Continuity of LF-algebra representations associated to representations of Lie groups

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Abstract Let G be a finite-dimensional Lie group, and let E be a locally convex topological G-module. If E is sequentially complete, then E and the space E^∞ of smooth vectors are $C_c^\infty(G)$-modules, but the module multiplication need not be continuous. The pathology can be ruled out if E is (or embeds into) a projective limit of Banach G-modules. Moreover, in this case E^ω (the space of analytic vectors) is a module for the algebra $\mathcal{A}(G)$ of superdecaying analytic functions introduced by Gimperlein, Krötz, and Schlichtkrull. We prove that E^ω is a *topological* $\mathcal{A}(G)$-module if E is a Banach space or, more generally, if every countable set of continuous seminorms on E has an upper bound. The same conclusion is obtained if G has a compact Lie algebra. The question of whether $C_c^\infty(G)$ and $\mathcal{A}(G)$ are topological algebras is also addressed.

Introduction and statement of results

We study continuity properties of algebra actions associated with representations of a (finite-dimensional, real) Lie group G. Throughout this note, E denotes a topological G-module, that is, a complex locally convex space endowed with a continuous left G-action $\pi\colon G \times E \to E$ by linear maps $\pi(g,\cdot)$.

Results concerning $C_c^\infty(G)$ and the space of smooth vectors

Our first results concern the convolution algebra $C_c^\infty(G)$ of complex-valued test functions on a Lie group G. As usual, $v \in E$ is called a *smooth vector* if the orbit map $\pi_v\colon G \to E$, $\pi_v(g) := \pi(g,v)$ is smooth. The space E^∞ is endowed with the initial topology $\mathcal{O}_{E^\infty}$ with respect to the map

$$\Phi\colon E^\infty \to C^\infty(G,E), \qquad \Phi(v) = \pi_v. \tag{1}$$

Let λ_G be a left Haar measure on G. If E is sequentially complete or has the metric convex compactness property (see [41] for information on this concept),* then the weak integral

$$\Pi(\gamma, v) := \int_G \gamma(x)\pi(x,v)\, d\lambda_G(x) \tag{2}$$

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*That is, each metrizable compact subset $K \subseteq E$ has a relatively compact convex hull.

exists in E for all $v \in E$ and $\gamma \in C_c^\infty(G)$ (see [29, Proposition 1.2.3] and [39, Theorem 3.27]). In this way, E becomes a $C_c^\infty(G)$-module. Moreover, $\Pi(\gamma, v) \in E^\infty$ for all $\gamma \in C_c^\infty(G)$ and $v \in E$, whence E^∞ is a $C_c^\infty(G)$-submodule in particular (as we recall in Lemma 1.9). It is natural to ask whether the module multiplication

$$C_c^\infty(G) \times E \to E, \qquad (\gamma, v) \mapsto \Pi(\gamma, v), \tag{3}$$

respectively,

$$C_c^\infty(G) \times E^\infty \to E^\infty, \qquad (\gamma, v) \mapsto \Pi(\gamma, v), \tag{4}$$

is continuous, that is, if E and $(E^\infty, \mathcal{O}_{E^\infty})$ are topological $C_c^\infty(G)$-modules. Contrary to a recent assertion (see [14, pp. 667–668]), this can fail even if E is Fréchet.

PROPOSITION A

If G is a noncompact Lie group and $E := C^\infty(G)$ with $\pi\colon G \times C^\infty(G) \to C^\infty(G)$, $\pi(g, \gamma)(x) := \gamma(g^{-1}x)$, then neither E nor E^∞ are topological $C_c^\infty(G)$-modules, that is, the maps (3) and (4) are discontinuous.

A continuous seminorm p on E is called *G-continuous* if $\pi\colon G \times (E, p) \to (E, p)$ is continuous (see [4, p. 7]). Varying terminology from [31], we call a topological G-module E *proto-Banach* if the topology of E is defined by a set of G-continuous seminorms.* If E is a Fréchet space, then E is proto-Banach if and only if there is a sequence $(p_n)_{n \in \mathbb{N}}$ of G-continuous seminorms defining the topology, that is, if and only if Π is an F-representation as in [4], [14], and [15].

PROPOSITION B

Let G be a Lie group, and let E be a proto-Banach G-module that is sequentially complete or has the metric convex compactness property. Then the map $\Pi\colon C_c^\infty(G) \times E \to E^\infty$ from (2) is continuous. In particular, E and E^∞ are topological $C_c^\infty(G)$-modules.

We mention that $C_c^\infty(G)$ is a topological algebra if and only if G is σ-compact (see [6, p. 3]; cf. [30, Proposition 2.3] for the special case $G = \mathbb{R}^n$).

Results concerning $\mathcal{A}(G)$ and the space of analytic vectors

Let G be a connected Lie group now. If E is a topological G-module, say that $v \in E$ is an *analytic vector* if the orbit map $\pi_v\colon G \to E$ is real analytic (in the sense recalled in Section 4). Write $E^\omega \subseteq E$ for the space of all analytic vectors. If $G \subseteq G_{\mathbb{C}}$ (which we assume henceforth for simplicity of the presentation), let $(V_n)_{n \in \mathbb{N}}$ be a basis of relatively compact, symmetric, connected identity neighborhoods in $G_{\mathbb{C}}$, such that $V_n \supseteq \overline{V_{n+1}}$ (e.g., we can choose V_n as in [15]). Then $v \in E$ is an analytic vector if and only if π_v admits a complex analytic extension $\widetilde{\pi}_v\colon GV_n \to$

*Namely, E embeds into a projective limit of Banach G-modules (cf. [4, Remark 2.5]).

E for some $n \in \mathbb{N}$ (see Lemma 4.4). We write $E_n \subseteq E^\omega$ for the space of all $v \in E^\omega$ such that π_v admits a $\mathbb{C}$-analytic extension to GV_n, and give E_n the topology making

$$\Psi_n \colon E_n \to \mathcal{O}(GV_n, E), \qquad v \mapsto \widetilde{\pi}_v,$$

a topological embedding, using the compact open topology on the space $\mathcal{O}(GV_n, E)$ of all E-valued $\mathbb{C}$-analytic maps on GV_n. Like [15], we give E^ω the topology making it the direct limit $E^\omega = \lim_{\longrightarrow} E_n$ as a locally convex space.*

We fix a left-invariant Riemannian metric $\mathbf{g}$ on G, let $\mathbf{d} \colon G \times G \to [0, \infty[$ be the associated left-invariant distance function, and set

$$d(g) := \mathbf{d}(g, 1) \quad \text{for } g \in G. \tag{5}$$

Following [15] and [14], we let $\mathcal{R}(G)$ be the Fréchet space of continuous functions $\gamma \colon G \to \mathbb{C}$ which are *superdecaying* in the sense that

$$\|\gamma\|_N := \sup\big\{ |\gamma(x)| e^{N d(x)} \colon x \in G \big\} < \infty \quad \text{for all } N \in \mathbb{N}_0. \tag{6}$$

Then $\mathcal{R}(G)$ is a topological algebra under convolution (see [15, Proposition 4.1(ii)]). If E is a sequentially complete proto-Banach G-module, then

$$\Pi(\gamma, v) := \int_G \gamma(x) \pi(x, v) \, d\lambda_G(x) \quad \text{for } \gamma \in \mathcal{R}(G), v \in E$$

exists in E as an absolutely convergent integral, and Π makes E a topological $\mathcal{R}(G)$-module (as for F-representations; see [15, Proposition 4.1(iii)]).

As $G \times \mathcal{R}(G) \to \mathcal{R}(G)$, $\pi(g, \gamma)(x) := \gamma(g^{-1}x)$ is an F-representation (see [15, Proposition 4.1(i)]), $\mathcal{A}(G) := \mathcal{R}(G)^\omega$ is the locally convex direct limit of the steps $\mathcal{A}_n(G) := \mathcal{R}(G)_n$. Since $\mathbb{C}$-analytic extensions of orbit maps can be multiplied pointwise in $(\mathcal{R}(G), *)$, both $\mathcal{A}_n(G)$ and $\mathcal{A}(G)$ are subalgebras of $\mathcal{R}(G)$.

If E is a sequentially complete proto-Banach G-module, then

$$\Pi(\gamma, v) \in E^\omega \quad \text{for all } \gamma \in \mathcal{A}(G), v \in E; \tag{7}$$

moreover,

$$\mathcal{A}_n(G) \times E \to E_n, \qquad (\gamma, v) \mapsto \Pi(\gamma, v) \quad \text{is continuous for each } n \in \mathbb{N}. \tag{8}$$

This can be shown as in the case of F-representations in [15, Proposition 4.6].

PROBLEM

The following assertions concerning F-representations and the algebras $\mathcal{A}(G)$ (stated in [15, Propositions 4.2(ii), 4.6]) seem to be open in general (in view of difficulties explained presently, in Remark 2).

*If G is an arbitrary connected Lie group, let $q \colon \widetilde{G} \to G$ be the universal covering group, and let $V_n \subseteq (\widetilde{G})_{\mathbb{C}}$ be as above. Then $\widetilde{G} \subseteq (\widetilde{G})_{\mathbb{C}}$. Define E_n now as the space of all $v \in E^\omega$ such that $\pi_v \circ q$ has a complex analytic extension to $\widetilde{G}V_n \subseteq (\widetilde{G})_{\mathbb{C}}$, and topologize E^ω as before. In this way, we could easily drop the condition that $G \subseteq G_{\mathbb{C}}$.

(a) Is $\Pi\colon \mathcal{A}(G) \times E \to E^\omega$ continuous for each F-representation (E, π) (or even for each sequentially complete proto-Banach G-module)?*

(b) Is $\Pi\colon \mathcal{A}(G) \times E^\omega \to E^\omega$ continuous in the situation of (a)?†

(c) Is the convolution $\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G)$ continuous?

To formulate a solution to these problems in special cases, recall that a preorder on the set $P(E)$ of all continuous seminorms p on a locally convex space E is obtained by declaring $p \preceq q$ if $p \le Cq$ pointwise for some $C > 0$. The space E is said to have the *countable neighborhood property* if every countable set $M \subseteq P(E)$ has an upper bound in $(P(E), \preceq)$ (see [8], [11], and the references therein).

PROPOSITION C

Let G be a connected Lie group with $G \subseteq G_{\mathbb{C}}$, and let E be a sequentially complete, proto-Banach G-module. If E is normable or E has the countable neighborhood property, then $\Pi\colon \mathcal{A}(G) \times E \to E^\omega$ is continuous. In particular, E^ω is a topological $\mathcal{A}(G)$-module.

REMARK 1

Recall that a metrizable locally convex space has the countable neighborhood property (c.n.p.) if and only if it is normable. Because the c.n.p. is inherited by countable locally convex direct limits (see [11]), it follows that every LB-space (i.e., every countable locally convex direct limit of Banach spaces) has the c.n.p. Also, locally convex spaces E which are k_ω-spaces have the c.n.p. (see [21, Corollary 8.1], [20, Example 9.4]; cf. [8]).‡ For example, the dual E' of any metrizable topological vector space E is a k_ω-space, when equipped with the compact open topology (cf. [3, Corollary 4.7]).

