{U}(\Omega)$.

The map $G = U(\mathcal{H}) \ni U \longrightarrow U \in U(\mathcal{H})$ gives a unitary representation of G. As a component of G_J, $U(\mathcal{H}) = G$ is b-complete.

In $\mathcal{H}$, we take a countable infinite orthonormal system $L \equiv \{v_1, v_2, \ldots\}$ and the closed subspace $\mathcal{H}_0$ spanned by L. We consider the unitary operator U_n which is identity on $(\mathcal{H}_0)^\perp$ and is defined on $\mathcal{H}_0$ as follows:

$$U_n\Big(\sum_j a_j v_j\Big) = a_n v_1 + \sum_{j=1}^{n-1} a_j v_{j+1} + \sum_{j=n+1}^{\infty} a_j v_j. \tag{6.4}$$

Then $U_m^{-1}(\sum_j a_j v_j) = \sum_{j=2}^{m} a_j v_{j-1} + a_1 v_m + \sum_{j=m+1}^{\infty} a_j v_j$. Therefore, for $n < m$,

$$\begin{aligned} &(U_m^{-1}U_n)\Big(\sum_j a_j v_j\Big) = \sum_{j=1}^{n-1} a_j v_j + \sum_{j=n}^{m-1} a_{j+1} v_j + a_n v_m + \sum_{j=m+1}^{\infty} a_j v_j, \\ &(U_m^{-1}U_n)\Big(\sum_j a_j v_j\Big) - \Big(\sum_j a_j v_j\Big) = \sum_{j=n}^{m-1} (a_{j+1} - a_j) v_j + (a_n - a_m) v_m. \end{aligned} \tag{6.5}$$

However,

$$\Big\|\sum_{j=n}^{m-1}(a_{j+1}-a_j)v_j+(a_n-a_m)v_m\Big\|$$
$$\le\Big\|\sum_{j=n}^{m-1}a_{j+1}v_j+a_nv_m\Big\|+\Big\|\sum_{j=n}^{m-1}a_jv_j+a_mv_m\Big\|$$
$$\le\Big(\sum_{j=n}^{m-1}\big|(a_{j+1})\big|^2+|a_n|^2\Big)^{1/2}+\Big(\sum_{j=n}^{m-1}|a_j|^2+|a_m|^2\Big)^{1/2}\le 2\Big(\sum_{j=n}^{\infty}|a_j|^2\Big)^{1/2}.$$

This shows that $\{U_n\}_n$ is a Cauchy sequence, and

$$U_n\Big(\sum_j a_jv_j\Big)-\sum_j a_jv_{j+1}=a_nv_1+\sum_{j=n+1}^{\infty}(a_j-a_{j-1})v_j$$

converges to zero as $n\to\infty$. So $\{U_n\}_n$ strongly (equivalently, weakly) converges to the operator

$$P\Big(\sum_j a_jv_j\Big)=\sum_j a_jv_{j+1}$$

which is valued in the subspace $\{v_1\}^{\perp}$ of $\mathcal{H}$. So P is not unitary.

This concludes that G is not complete. □

PROPOSITION 6.1

The group of unitary operators $G\equiv U(\mathcal{H})$ is a T-type group and not strongly T-type; therefore it satisfies u-duality but not i-duality.

EXAMPLE 2

Let $G_0\equiv\{U_g\}$ be a group of unitary operators on some Hilbert space $\mathcal{H}$. Introduce the strong (resp., weak) topology on G_0; then G_0 becomes a topological group.

Take b-completion G of G_0 in $J(\mathcal{H})$. Lemma 1.3(2) shows that $G\subset U(\mathcal{H})$. Therefore G is a b-complete topological group.

A similar argument as for Lemma 6.1 leads to the fact that this G has an SSUR.

Therefore G is a T-type group and satisfies the weak Tannaka-type u-duality.

EXAMPLE 3

We consider a sequence of topological groups as

$$G\supset G_1\supset G_2\supset\cdots\supset G_n\supset\cdots,\quad \bigcap G_n=\{e\}. \tag{6.6}$$

Here for each n, G_n is a closed normal subgroup of G and the factor group $G^{(n)}\equiv G/G_n$ is a locally compact topological group.

We give the topology on G the projective limit topology of $\{G^{(n)}\}$, that is, the weakest topology for which the canonical map

$$\varphi_n : G \longrightarrow G^{(n)} \tag{6.7}$$

is continuous and open.

