Krein's strings whose spectral functions are of polynomial growth

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Abstract In the case of Krein's strings with spectral functions of polynomial growth a necessary and sufficient condition for the Krein's correspondence to be continuous is given.

1. Introduction

Let $\mathcal{M}$ be the totality of nondecreasing, right-continuous functions on $[0,\infty)$ satisfying

$$m(0-)=0, \qquad m(x)\le\infty,$$

and set

$$\begin{cases} l=\inf\{x\ge 0; m(x)=\infty\},\\ a=\inf\{x\ge 0; m(x)>0\}.\end{cases}$$

For $m\in\mathcal{M}$ denote $\varphi_\lambda(x),\psi_\lambda(x)$ the solutions to

$$\begin{cases} \varphi_\lambda(x)=1-\lambda\int_0^x (x-y)\varphi_\lambda(y)\,dm(y),\\ \psi_\lambda(x)=x-\lambda\int_0^x (x-y)\psi_\lambda(y)\,dm(y),\end{cases}$$

and define

$$h(\lambda)=\lim_{x\to l}\frac{\psi_\lambda(x)}{\varphi_\lambda(x)}=\int_0^l \varphi_\lambda(x)^{-2}\,dx.$$

Then it is known that there exists a unique measure σ on $[0,\infty)$ satisfying

$$h(\lambda)=a+\int_0^\infty \frac{1}{\xi-\lambda}\,d\sigma(\xi),$$

and conversely, h determines m uniquely. Conventionally it is understood that for $m\in\mathcal{M}$ taking ∞ identically on $[0,\infty)$ the h vanishing identically corresponds, and for $m\in\mathcal{M}$ vanishing identically on $[0,\infty)$ the h taking identically ∞ corresponds. This is the theorem obtained by Krein [8], and m is called Krein's (regular) string. Later Kasahara [1] established the continuity for the correspondence and applied it to show limit theorems for 1D diffusion processes with

Kyoto Journal of Mathematics, Vol. 53, No. 4 (2013), 787–814
DOI 10.1215/21562261-2366094,
Received February 27, 2012. Revised September 13, 2012. Accepted September 27, 2012.
2010 Mathematics Subject Classification: Primary 34L05, 34B20; Secondary 60J60, 60J55.

m their speed measures. Recently Kotani [7] extended Kasahara's result to a certain kind of singular strings m, namely, to m which is a nondecreasing and right-continuous function on $(-\infty,\infty)$ satisfying

$$m(-\infty)=0, \qquad m(x)\le\infty,$$

and

$$\int_{-\infty}^{a} x^2\,dm(x)<\infty \tag{1.1}$$

for some a. When the condition (1.1) is satisfied, the boundary $-\infty$ is called the limit circle type for the associated generalized second-order differential operator $d^2/dm\,dx$. In this case he introduced a new h by

$$h(\lambda)=\lim_{x\to-\infty}\Big(x+\varphi_\lambda(x)\int_x^l \frac{dy}{\varphi_\lambda(y)^2}\Big)=a+\int_0^\infty\Big(\frac{1}{\xi-\lambda}-\frac{\xi}{\xi^2+1}\Big)\,d\sigma(\xi),$$

which satisfies

$$h'(\lambda)=\int_{-\infty}^{l}\frac{\partial}{\partial\lambda}\varphi_\lambda(x)^{-2}\,dx,$$

and proved the continuity of the correspondence between m and h. Probabilistic applications of this result were given by Kasahara and Watanabe [2], [3], and it was interpreted from the point of view of the excursion theory by Yano [9]. In this article we consider m satisfying a milder condition than (1.1), namely,

$$\int_{-\infty}^{a}|x|\,dm(x)<\infty,$$

and obtain the continuity result under additional conditions on m, which allows any power growth of the spectral measures at ∞.

2. Preliminaries

Let $m(x)$ be a nondecreasing and right-continuous function on $(-\infty,\infty)$ satisfying

$$m(-\infty)=0, \qquad m(\infty)\le\infty.$$

Set

$$l=\sup\{x>-\infty, m(x)<+\infty\}, \qquad l_+=\sup\operatorname{supp}dm, \qquad l_-=\inf\operatorname{supp}dm.$$

Note that $m(l)=\infty$ if $l<\infty$. Assume that

$$\int_{-\infty}^{a}|x|\,dm(x)<\infty \tag{2.1}$$

with some $a\in(l_-,l_+)$. Let $\mathcal{E}$ be the totality of nondecreasing functions m satisfying (2.1). We exclude m vanishing identically on $(-\infty,\infty)$ from $\mathcal{E}$. One can regard dm as a distribution of weight, and in this case m works as a string. On the other hand, one can associate a generalized diffusion process with generator L:

$$L = \frac{d}{dm}\frac{d}{dx}$$

if we impose a suitable boundary condition if necessary. The condition (2.1) is called an entrance condition in 1D diffusion theory developed by W. Feller, so we say that m satisfies (2.1), a string of entrance type. For an entrance type m, it is easy to show that for $\lambda \in \mathbf{C}$ an integral equation

$$\varphi(x) = 1 - \lambda \int_{-\infty}^{x} (x-y)\varphi(y)\, dm(y)$$

has a unique solution, which is denoted by $\varphi_\lambda(x)$. Introduce a subspace

$$L_0^2(dm) = \{ f \in L^2(dm); \operatorname{supp} f \subset (-\infty, l) \},$$

and for $f \in L_0^2(dm)$ define a generalized Fourier transform by

$$\widehat{f}(\lambda) = \int_{-\infty}^{l} f(x)\varphi_\lambda(x)\, dm(x).$$

Krein's spectral theory implies that there exists a measure σ on $[0,\infty)$ satisfying

$$\int_{-\infty}^{l} |f(x)|^2\, dm(x) = \int_0^\infty |\widehat{f}(\xi)|^2\, d\sigma(\xi) \quad \text{for any } f \in L_0^2(dm); \tag{2.2}$$

σ is called a spectral measure for the string m. The nonuniqueness of such σ occurs if and only if

$$l_+ + m(l_+) < \infty. \tag{2.3}$$

The number $l(\geq l_+)$ possesses its meaning only when (2.3) is satisfied, and in this case there exists a σ satisfying (2.2) with the boundary condition

$$f(l_+) + (l - l_+) f^+(l_+) = 0$$

at l_+. Here f^+ is the derivative from the right-hand side. If $l = \infty$, then this should be interpreted as

$$f^+(l_+) = 0.$$

At the left boundary l_- no boundary condition is necessary if $l_- = -\infty$, and if $l_- > -\infty$, then we impose the reflective boundary condition, namely,

$$f^-(l_-) = 0 \quad \text{the derivative from the left.}$$

Generally, for a string m of entrance type it is known that for $\lambda < 0$ there exists uniquely f such that

$$\begin{cases} -Lf = \lambda f, \quad f > 0, f^+ \leq 0, f(l-) = 0, \\ f(x)\varphi_\lambda^+(x) - f^+(x)\varphi_\lambda(x) = 1. \end{cases}$$

This unique f is denoted by f_λ and contains information of the boundary condition we are imposing on $-L$ at the right boundary l_+, and f_λ can be represented by φ_λ as

$$f_\lambda(x) = \varphi_\lambda(x) \int_x^l \frac{dy}{\varphi_\lambda(y)^2}. \tag{2.4}$$

The right-hand-side integral is always convergent for $\lambda < 0$, because if $\operatorname{supp} dm \neq \phi$, then choosing $a \in \operatorname{supp} dm$, we see for $x > a$,

$$\varphi_\lambda(x) \geq 1 - \lambda \int_{-\infty}^{x} (x-y)\, dm(y) \geq 1 - \lambda \int_{-\infty}^{a} (x-y)\, dm(y) \geq 1 - \lambda(x-a)m(a);$$

hence

$$\int_x^l \frac{dy}{\varphi_\lambda(y)^2} \leq \int_x^l \frac{dy}{(1-\lambda(y-a)m(a))^2} < \infty$$

for $x > a$. If $\operatorname{supp} dm = \phi$, then $l < \infty$ and

$$m(x) = \begin{cases} 0 & \text{for } x < l, \\ \infty & \text{for } x > l, \end{cases} \tag{2.5}$$

which implies

$$\varphi_\lambda(x) = \begin{cases} 1 & \text{for } x < l, \\ \infty & \text{for } x > l, \end{cases}$$

and

$$\int_x^l \frac{dy}{\varphi_\lambda(y)^2} = l - x < \infty.$$

Here note that we have excluded $m = 0$ identically on $(-\infty, \infty)$; hence $l < \infty$. If m is a nondecreasing function of (2.5) the spectral measure vanishes identically on $[0, \infty)$. If m is ∞ identically on $(-\infty, \infty)$, then the spectral function σ is defined to be 0 identically on $[0, \infty)$. Conversely, if a spectral measure vanishes identically on $[0, \infty)$, then the associated string m should be of (2.5). We note that $\varphi_\lambda(x)$ is an entire function of minimal exponential type as a function of λ and the zeros of $\varphi_\lambda(x)$ coincide with the eigenvalues of $-L$ defined as a self-adjoint operator on $L^2(dm, (-\infty, x])$ with the Dirichlet boundary condition at x, which means that $\varphi_\lambda(x)$ has simple zeros on $(0, \infty)$. The Green function g_λ for $-L$ on $L^2(dm)$ is given by