For G a compact, connected Lie group, the convolution $\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G)$ is continuous, and thus $\mathcal{A}(G)$ is a topological algebra. In fact, $\mathcal{R}(G) = C(G)$ is normable in this case (as each $\|\cdot\|_N$ is equivalent to $\|\cdot\|_\infty$ then). Since $\mathcal{A}(G) = \mathcal{R}(G)^\omega$, Proposition C applies.

The same conclusion can be obtained by an alternative argument, which shows also that $(\mathcal{A}(G), *)$ is a topological algebra for each abelian connected Lie group G. In contrast to the setting of Proposition C, quite general spaces E are allowed now, but conditions are imposed on G. Recall that a real Lie algebra $\mathfrak{g}$ is said to be *compact* if it admits an inner product making $e^{\mathrm{ad}(x)}$ an isometry for each $x \in \mathfrak{g}$ (where $\mathrm{ad}(x) := [x, \cdot]$ as usual). If G is compact or abelian, then its Lie algebra $L(G)$ is compact.

*By Lemma 4.14, $\Pi\colon \mathcal{A}(G) \times E \to E^\omega$ is always separately continuous, hypocontinuous in its second argument, and sequentially continuous (hence also the maps in (b) and (c)).

†(b) follows from (a) as the inclusion $E^\omega \to E$ is continuous linear.

‡A topological space X is k_ω if $X = \varinjlim K_n$ with compact spaces $K_1 \subseteq K_2 \subseteq \cdots$ (see [12], [24]).

PROPOSITION D

Let G be a connected Lie group with $G \subseteq G_{\mathbb{C}}$, whose Lie algebra $L(G)$ is compact. Then E^ω is a topological $\mathcal{A}(G)$-module for each sequentially complete, proto-Banach G-module E. In particular, convolution is jointly continuous, and thus $(\mathcal{A}(G), *)$ is a topological algebra.

REMARK 2

Note that, due to the continuity of the maps (8), the map

$$\Pi\colon \mathcal{A}(G) \times E \to E^\omega$$

is continuous with respect to the topology $\mathcal{O}_{\mathrm{DL}}$ on $\mathcal{A}(G) \times E$ which makes it the direct limit $\lim_{\longrightarrow}(\mathcal{A}_n(G) \times E)$ as a topological space. On the other hand, there is the topology $\mathcal{O}_{\mathrm{lcx}}$ making $\mathcal{A}(G) \times E$ the direct limit $\lim_{\longrightarrow}(\mathcal{A}_n(G) \times E)$ as a locally convex space. Since locally convex direct limits and two-fold products commute (see [30, Theorem 3.4]), $\mathcal{O}_{\mathrm{lcx}}$ coincides with the product topology on $(\lim_{\longrightarrow}\mathcal{A}_n(G)) \times E = \mathcal{A}(G) \times E$ and hence is the topology we are interested in. Unfortunately, as Π is not linear, it is *not* possible to deduce continuity of Π on $(\mathcal{A}(G) \times E, \mathcal{O}_{\mathrm{lcx}})$ from the continuity of the maps (8).[*,†]

Of course, whenever $\mathcal{O}_{\mathrm{DL}} = \mathcal{O}_{\mathrm{lcx}}$, we obtain continuity of $\Pi\colon \mathcal{A}(G) \times E \to E^\omega$ with respect to $\mathcal{O}_{\mathrm{lcx}}$. Now $\mathcal{O}_{\mathrm{lcx}} \subseteq \mathcal{O}_{\mathrm{DL}}$ always, but equality $\mathcal{O}_{\mathrm{lcx}} = \mathcal{O}_{\mathrm{DL}}$ only occurs in exceptional situations. In the prime case of an F-representation (E, π) of G, we have $\mathcal{O}_{\mathrm{DL}} \neq \mathcal{O}_{\mathrm{lcx}}$ in all cases of interest, as we shall presently see. Thus, Problems (a)–(c) remain open in general (apart from the special cases settled in Propositions C and D).

The following observation pinpoints the source of these difficulties.

PROPOSITION E

If E is an infinite-dimensional Fréchet space and G a connected Lie group with $G \subseteq G_{\mathbb{C}}$ and $G \neq \{1\}$, then $\mathcal{O}_{\mathrm{DL}} \neq \mathcal{O}_{\mathrm{lcx}}$ on $\mathcal{A}(G) \times E$.

Proposition E will be deduced from a new result on direct limits of topological groups (see Proposition 7.1), which is a variant of Yamasaki's theorem [42, Theorem 4] for direct sequences which need not be strict but are sequentially compact regular.

We mention that $\mathcal{A}(G)$ also is an algebra under *pointwise* multiplication (instead of convolution) and in fact is a topological algebra (see Section 8). Sections 1–3 are devoted to Propositions A and B; Sections 4–7 are devoted to the proofs of Propositions C, D, and E (and the respective preliminaries). The proofs of some auxiliary results have been relegated to the appendix (see also

[*]This problem was overlooked in [15, proof of Proposition 4.6].

[†]Note that, in Proposition A, the convolution $C_c^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is discontinuous, although its restriction to $C^\infty_{[-n,n]}(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is continuous for all $n \in \mathbb{N}$.

[35], [36] for recent studies of smooth and analytic vectors, with a view towards infinite-dimensional groups).

BASIC NOTATION

We write $\mathbb{N}_0 := \{0,1,2,\ldots\}$. If E is a vector space and q a seminorm on E, we set $B^q_\varepsilon(x) := \{y \in E\colon q(y-x) < \varepsilon\}$ for $x \in E$, $\varepsilon > 0$. $L(G)$ is the Lie algebra of a Lie group G, and $\operatorname{im}(f)$ is the image of a map f. If X is a set and $f\colon X \to \mathbb{C}$ a map, as usual $\|f\|_\infty := \sup\{|f(x)|\colon x \in X\}$. If q is a seminorm on a vector space E and $f\colon X \to E$, we write $\|f\|_{q,\infty} := \sup\{q(f(x))\colon x \in X\}$.

1. Preliminaries for Propositions A and B

We shall use concepts and basic tools from calculus in locally convex spaces.

1.1.

Let E, F be real locally convex spaces, let $U \subseteq E$ be open, and let $r \in \mathbb{N}_0 \cup \{\infty\}$. Call $f\colon U \to F$ a C^r-map if f is continuous; the iterated directional derivatives

$$d^{(k)} f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1} f)(x)$$

exist in E for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$, and $y_1, \ldots, y_k \in E$; and each $d^{(k)} f\colon U \times E^k \to F$ is continuous. The C^∞-maps are also called *smooth*.

See [17], [26], [28], [37], and [38]. Since compositions of C^r-maps are C^r, one can define C^r-manifolds modelled on locally convex spaces as expected.

1.2.

Given a Hausdorff space M and locally convex space E, we endow the space $C^0(M,E)$ of continuous E-valued functions on M with the compact open topology. If M is a C^r-manifold modeled on a locally convex space X, we give $C^r(M,E)$ the compact open C^r-topology, that is, the initial topology with respect to the maps $C^r(M,E) \to C^0(V \times X^k, E)$, $\gamma \mapsto d^{(k)}(\gamma \circ \phi^{-1})$ for all charts $\phi\colon U \to V$ of M and $k \in \mathbb{N}_0$ such that $k \leq r$. If M is finite-dimensional and $K \subseteq M$ is compact, as usual we endow $C^r_K(M,E) := \{\gamma \in C^r(M,E)\colon \gamma|_{M\setminus K} = 0\}$ with the topology induced by $C^r(M,E)$, and give $C^r_c(M,E) = \bigcup_K C^r_K(M,E)$ the locally convex direct limit topology. We abbreviate $C^r(M) := C^r(M,\mathbb{C})$, and so forth.

The following variant is essential for our purposes.

1.3.

Let E_1, E_2, and F be real locally convex spaces, let $U \subseteq E_1 \times E_2$ be open, and let $r, s \in \mathbb{N}_0 \cup \{\infty\}$. A map $f\colon U \to F$ is called a $C^{r,s}$-*map* if the derivatives

$$d^{(k,\ell)} f(x,y,a_1,\ldots,a_k,b_1,\ldots,b_\ell) := (D_{(a_k,0)} \cdots D_{(a_1,0)} D_{(0,b_\ell)} \cdots D_{(0,b_1)} f)(x,y)$$

exist for all $k, \ell \in \mathbb{N}_0$ such that $k \leq r$ and $\ell \leq s$, $(x,y) \in U$ and $a_1,\ldots,a_k \in E_1$, $b_1,\ldots,b_\ell \in E_2$, and $d^{(k,\ell)} f\colon U \times E_1^k \times E_2^\ell \to F$ is continuous.

We refer to [1] for a detailed development of the theory of $C^{r,s}$-maps. Notably, f as in (1.3) is $C^{\infty,\infty}$ if and only if it is smooth. If $h \circ f \circ (g_1 \times g_2)$ is defined, where

h is C^{r+s}, f is $C^{r,s}$, g_1 is C^r, and g_2 is C^s, then the map $h \circ f \circ (g_1 \times g_2)$ is $C^{r,s}$. As a consequence, we can speak of $C^{r,s}$-maps $f\colon M_1 \times M_2 \to M$ if M, M_1, M_2 are smooth manifolds (likewise for $f\colon U \to M$ on an open set $U \subseteq M_1 \times M_2$). See [1] and [2] for these basic facts, as well as the following aspect of the exponential law for $C^{r,s}$-maps, which is essential for us.*

LEMMA 1.4

Let $r, s \in \mathbb{N}_0 \cup \{\infty\}$, let E be a locally convex space, let M be a C^r-manifold, and let N be a C^s-manifold (both modeled on some locally convex space). If $f\colon M \times N \to E$ is a $C^{r,s}$-map, then

$$f^\vee\colon M \to C^s(N, E), \qquad f^\vee(x)(y) := f(x, y)$$

is a C^r-map. Hence, if $g\colon M \to C^s(N, E)$ is a map such that $\widehat{g}\colon M \times N \to E$, $\widehat{g}(x, y) := g(x)(y)$ is $C^{r,s}$, then g is C^r.

In particular, we encounter $C^{\infty,0}$-maps of the following form.

LEMMA 1.5

Let E_1, E_2, E_3 be real locally convex spaces, let F be a complex locally convex space, let $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ be open, let $g\colon U_1 \times U_2 \to \mathbb{C}$ be a smooth map, let $h\colon U_1 \to E_3$ be a smooth map, and let $\pi\colon U_2 \times E_3 \to F$ be a continuous map such that $\pi(y, \cdot)\colon E_3 \to F$ is real linear for each $y \in U_2$. Then the following map is $C^{\infty,0}$:

$$f\colon U_1 \times U_2 \to F, \qquad f(x, y) := g(x, y)\pi\big(y, h(x)\big).$$

For the proof of Lemma 1.5 (and those of the next four lemmas), the reader is referred to Appendix A.

LEMMA 1.6

For each Lie group G, the left translation action

$$\pi\colon G \times C_c^\infty(G) \to C_c^\infty(G), \qquad \pi(g, \gamma)(x) := \gamma(g^{-1}x)$$

is a smooth map.

We mention that G is not assumed to be σ-compact in Lemma 1.6. (Of course, σ-compact groups are the case of primary interest.)