We call such a G a projective limit group of $\{G^{(n)}\}$ and write

$$G \equiv \underset{n}{\mathbf{Proj\text{-}lim}}\{G^{(n)}\}. \tag{6.8}$$

Under such a situation, for any neighborhood V of e in G, there exists an n and a neighborhood V_n of e in $G^{(n)}$ such that

$$(\varphi_n)^{-1}(V_n) \subset V.$$

Any locally compact group has an SSUR, and any unitary representation of $G^{(n)}$ gives naturally a unitary representation of G through the canonical map φ_n. So G has an SSUR too.

Any element g in G corresponds to a sequence $\{g^{(n)}(\in G^{(n)})\}_n$ such that

$$g^{(n)} = \varphi_n(g),$$

and, for any Cauchy filter base $\mathcal{F} \equiv \{F_\alpha\}$ on G, $\mathcal{F}_n \equiv \{\varphi_n(F_\alpha)\}$ gives also a Cauchy filter base on a locally compact group $G^{(n)}$ which is complete. So $\mathcal{F}$ converges in G and G is complete.

Thus we get the following.

PROPOSITION 6.2

The limit group $G \equiv \mathbf{Proj\text{-}lim}_n\{G^{(n)}\}$ is strongly-T-type; therefore it satisfies i-duality.

EXAMPLE 4

In the papers [7] and [8], we have shown that inductive limit groups $G \equiv \lim_{n\to\infty} G_n$, where each G_n is locally compact and G_n is embedded as a closed subgroup into G_{n+1} but not locally homeomorphic, satisfies the i-duality theorem. In this situation, G is not locally compact.

So, this group gives an example which satisfies i-duality but not b-duality.

EXAMPLE 5

C. Chevalley gave a complexification of compact Lie groups in his book [1] using Tannaka duality for compact groups.

His argument is as follows. He gives a lemma for compact Lie groups.

LEMMA 6.3

A compact group is a Lie group if and only if it has a faithful finite-dimensional unitary representation $D \equiv \{\mathcal{H}, T_g\}$.

By this lemma, G is imbedded isomorphically into $\mathrm{GL}(d, \mathbf{C})$. We will write it as G_0. He considered the algebra $\mathfrak{F}$ of functions generated by all matrix elements $\mathfrak{F} \equiv \{\langle T_g u, v\rangle, \overline{\langle T_g u, v\rangle}\ (u, v \in \mathcal{H})\}$. That is, the same as $\mathfrak{F}$ is a *-algebra of functions generated by matrix elements of unitary representations through tensor products and direct sums of representations.

He considered the set $(G_0)^{\mathbf{C}} = \mathbf{Hom}(\mathfrak{F}, \mathbf{C})$ and showed that $(G_0)^{\mathbf{C}}$ must be an algebraic subgroup of $\mathrm{GL}(d, \mathbf{C})$ topologically isomorphic to $G \times \mathbf{R}^{\dim G}$.

We can extend any irreducible representation ω of G to a representation $\widetilde{\omega} \equiv \{\mathcal{H}^\omega, T^\omega_{g_0}\}$ of $(G_0)^{\mathbf{C}}$. But it may not be a unitary representation of $(G_0)^{\mathbf{C}}$.

Any unitary representation D of G is decomposable to a discrete direct sum as $D \sim \sum^\oplus_\alpha \omega_\alpha$, and according to this decomposition, we can consider $\widetilde{D} \sim \sum^\oplus_\alpha \widetilde{\omega}_\alpha$.

On the space $\widetilde{\mathcal{H}}^D \equiv \sum^\oplus_\alpha \mathcal{H}^{\omega_\alpha}$, the operator $(\widetilde{T}^D_{g_0})_0 = \sum^\oplus_\alpha T^{\omega_\alpha}_{g_0}$ is closable and $\{(\widetilde{T}^D_{g_0})_0\}_{g_0 \in (G_0)^{\mathbf{C}}}$ gives a representation of $(G_0)^{\mathbf{C}}$. Take the closure $\widetilde{T}^D_{g_0}$ of each $(\widetilde{T}^D_{g_0})_0$ on $\mathcal{H}^D$; then $\mathbf{T}^D \equiv \{\widetilde{T}^D_{g_0}\}_{g_0 \in (G_0)^{\mathbf{C}}}$ is also a representation of $(G_0)^{\mathbf{C}}$.

Moreover, the relations of elements of the algebra $\mathfrak{F}$ shows that these representations of $(G_0)^{\mathbf{C}}$ satisfy (B-1)–(B-4) in Section 1.

Consequently, we see that $\mathbf{T}_{g_0} \equiv \{\widetilde{T}^D_{g_0}\}_{D \in \Omega}$ gives a cl-birepresentation.

This complexification of a compact Lie group is an example of a group for which b-duality is valid but c-duality is not valid.

EXAMPLE 6

A totally disconnected locally compact group gives an example for which c-duality is valid, since it is a locally compact NOS-group.

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