$$g_\lambda(x, y) = g_\lambda(y, x) = f_\lambda(y)\varphi_\lambda(x)$$

for $x \leq y$. The relationship between σ and g_λ is described by an identity

$$\int_{-\infty}^{l} \int_{-\infty}^{l} g_\lambda(x, y) f(x) \overline{f(y)}\, dm(x)\, dm(y) = \int_0^\infty \frac{|\widehat{f}(\xi)|^2}{\xi - \lambda} \sigma(d\xi)$$

for any $f \in L^2(dm)$, and

$$g_\lambda(x, y) = \int_0^\infty \frac{\varphi_\xi(x)\varphi_\xi(y)}{\xi - \lambda}\, d\sigma(\xi),$$

through which σ is determined uniquely from the string m. Distinct m's may give an equal σ; namely, for $a \in \mathbf{R}$ a new string

$$m_a(x) = m(x + a)$$

defines the same σ, because

$$\varphi_\lambda^a(x) = \varphi_\lambda(x + a), \qquad f_\lambda^a(x) = f_\lambda(x + a),$$

and hence

$$g_\lambda^a(x,x)=\varphi_\lambda^a(x)f_\lambda^a(x)=g_\lambda(x+a,x+a)=\int_0^\infty \frac{\varphi_\xi(x+a)^2}{\xi-\lambda}\,d\sigma(\xi).$$

On the other hand,

$$g_\lambda^a(x,x)=\int_0^\infty \frac{\varphi_\xi^a(x)\varphi_\xi^a(x)}{\xi-\lambda}\,d\sigma_a(\xi)=\int_0^\infty \frac{\varphi_\xi(x+a)^2}{\xi-\lambda}\,d\sigma_a(\xi);$$

hence an identity

$$\sigma_a(\xi)=\sigma(\xi)$$

should be held. Conversely, we have the following.

THEOREM 1 (SEE KOTANI [5], [6])
If two strings m_1 and m_2 of $\mathcal{E}$ have the same spectral measure σ, then $m_1(x+c)=m_2(x)$ for a $c\in\mathbf{R}$.

If we hope to obtain the continuity of the correspondence between m and σ, we have to keep the nonuniqueness in mind. Namely, for m of $\mathcal{E}$ a sequence $\{m_n\}_{n\geq 1}$ of $\mathcal{E}$ defined by

$$m_n(x)=m(x-n)$$

converges to the trivial function 0 as $n\to\infty$. However, the associated σ's are independent of n. Therefore, we shall give several alternative definitions of convergence by imposing certain extra conditions (related to tightness) in addition to pointwise convergence. Set

$$M(x)=\int_{-\infty}^x (x-y)\,dm(y)=\int_{-\infty}^x m(y)\,dy.$$

Then, the condition (2.1) is equivalent to

$$M(x)<\infty$$

for $x<l$. Using a convention

$$[-\infty,a)=(-\infty,a),\qquad (a,\infty]=(a,\infty)\quad\text{and so on,}$$

we can see that M is a nondecreasing convex function on $(-\infty,\infty)$ satisfying

$$\begin{cases} M(x)=0 \text{ on } (-\infty,l_-],\\ \text{continuous and strictly increasing on } [l_-,l),\\ M(x)=\infty \text{ on } (l,\infty).\end{cases}$$

For a fixed positive number c, we assume that

$$0\in(l_-,l]\qquad\text{and}\qquad M(l)\geq c \tag{2.6}$$

and normalize such an m by

$$M(0)=c. \tag{2.7}$$

Denote by $\mathcal{E}^{(c)}$ the set of all elements of $\mathcal{E}$ satisfying (2.6), (2.7), and set

$$\mathcal{E}_+ = \bigcup_{c>0} \mathcal{E}^{(c)}.$$

In this definition of $\mathcal{E}_+$ among functions satisfying (2.1) any function m defined by (2.5) for some $l \le \infty$ is excluded from $\mathcal{E}_+$. Therefore, $\mathcal{E}\backslash\mathcal{E}_+$ consists of m satisfying (2.5) for some $l < \infty$. The uniqueness of the correspondence between m and σ holds under this normalization. Set

$$\mathcal{S} = \text{the set of all spectral measures for strings of } \mathcal{E}.$$

Any suitable characterization of $\mathcal{S}$ is not known yet; however, any measure on $[0,\infty)$ with polynomial growth at ∞ belongs to $\mathcal{S}$.

We prepare a basic estimate for φ_λ; φ_λ can be represented as

$$\varphi_\lambda(x) = \sum_{n=0}^{\infty} (-\lambda)^n \phi_n(x), \tag{2.8}$$

where $\{\phi_n\}_{n\ge 0}$ are

$$\phi_n(x) = \int_{-\infty}^{x} (x-y)\phi_{n-1}(y)\,dm(y), \qquad \phi_0(x) = 1.$$

Then, the convergence of the above series can be shown by the following lemma.

LEMMA 1

φ_λ is given by an absolute convergent series (2.8) and satisfies

$$\left|\varphi_\lambda(x)\right| \le \exp\bigl(|\lambda| M(x)\bigr).$$

Proof

First we show that for any $k \ge 0$,

$$\phi_k(x) \le \frac{M(x)^k}{k!} \tag{2.9}$$

holds. Observe that

$$\phi_1(x) = \int_{-\infty}^{x} (x-y)\,dm(y) = M(x).$$

Assuming (2.9) for some k, we have

$$\begin{aligned}
\phi_{k+1}(x) &\le \frac{1}{k!}\int_{-\infty}^{x} (x-y) M(y)^k\,dm(y) \\
&= \frac{1}{k!}\int_{-\infty}^{x} \bigl(M(y) - k(x-y)M'(y)\bigr) M(y)^{k-1} M'(y)\,dy \\
&\le \frac{1}{k!}\int_{-\infty}^{x} M'(y) M(y)^k\,dy = \frac{M(x)^{k+1}}{(k+1)!},
\end{aligned}$$

which proves (2.9) for general k. Then the estimate of φ_λ is clear. □

Here we clarify the convergence of a sequence of monotone functions taking value ∞. For a nonnegative and nondecreasing function m which may take ∞, set

$$\widehat{m}(x) = \frac{2}{\pi}\tan^{-1} m(x), \quad x \in \mathbf{R}.$$

Then we have

$$\widehat{m}(x) \in [0,1]$$

and a right-continuous nondecreasing function satisfying

$$0 \le \widehat{m}(-\infty) \le \widehat{m}(x) \le \widehat{m}(l-) \le \widehat{m}(l) = 1,$$

if $l < \infty$. A sequence of nonnegative and nondecreasing functions m_n is defined to converge to m as $n \to \infty$ if

$$\widehat{m}_n(x) \to \widehat{m}(x) \tag{2.10}$$

holds at any point of continuity of $\widehat{m}(x)$.

LEMMA 2

Suppose that $m_n \in \mathcal{E}$ converges to $m \in \mathcal{E}$ as $n \to \infty$. Then it holds that

$$\varliminf_{n\to\infty} l_n \ge l.$$

Proof

Let $x < l$ be a point of continuity for $\widehat{m}$. Then

$$\widehat{m}_n(x) \to \widehat{m}(x) < 1;$$

hence

$$\widehat{m}_n(x) < 1$$

for every sufficiently large n, which implies $x < l_n$ and completes the proof. □

The continuity of the correspondence from $\mathcal{E}$ to $\mathcal{S}$ is not hard to show. Let m_n, m be strings of $\mathcal{E}$, and define the convergence of m_n to m by

(A) $m_n(x) \to m(x)$ for every point of continuity of m,
(B) $\lim_{x\to-\infty} \sup_{n\ge 1} M_n(x) = 0$.

THEOREM 2

Suppose that $m_n \in \mathcal{E}$ converge to $m \in \mathcal{E}$. Then, for every $\lambda < 0$ the Green functions $g_\lambda^{(n)}(x,y)$ of the string m_n converge to the Green function $g_\lambda(x,y)$ of m for any $x, y < l$. In particular the spectral functions $\sigma_n(\xi)$ converge to $\sigma(\xi)$ at every point of continuity of σ.