LEMMA 1.7

Let X be a locally compact space, let E be a comlex locally convex space, and let $f \in C^0(X, E)$. Then the multiplication operator $m_f\colon C_c^0(X) \to C_c^0(X, E)$, $m_f(\gamma)(x) := \gamma(x)f(x)$ is continuous linear.

*Exponential laws for smooth functions are basic tools of infinite-dimensional analysis (see, e.g., [22]; cf. [34] for related bornological results).

We also need a lemma on the parameter dependence of weak integrals. Note that the definition of $C^{r,0}$-maps does not use the vector space structure on E_2 and makes perfect sense if E_2 is merely a topological space.

LEMMA 1.8

Let X, E be locally convex spaces, let $P \subseteq X$ be open, let $r \in \mathbb{N}_0 \cup \{\infty\}$, let K be a compact topological space, let μ be a finite measure on the σ-algebra of Borel sets of K, and let $f\colon P \times K \to E$ be a $C^{r,0}$-map. Assume that the weak integral $g(p) := \int_K f(p,x)\,d\mu(x)$ exists in E, as well as the weak integrals

$$\int_K d^{(k,0)} f(p,x,q_1,\ldots,q_k)\,d\mu(x), \tag{9}$$

for all $k \in \mathbb{N}$ such that $k \leq r$, $p \in P$, and $q_1,\ldots,q_k \in X$. Then $g\colon P \to E$ is a C^r-map, with $d^{(k)}g(p,q_1,\ldots,q_k)$ given by (9).

LEMMA 1.9

Let G be a Lie group, and let $\pi\colon G \times E \to E$ be a topological G-module which is sequentially complete or has the metric convex compactness property. Then $w := \Pi(\gamma,v) \in E^\infty$ for all $\gamma \in C^\infty_c(G)$ and $v \in E$. In particular, E and E^∞ are $C^\infty_c(G)$-modules.

2. Proof of Proposition A

The evaluation map $\varepsilon\colon C^\infty(G) \times G \to \mathbb{C}$, $(\gamma,x) \mapsto \gamma(x)$ is smooth (see, e.g., [26] or [22, Proposition 11.1]). In view of Lemma 1.4, the mapping $\pi\colon G \times C^\infty(G) \to C^\infty(G)$, $\pi(g,\gamma)(x) = \gamma(g^{-1}x)$ is smooth, because

$$\widehat{\pi}\colon G \times C^\infty(G) \times G \to \mathbb{C}, \qquad (g,\gamma,x) = \gamma(g^{-1}x) = \varepsilon(\gamma, g^{-1}x)$$

is smooth. Hence each $\gamma \in C^\infty(G)$ is a smooth vector. Using Lemma 1.4 again, we see that the linear map

$$\Phi\colon C^\infty(G) \to C^\infty\big(G, C^\infty(G)\big), \qquad \Phi(\gamma) = \pi_\gamma$$

is smooth (and hence continuous) because $\widehat{\Phi}\colon C^\infty(G) \times G \to C^\infty(G)$, $\widehat{\Phi}(\gamma,g) = \pi_\gamma(g) = \pi(g,\gamma)$ is smooth. As a consequence, $C^\infty(G)$ and the space $C^\infty(G)^\infty$ of smooth vectors coincide as locally convex spaces.

Now $\Pi(\gamma,\eta) = \gamma * \eta$ for $\gamma \in C^\infty_c(G)$ and $\eta \in C^\infty(G)$. In fact, for each $x \in G$, the point evaluation $\varepsilon_x\colon C^\infty(G) \to \mathbb{C}$, $\theta \mapsto \theta(x)$ is continuous linear. Hence

$$\Pi(\gamma,\eta)(x) = \Big(\int_G \gamma(y)\eta(y^{-1}\cdot)\,d\lambda_G(y)\Big)(x) = \int_G \gamma(y)\eta(y^{-1}x)\,d\lambda_G(y) = (\gamma * \eta)(x).$$

Thus Π is the map $C^\infty_c(G) \times C^\infty(G) \to C^\infty(G)$, $(\gamma,\eta) \mapsto \gamma * \eta$, which is discontinuous by [6, Proposition 7.1].

3. Proof of Proposition B

LEMMA 3.1

In the situation of Lemma 1.9, the bilinear mapping $\Pi\colon C^\infty_c(G) \times E \to E^\infty$ is *separately continuous, hypocontinuous in its second argument, and sequentially continuous. If E is barreled (e.g., if E is a Fréchet space), then* Π *is hypocontinuous in both arguments.*

Proof
We need only show that Π is separately continuous. In fact, $C_c^\infty(G)$ is barreled, being a locally convex direct limit of Fréchet spaces (see [40, II.7.1, II.7.2]. Hence, if Π is separately continuous, it automatically is hypocontinuous in its second argument (see [40, II.5.2]) and hence sequentially continuous (see [33, Remark following §40, 1. (5), p. 157]).

Fix $\gamma \in C_c^\infty(G)$, and let K be its support. Let $\Phi\colon E^\infty \to C^\infty(G,E)$ be as in (1). The map $\Pi(\gamma,\cdot)$ will be continuous if we can show that $h := \Phi \circ \Pi(\gamma,\cdot) : E \to C^\infty(G,E)$ is continuous. By Lemma 1.4, this will hold if $\widehat{h}\colon E \times G \to E$,

$$(v,g) \mapsto \pi(g)\int_G \gamma(x)\pi(x,v)\,d\lambda_G(x) = \int_G \gamma(g^{-1}y)\pi(y,v)\,d\lambda_G(y) \tag{10}$$

is $C^{0,\infty}$. It suffices to show that $\widehat{h}$ is C^∞. Given $g_0 \in G$, let $U \subseteq G$ be a relatively compact, open neighborhood of g_0. We show that $\widehat{h}$ is smooth on $E \times U$. For $g \in U$, the domain of integration can be replaced by the compact set $\overline{U}K \subseteq G$ without changing the second integral in (10). By Lemma 1.5,

$$(E \times G) \times G \to E, \qquad \big((v,g),y\big) \mapsto \gamma(g^{-1}y)\pi(y,v)$$

is $C^{\infty,0}$. Its restriction to $(E \times U) \times \overline{U}K$ therefore satisfies the hypotheses of Lemma 1.8, and hence the parameter-dependent integral $\widehat{h}|_{E\times U}$ is smooth.

Next, fix $v \in E$. For $\gamma \in C_c^\infty(G)$, define $\psi(\gamma)\colon G \to C_c^0(G,E)$ via $\psi(\gamma)(g)(y) := \gamma(g^{-1}y)\pi(y,v)$. We claim the following:

(a) $\psi(\gamma) \in C^\infty(G, C_c^0(G,E))$ for each $\gamma \in C_c^\infty(G)$; and
(b) the linear map $\psi\colon C_c^\infty(G) \to C^\infty(G, C_c^0(G,E))$ is continuous.

Note that the integration operator $I\colon C_c^0(G,E) \to E$, $\eta \mapsto \int_G \eta(y)\,d\lambda_G(y)$ is continuous linear,* entailing that also

$$C^\infty(G,I)\colon C^\infty\big(G, C_c^0(G,E)\big) \to C^\infty(G,E), \qquad f \mapsto I \circ f$$

is continuous linear (see [26] or [22, Lemma 4.13]). If the claim holds, then the formula $\Phi \circ \Pi(\cdot,v) = C^\infty(G,I) \circ \psi$ shows that $\Phi \circ \Pi(\cdot,v)$ is continuous, and thus $\Pi(\cdot,v)$ is continuous. Hence, it only remains to establish the claim.

To prove (a), fix $\gamma \in C_c^\infty(G)$, and let K be its support. It suffices to show that, for each $g_0 \in G$ and relatively compact, open neighborhood U of g_0 in G, the restriction $\psi(\gamma)|_U$ is smooth. As the latter has its image in $C^0_{\overline{U}K}(G,E)$, which is a closed vector subspace of both $C_c^0(G,E)$ and $C^0(G,E)$ with the same induced topology, it suffices to show that $h := \psi(\gamma)|_U$ is smooth as a map to $C^0(G,E)$

*In fact, the restriction of I to $C_K^0(G,E)$ is continuous for each compact set $K \subseteq G$, because $q(I(\gamma)) \leq \|\gamma\|_{q,\infty}\lambda_G(K)$ for each continuous seminorm q on E and $\gamma \in C_K^0(G,E)$.

(see [5, Lemma 10.1]). But this is the case (by Lemma 1.4), as

$$\widehat{h}\colon U \times G \to E, \qquad (g,y) \mapsto \gamma(g^{-1}y)\pi(y,v)$$

is $C^{\infty,0}$ (by Lemma 1.5). By Lemma 1.4, to prove (b) we need to check that

$$\widehat{\psi}\colon C^\infty_c(G) \times G \to C^0_c(G,E), \qquad \widehat{\psi}(\gamma,g)(y) = \gamma(g^{-1}y)\pi(y,v)$$

is $C^{0,\infty}$. We show that $\widehat{\psi}$ is C^∞. By Lemma 1.7, the map

$$\theta\colon C^\infty_c(G) \to C^0_c(G,E), \qquad \theta(\gamma)(y) := \gamma(y)\pi(y,v)$$

is continuous linear. The map $\tau\colon G \times C^\infty_c(G) \to C^\infty_c(G)$, $\tau(g,\gamma)(x) = \gamma(g^{-1}x)$ is smooth, by Lemma 1.6. Since $\widehat{\psi}(\gamma,g) = \theta(\tau(g,\gamma))$, also $\widehat{\psi}$ is smooth. □

Proof of Proposition B

As the inclusion map $E^\infty \to E$ is continuous, the final assertions follow once we have continuity of $\Pi\colon C^\infty_c(G) \times E \to E^\infty$.

We first assume that E is a Banach space. By Lemma 3.1, Π is hypocontinuous in the second argument. As the unit ball $B \subseteq E$ is bounded, it follows that $\Pi|_{C^\infty_c(G)\times B}$ is continuous (see Proposition 4 in [11, Chapter III, §5, no. 3]). Since B is a 0-neighborhood, we conclude that Π is continuous.

If E is a proto-Banach G-module, then the topology on E is initial with respect to a family $f_j\colon E \to E_j$ of continuous linear G-equivariant maps to certain Banach G-modules (E_j, π_j). As a consequence, the topology on $C^\infty(G,E)$ is initial with respect to the mappings

$$h_j := C^\infty(G, f_j)\colon C^\infty(G,E) \to C^\infty(G,E_j), \qquad \gamma \mapsto f_j \circ \gamma$$

(see [26]). Therefore, the topology on E^∞ is initial with respect to the maps $h_j \circ \Phi$ (with Φ as in (1)). Now consider $\Phi_j\colon E^\infty_j \to C^\infty(G,E_j)$, $w \mapsto (\pi_j)_w$. Since $f_j \circ \pi_v = (\pi_j)_{f_j(v)}$, we have $f_j(E^\infty) \subseteq (E_j)^\infty$. Moreover, the topology on E^∞ is initial with respect to the maps $h_j \circ \Phi = \Phi_j \circ f_j$. By the Banach case already discussed, $\Pi_j\colon C^\infty_c(G) \times E_j \to (E_j)^\infty$, $\Pi_j(\gamma,w) := \int_G \gamma(y)\pi_j(y,w)\,d\lambda_G(y)$ is continuous for each $j \in J$. Since $\Phi_j \circ f_j \circ \Pi = \Phi_j \circ \Pi_j \circ (\mathrm{id}_{C^\infty_c(G)} \times f_j)$ is continuous for each j, so is Π. □

4. Preliminaries for Propositions C, D, and E

If E is a vector space and $(U_j)_{j\in J}$ a family of subsets $U_j \subseteq E$, we abbreviate

$$\sum_{j\in J} U_j := \bigcup_F \sum_{j\in F} U_j,$$

for F ranging through the set of finite subsets of J.