Proof

Under the conditions it is easy to see that the φ-functions $\varphi_\lambda^{(n)}(x)$ of m_n converge to the φ-function $\varphi_\lambda(x)$ of m compact uniformly with respect to $(x, \lambda) \in$

$(-\infty, l)\times\mathbf{C}$ from the uniform bound for $\varphi_\lambda^{(n)}$ due to Lemma 1. Moreover, if $m(a)>0$ at some a, a point of continuity of m, then there exists a positive constant C such that

$$\varphi_\lambda^{(n)}(y)\geq 1-\lambda M_n(y)\geq 1+C(y-a)$$

holds for any $y>a$; hence

$$f_\lambda^{(n)}(x)=\varphi_\lambda^{(n)}(x)\int_x^{l_n}\frac{1}{\varphi_\lambda^{(n)}(y)^2}\,dy$$

also converge to $f_\lambda(x)$. If $\operatorname{supp} m=\phi$, namely, $m(x)=0$ identically on $(-\infty,l)$, then

$$f_\lambda^{(n)}(x)\to l-x$$

if $l<\infty$. The case $l=\infty$ is excluded. Consequently, we have

$$g_\lambda^{(n)}(x,y)=\varphi_\lambda^{(n)}(y)f_\lambda^{(n)}(x)\to\varphi_\lambda(y)f_\lambda(x)=g_\lambda(x,y)$$

for any $y\leq x<l$. The identity

$$g_\lambda^{(n)}(x,y)=\int_0^\infty\frac{\varphi_\xi^{(n)}(x)\varphi_\xi^{(n)}(y)}{\xi-\lambda}\,d\sigma_n(\xi)$$

shows the last statement of the theorem. □

3. Scales and estimates by trace

The straight converse statement of Theorem 2 is hopeless to be true, because there is no characterization for a measure σ on $[0,\infty)$ to be a spectral measure of a string $m\in\mathcal{E}$. Therefore we prove the converse continuity of the correspondence by imposing a condition on $\{\sigma_n\}$. In the process of the proof we have to estimate $\varphi_\lambda(x)^{-2}$ in terms of m. A better way to investigate $\varphi_\lambda(x)^{-2}$ is to use probabilistic methods. Recall that for each fixed a, $\varphi_\lambda(a)$ has simple zeros $\{\mu_n\}_{n\geq1}$ which are eigenvalues of $-L$ on $(-\infty,a]$ with Dirichlet boundary condition at $x=a$. Since the Green function for this operator is

$$a-(x\vee y),$$

we see that

$$\sum_{n=1}^\infty\mu_n^{-1}=\operatorname{tr}(-L)^{-1}=\int_{-\infty}^a(a-x)\,dm(x)=M(a)<\infty. \tag{3.1}$$

Choosing a $b<l$, we denote by $\phi_\lambda^b(\psi_\lambda^b)$ the solutions of

$$-\frac{d}{dm}\frac{d}{dx}f=\lambda f,\quad\text{with } f(b)=1, f'(b)=0\ \big(f(b)=0, f'(b)=1\big),\ \text{respectively.}$$

Then we see that an identity

$$\varphi_\lambda(x)=\varphi_\lambda(b)\phi_\lambda^b(x)+\varphi_\lambda'(b)\psi_\lambda^b(x)$$

holds. Lemma 1 implies that $\varphi_\lambda(b)$ and $\varphi_\lambda'(b)$ are entire functions of at most exponential type $M(b)$ as functions of λ. On the other hand, $\phi_\lambda^b(x)$ and $\psi_\lambda^b(x)$

are entire functions of order at most 1/2 as functions of λ. Therefore we know that $\varphi_\lambda(x)$ is an entire function of minimal exponential type, which combined with (3.1) shows that

$$\varphi_\lambda(a)=\prod_{n=1}^{\infty}\Big(1-\frac{\lambda}{\mu_n}\Big).$$

For the detail refer to [5, p. 441]. Now let $\{X_n\}_{n\geq 1}$ be independent random variables, each of which has an exponential distribution of mean 1. Then, an identity

$$E\exp\Big(\lambda\sum_{n=1}^{\infty}\mu_n^{-1}X_n\Big)=\prod_{n=1}^{\infty}\Big(1-\frac{\lambda}{\mu_n}\Big)^{-1}=\varphi_\lambda(a)^{-1}$$

holds. Therefore, letting $\{\widetilde{X}_n\}_{n\geq 1}$ be independent copies of $\{X_n\}_{n\geq 1}$ and setting

$$Y_n=X_n+\widetilde{X}_n,\qquad X=\sum_{n=1}^{\infty}\mu_n^{-1}Y_n,$$

we have

$$\varphi_\lambda(a)^{-2}=E\exp(\lambda X). \tag{3.2}$$

We denote $X=X(a)$ if necessary, because the eigenvalue $\{\mu_n\}_{n\geq 1}$ depends on the boundary a.

LEMMA 3

Suppose that the spectral measure σ of an $m\in\mathcal{E}$ satisfies

$$p(t)=\int_0^\infty e^{-t\xi}\,d\sigma(\xi)<\infty\quad\textit{for any } t>0.$$

Then, for any nonnegative Borel measurable function f on $[0,\infty)$,

$$\int_{-\infty}^{l}Ef\big(X(x)\big)\,dx=\int_0^\infty p(t)f(t)\,dt \tag{3.3}$$

holds by permitting the integrals to take the value ∞ simultaneously.

Proof

From (2.4) it follows that for any $x<l$ and $\lambda<0$.

$$\int_x^l Ee^{\lambda X(y)}\,dy=Ee^{\lambda X(x)}\int_0^\infty\frac{\varphi_\xi(x)^2}{\xi-\lambda}\,d\sigma(\xi)=\int_0^\infty Ee^{\lambda(X(x)+t)}p(t,x,x)\,dt$$

holds, where $p(t,x,y)$ is the transition probability density defined by

$$p(t,x,y)=\int_0^\infty e^{-t\xi}\varphi_\xi(x)\varphi_\xi(y)\,d\sigma(\xi).$$

Hence a functional monotone class theorem shows that the identity below holds for any nonnegative bounded continuous function f on $[0,\infty)$:

$$\int_x^l E\big(f\big(X(y)\big)e^{\lambda X(y)}\big)\,dy=\int_0^\infty E\big(f\big(X(x)+t\big)e^{\lambda(X(x)+t)}\big)p(t,x,x)\,dt. \tag{3.4}$$

Since, for $t > 2M(x)$,

$$p(t,x,x) = \int_0^\infty e^{-\xi t}\varphi_\xi(x)^2 \, d\sigma(\xi) \le \int_0^\infty e^{-\xi t} e^{2\xi M(x)} \, d\sigma(\xi) = p\big(t - 2M(x)\big)$$

holds, assuming $f(t) = 0$ for $t < \epsilon$, we see that

$$\int_{-\infty}^l E\big(f\big(X(y)\big)e^{\lambda X(y)}\big) \, dy = \int_0^\infty f(t)e^{\lambda t} p(t) \, dt$$

by letting $x \to -\infty$. Here we have used the fact that

$$X(x) \to 0 \quad \text{as } x \to -\infty.$$

The rest of the proof is routine. □

Now we define a scale function ϕ on $[0,1]$ as a function satisfying the following properties:

(S.1) ϕ is strictly increasing, convex and $\phi(0) = 0$, $\phi'(1-) < \infty$;

(S.2) for each $x > 0$,

$$\varlimsup_{y \downarrow 0} \frac{\phi(xy)}{\phi(y)} < \infty;$$

(S.3) for each $x \in (0,1]$,

$$\varliminf_{y \downarrow 0} \frac{\phi(xy)}{\phi(y)} > 0.$$

The property (S.1) enables us to extend ϕ linearly to $[1,\infty)$, namely,

$$\phi(x) = \phi(1) + \phi'(1-)(x-1)$$

for $x > 1$. Then ϕ becomes a nonnegative, convex, and nondecreasing function on $[0,\infty)$. Throughout the paper ϕ is always extended to $[1,\infty)$ linearly in this way. A regularly varying function at 0 satisfies the conditions (S.2) and (S.3). Set

$$C_+(x) = \sup_{y>0} \frac{\phi(xy)}{\phi(y)} < \infty, \qquad C_-(x) = \inf_{y \in (0,1]} \frac{\phi(xy)}{\phi(y)} > 0.$$

Then C_+ becomes nonnegative, convex, and nondecreasing on $[0,\infty)$. It satisfies the submultiplicative property

$$C_+(xy) \le C_+(x)C_+(y)$$

for any $x, y > 0$; hence, setting

$$\alpha_+ = \sup_{x>1} \frac{\log C_+(e^x)}{x} \in [0,\infty)$$

we see that

$$C_+(x) \le x^{\alpha_+}$$

holds for any $x \geq e$. We note that α_+ should not be less than 1 due to the convexity of C_+. Since

$$\phi(xy) \leq C_+(x)\phi(y)$$

holds for any $x, y > 0$, we have

$$\phi(x) \geq \frac{\phi(1)}{C_+(1/x)} \geq \phi(1)x^{\alpha_+}$$

for any $x \in [0, 1/e]$. Therefore the property (S.2) restricts ϕ not to decay faster than with a power order:

$$\phi(xy) \leq C_+(x)\phi(y). \tag{3.5}$$

C_- satisfies

$$C_-(xy) \geq C_-(x)C_-(y)$$

for any $x, y \in [0, 1]$. Typical examples for functions satisfying (S.1)∼(S.3) are

$$\phi(x) = x^\alpha, \qquad x^\alpha(c - \log x),$$

where $\alpha \geq 1$ and c is a sufficiently large positive constant.

LEMMA 4

Let $\{Y_n\}_{n\geq 1}$ be a sequence of identically distributed nonnegative random variables with mean μ, let $\{\lambda_n\}_{n\geq 1}$ be a nonnegative sequence satisfying

$$\sum_{n=1}^{\infty} \lambda_n < \infty,$$

and set

$$X = \sum_{n=1}^{\infty} \lambda_n Y_n.$$

Then, we have the following.

(1) *If ϕ satisfies (S.1), then*

$$\phi(EX) \leq E\phi(X).$$

(2) *If ϕ satisfies (S.1) and (S.2), then*

$$E\phi(X) \leq \Big(EC_+\Big(\frac{Y_1}{\mu}\Big)\Big)\phi(EX).$$

Proof

Jensen's inequality implies the inequality in (1). To show the second inequality we set

$$m_n = m^{-1}\lambda_n, \quad m = \sum_{k=1}^{\infty} \lambda_k.$$

Then (3.5) implies

$$\phi(X) = \phi\Big(m\mu \sum_{n=1}^{\infty} m_n \frac{Y_n}{\mu}\Big) \le \phi(m\mu) C_+\Big(\sum_{n=1}^{\infty} m_n \frac{Y_n}{\mu}\Big).$$

Since the function C_+ is convex, we have

$$C_+\Big(\sum_{n=1}^{\infty} m_n \frac{Y_n}{\mu}\Big) \le \sum_{n=1}^{\infty} m_n C_+\Big(\frac{Y_n}{\mu}\Big)$$

and

$$E\phi(X) \le \phi(m\mu) \sum_{n=1}^{\infty} m_n E C_+\Big(\frac{Y_n}{\mu}\Big) = \phi(EX) E C_+\Big(\frac{Y_1}{\mu}\Big).$$ □

Let X be defined as in (3.2), and set

$$C_\phi = E C_+\Big(\frac{Y_1}{\mu}\Big) = \int_0^{\infty} t e^{-t} C_+(t/2)\, dt < \infty.$$

LEMMA 5

We have the following.

(1) *If ϕ satisfies (S.1), then*

$$E\big(\phi(X)e^{\lambda X}\big) \ge \varphi_\lambda(a)^{-2} \phi\Big(\int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy\Big).$$

(2) *If ϕ satisfies (S.1) and (S.2), then*

$$E\big(\phi(X)e^{\lambda X}\big) \le C_\phi \varphi_\lambda(a)^{-2} \phi\Big(\int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy\Big).$$

Proof

For a fixed $\lambda < 0$ let Z be a nonnegative random variable satisfying

$$E e^{\mu Z} = \varphi_{\lambda+\mu}(a)^{-2} \varphi_\lambda(a)^2 = \prod_{n=1}^{\infty} \Big(1 - \frac{\mu}{\mu_n - \lambda}\Big)^{-2}.$$

Then, note an identity

$$E\big(\phi(X)e^{\lambda X}\big) = \varphi_\lambda(a)^{-2} E\big(\phi(Z)\big), \tag{3.6}$$

which can be shown from the observation

$$\frac{\partial^k}{\partial \lambda^k} \varphi_\lambda(a)^{-2} = \varphi_\lambda(a)^{-2} \frac{\partial^k}{\partial \mu^k}\big(\varphi_{\lambda+\mu}(a)^{-2} \varphi_\lambda(a)^2\big)|_{\mu=0}$$

for any $k \ge 0$, because this implies the identity when $\phi(x) = x^k$. To apply Lemma 4 to Z we need to compute EZ. If we denote the Green operator for L on $(-\infty, a]$ with Dirichlet boundary condition at a by G_λ, then

$$G_\lambda(x,y) = \varphi_\lambda(y)\varphi_\lambda(x) \int_x^a \varphi_\lambda(z)^{-2}\, dz \quad \text{for } x \ge y.$$

Hence

$$EZ = \sum_{j=1}^{\infty} \frac{1}{\mu_j - \lambda} = \operatorname{tr} G_\lambda = \int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy$$

holds, and we have the inequalities in the statement. □

The right-hand side of the inequalities in Lemma 5 can be estimated further.

LEMMA 6

For $\lambda < 0$ the following inequalities are valid:

(1) $\int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy \geq M(a)\varphi_\lambda(a)^{-2}$,

(2) $\int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy \leq M(a) \wedge (\frac{\log \varphi_\lambda(a)}{-\lambda})$.

Proof

Inequality (1) and the first inequality of (2) follow from the monotonicity of $\varphi_\lambda(z)$; namely, we have

$$\int_x^a \varphi_\lambda(y)^{-2}\, dy \geq \varphi_\lambda(a)^{-2}(a-x), \qquad \int_x^a \varphi_\lambda(y)^{-2}\, dy \leq \varphi_\lambda(x)^{-2}(a-x),$$

which implies

$$\int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy \geq \varphi_\lambda(a)^{-2} \int_{-\infty}^{a} (a-x)\, dm(x) = \varphi_\lambda(a)^{-2} M(a)$$

and

$$\int_{-\infty}^{a} \varphi_\lambda(x)^2\, dm(x) \int_x^a \varphi_\lambda(y)^{-2}\, dy \leq \int_{-\infty}^{a} (a-x)\, dm(x) = M(a).$$

The second inequality of (2) follows by using the equation satisfied by $\varphi_\lambda(x)$,

$$d\varphi_\lambda'(y) = -\lambda \varphi_\lambda(y)\, dm(y),$$

which yields

$$\begin{aligned}
&-\lambda \int_{-\infty}^{a} \varphi_\lambda(y)^2\, dm(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&\quad = \int_{-\infty}^{a} \varphi_\lambda(y)\, d\varphi_\lambda'(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&\quad = \varphi_\lambda(y)\varphi_\lambda'(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \Big|_{-\infty}^{a} \\
&\qquad - \int_{-\infty}^{a} \varphi_\lambda'(y)^2\, dy \int_y^a \varphi_\lambda(z)^{-2}\, dz + \int_{-\infty}^{a} \frac{\varphi_\lambda'(y)}{\varphi_\lambda(y)}\, dy.
\end{aligned}$$

Noting

$$\varphi_\lambda(y)\varphi_\lambda'(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \underset{y \to -\infty}{\sim} -\lambda m(y)(a-y) \underset{y \to -\infty}{\to} 0,$$

we see that

$$-\lambda\int_{-\infty}^{a}\varphi_\lambda(y)^2\,dm(y)\int_y^a\varphi_\lambda(z)^{-2}\,dz=\log\varphi_\lambda(a)-\int_{-\infty}^{a}\varphi_\lambda'(y)^2\,dy\int_y^a\varphi_\lambda(z)^{-2}\,dz$$
$$\le\log\varphi_\lambda(a),$$

which completes the proof. □

As the last lemma in this section, we have the following.

LEMMA 7

The following two estimates hold.

(1) *For any function ϕ satisfying (S.1) it holds that*

$$\int_0^\infty p(t)\phi(t)e^{\lambda t}\,dt\ge\int_{-\infty}^{l}\phi\big(M(x)\varphi_\lambda(x)^{-2}\big)\varphi_\lambda(x)^{-2}\,dx.$$

(2) *For any function ϕ satisfying (S.1), (S.2) it holds that*

$$\int_0^\infty p(t)\phi(t)e^{\lambda t}\,dt\le C_\phi\int_{-\infty}^{l}\phi\Big(M(x)\wedge\Big(\frac{\log\varphi_\lambda(x)}{-\lambda}\Big)\Big)\varphi_\lambda(x)^{-2}\,dx$$
$$\le C_\phi\int_{-\infty}^{a}\phi\big(M(x)\big)\varphi_\lambda(x)^{-2}\,dx+C_\phi\frac{-\lambda}{\varphi_\lambda'(a)}\int_0^\infty\phi(t)e^{\lambda t}\,dt.$$

Proof

All we have to show is an estimate of the integral

$$\int_a^l\phi\Big(\frac{\log\varphi_\lambda(x)}{-\lambda}\Big)\varphi_\lambda(x)^{-2}\,dx.$$

Noting the monotonicity of $\varphi_\lambda(x),\varphi_\lambda'(x)$, and $\varphi_\lambda(x)\ge 1$, we see that

$$\int_a^l\phi\Big(\frac{\log\varphi_\lambda(x)}{-\lambda}\Big)\varphi_\lambda(x)^{-2}\,dx$$
$$=\int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)}\phi\Big(\frac{\log z}{-\lambda}\Big)\frac{1}{z^2\varphi_\lambda'(\varphi_\lambda^{-1}(z))}\,dz$$
$$\le\frac{1}{\varphi_\lambda'(\varphi_\lambda^{-1}(\varphi_\lambda(a)))}\int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)}\phi\Big(\frac{\log z}{-\lambda}\Big)\frac{dz}{z^2}\le\frac{-\lambda}{\varphi_\lambda'(a)}\int_0^\infty\phi(t)e^{\lambda t}\,dt.$$
□

4. Continuity of the correspondence from $\mathcal{S}$ to $\mathcal{E}$

In this section we give a partial converse of Theorem 2. The lemma below will be useful later.

LEMMA 8

Let $m_n\in\mathcal{E}$, and let σ_n be its spectral function. Suppose that

$$\lim_{n\to\infty}M_n(l_n)=0\tag{4.1}$$

holds. Then $\sigma_n(\xi)\to 0$ for any $\xi>0$.