4.1.

If E and F are complex locally convex spaces, then a function $f\colon U \to F$ on an open set $U \subseteq E$ is called *complex analytic* (or $\mathbb{C}$-analytic) if f is continuous and each $x \in U$ has a neighborhood $Y \subseteq U$ such that

$$(\forall y \in Y) \quad f(y) = \sum_{n=0}^{\infty} p_n(y-x) \tag{11}$$

pointwise, for some continuous homogeneous complex polynomials $p_n\colon E \to F$ of degree n (see [7], [17], [26], [38] for further information).

4.2.
If E and F are real locally convex spaces, following [38], [17], and [26] we call a function $f\colon U \to F$ on an open set $U \subseteq E$ *real analytic* (or $\mathbb{R}$-analytic) if it extends to a $\mathbb{C}$-analytic function $\widetilde{U} \to F_{\mathbb{C}}$ on an open set $\widetilde{U} \subseteq E_{\mathbb{C}}$.

4.3.
Both concepts are chosen in such a way that compositions of $\mathbb{K}$-analytic maps are $\mathbb{K}$-analytic (for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). They therefore give rise to notions of $\mathbb{K}$-analytic manifolds modeled on locally convex spaces and $\mathbb{K}$-analytic mappings between them. If E is finite-dimensional (or Fréchet) and F is sequentially complete (or Mackey-complete),* then a map $f\colon E \supseteq U \to F$ as in Section 4.2 is $\mathbb{R}$-analytic if and only if it is continuous and admits local expansions (11) into continuous homogeneous real polynomials (cf. [7, Theorem 7.1] and [25, Lemma 1.1]), that is, if and only if it is real analytic in the sense of [7].

By the next lemma (proved in Appendix A, like all other lemmas from this section), our notion of analytic vector coincides with that in [15].

LEMMA 4.4
Let G be a connected Lie group with $G \subseteq G_{\mathbb{C}}$, let $\pi\colon G \times E \to E$ be a topological G-module, and let $v \in E$. Then $v \in E^{\omega}$ if and only if the orbit map π_v admits a $\mathbb{C}$-analytic extension $GV \to E$ for some open identity neighborhood $V \subseteq G_{\mathbb{C}}$.

4.5.
The map $d\colon G \to [0, \infty[$ from (5) has the following elementary properties:

$$(\forall x, y \in G) \quad d(xy) \le d(x) + d(y) \qquad \text{and} \qquad d(x^{-1}) = d(x). \tag{12}$$

It is essential for us that

$$\int_G e^{-\ell d(g)}\, d\lambda_G(g) < \infty \tag{13}$$

for some $\ell \in \mathbb{N}_0$, by [13, Section 1, Lemme 2]. For each G-continuous seminorm p on a topological G-module $\pi\colon G \times E \to E$, there exist $C, c > 0$ such that

$$(\forall g \in G)(\forall v \in E) \quad p\big(\pi(g,v)\big) \le C e^{c d(g)} p(v), \tag{14}$$

as a consequence of [13, Section 2, Lemme 1].

*In the sense that each Mackey–Cauchy sequence in F converges (see [34]).

4.6.
Given a connected Lie group G with $G \subseteq G_{\mathbb{C}}$, let $\widetilde{A}_n(G)$ be the space of all $\mathbb{C}$-analytic functions $\eta\colon V_nG \to \mathbb{C}$ such that

$$\|\eta\|_{K,N} := \sup\big\{\big|\eta(z^{-1}g)\big|e^{Nd(g)}\colon z \in K, g \in G\big\} < \infty \tag{15}$$

for each $N \in \mathbb{N}_0$ and compact set $K \subseteq V_n$ (for V_n as in the introduction). Make $\widetilde{A}_n(G)$ a locally convex space using the norms $\|\cdot\|_{K,N}$. It is essential for us that the map

$$\widetilde{\mathcal{A}}_n(G) \to \mathcal{A}_n(G), \qquad \eta \mapsto \eta|_G$$

is an isomorphism of topological vector spaces. Its inverse is the map $\gamma \mapsto \widetilde{\gamma}$ taking γ to its unique $\mathbb{C}$-analytic extension $\widetilde{\gamma}\colon V_nG \to \mathbb{C}$ (see [15, Lemma 4.3]). Given $\gamma \in \mathcal{A}_n(G)$ and K, N as before, we abbreviate $\|\gamma\|_{K,N} := \|\widetilde{\gamma}\|_{K,N}$.

The next two lemmas show that the space $\widetilde{\mathcal{A}}_n(G)$ and its topology remain unchanged if, instead, one requires (15) for all compact subsets $K \subseteq GV_n$.

LEMMA 4.7
If $K \subseteq GV_n$ is compact, then there exists a compact set $L \subseteq V_n$ such that $GK \subseteq GL$.

LEMMA 4.8
If $K, L \subseteq GV_n$ are compact sets such that $GK \subseteq GL$, let $\theta := \max\{d(h)\colon h \in KL^{-1}\} < \infty$. Then $\|\gamma\|_{K,N} \le e^{N\theta}\|\gamma\|_{L,N}$, for all $\gamma \in \mathcal{A}_n(G)$ and $N \in \mathbb{N}_0$.

We set up a notation for seminorms on E_n which define its topology.

4.9.
Let G be a connected Lie group with $G \subseteq G_{\mathbb{C}}$, and let E be a topological G-module. If $K \subseteq GV_n$ is compact and p a continuous seminorm on E, set

$$\|v\|_{K,p} := \sup\big\{p\big(\widetilde{\pi}_v(z)\big)\colon z \in K\big\} \quad \text{for } v \in E_n. \tag{16}$$

We need a variant of Lemma 1.8 ensuring complex analyticity. The $C^{1,0}_{\mathbb{C}}$-maps encountered here are defined as in Section 1.3, except that complex directional derivatives are used in the first factor.

LEMMA 4.10
Let Z, E be complex locally convex spaces, let $U \subseteq Z$ be open, let Y be a σ-compact locally compact space, let μ be a Borel measure on Y which is finite on compact sets, and let $f\colon U \times Y \to E$ be a $C^{1,0}_{\mathbb{C}}$-map. Assume that E is sequentially complete, and assume that, for each continuous seminorm q on E, there exists a μ-integrable function $m_q\colon Y \to [0,\infty]$ such that $q(f(z,y)) \le m_q(y)$ for all $(z,y) \in U \times Y$. Then $g(z) := \int_Y f(z,y)\,d\mu(y)$ exists in E as an absolutely convergent integral, for all $z \in U$, and $g\colon U \to E$ is $\mathbb{C}$-analytic.

Also, the following fact from [15] will be used.

LEMMA 4.11

Let G be a connected Lie group with $G \subseteq G_{\mathbb{C}}$, and let (E, π) be a sequentially complete proto-Banach G-module. Let $n \in \mathbb{N}$. Then $w := \Pi(\gamma, v) \in E_n$ for all $\gamma \in \mathcal{A}_n(G)$ and $v \in E$. The $\mathbb{C}$-analytic extension of the orbit map π_w of w is given by

$$\widetilde{\pi}_w \colon GV_n \to E, \qquad z \mapsto \int_G \widetilde{\gamma}(z^{-1}y)\pi(y, v)\, d\lambda_G(y). \tag{17}$$

The E-valued integrals in (17) converge absolutely.

The next two lemmas will enable a proof of Proposition D.

LEMMA 4.12

Let G be a connected Lie group such that $G \subseteq G_{\mathbb{C}}$ and $L(G)$ is a compact Lie algebra. Then there exists a basis $(V_n)_{n\in\mathbb{N}}$ of open, connected, relatively compact identity neighborhoods $V_n \subseteq G_{\mathbb{C}}$ such that $\overline{V_{n+1}} \subseteq V_n$ and $gV_ng^{-1} = V_n$ for all $n \in \mathbb{N}$ and $g \in G$. In addition, one can achieve that $\{gxg^{-1} \colon g \in G, x \in K\}$ has compact closure in V_n, for each $n \in \mathbb{N}$ and each compact subset $K \subseteq V_n$.

LEMMA 4.13

Let G be a connected Lie group with a compact Lie algebra, and let $G \subseteq G_{\mathbb{C}}$. If (E, π) is a sequentially complete proto-Banach G-module, let $(V_n)_{n\in\mathbb{N}}$ be as in Lemma 4.12, and define E_n using V_n, for each $n \in \mathbb{N}$. Then $w := \Pi(\gamma, v) \in E_n$ for all $\gamma \in \mathcal{A}(G)$ and $v \in E_n$. The $\mathbb{C}$-analytic extension of the orbit map π_w of w is given by

$$\widetilde{\pi}_w \colon GV_n \to E, \qquad z \mapsto \int_G \gamma(y)\widetilde{\pi}_v(zy)\, d\lambda_G(y). \tag{18}$$

The E-valued integrals in (18) converge absolutely.

LEMMA 4.14

In Lemma 4.11, the bilinear map $\Pi \colon \mathcal{A}(G) \times E \to E^{\omega}$ is separately continuous, hypocontinuous in the second argument, and sequentially continuous. If E is barreled (e.g., if E is a Fréchet space), then Π is hypocontinuous in both arguments.

Recall that a topological space X is said to be sequentially compact if it is Hausdorff and every sequence in X has a convergent subsequence (see [10, p. 208]).

LEMMA 4.15

If E is a locally convex space and $K \subseteq E$ a sequentially compact subset, then K is bounded in E.

The following fact has also been used in [15, Appendix B] (without proof).

LEMMA 4.16
$\mathcal{A}_n(G)$ is a Montel space, for each Lie group G such that $G \subseteq G_{\mathbb{C}}$ and $n \in \mathbb{N}$.

5. Proof of Proposition C

Let W be a 0-neighborhood in E^ω. Then there are 0-neighborhoods $S_n \subseteq E_n$ for $n \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} S_n \subseteq W$. Shrinking S_n if necessary, we may assume that $S_n = \{v \in E_n : \|v\|_{K_n,p_n} < 1\}$ for some compact subset $K_n \subseteq GV_n$ and G-continuous seminorm p_n on E (with notation as in Section 4.9).

For the intermediate steps of the proof, we can proceed similarly as in [15, proof of Proposition 4.6]: By 4.5, there exist $C_n > 0$, $m_n \in \mathbb{N}_0$ such that $p_n(\pi(g,v)) \leq p_n(v)C_n e^{m_n d(g)}$ for all $g \in G$ and $v \in E$. Pick $\ell \in \mathbb{N}_0$ with $C := \int_G e^{-\ell d(y)}\, d\lambda_G(y) < \infty$ (see (13)), and set $N_n := m_n + \ell$. For $\gamma \in \mathcal{A}_n(G)$ and $v \in E$, we have $w := \Pi(\gamma, v) \in E_n$ by Lemma 4.11, and $\widetilde{\pi}_w$ is given by (17). The integrand in (17) admits the estimate $p_n(\widetilde{\gamma}(z^{-1}y)\pi(y,v)) \leq p_n(v)C_n\|\gamma\|_{K_n,N_n} e^{-\ell d(y)}$ for all $z \in K_n$, $y \in G$ (cf. (28) with $x = 1$). Hence

$$p_n\big(\widetilde{\pi}_w(z)\big) \leq p_n(v) C C_n \|\gamma\|_{K_n,N_n}.$$

By Lemma 4.7, there exists a compact subset $L_n \subseteq V_n$ such that $GK_n \subseteq GL_n$. Let $\theta_n := \max\{d(h) : h \in K_n L_n^{-1}\}$. If E has the countable neighborhood property, then there exists a continuous seminorm p on E and constants $a_n \geq 0$ such that $p_n \leq a_n p$ for all $n \in \mathbb{N}$. Thus, using Lemma 4.8,

$$p_n\big(\widetilde{\pi}_w(z)\big) \leq a_n p(v) C C_n e^{\theta_n N_n} \|\gamma\|_{L_n,N_n}.$$

Choose $\varepsilon_n > 0$ so small that $\varepsilon_n a_n C C_n e^{\theta_n N_n} < 1$, and set $P_n := \{\gamma \in \mathcal{A}_n(G) : \|\gamma\|_{L_n,N_n} < \varepsilon_n\}$. Then $\|\Pi(\gamma,v)\|_{K_n,p_n} \leq \varepsilon_n a_n C C_n e^{\theta_n N_n} < 1$ for all $v \in B_1^p(0)$ and $\gamma \in P_n$, and thus $\Pi(P_n \times B_1^p(0)) \subseteq S_n$. Then $P := \sum_{n\in\mathbb{N}} P_n$ is a 0-neighborhood in $\mathcal{A}(G)$ and

$$\Pi\big(P \times B_1^p(0)\big) \subseteq \sum_{n\in\mathbb{N}} \Pi\big(P_n \times B_1^p(0)\big) \subseteq \sum_{n\in\mathbb{N}} S_n \subseteq W.$$

Hence Π is continuous at $(0,0)$ and hence continuous (as Π is bilinear).

6. Proof of Proposition D

Choose $(V_n)_{n\in\mathbb{N}}$ as in Lemma 4.12. Let W be a 0-neighborhood in E^ω. Then there are 0-neighborhoods $S_n \subseteq E_n$ for $n \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} S_n \subseteq W$. Shrinking S_n if necessary, we may assume that $S_n = \{v \in E_n : \|v\|_{K_n,p_n} < 1\}$ for some compact subset $K_n \subseteq GV_n$ and G-continuous seminorm p_n on E (with notation as in Section 4.9). After increasing K_n, we may assume that $K_n = A_n B_n$ with compact subsets $A_n \subseteq G$ and $B_n \subseteq V_n$.

By Section 4.5, there exist $C_n > 0$, $m_n \in \mathbb{N}_0$ such that $p_n(\pi(g,v)) \leq p_n(v) \times C_n e^{m_n d(g)}$ for all $g \in G$ and $v \in E$. Then $R_n := \sup\{e^{m_n d(x)} : x \in A_n\} < \infty$. Pick $\ell \in \mathbb{N}_0$ with $C := \int_G e^{-\ell d(y)}\, d\lambda_G(y) < \infty$ (see (13)).

For $i \in \mathbb{N}$, let N_i be the maximum of $m_1 + \ell, \ldots, m_i + \ell$. Pick $\varepsilon_i \in]0, 2^{-i}[$ so small that $R_i C_i C \varepsilon_i < 2^{-i}$. Set $P_i := \{\gamma \in \mathcal{A}_i(G) \colon \|\gamma\|_{B_i, N_i} < \varepsilon_i\}$. Then $P := \sum_{i \in \mathbb{N}} P_i$ is a 0-neighborhood in $\mathcal{A}(G)$.

For $j \in \mathbb{N}$, let q_j be the pointwise maximum of $p_1, \ldots, p_j$. Let H_j be the closure of $\{gyg^{-1} \colon g \in G, y \in B_j\}$ in V_j. Choose $\delta_j \in]0, 2^{-j}[$ so small that $CC_j R_j \delta_j < 2^{-j}$. Set $Q_j := \{v \in E_j \colon \|v\|_{H_j, q_j} < \delta_j\}$. Then $Q := \sum_{j \in \mathbb{N}} Q_j$ is a 0-neighborhood in E^ω.

We now verify that $\Pi(P \times Q) \subseteq W$, entailing that the bilinear map Π is continuous at $(0,0)$ and thus continuous. To this end, let $\gamma \in P$, $v \in Q$. Then $\gamma = \sum_{i=1}^\infty \gamma_i$ and $v = \sum_{j=1}^\infty v_j$ with suitable $\gamma_i \in P_i$ and $v_j \in Q_j$, such that $\gamma_i \neq 0$ for only finitely many i and $v_j \neq 0$ for only finitely many j. Abbreviate $w_{i,j} := \Pi(\gamma_i, v_j)$.

If $j < i$, then $w_{i,j} \in E_j$ by Lemma 4.13. Moreover, (29) shows that

$$\|w_{i,j}\|_{K_j, p_j} \leq CC_j R_j \|\gamma_i\|_{m_j + \ell} \|v_j\|_{H_j, p_j} < CC_j R_j \varepsilon_i \delta_j < 2^{-i} 2^{-j}. \tag{19}$$

If $i \leq j$, then $w_{i,j} \in E_i$ by Lemma 4.11, and (28) implies that

$$\|w_{i,j}\|_{K_i, p_i} \leq R_i C_i C \|\gamma_i\|_{B_i, m_i + \ell} p_i(v_j) \leq R_i C_i C \varepsilon_i \delta_j < 2^{-i} 2^{-j}. \tag{20}$$

For each $n \in \mathbb{N}$, we have $\sum_{\min\{i,j\} = n} w_{i,j} \in S_n$, since (by (19) and (20))

$$\Big\| \sum_{\min\{i,j\}=n} w_{i,j} \Big\|_{K_n, p_n} \leq \sum_{\min\{i,j\}=n} 2^{-i} 2^{-j} < 1.$$

Hence $\Pi(\gamma, v) = \sum_{n=1}^\infty \sum_{\min\{i,j\}=n} w_{i,j} \in \sum_{n \in \mathbb{N}} S_n \subseteq W$, as required.

7. Proof of Proposition E

We use a variant of [42, Theorem 4], which does not require that the direct sequence be strict.

PROPOSITION 7.1

Let $G_1 \subseteq G_2 \subseteq \cdots$ be a sequence of metrizable topological groups such that all inclusion maps $G_n \to G_{n+1}$ are continuous homomorphisms. Let $\mathcal{O}_{\mathrm{DL}}$ be the direct limit topology on $G := \bigcup_{n \in \mathbb{N}} G_n$, and let $\mathcal{O}_{\mathrm{TG}}$ be the topology making G the direct limit $\lim_{\longrightarrow} G_n$ as a topological group. Assume the following:

(a) *for each $n \in \mathbb{N}$, there is $m > n$ such that the set G_n is not open in G_m;*

(b) *there exists $n \in \mathbb{N}$ such that, for all identity neighborhoods $U \subseteq G_n$ and $m > n$, the closure of U in G_m is not compact; and*

(c) *there exists a Hausdorff topology $\mathcal{T}$ on G making each inclusion map $G_n \to G$ continuous, and such that every sequentially compact subset of $(G, \mathcal{T})$ is contained in some G_n and compact in there.*

Then $\mathcal{O}_{\mathrm{DL}}$ does not make the group multiplication $G \times G \to G$ continuous, and hence $\mathcal{O}_{\mathrm{DL}} \neq \mathcal{O}_{\mathrm{TG}}$.

REMARK 7.2

By definition, a set $M \subseteq G$ is open (resp., closed) in $(G, \mathcal{O}_{\mathrm{DL}})$ if and only if $M \cap G_n$ is open (resp., closed) in G_n for each $n \in \mathbb{N}$. By contrast, $\mathcal{O}_{\mathrm{TG}}$ is defined as the finest among the topologies on G making G a topological group, and each inclusion map $G_n \to G$ continuous (see [20], [24], [30], [42] for comparative discussions of $\mathcal{O}_{\mathrm{DL}}$ and $\mathcal{O}_{\mathrm{TG}}$).

Proof of Proposition 7.1

If G_n is not open in G_m for some $m > n$, then G_n also fails to be open in G_k for all $k > m$. In fact, let $i_{m,k} \colon G_m \to G_k$ be the continuous inclusion map. If G_n were open in G_k, then $G_n = i_{m,k}^{-1}(G_n)$ would also be open in G_m, a contradiction. Similarly, if n is as in (b) and $k > n$, then also G_k does not have an identity neighborhood which has compact closure in G_ℓ for some $\ell > k$. In fact, if U were such a neighborhood, then $i_{k,n}^{-1}(U)$ would be an identity neighborhood in G_n whose closure in G_ℓ is contained in $\overline{U}$ and hence compact, a contradiction. After passing to a subsequence, we may hence assume that G_n is not open in G_{n+1} (and hence not an identity neighborhood), for each $n \in \mathbb{N}$. And we can assume that, for each $n \in \mathbb{N}$ and identity neighborhood $U \subseteq G_n$, for each $m > n$ the closure of U in G_m is not compact.

If $\mathcal{O}_{\mathrm{DL}}$ makes the group multiplication continuous, then for every identity neighborhood $U \subseteq (G, \mathcal{O}_{\mathrm{DL}})$, there exists an identity neighborhood $W \subseteq (G, \mathcal{O}_{\mathrm{DL}})$ such that $WW \subseteq U$. Then

$$(\forall n \in \mathbb{N}) \quad (W \cap G_1)(W \cap G_n) \subseteq U \cap G_n. \tag{21}$$

Thus, assuming (a)–(c), $\mathcal{O}_{\mathrm{DL}}$ will not be a group topology if we can construct an identity neighborhood $U \subseteq (G, \mathcal{O}_{\mathrm{DL}})$ such that (21) fails for each W.

To achieve this, let d_n be a metric on G_n defining its topology, for $n \in \mathbb{N}$. Let $V_1 \supseteq V_2 \supseteq \cdots$ be a basis of identity neighborhoods in G_1.

Since G_n is metrizable and G_{n-1} is not an identity neighborhood in G_n, for each $n \geq 2$ we find a sequence $(x_{n,k})_{k \in \mathbb{N}}$ in $G_n \setminus G_{n-1}$ such that $x_{n,k} \to 1$ in G_n as $k \to \infty$. Let $K := \overline{V_{n-1}}$ be the closure of V_{n-1} in G_n. Then K cannot be sequentially compact in $(G, \mathcal{T})$, as otherwise K would be compact in G_m for some $m \in \mathbb{N}$ (by (c)), contradicting (b). Hence K contains a sequence $(w_{n,k})_{k \in \mathbb{N}}$ which does not have a convergent subsequence in $(G, \mathcal{T})$ and hence does not have a convergent subsequence in G_m for any $m \geq n$. Pick $z_{n,k} \in V_{n-1}$ such that

$$d_n(w_{n,k}, z_{n,k}) < \frac{1}{k}. \tag{22}$$

Then also $(z_{n,k})_{k \in \mathbb{N}}$ does not have a convergent subsequence in G_m for any $m \geq n$. (If z_{n,k_ℓ} were convergent, then w_{n,k_ℓ} would converge to the same limit, by (22).) Moreover, $(z_{n,k} x_{n,k})_{k \in \mathbb{N}}$ does not have a convergent subsequence $(z_{n,k_\ell} x_{n,k_\ell})_{\ell \in \mathbb{N}}$ in G_m for any $m \geq n$ (because then $z_{n,k_\ell} = (z_{n,k_\ell} x_{n,k_\ell}) x_{n,k_\ell}^{-1}$ would converge, a contradiction).