Proof
Since $M_n(l_n) < \infty$, we have $l_n < \infty$. Set $\widetilde{m}_n(x) = m_n(x + l_n)$. Then

$$\widetilde{M}_n(0) \to 0, \tag{4.2}$$

and its spectral measure coincides with σ_n. Since the condition (4.2) implies

$$\widetilde{m}_n(x) \to \begin{cases} 0 & \text{for } x < 0, \\ \infty & \text{for } x > 0 \end{cases}$$

in $\mathcal{E}$, Theorem 2 shows $\sigma_n \to 0$. □

THEOREM 3
Let $m_n \in \mathcal{E}$, and let σ_n be its spectral function satisfying

$$p_n(t) = \int_0^\infty e^{-t\xi}\, d\sigma_n(\xi) < \infty \quad \textit{for any } t > 0$$

and

$$\sup_{n \geq 1} \int_0^1 p_n(t)\phi(t)\, dt < \infty \tag{4.3}$$

for a function ϕ satisfying (S.1). Assume that there exists a nontrivial measure σ on $[0, \infty)$ satisfying

$$\sigma_n(\xi) \to \sigma(\xi)$$

at every point of continuity of σ. Then

$$\lim_{n\to\infty} p_n(t) = p(t), \qquad \varliminf_{n\to\infty} M_n(l_n) > 0$$

hold. Choose c such that

$$0 < c < \varliminf_{n\to\infty} M_n(l_n),$$

and define a_n by the solution $M_n(a_n) = c$. Then there exists a unique $m \in \mathcal{E}^{(c)}$ with spectral measure σ, and it holds that $m_n(\cdot + a_n) \to m$ in $\mathcal{E}$; hence $\sigma \in \mathcal{S}$.

Proof
For any $\epsilon > 0$ and any $N > 0$,

$$\int_0^\epsilon p_n(t)\phi(t)\, dt = \int_0^\epsilon \phi(t)\, dt \int_0^\infty e^{-t\xi}\, d\sigma_n(\xi) \geq e^{\epsilon N} \int_N^\infty e^{-2\epsilon\xi}\, d\sigma_n(\xi) \int_0^\epsilon \phi(t)\, dt$$

holds, and the condition (4.3) implies that there exists a constant C_ϵ such that

$$\int_N^\infty e^{-2\epsilon\xi}\, d\sigma_n(\xi) \leq e^{-\epsilon N} \frac{\int_0^\epsilon p_n(t)\phi(t)\, dt}{\int_0^\epsilon \phi(t)\, dt} \leq e^{-\epsilon N} C_\epsilon$$

is valid for any n, N, which yields

$$p_n(t) \to p(t) = \int_0^\infty e^{-t\xi}\, d\sigma(\xi) \tag{4.4}$$

as $n \to \infty$. Applying (3.3) to $\phi(t)e^{\lambda t}$ for $\lambda < 0$ shows that

$$\int_{-\infty}^{l_n} E\big(\phi\big(X_n(x)\big)e^{\lambda X_n(x)}\big)\,dx = \int_0^\infty p_n(t)e^{\lambda t}\phi(t)\,dt$$
$$\le \int_0^1 p_n(t)e^{\lambda t}\phi(t)\,dt + p_n(1)\int_1^\infty e^{\lambda t}\phi(t)\,dt \le C$$

with a constant C, where $X_n(x)$ is defined by the eigenvalues $\{\mu_j(x)\}_{j\ge 1}$ corresponding to m_n. Since we assume that the limiting spectral measure σ is non-trivial, Lemma 8 shows

$$\varliminf_{n\to\infty} M_n(l_n) > 0.$$

For c such that

$$0 < c < \varliminf_{n\to\infty} M_n(l_n),$$

define a_n by the solution $M_n(a_n) = c$, and set

$$\widetilde{m}_n(x) = m_n(x + a_n).$$

Then $\widetilde{m}_n \in \mathcal{E}^{(c)}$ and an inequality (1) of Lemma 7 imply

$$\int_{-\infty}^{l_n} E\big(\phi\big(X_n(x)\big)e^{\lambda X_n(x)}\big)\,dx \ge \int_{-\infty}^{\widetilde{l}_n} \widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\phi\big(\widetilde{M}_n(x)\widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\big)\,dx,$$

and from Lemma 1 we have

$$\int_{-\infty}^{\widetilde{l}_n} \widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\phi\big(\widetilde{M}_n(x)\widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\big)\,dx \ge \int_{-\infty}^{l_n} e^{2\lambda \widetilde{M}_n(x)}\phi\big(\widetilde{M}_n(x)e^{2\lambda \widetilde{M}_n(x)}\big)\,dx$$
$$\ge \int_{-\infty}^{0} e^{2\lambda c}\phi\big(\widetilde{M}_n(x)e^{2\lambda c}\big)\,dx.$$

Thus

$$\int_{-\infty}^{0} e^{2\lambda c}\phi\big(\widetilde{M}_n(x)e^{2\lambda c}\big)\,dx \le C$$

is valid for any $n \ge 1$. Therefore, for any $x < 0$,

$$e^{2\lambda c}(-x)\phi\big(\widetilde{M}_n(x)e^{2\lambda c}\big) \le \int_x^0 e^{2\lambda c}\phi\big(\widetilde{M}_n(y)e^{2\lambda c}\big)\,dy \le C,$$

which implies that $\{\widetilde{m}_n\}_{n\ge 1}$ has a convergent subsequence in the sense of the convergence in $\mathcal{E}$, namely, the convergence under conditions (A) and (B). Since we have proved (4.4), the uniqueness of the spectral measure in $\mathcal{E}^{(c)}$ and Theorem 2 complete the proof. □

COROLLARY 1

Suppose that a measure σ on $[0,\infty)$ satisfies

$$\int_0^1 p(t)\phi(t)\,dt < \infty$$

with a function ϕ on $[0,1]$ satisfying (S.1). Then, there exists an $m \in \mathcal{E}$ with spectral measure σ in $\mathcal{S}$.

Proof
Define σ_n by

$$\sigma_n(\xi) = \begin{cases} \sigma(\xi) & \text{for } \xi < n, \\ \sigma(n) & \text{for } \xi \geq n. \end{cases}$$

Then this σ_n satisfies all the conditions of Theorem 3. Applying the theorem we easily obtain the corollary. □

This corollary provides a plenty of spectral measures in $\mathcal{S}$ growing faster than any power order at ∞.

5. Continuity of the correspondence between $\mathcal{E}_\phi$ and $\mathcal{S}_\phi$

In this section we give a necessary and sufficient condition for the continuity of the correspondence by restricting the order of growth of spectral measures at ∞. We call ϕ a scale function if it satisfies conditions (S.1), (S.2), and (S.3). For a scale function ϕ set

$$\mathcal{E}_\phi = \Big\{ m \in \mathcal{E};\int_{-\infty}^{a} \phi\big(M(x)\big)\,dx < \infty \text{ for } \exists a \in (l_-, l_+)\Big\},$$

and

$$\mathcal{S}_\phi = \Big\{ \sigma;\int_{1}^{\infty} \widetilde{\phi}(\xi)\sigma(d\xi) < \infty \Big\},$$

where

$$\widetilde{\phi}(\xi) = \int_0^\infty e^{-t\xi}\phi(t)\,dt.$$

It is easy to see that

$$\int_1^\infty \widetilde{\phi}(\xi)\sigma(d\xi) < \infty \Longleftrightarrow \int_0^1 p(t)\phi(t)\,dt < \infty.$$

Moreover, from the properties of scales, it is always valid that for $\sigma \in \mathcal{S}_\phi$,

$$\int_1^\infty \xi^{-\alpha-1}\sigma(d\xi) < \infty$$

for an $\alpha \geq 1$. On the other hand, if $m \in \mathcal{E}_\phi$, Lemma 7(2) yields

$$\begin{aligned}\int_0^\infty p(t)\phi(t)e^{\lambda t}\,dt &\leq C_\phi \int_{-\infty}^{a} \phi\big(M(x)\big)\varphi_\lambda(x)^{-2}\,dx + C_\phi \frac{-\lambda}{\varphi'_\lambda(a)}\int_0^\infty \phi(t)e^{\lambda t}\,dt \\ &\leq C_\phi \int_{-\infty}^{a} \phi\big(M(x)\big)\,dx + C_\phi \frac{-\lambda}{\varphi'_\lambda(a)}\int_0^\infty \phi(t)e^{\lambda t}\,dt.\end{aligned}$$

Therefore, it holds that

$$\int_0^\infty p(t)\phi(t)e^{\lambda t}\,dt < \infty,$$

which shows $\sigma \in \mathcal{S}_\phi$. Conversely, assume $\sigma \in \mathcal{S}_\phi$. Then

$$\int_0^\infty p(t)\phi(t)e^{\lambda t}\,dt = \int_0^1 p(t)\phi(t)e^{\lambda t}\,dt + \int_1^\infty p(t)\phi(t)e^{\lambda t}\,dt$$
$$\leq \int_0^1 p(t)\phi(t)\,dt + p(1)\int_1^\infty \phi(t)e^{\lambda t}\,dt < \infty$$

for any $\lambda < 0$. Therefore,

$$\int_{-\infty}^{l} \phi\big(M(x)\varphi_\lambda(x)^{-2}\big)\varphi_\lambda(x)^{-2}\,dx < \infty$$

holds. Since ϕ satisfies the condition (S.3)

$$\phi\big(M(x)\varphi_\lambda(x)^{-2}\big) \geq C_-\big(\varphi_\lambda(x)^{-2}\big)\phi\big(M(x)\big)$$

holds. Due to

$$\varphi_\lambda(x)^{-2} \geq e^{2\lambda M(x)}$$

and $M(x) \to 0$ as $x \to -\infty$, we easily see that

$$\int_{-\infty}^{a} \phi\big(M(x)\big)\,dx < \infty,$$

which implies that $m \in \mathcal{E}_\phi$. Therefore, a string belongs to $\mathcal{E}_\phi$ if and only if its spectral measure is an element of $\mathcal{S}_\phi$.