As a consequence, the set $C_n := \{z_{n,k} x_{n,k} \colon k \in \mathbb{N}\}$ is closed in G_m for each $m \geq n$. Also note that $z_{n,k} x_{n,k} \in G_n \setminus G_{n-1}$ and thus $C_n \subseteq G_n \setminus G_{n-1}$. Hence

$A_n := \bigcup_{\nu=2}^{n} C_\nu$ is a closed subset of G_n for each $n \geq 2$, and $A := \bigcup_{n\geq 2} A_n$ is closed in $(G, \mathcal{O}_{\mathrm{DL}})$ because $A \cap G_n = A_n$ is closed for each $n \geq 2$. Thus $U := G \setminus A$ is open in $(G, \mathcal{O}_{\mathrm{DL}})$, and $U \cap G_n = G_n \setminus A_n$. We show that $WW \not\subseteq U$ for any 0-neighborhood $W \subseteq G$. In fact, there is $n \geq 2$ such that $V_{n-1} \subseteq W \cap G_1$. Since $x_{n,k} \to 0$ in G_n as $k \to \infty$, there is $k_0 \in \mathbb{N}$ such that $x_{n,k} \in W \cap G_n$ for all $k \geq k_0$. Also, $z_{n,k_0} \in V_{n-1}$. Hence $z_{n,k_0} x_{n,k_0} \in (W \cap G_1)(W \cap G_n)$. But $z_{n,k_0} x_{n,k_0} \in A_n$, and thus $z_{n,k_0} x_{n,k_0} \notin U \cap G_n$. As a consequence, $WW \not\subseteq U$. □

Because the locally convex direct limit topology $\mathcal{O}_{\mathrm{lcx}}$ on an ascending union of locally convex spaces coincides with $\mathcal{O}_{\mathrm{TG}}$ (see [30, Proposition 3.1]), we obtain the following.

COROLLARY 7.3

Let $E_1 \subseteq E_2 \subseteq \cdots$ be metrizable locally convex spaces such that all inclusion maps $E_n \to E_{n+1}$ are continuous linear. On $E := \bigcup_{n\in\mathbb{N}} E_n$, let $\mathcal{O}_{\mathrm{DL}}$ be the direct limit topology, and let $\mathcal{O}_{\mathrm{lcx}}$ be the locally convex direct limit topology. Then $\mathcal{O}_{\mathrm{DL}} \neq \mathcal{O}_{\mathrm{lcx}}$ if (a)–(c) are satisfied.

(a) *For each $n \in \mathbb{N}$, there exists $m > n$ such that $E_m \setminus E_n \neq \emptyset$.*

(b) *There exists $n \in \mathbb{N}$ such that, for each 0-neighborhood $U \subseteq E_n$ and $m > n$, the closure of U in E_m is not compact.*

(c) *$\mathcal{O}_{\mathrm{lcx}}$ is Hausdorff, and every sequentially compact subset of $(E, \mathcal{O}_{\mathrm{lcx}})$ is contained in some E_n and compact in there.*

It is convenient to make the special choice of the V_n proposed in [15] now. To this end, extend $\mathbf{g}$ to a left-invariant Riemannian metric on $G_{\mathbb{C}}$, write $\mathbf{d}_{\mathbb{C}} \colon G_{\mathbb{C}} \times G_{\mathbb{C}} \to [0, \infty[$ for the associated distance function, and set $d_{\mathbb{C}}(z) := \mathbf{d}_{\mathbb{C}}(z, 1)$ for $z \in G_{\mathbb{C}}$. For $\rho > 0$, let

$$B_\rho := \{ z \in G_{\mathbb{C}} \colon d_{\mathbb{C}}(z) < \rho \}$$

be the respective open ball around 1. Then the sets $V_n := B_{1/n}$, for $n \in \mathbb{N}$, have properties as described in the introduction. Notably, $\overline{B_\rho}$ is compact for each $\rho > 0$ and hence also each $\overline{V_n}$ (see [13, p. 74]).

LEMMA 7.4

Let G be a connected Lie group with $G \subseteq G_{\mathbb{C}}$, and let $G \neq \{1\}$. Then the sequence $\mathcal{A}_1(G) \subseteq \mathcal{A}_2(G) \subseteq \cdots$ does not become stationary.

Proof

Step 1. If G is compact, then G is isomorphic to a closed subgroup of some unitary group. Hence G can be realized as a closed $\mathbb{R}$-analytic submanifold of some $\mathbb{R}^k$ (which is also clear from [27, Theorem 3]), entailing that $\mathbb{R}$-analytic functions (like restrictions of linear functionals) separate points on G. In particular, there exists a nonconstant $\mathbb{R}$-analytic function $\gamma \colon G \to \mathbb{R}$, and the latter then extends to a $\mathbb{C}$-analytic function $\widetilde{\gamma}$ on some neighborhood of G in $G_{\mathbb{C}}$, which (since G

is compact) can be assumed to be of the form GV_m for some $m \in \mathbb{N}$. Then $\widetilde{\gamma} \in \widetilde{A}_m(G)$ (noting that d is bounded).

Step 2. If G is not compact, we recall the regularized distance function: There exist $m \in \mathbb{N}$ and a $\mathbb{C}$-analytic function $\widetilde{d}\colon GV_m \to \mathbb{C}$ such that

$$C := \sup\{\big|\widetilde{d}(gz) - d(g)\big| : g \in G, z \in V_m\} < \infty$$

(see [14, Lemma 4.3]). Then also $\theta\colon V_mG \to \mathbb{C}$, $\theta(z) := \widetilde{d}(z^{-1})$ is $\mathbb{C}$-analytic, and $|\theta(zg) - d(g)| = |\widetilde{d}(g^{-1}z^{-1}) - d(g^{-1})| \leq C$ for all $z \in V_m$ and $g \in G$. Since $\{x \in G\colon d(g) \leq R\}$ is compact for each $R > 0$ (see [13]), for each $R > 0$ there exists $g \in G$ such that $d(g) > R$ and thus $|\theta(g)| > R - C$. Hence θ is not constant, and hence also θ^2 and $\mathrm{Re}(\theta^2)$ are not constant. If $N \in \mathbb{N}_0$, there is $r_N > 0$ such that

$$a^2 - 2aC - C^2 \geq Na \quad \text{for all } a \geq r_N.$$

Since $\theta(zg)^2 = d(g)^2 + 2(\theta(zg) - d(g))d(g) + (\theta(zg) - d(g))^2$, we deduce that

$$\mathrm{Re}\big(\theta(zg)^2\big) \geq d(g)^2 - 2Cd(g) - C^2 \geq Nd(g)$$

for all $z \in V_m$ and all $g \in G$ such that $d(g) \geq r_N$. We also have

$$\mathrm{Re}\big(\theta(zg)^2\big) \geq -\big|\theta(zg)^2\big| \geq -\big(d(g) + C\big)^2 \quad \text{for all } z \in V_m,\ g \in G.$$

Thus $\gamma\colon V_mG \to \mathbb{C}$, $\gamma(z) := e^{-\theta(z)^2}$ is nonconstant, $\mathbb{C}$-analytic, and

$$\big|\gamma(zg)\big|e^{Nd(g)} = e^{-\mathrm{Re}(\theta(zg)^2)}e^{Nd(g)} \leq e^{(r_N + C)^2 + Nr_N}$$

for all $z \in V_m$, $g \in G$. Hence $\|\gamma\|_{K,N} < \infty$ for each compact set $K \subseteq V_m$ and $N \in \mathbb{N}_0$. Thus $\gamma \in \widetilde{A}_m(G)$, and hence $\widetilde{A}_n(G) \neq \{0\}$ for all $n \geq m$.

Step 3. In either case, let $\widetilde{\gamma} \in \widetilde{\mathcal{A}}_m(G)$ be a nonconstant function, and let $n > m$. Then also $\gamma := \widetilde{\gamma}|_G$ is nonconstant (as G is totally real in $G_{\mathbb{C}}$). If G is compact, then $|\gamma|$ attains a maximum $a > 0$. If G is noncompact, then γ vanishes at infinity. Hence $|\gamma(G)| \cup \{0\}$ is compact, and hence $|\gamma|$ attains a maximum $a > 0$. In either case, because $\widetilde{\gamma}$ is an open map, there exists $z_0 \in V_nG$ such that $|\widetilde{\gamma}(z_0)| > a$. Set $b := \widetilde{\gamma}(z_0)$. The set $K := \{(v,g) \in \overline{V_n} \times G\colon \widetilde{\gamma}(vg) = b\}$ is compact. After replacing z_0 with v_0g_0 for suitable $(v_0, g_0) \in K$, we may assume that z_0 is of the form v_0g_0 with $\rho := d_{\mathbb{C}}(v_0) = \min\{d_{\mathbb{C}}(v)\colon (v,g) \in K\} > 0$. Then $W := \{z \in V_nG\colon \widetilde{\gamma}(z) \neq b\}$ is an open subset of V_nG such that $G \subseteq W$, and

$$\theta\colon W \to \mathbb{C}, \qquad \theta(z) := \frac{\widetilde{\gamma}(z)}{\widetilde{\gamma}(z) - b}$$

is a $\mathbb{C}$-analytic function. Set $B_\rho := \{z \in G_{\mathbb{C}}\colon d_{\mathbb{C}}(z) < \rho\}$. Then $B_\rho G \subseteq W$, by the minimality of $d_{\mathbb{C}}(v_0)$. Also $\rho < 1/n$ (as $z_0 \in V_nG$). Let $k \in \mathbb{N}$ such that $1/k < \rho$ (and thus $k > n$). Then $V_kG \subseteq W$. We show that $\eta := \theta|_G \in \mathcal{A}_k(G)$ but $\eta \notin \mathcal{A}_n(G)$. Let $K \subseteq V_k$ be compact. Since $|\widetilde{\gamma}(z^{-1}g)| \leq \|\widetilde{\gamma}\|_{K,1}e^{-d(g)}$ and $d(g) \to \infty$ as $g \to \infty$, there exists a compact subset $L \subseteq G$ such that

$$(\forall z \in K, g \in G \setminus L) \quad \big|\widetilde{\gamma}(z^{-1}g)\big| \leq a.$$

Hence $|\widetilde{\gamma}(z^{-1}g)-b|\geq |b|-|\widetilde{\gamma}(z^{-1}g)|\geq |b|-a>0$, and thus, for each $N\in\mathbb{N}_0$,

$$\big|\theta(z^{-1}g)\big|e^{Nd(g)}\leq \frac{|\widetilde{\gamma}(z^{-1}g)|e^{Nd(g)}}{|b|-a}\leq \frac{\|\widetilde{\gamma}\|_{K,N}}{|b|-a}$$

for all $z\in K$ and $g\in G\setminus L$. Since $|\theta(z^{-1}g)|^{Nd(g)}$ is bounded for (z,g) in the compact set $K\times L$, we deduce that $\|\theta\|_{K,N}<\infty$. Hence $\widetilde{\eta}:=\theta|_{V_kG}\in\mathcal{A}_k(G)$ and $\eta:=\widetilde{\eta}|_G\in\mathcal{A}_k(G)$.

We have $\eta\notin\mathcal{A}_n(G)$. If η were in $\mathcal{A}_n(G)$, we could find $\sigma\in\widetilde{\mathcal{A}}_n(G)$ with $\sigma|_G=\eta$. Then $\theta|_{B_\rho G}=\sigma|_{B_\rho G}$, as $B_\rho G$ is a connected open set in $G_{\mathbb{C}}$, and θ coincides with σ on the totally real submanifold G of $B_\rho G$. Given $\varepsilon\in]0,\rho[$, let $c_\varepsilon\colon[0,1]\to G_{\mathbb{C}}$ be a piecewise C^1-path with $c_\varepsilon(0)=1$ and $c_\varepsilon(1)=v_0$, of length $<d_{\mathbb{C}}(v_0)+\varepsilon=\rho+\varepsilon$. Let $t_\varepsilon\in[0,1]$ such that $c_\varepsilon|_{[t_\varepsilon,1]}$ has length ε. Then

$$\mathbf{d}_{\mathbb{C}}\big(c_\varepsilon(t_\varepsilon),v_0\big)=\mathbf{d}_{\mathbb{C}}\big(c_\varepsilon(t_\varepsilon),c_\varepsilon(1)\big)\leq\varepsilon. \tag{23}$$

Likewise, $d_{\mathbb{C}}(c_\varepsilon(t_\varepsilon))=\mathbf{d}_{\mathbb{C}}(c_\varepsilon(t_\varepsilon),1)$ is bounded by the length of $c_\varepsilon|_{[0,t_\varepsilon]}$, and hence $<\rho+\varepsilon-\varepsilon=\rho$. Hence $c_\varepsilon(t_\varepsilon)\in B_\rho$, and thus

$$\theta\big(c_\varepsilon(t_\varepsilon)g_0\big)=\sigma\big(c_\varepsilon(t_\varepsilon)g_0\big)\to\sigma(v_0g_0)=\sigma(z_0)$$

as $\varepsilon\to 0$ (noting that $c_\varepsilon(t_\varepsilon)\to v_0$ by (23)). But $|\theta(z)|\to\infty$ as $z\in W$ tends to z_0, a contradiction. □

Proof of Proposition E

Each step $H_n:=\mathcal{A}_n(G)\times E$ is metrizable. For each $n\in\mathbb{N}$, there is $m>n$ such that $H_n\neq H_m$ as a set (by Lemma 7.4). Hence condition (a) in Corollary 7.3 is satisfied. Also (b) is satisfied: Given n and a 0-neighborhood $U\subseteq H_n$, we cannot find $m\geq n$ such that the closure $\overline{U}$ of U in H_m is compact, because $(\{0\}\times E)\cap\overline{U}$ would be a compact 0-neighborhood in $\{0\}\times E\cong E$ then, and thus E is finite-dimensional (contradiction). To verify (c), let $K\subseteq\mathcal{A}(G)\times E$ be a sequentially compact set (with respect to the locally convex direct limit topology). Then the projections K_1 and K_2 of K to the factors $\mathcal{A}(G)$ and E, respectively, are sequentially compact sets. Since E is metrizable, $K_2\subseteq E$ is compact. Now, the sequentially compact set $K_1\subseteq\mathcal{A}(G)$ is bounded (see Lemma 4.15). Because the locally convex direct limit $\mathcal{A}(G)=\lim_{\longrightarrow}\mathcal{A}_n(G)$ is regular (see [15, Theorem B.1]), it follows that $K_1\subseteq\mathcal{A}_n(G)$ for some $n\in\mathbb{N}$, and K_1 is bounded in $\mathcal{A}_n(G)$. As $\mathcal{A}_n(G)$ is a Montel space (see Lemma 4.16), it follows that K_1 has compact closure $\overline{K_1}$ in $\mathcal{A}_n(G)$. Now K is a sequentially compact subset of the compact metrizable set $\overline{K_1}\times K_2\subseteq\mathcal{A}(G)\times E$ and hence compact in the induced topology. As $\mathcal{A}_n(G)\times E$ and $\mathcal{A}(G)\times E$ induce the same topology on the compact set $\overline{K_1}\times K_2$, it follows that K is also compact in $\mathcal{A}_n(G)\times E$. Thus all conditions of Corollary 7.3 are satisfied, and thus $\mathcal{O}_{\mathrm{DL}}\neq\mathcal{O}_{\mathrm{lcx}}$. □

8. $(\mathcal{A}(G),\cdot)$ as a topological algebra

If $n,m\in\mathbb{N}$, $\gamma\in\mathcal{A}_n(G)$, and $\eta\in\mathcal{A}_m(G)$, then the pointwise product $\widetilde{\gamma}\cdot\widetilde{\eta}$ of the complex analytic extensions is defined on V_kG with $k:=n\vee m:=\max\{n,m\}$.

If $K \subseteq V_k$ is a compact subset and $N, M \in \mathbb{N}_0$, then $|\widetilde{\gamma}\widetilde{\eta}(z^{-1}g)|e^{(N+M)d(g)} = |\widetilde{\gamma}(z^{-1}g)|e^{Nd(g)}|\widetilde{\eta}(z^{-1}g)|e^{Md(g)}$ for all $z \in K$, $g \in G$, and thus

$$\|\widetilde{\gamma} \cdot \widetilde{\eta}\|_{K,N+M} \leq \|\widetilde{\gamma}\|_{K,N}\|\widetilde{\eta}\|_{K,M} < \infty, \tag{24}$$

whence $\widetilde{\gamma} \cdot \widetilde{\eta} \in \widetilde{\mathcal{A}}_k(G)$, and hence $\gamma \cdot \eta \in \mathcal{A}_k(G)$. Thus pointwise multiplication makes $\mathcal{A}(G)$ an algebra.

To see that the multiplication is continuous at $(0,0)$, let $W \subseteq \mathcal{A}(G)$ be a 0-neighborhood. There are 0-neighborhoods $W_n \subseteq \mathcal{A}_n(G)$ such that $\sum_{n\in\mathbb{N}} W_n \subseteq W$. We have to find 0-neighborhoods $Q_n \subseteq \mathcal{A}_n(G)$ such that

$$\sum_{(n,m)\in\mathbb{N}^2} Q_n \cdot Q_m \subseteq W.$$

This will be the case if we can achieve

$$(\forall k \in \mathbb{N}) \quad \sum_{n\vee m=k} Q_n \cdot Q_m \subseteq W_k. \tag{25}$$

We may assume that $W_n = \{\gamma \in \mathcal{A}_n(G)\colon \|\gamma\|_{K_n,N_n} < \varepsilon_n\}$ for some compact subset $K_n \subseteq V_n$, $N_n \in \mathbb{N}_0$ and $\varepsilon_n \in]0,1]$. After replacing K_n with $K_n \cup \overline{V_{n+1}}$, we may assume that $K_n \supseteq V_{n+1}$, and thus $K_n \supseteq K_{n+1}$, for each $n \in \mathbb{N}$. Thus

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots .$$

Then the 0-neighborhoods

$$Q_n := \Big\{\gamma \in \mathcal{A}_n(G)\colon \|\gamma\|_{K_n,N_n} < \frac{\varepsilon_n}{n^2}\Big\}$$

satisfy (25). To see this, let $k \in \mathbb{N}$ and $(n,m) \in \mathbb{N}^2$ be such that $n \vee m = k$. If $n = k$, using (24) and $K_n \subseteq K_m$ we estimate

$$\begin{aligned}\|\gamma \cdot \eta\|_{K_k,N_k} &= \|\gamma \cdot \eta\|_{K_n,N_n} = \|\gamma\|_{K_n,N_n}\|\eta\|_{K_n,0} \leq \|\gamma\|_{K_n,N_n}\|\eta\|_{K_m,0} \\ &< \frac{\varepsilon_n\varepsilon_m}{n^2m^2} \leq \frac{\varepsilon_n}{n^2} = \frac{\varepsilon_k}{k^2}.\end{aligned}$$

Likewise, $\|\gamma \cdot \eta\|_{K_k,N_k} \leq \|\gamma\|_{K_n,0}\|\eta\|_{K_m,N_m} < \varepsilon_k/k^2$ if $m = k$. Since there are at most k^2 pairs (n,m) with $n \vee m = k$, for all choices of $\gamma_{n,m} \in Q_n$, $\eta_{n,m} \in Q_m$ the triangle inequality yields

$$\Big\|\sum_{n\vee m=k} \gamma_{n,m} \cdot \eta_{n,m}\Big\|_{K_k,N_k} < k^2\frac{\varepsilon_k}{k^2},$$

and thus $\sum_{n\vee m=k} \gamma_{n,m} \cdot \eta_{n,m} \in W_k$, verifying (25).

Appendix A: Proofs of the lemmas in Sections 1 and 4

Proof of Lemma 1.5
For fixed $y \in U_2$, the map $s\colon U_1 \times U_1 \to F$, $s(x_1,x_2) := g(x_1,y)\pi(y,h(x_2))$ is $C^{1,0}$ and $C^{0,1}$ and hence C^1. By linearity, $ds((x_1,x_2),\cdot)$ is the sum of the partial differentials and hence is given by

$$\begin{aligned}ds\big((x_1,x_2),(u_1,u_2)\big) &= d^{(1,0)}g(x_1,y,u_1)\pi\big(y,h(x_2)\big) \\ &\quad + g(x_1,y)\, d^{(0,1)}\pi\big(y,dh(x_2,u_2)\big)\end{aligned}$$

for all $x_1, x_2 \in U_2$ and $u_1, u_2 \in E_1$. Thus $d^{(1,0)}f(x,y,u)$ exists for all $(x,y,u) \in U_1 \times U_2 \times E_1$ and is given by

$$d^{(1,0)}f(x,y,u) = g_1\big((x,u),y\big)\pi\big(y,h(x)\big) + g(x,y)\pi\big(y,dh(x,u)\big)$$

with $g_1((x,u),y) := d^{(1,0)}g(x,y,u)$. Set $f_1((x,u),y) := d^{(1,0)}f(x,y,u)$. By induction, $f_1\colon (U_1 \times E_1) \times U_2 \to F$ is $C^{k,0}$, whence $d^{(j+1,0)}f(x,y,u,u_1,\ldots,u_j) = d^{(j,0)}f_1((x,u),y,(u_1,0),\ldots,(u_j,0))$ exists for all $j \in \mathbb{N}_0$ with $j \leq k$ and $u_1,\ldots, u_j \in E_1$, and is continuous in $(x,y,u,u_1,\ldots,u_j)$. Thus f is $C^{k+1,0}$. □

Direct sums of locally convex spaces are always endowed with the locally convex direct sum topology in this article (as in [9]; see also [32]). To enable the proof of Lemma 1.6, we shall need the following fact.