For ϕ let m_n, m be strings of $\mathcal{E}_\phi$, and define the convergence of m_n to m in $\mathcal{E}_\phi$ by

(C) $\lim_{x\to-\infty}\sup_{n\geq 1}\int_{-\infty}^{x}\phi(M_n(y))\,dy = 0$,

in addition to the condition (A). The convergence of spectral measures in $\mathcal{S}_\phi$ is defined by

(A′) $\sigma_n(\xi) \to \sigma(\xi)$ at every point of continuity of σ,
(C′) $\lim_{N\to\infty}\sup_{n\geq 1}\int_N^\infty \widetilde{\phi}(\xi)\sigma_n(d\xi) = 0$.

An equivalent statement is possible by $p(t)$:

(A″) $p_n(t) \to p(t)$ for any $t > 0$,
(C″) $\lim_{\epsilon\downarrow 0}\sup_{n\geq 1}\int_0^\epsilon p_n(t)\phi(t)\,dt = 0$.

Set

$$\mathcal{E}_\phi^{(c)} = \mathcal{E}_\phi \cap \mathcal{E}^{(c)}.$$

THEOREM 4

Let $\{\sigma_n\}_{n\geq 1}, \sigma$ be elements of $\mathcal{S}_\phi$, and let $m_n, m \in \mathcal{E}_\phi^{(c)}$ be the strings corresponding to σ_n, σ, respectively. Then, $m_n \to m$ in $\mathcal{E}_\phi^{(c)}$ if and only if $\sigma_n \to \sigma$ in $\mathcal{S}_\phi$.

Proof

Suppose that $m_n \to m$ in $\mathcal{E}_\phi^{(c)}$. Then, Theorem 2 shows the validity of the condition (A′). Therefore we have only to check the condition (C″). From Lemma 7(2),

$$\int_0^\infty p_n(t)\phi(t)e^{\lambda t}\,dt \le C_\phi \int_{-\infty}^0 \phi\big(M_n(x)\big)\varphi_\lambda^{(n)}(x)^{-2}\,dx + C_\phi \frac{-\lambda}{\varphi_\lambda^{(n)\prime}(0)}\int_0^\infty \phi(t)e^{\lambda t}\,dt$$

is valid. Fix $\epsilon > 0$, and choose $a < 0$ such that

$$C_\phi \int_{-\infty}^a \phi\big(M_n(x)\big)\,dx < \epsilon$$

for any $n \ge 1$. Since $M_n(x), \varphi_\lambda^{(n)}(x)$ converge to $M(x), \varphi_\lambda(x)$ uniformly on $(-\infty, 0]$, and the estimate

$$\varphi_\lambda^{(n)}(0) \ge 1 - \lambda M_n(0) = 1 - \lambda c$$

shows that if $-\lambda$ is sufficiently large, then

$$C_\phi \int_a^0 \phi\big(M_n(x)\big)\varphi_\lambda^{(n)}(x)^{-2}\,dx < \epsilon$$

is valid for any $n \ge 1$. Moreover, due to $c > 0$,

$$\varliminf_{n\to\infty} m_n(0) \ge m(0-) > 0$$

holds; hence

$$C_\phi \frac{-\lambda}{\varphi_\lambda^{(n)\prime}(0)}\int_0^\infty \phi(t)e^{\lambda t}\,dt \le C_\phi \frac{1}{m_n(0)}\int_0^\infty \phi(t)e^{\lambda t}\,dt < \epsilon$$

also holds for any $n \ge 1$ if we choose sufficiently large $-\lambda$, which implies that

$$\sup_{n\ge 1}\int_0^\infty p_n(t)\phi(t)e^{\lambda t}\,dt \le 3\epsilon.$$

From

$$\int_0^{-1/\lambda} p_n(t)\phi(t)\,dt \le e\int_0^\infty p_n(t)\phi(t)e^{\lambda t}\,dt \le 3e\epsilon$$

the condition (C$''$) is confirmed. Conversely, assume that $\sigma_n \to \sigma$ in $\mathcal{S}_\phi$. Then Theorem 3 shows that the condition (A) holds. Hence we have only to check the condition (C). From Lemma 7(1),

$$\int_0^\infty p_n(t)\phi(t)e^{\lambda t}\,dt \ge \int_{-\infty}^l \phi\big(M_n(x)\varphi_\lambda^{(n)}(x)^{-2}\big)\varphi_\lambda^{(n)}(x)^{-2}\,dx$$

follows. The property (S.3) implies

$$\phi\big(M_n(x)\varphi_\lambda^{(n)}(x)^{-2}\big) \ge C_-\big(\varphi_\lambda^{(n)}(x)^{-2}\big)\phi\big(M_n(x)\big),$$

and, as was pointed out in the proof of Theorem 3, $\varphi_\lambda^{(n)}(x) \to 1$ as $x \to -\infty$ uniformly with respect to n. Therefore, the condition (C$''$) guarantees the condition (C). □

The last theorem can be restated as the convergence in $\mathcal{E}_\phi$.

THEOREM 5

Let $\{\sigma_n\}_{n\ge1}, \sigma$ be elements of $\mathcal{S}_\phi$, and let $m_n, m \in \mathcal{E}_\phi$ be the strings corresponding *to σ_n, σ respectively. Assume that $\sigma_n \to \sigma$ in $\mathcal{S}_\phi$ and σ is nontrivial. Then, there exist a sequence $\{a_n\}_{n\ge 1}$ in $\mathbf{R}$ and $c>0$ with $m_n(\cdot + a_n) \in \mathcal{E}_\phi^{(c)}$, and*

$$m_n(\cdot + a_n) \to m$$

holds in $\mathcal{E}_\phi$.

Proof
Since σ is nontrivial, we can apply Lemma 8. The rest of the proof is clear from Theorem 4. □

For applications it will be helpful to rewrite the condition (C) as Kasahara and Watanabe did in [4].

LEMMA 9
Assume that ϕ satisfies (S.1). Then a sequence $\{m_n\}_{n\ge 1}$ converges to m in $\mathcal{E}_\phi$ if and only if $\{m_n\}_{n\ge 1}$ and m satisfy the condition below.

(D) *For any $x \in \mathbf{R}$,*

$$\int_{-\infty}^{x} \phi\big(M_n(y)\big)\,dy \to \int_{-\infty}^{x} \phi\big(M(y)\big)\,dy.$$

Similarly the set of conditions (A′) and (C′) is equivalent to (D′), and that of (A″) and (C″) is equivalent to (D″).

(D′) *For any $\lambda < 0$,*

$$\int_0^\infty \widetilde{\phi}(\xi - \lambda)\sigma_n(d\xi) \to \int_0^\infty \widetilde{\phi}(\xi - \lambda)\sigma(d\xi).$$

(D″) *For any $\lambda < 0$,*

$$\int_0^\infty p_n(t)\phi(t)e^{t\lambda}\,dt \to \int_0^\infty p(t)\phi(t)e^{t\lambda}\,dt.$$

Proof
Assume that $m_n \to m$ in $\mathcal{E}_\phi$. Then, (C) implies that there exists $c < l$ such that

$$\int_{-\infty}^{c} \phi\big(M_n(y)\big)\,dy \le 1.$$

Hence for any $x < c$,

$$\phi\big(M_n(x)\big)(c-x) \le \int_x^c \phi\big(M_n(y)\big)\,dy \le \int_{-\infty}^{c} \phi\big(M_n(y)\big)\,dy \le 1,$$

which shows that

$$M_n(x) \le \phi^{-1}\Big(\frac{1}{c-x}\Big)$$

for any $n \ge 1$ and $x < c$. Then it is easy to see that $M_n(x) \to M(x)$ at every point x, and this together with (C) implies (D). Conversely, for any $\epsilon > 0$, choose $c < l$

such that

$$\int_{-\infty}^{c} \phi\big(M(y)\big)\,dy < \epsilon.$$

Then, clearly (C) follows from (D). The condition (A) can be derived from (D) by the monotonicity of ϕ and M_n. We omit the proof for (D$'$) and (D$''$). □

6. Application

Typical examples of m belonging to $\mathcal{E}$ are

$$m_\alpha(x) = \begin{cases} C_\alpha x^{-\beta} & x>0 \text{ if } 0<\alpha<1, \\ e^x & x\in \mathbf{R} \text{ if } \alpha=1 \\ C_\alpha(-x)^{-\beta} & x<0 \text{ if } \alpha>1, \end{cases} \qquad \text{with } \beta = \frac{\alpha}{\alpha-1},$$

and the spectral measures and $p(t)$ are

$$\sigma_\alpha(d\xi) = \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\,d\xi^\alpha, \qquad p_\alpha(t) = \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)}t^{-\alpha},$$

where

$$C_\alpha = \begin{cases} \big(\frac{1-\alpha}{\alpha}\big)^{\frac{\alpha}{1-\alpha}}, & 0<\alpha<1, \\ \big(\frac{\alpha-1}{\alpha}\big)^{-\frac{\alpha}{\alpha-1}}, & 1<\alpha. \end{cases}$$

In this section we consider the asymptotic behavior of the spectral measures and the transition probability densities when strings are close to the above typical ones. If $\alpha \in (0,2)$, the following results are already known. Here we denote

$$f(x) \sim g(x) \quad \text{as } x\uparrow 0 \ (x\to\infty)$$

if

$$\lim_{x\uparrow 0}\frac{f(x)}{g(x)} = 1 \qquad \Big(\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1\Big)$$

hold, respectively. Let φ be a function regularly varying at 0 with exponent $\alpha-1$.

THEOREM 6 (SEE [1], [4])

The following asymptotic relationship between m and p is valid.

(1) *If $\alpha\in(0,1)$, then*

$$m(x) \sim \frac{(-\beta)^\beta}{x\varphi^{-1}(x)} \quad \text{as } x\uparrow\infty$$

holds if and only if

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)}\frac{1}{t}\varphi\Big(\frac{1}{t}\Big) \quad \text{as } t\to\infty.$$

(2) *If $\alpha\in(1,2)$, then*

$$m(x) \sim \frac{\beta^\beta}{-x\varphi^{-1}(-x)} \quad \text{as } x\uparrow 0$$

holds if and only if

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)} \frac{1}{t} \varphi\Big(\frac{1}{t}\Big) \quad \textit{as } t \to \infty.$$

They showed an analogous result in the case $\alpha = 1$ in Kasahara and Watanabe [4]. In this section we extend their results to the case $\alpha \geq 2$ by applying Theorem 4. The basic idea, which was first employed by Kasahara [1], is to use the continuity between m and p and the scaling relationship

$$abm(ax) \leftrightarrow \frac{1}{ab} p(b^{-1}t) \tag{6.1}$$

for any $a, b > 0$. The proof proceeds as in [4], especially in the case $\alpha = 1$.

Let $m \in \mathcal{E}$ be a nondecreasing function with $l = 0$; namely, let

$$m(x) < \infty \quad \text{on } (-\infty, 0) \qquad \text{and} \qquad m(x) = \infty \quad \text{on } (0, \infty).$$

Let φ be a regularly varying function at 0 with exponent $\alpha - 1$, and set

$$m_\nu(x) = \nu \varphi^{-1}(\nu) m(\nu x).$$

Then from (6.1) we have

$$\begin{cases} M_\nu(x) = \varphi^{-1}(\nu) M(\nu x), \\ p_\nu(t) = \nu^{-1} \varphi^{-1}(\nu)^{-1} p(\varphi^{-1}(\nu)^{-1} t), \\ \sigma_\nu(\xi) = \nu^{-1} \varphi^{-1}(\nu)^{-1} \sigma(\varphi^{-1}(\nu)\xi). \end{cases} \tag{6.2}$$

To consider an extension of Theorem 6 we introduce conditions on m and σ:

$$m(x) \sim \frac{\beta^\beta}{-x\varphi^{-1}(-x)} \quad \text{as } x \uparrow 0, \tag{6.3}$$

which means that

$$M(x) \sim \frac{\beta^\beta}{(\beta - 1)\varphi^{-1}(-x)} \quad \text{as } x \uparrow 0, \tag{6.4}$$

and

$$p(t) = \int_0^\infty e^{-t\xi} \, d\sigma(\xi) < \infty \quad \text{for any } t > 0. \tag{6.5}$$

PROPOSITION 1

If $m \in \mathcal{E}$ satisfies (6.3), then it holds that

$$\sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2} \xi^\alpha \quad \textit{as } \xi \downarrow 0. \tag{6.6}$$

Moreover, if m satisfies (6.5) as well, then (6.7) below holds:

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)} \frac{1}{t} \varphi\Big(\frac{1}{t}\Big) \quad \textit{as } t \to \infty. \tag{6.7}$$

Proof

Since

$$M_\nu(x) = \int_{-\infty}^x m_\nu(y) \, dy = \varphi^{-1}(\nu) M(\nu x)$$

holds, from (6.4) we know that

$$M_\nu(x) \to \frac{\beta^\beta}{\beta - 1}(-x)^{1-\beta} \quad \text{as } \nu \to 0$$

for any $x < 0$, which means that $\{M_\nu\}$ satisfies the condition (B). Applying Theorem 2 yields

$$\sigma_\nu(\xi) \to \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^\alpha \quad \text{for any } \xi > 0$$

as $\nu \to 0$, which is equivalent to (6.6) due to (6.2). If we assume the condition (6.5) as well on σ, the Abelian theorem for the Laplace transform shows the property (6.7). □

To obtain a converse statement to the above proposition we need the following.

LEMMA 10

Assume that $\sigma \in \mathcal{S}$ satisfies condition (6.5) and a condition

$$\int_0^1 p(t)\phi(t)\,dt < \infty \tag{6.8}$$

for a positive function ϕ on $[0,1]$ satisfying

$$\phi(st) \le Ct^k\phi(s) \quad \textit{for any } s,t \le 1 \textit{ for some } k > \alpha - 1. \tag{6.9}$$

Then $\{p_\nu(t)\}$ satisfies condition (4.3), namely,

$$\sup_{\nu>0}\int_0^1 p_\nu(t)\phi(t)\,dt < \infty. \tag{6.10}$$

Proof

Since φ is a regularly varying function at 0 with exponent $\alpha - 1$, $t^{-1}\varphi(t^{-1})$ is regularly varying at ∞ with exponent $-\alpha$, and there exists a slowly varying function $l(t)$ such that

$$\frac{1}{t}\varphi\Big(\frac{1}{t}\Big) = t^{-\alpha}l(t).$$

Generally a slowly varying function $l(t)$ has an expression

$$l(t) = c(t)\exp\Big(\int_a^t \frac{\epsilon(u)}{u}\,du\Big) \tag{6.11}$$

with a positive constant a and functions $c(t)$, $\epsilon(t)$ behaving as

$$c(t) \to c > 0, \qquad \epsilon(t) \to 0 \quad \text{as } t \to \infty.$$

Now we decompose the integral in (6.10) into two parts:

$$\int_0^1 p_\nu(t)\phi(t)\,dt = \nu^{-1}\varphi^{-1}(\nu)^{-1}\int_0^1 p\big(\varphi^{-1}(\nu)^{-1}t\big)\phi(t)\,dt = I_1 + I_2$$

with

$$\begin{cases} I_1 = \nu^{-1}\varphi^{-1}(\nu)^{-1} \int_{N\varphi^{-1}(\nu)}^{1} p(\varphi^{-1}(\nu)^{-1}t)\phi(t)\,dt, \\ I_2 = \nu^{-1}\varphi^{-1}(\nu)^{-1} \int_{0}^{N\varphi^{-1}(\nu)} p(\varphi^{-1}(\nu)^{-1}t)\phi(t)\,dt, \end{cases}$$

where N is chosen so that

$$\left|\epsilon(u)\right| \le \delta \quad \text{for any } u \ge N$$

holds with a positive δ satisfying $\delta < k-(\alpha-1)$. Since the condition (6.7) implies

$$0 < \frac{p(\varphi^{-1}(\nu)^{-1}t)}{\varphi^{-1}(\nu)^{\alpha}t^{-\alpha}l(\varphi^{-1}(\nu)^{-1}t)} \le C'$$

for any $t \ge N\varphi^{-1}(\nu)$ with some constant C', we have

$$I_1 \le C'\nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}l\big(\varphi^{-1}(\nu)^{-1}\big)\int_{N\varphi^{-1}(\nu)}^{1} t^{-\alpha}\frac{l(\varphi^{-1}(\nu)^{-1}t)}{l(\varphi^{-1}(\nu)^{-1})}\phi(t)\,dt.$$

First note that

$$\nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}l\big(\varphi^{-1}(\nu)^{-1}\big) = \nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}\varphi^{-1}(\nu)^{1-\alpha}\varphi\big(\varphi^{-1}(\nu)\big) = 1,$$

and (6.11) shows, for $t \ge N\varphi^{-1}(\nu)$, that

$$\begin{aligned} \frac{l(\varphi^{-1}(\nu)^{-1}t)}{l(\varphi^{-1}(\nu)^{-1})} &= \exp\Big(\int_a^{\varphi^{-1}(\nu)^{-1}t}\frac{\epsilon(u)}{u}\,du - \int_a^{\varphi^{-1}(\nu)^{-1}}\frac{\epsilon(u)}{u}\,du\Big) \\ &= \exp\Big(-\int_{\varphi^{-1}(\nu)^{-1}t}^{\varphi^{-1}(\nu)^{-1}}\frac{\epsilon(u)}{u}\,du\Big) \\ &\le \exp(\delta\log t^{-1}) = t^{-\delta}. \end{aligned}$$

In (6.9) setting $s=1$, we have $\phi(t) \le C\phi(1)t^{k}$; hence

$$I_1 \le C\int_{N\varphi^{-1}(\nu)}^{1} t^{-\alpha}t^{-\delta}\phi(t)\,dt \le CC'\phi(1)\int_0^1 t^{-\alpha-\delta+k}\,dt$$

is valid. Due to (6.9) I_2 can be estimated as

$$\begin{aligned} I_2 &= \nu^{-1}\int_0^N p(s)\phi\big(\varphi^{-1}(\nu)s\big)\,ds \\ &\le C\nu^{-1}\big(\varphi^{-1}(\nu)\big)^k\int_0^N p(s)\phi(s)\,ds \le C''\nu^{-1+\frac{k}{\alpha-1}-\delta'}\int_0^N p(s)\phi(s)\,ds, \end{aligned}$$

where $\delta' > 0$ can be chosen so that

$$-1+\frac{k}{\alpha-1}-\delta' > 0$$

holds. Consequently, we have

$$\int_0^1 p_\nu(t)\phi(t)\,dt \le CC'\phi(1)\int_0^1 t^{-\alpha-\delta+k}\,dt + C''\nu^{-1+\frac{k}{\alpha-1}-\delta'}\int_0^N p(s)\phi(s)\,ds,$$

and (6.8) implies the second assertion of (6.10). $\square$

REMARK 1

The property (6.9) is satisfied not only by $\phi(t) = t^k$ with $k > \alpha - 1$ but also by subexponential functions: for $p > 1, c > 0$,

$$\phi(t) = \exp\bigl(-c(-\log t)^p\bigr).$$

Within the knowledge of the previous sections the best converse statement to Proposition 1 is as follows.

PROPOSITION 2

Let $m \in \mathcal{E}$ be a nondecreasing function with $l = 0$ and $M(0) = \infty$. Assume that $\sigma \in \mathcal{S}$ satisfies the conditions (6.5) and (6.8) with a positive function ϕ on $[0,1]$ satisfying (S.1) and (6.9). Then the property (6.6) (equivalently, (6.7)) implies (6.3).

Proof

First note that (6.6) is equivalent to

$$\sigma_\nu(\xi) \to \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^\alpha \quad \text{for any } \xi > 0,$$

as $\nu \to 0$. Since we are assuming that $M(0) = \infty$,

$$M_\nu(0) = \varphi^{-1}(\nu)M(0) = \infty$$

holds for any $\nu > 0$, and there exists uniquely $a_\nu < 0$ such that

$$M_\nu(a_\nu) = \frac{\beta^\beta}{\beta - 1} \equiv c.$$

Set

$$\widetilde{M}_\nu(x) = M_\nu(x + a_\nu + 1).$$

Then, taking -1 instead of 0 as a normalization point, Lemma 10 makes it possible to apply Theorem 3, and we have the following:

$$\widetilde{M}_\nu(x) \to \begin{cases} c(-x)^{1-\beta} & \text{for } x < 0, \\ \infty & \text{for } x > 0 \end{cases}$$

holds in $\mathcal{E}$ as $\nu \to 0$, from which

$$\varphi^{-1}(\nu)M\bigl(\nu(x + a_\nu + 1)\bigr) \to \begin{cases} c(-x)^{1-\beta} & \text{for } x < 0, \\ \infty & \text{for } x > 0 \end{cases} \tag{6.12}$$

follows. To simplify the involved formula (6.12) we take their inverse. Set

$$u = M\bigl(\nu(x + a_\nu + 1)\bigr), \qquad \lambda = c\varphi^{-1}(\nu)^{-1}.$$

Since $\varphi^{-1}(\nu)M(\nu a_\nu) = c$, we easily see that

$$\varphi(c\lambda^{-1})(x+1) + M^{-1}(\lambda) = M^{-1}(u).$$

Denoting $y = \lambda^{-1}u$, (6.12) is equivalent to

$$y \to (-x)^{1-\beta},$$

from which

$$\frac{M^{-1}(\lambda y)-M^{-1}(\lambda)}{\varphi(c\lambda^{-1})}=x+1\to 1-y^{-(\beta-1)}$$

follows for any $y>0$ as $\lambda\to\infty$. Since φ is regularly varying at 0 with exponent $\alpha-1$,

$$\frac{\varphi(c\lambda^{-1})}{\varphi(\lambda^{-1})}\to c^{\alpha-1}=(\alpha-1)^{-1}\alpha^{\alpha}\quad\text{as } \lambda\to\infty,$$

and

$$\lim_{\lambda\to\infty}\frac{M^{-1}(\lambda x)-M^{-1}(\lambda)}{\varphi(\lambda^{-1})}=(\alpha-1)^{-1}\alpha^{\alpha}(1-x^{-(\alpha-1)}) \tag{6.13}$$

follows. Then Lemma 11 below shows (6.3). □

LEMMA 11

(6.13) implies (6.3).

Proof

Assume (6.13). Since $M^{-1}(x)$ has a monotone density

$$\big(M^{-1}(x)\big)'=\frac{1}{m(M^{-1}(x))},$$

the monotone density theorem implies

$$\lim_{\lambda\to\infty}\Big(\frac{M^{-1}(\lambda x)-M^{-1}(\lambda)}{\varphi(\lambda^{-1})}\Big)'=\big((\alpha-1)^{-1}\alpha^{\alpha}(1-x^{-(\alpha-1)})\big)',$$

which is

$$\lim_{\lambda\to\infty}\frac{\lambda}{\varphi(\lambda^{-1})m(M^{-1}(\lambda x))}=\alpha^{\alpha}x^{-\alpha}.$$

Setting $x=1$ and $u=M^{-1}(\lambda)$, we have

$$\lim_{u\to 0}\frac{M(u)}{\varphi(M(u)^{-1})m(u)}=\alpha^{\alpha}.$$

For any $\epsilon>0$ there exists $\delta>0$ such that for any $u\in(-\delta,0)$,

$$\alpha^{-\alpha}-\epsilon\le\frac{\varphi(M(u)^{-1})m(u)}{M(u)}\le\alpha^{-\alpha}+\epsilon$$

is valid. Noting $m(u)=M(u)'$, we see that

$$(\alpha^{-\alpha}-\epsilon)(-x)\le\int_x^0\frac{\varphi(M(u)^{-1})}{M(u)}\,dM(u)\le(\alpha^{-\alpha}+\epsilon)(-x)$$

for any $x\in(-\delta,0)$; hence

$$(\alpha^{-\alpha}-\epsilon)(-x)\le\int_{M(x)}^{\infty}\frac{\varphi(y^{-1})}{y}\,dy=\int_0^{M(x)^{-1}}\frac{\varphi(z)}{z}\,dz\le(\alpha^{-\alpha}+\epsilon)(-x).$$

Since $\varphi(z)/z$ is a regularly varying function at 0 with exponent $\alpha-2$,

$$\int_0^y \frac{\varphi(z)}{z}\,dz \sim \frac{\varphi(y)}{\alpha-1} \quad \text{as } y \downarrow 0$$

is valid, which implies

$$\frac{\varphi(M(x)^{-1})}{\alpha-1} \sim \alpha^{-\alpha}(-x) \quad \text{as } x \uparrow 0;$$

hence

$$M(x) \sim \frac{\beta^\beta}{\beta-1}\varphi^{-1}(-x)^{-1} \quad \text{as } x \uparrow 0.$$

This is equivalent to (6.3). □

In the above two propositions we stated the conditions which should be satisfied by $m \in \mathcal{E}$ in terms of its spectral function σ. It may be preferable to describe the result by m itself directly. To do so, unfortunately we have to impose a more restrictive condition on m, and combining Propositions 1 and 2 we have the following.

THEOREM 7

Let $\alpha \geq 2$, let $k > \alpha - 1$, and let φ be a regularly varying function at 0 with exponent $\alpha - 1$. Let $m \in \mathcal{E}$ be a nondecreasing function with $l = 0$ and $M(0) = \infty$. Assume that m satisfies

$$\int_{-\infty}^{-1} M(x)^k\,dx < \infty.$$

Then, the property

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)}\frac{1}{t}\varphi\Big(\frac{1}{t}\Big) \quad \textit{as } t \to \infty$$

holds if and only if the asymptotics below is valid:

$$m(x) \sim \frac{\beta^\beta}{-x\varphi^{-1}(-x)} \quad \textit{as } x \uparrow 0.$$

Proof

The proof is immediate from the above two propositions if we observe $\phi(t) = t^k$ satisfies all the requirements needed in Proposition 2. □

Acknowledgment. The author would like to express his hearty thanks to Professor Y. Kasahara, who allowed him to read a preprint which was very helpful in the course of proving Proposition 2 and Lemma 11.

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