LEMMA A.1

Let E be a locally convex space, let $r \in \mathbb{N}_0 \cup \{\infty\}$, let M be a paracompact, finite-dimensional C^r-manifold, and let $(U_j)_{j\in J}$ be a locally finite cover of M by relatively compact, open sets U_j. Then the following map is linear and a topological embedding:

$$\Psi\colon C^r_c(M,E) \to \bigoplus_{j\in J} C^r(U_j,E), \qquad \Psi(\gamma) = (\gamma|_{U_j})_{j\in J}. \tag{26}$$

Proof

The linearity is clear. If $K \subseteq M$ is a compact set, then $J_0 := \{j \in J\colon K \cap U_j \neq \emptyset\}$ is finite. The restriction Ψ_K of Ψ to $C^r_K(M,E)$ has image in $\bigoplus_{j\in J_0} C^r(U_j, E) \cong \prod_{j\in J_0} C^r(U_j,E)$ and is continuous as its components $C^r_c(M,E) \to C^r(U_j,E)$, $\gamma \mapsto \gamma|_{U_j}$ are continuous (cf. [16, Lemma 3.7]). Since $C^r_c(M,E) = \lim_{\longrightarrow} C^r_K(M,E)$ as a locally convex space, it follows that Ψ is continuous. Now pick a C^r-partition of unity $(h_j)_{j\in J}$ with $K_j := \operatorname{supp}(h_j) \subseteq U_j$. Then each $m_{h_j}\colon C^r(M,E) \to C^r_{K_j}(M,E)$, $\gamma \mapsto h_j \cdot \gamma$ is continuous linear (e.g., as a special case of [22, Proposition 4.16]), and hence so is the map $\mu\colon \bigoplus_{j\in J} C^r(U_j,E) \to \bigoplus_{j\in J} C^r_{K_j}(U_j,E)$, $(\gamma_j)_{j\in J} \mapsto (h_j\gamma_j)_{j\in J}$. Since $\mu \circ \Psi$ is an embedding (see [6, Lemma 1.3]), also Ψ is a topological embedding. □

We also use a tool from [23], which is a version of [19, Proposition 7.1] with parameters in a set U (for countable J, see [22, Proposition 6.10]).

LEMMA A.2

Let X be a finite-dimensional vector space, let $U \subseteq X$ be open, and let $(E_j)_{j\in J}$ and $(F_j)_{j\in J}$ be families of locally convex spaces. Let $U_j \subseteq E_j$ be open, let $r \in \mathbb{N}_0 \cup \{\infty\}$, and let $f_j\colon U \times U_j \to F_j$ be a map. Assume that there is a finite set $J_0 \subseteq J$ such that $0 \in U_j$ and $f_j(x,0) = 0$ for all $j \in J \setminus J_0$ and $x \in U$. Then $\bigoplus_{j\in J} U_j := (\bigoplus_{j\in J} E_j) \cap \prod_{j\in J} U_j$ is open in $\bigoplus_{j\in J} E_j$, and we can consider

$$f\colon U \times \bigoplus_{j\in J} U_j \to \bigoplus_{j\in J} F_j, \qquad f\big(x,(x_j)_{j\in J}\big) := \big(f_j(x,x_j)\big)_{j\in J}.$$

(a) *If J is countable and each f_j is C^r, then f is C^r.*
(b) *If J is uncountable and each f_j is C^{r+1}, then f is C^r.*

The conclusion of (b) remains valid if each f_j is $C^{0,1}$ and the mappings f_j and $d^{(0,1)}f_j\colon U \times U_j \times E_j \to F_j$ are C^r.

Proof of Lemma 1.6
Given $g_0 \in G$, let $U \subseteq G$ be a relatively compact, open neighborhood of g_0. We show that $\pi_U\colon U \times C_c^\infty(G) \to C_c^\infty(G)$, $(g,\gamma) \mapsto \pi(g,\gamma)$ is smooth. To this end, let $(U_j)_{j\in J}$ be a locally finite cover of G by relatively compact, open sets U_j. Then also $(U^{-1}U_j)_{j\in J}$ is locally finite.* As a consequence, both $\Psi\colon C_c^\infty(G) \to \bigoplus_{j\in J} C^\infty(U_j)$, $\Psi(\gamma) := (\gamma|_{U_j})_{j\in J}$ and the corresponding restriction map $\Theta\colon C_c^\infty(G) \to \bigoplus_{j\in J} C^\infty(U^{-1}U_j)$ are linear topological embeddings (see Lemma A.1). Since

$$\operatorname{im}(\Psi) = \big\{(\gamma_j)_{j\in J}\colon (\forall i,j \in J)(\forall x \in U_i \cap U_j)\gamma_i(x) = \gamma_j(x)\big\}$$

is a closed vector subspace of $\bigoplus_{j\in J} C^\infty(U_j)$, the map π_U will be smooth if we can show that $\Psi \circ \pi_U$ is smooth (cf. [5, Lemma 10.1]). For each $j \in J$, the evaluation map $\varepsilon_j\colon C^\infty(U^{-1}U_j) \times U^{-1}U_j \to \mathbb{C}$, $\varepsilon_j(\gamma,x) := \gamma(x)$ is smooth (see [26] or [22, Proposition 11.1]). Lemma 1.4 shows that

$$\Xi_j\colon U \times C^\infty(U^{-1}U_j) \to C^\infty(U_j), \qquad \Xi_j(g,\gamma)(x) := \gamma(g^{-1}x)$$

is C^∞, as $\widehat{\Xi_j}\colon U \times C^\infty(U^{-1}U) \times U_j \to \mathbb{C}$, $\widehat{\Xi_j}(g,\gamma,x) := \gamma(g^{-1}x) = \varepsilon_j(\gamma, g^{-1}x)$ is smooth. Then

$$\Xi\colon U \times \bigoplus_{j\in J} C^\infty(UU_j) \to \bigoplus_{j\in J} C^\infty(U_j), \qquad \Xi\big(x,(\gamma_j)_{j\in J}\big) := \big(\Xi_j(x,\gamma_j)\big)_{j\in J}$$

is C^∞, by Lemma A.2. Hence $\Psi \circ \pi_U = \Xi \circ (\mathrm{id}_U \times \Theta)$ (and hence π_U) is C^∞. □

Proof of Lemma 1.7
Since $C_c^0(M) = \lim_{\longrightarrow} C_K^0(M)$ as a locally convex space, the linear map m_f will be continuous if $C_K^0(M) \to C_K^0(M,E)$, $\gamma \mapsto \gamma f$ is continuous. This is the case by [16, Lemma 3.9]. □

Proof of Lemma 1.8
It suffices to prove the lemma for $r \in \mathbb{N}_0$. By [6, Lemma A.2], g is continuous. If $r > 0$, $k \in \mathbb{N}_0$ with $k \le r$, $p \in P$, and $q_1, \ldots, q_k \in X$, there is $\varepsilon > 0$ such that

$$h(t_1, \ldots, t_k) := g\Big(p + \sum_{j=1}^{k} t_k q_j\Big)$$

is defined for $(t_1,\ldots,t_k)$ in some open 0-neighborhood $W \subseteq \mathbb{R}^k$. By [6, Lemma A.3], $h\colon W \to E$ is C^k, and $d^{(k,0)}g(x,p,q_1,\ldots,q_k) = \partial^{(1,\ldots,1)}h(0,\ldots,0) = \int_K (D_{(q_k,0)} \cdots D_{(q_1,0)} f)(p,x)\, d\mu(x) = \int_K d^{(k,0)} f(p,x,q_1,\ldots,q_k)\, d\mu(x)$. By the case $r = 0$, the right-hand side is continuous in $(p,q_1,\ldots,q_k)$. So g is C^r. □

*If $K \subseteq G$ is compact, then $U^{-1}U_j \cap K \neq \emptyset \Leftrightarrow U_j \cap UK \neq \emptyset$.

Proof of Lemma 1.9
Let $K := \mathrm{supp}(\gamma) \subseteq G$. For $g \in G$, we have $\pi_w(g) = \pi(g, w) = \int_G \gamma(y)\pi(g, \pi(y,v))\, d\lambda_G(y) = \int_G \gamma(y)\pi(gy, v)\, d\lambda_G(y) = \int_G \gamma(g^{-1}y)\pi(y,v)\, d\lambda_G(y)$, using left invariance of the Haar measure for the last equality. Given $g_0 \in G$, let $U \subseteq G$ be an open, relatively compact neighborhood of g_0. As $g^{-1}y \in K$ implies $y \in \overline{U}K$ for $g \in U$ and $y \in G$, we get

$$\pi_w(g) = \int_{\overline{U}K} \gamma(g^{-1}y)\pi(y,v)\, d\lambda_G(y) \quad \text{for all } g \in U.$$

Since $U \times \overline{U}K \to E$, $(g, y) \mapsto \gamma(g^{-1}y)\pi(y,v)$ is a $C^{\infty,0}$-map, Lemma 1.8 shows that $\pi_w|_U$ is smooth. Hence π_w is smooth, and thus $w \in E^\infty$ indeed. Testing equality with continuous linear functionals and using Fubini's theorem and then left invariance of the Haar measure, one verifies that

$$\begin{aligned}
\Pi(\gamma * \eta, v) &= \int_G \int_G \gamma(z)\eta(z^{-1}y)\pi(y,v)\, d\lambda_G(z)\, d\lambda_G(y) \\
&= \int_G \gamma(z) \int_G \eta(z^{-1}y)\pi(y,v)\, d\lambda_G(y)\, d\lambda_G(z) \\
&= \int_G \gamma(z) \int_G \eta(y)\pi(zy,v)\, d\lambda_G(y)\, d\lambda_G(z) \\
&= \int_G \gamma(z)\pi\big(z, \Pi(\eta, v)\big)\, d\lambda_G(z) \\
&= \Pi\big(\gamma, \Pi(\eta, v)\big).
\end{aligned}$$

Hence E (and E^∞) are $C_c^\infty(G)$-modules. □

Proof of Lemma 4.4
If π_v has a $\mathbb{C}$-analytic extension $\widetilde{\pi}_v$ to GV for some open identity neighborhood $V \subseteq G_{\mathbb{C}}$, then (like any $\mathbb{C}$-analytic map) $\widetilde{\pi}_v$ is $\mathbb{R}$-analytic (see [26]). As inclusion $j\colon G \to G_{\mathbb{C}}$ is $\mathbb{R}$-analytic, so is $\pi_v = \widetilde{\pi}_v \circ j$.

Conversely, assume that π_v is $\mathbb{R}$-analytic. There is an open 0-neighborhood $W \subseteq L(G)_{\mathbb{C}}$ such that $\phi := \exp_{G_{\mathbb{C}}}|_W$ is a $\mathbb{C}$-analytic diffeomorphism onto an open subset $\phi(W)$ in $G_{\mathbb{C}}$, $\phi(W \cap L(G)) = G \cap \phi(W)$, and $\psi := \phi|_{W \cap L(G)}$ is an $\mathbb{R}$-analytic diffeomorphism onto its image in G. Then $\pi_v \circ \psi$ is $\mathbb{R}$-analytic and hence extends to a $\mathbb{C}$-analytic map $f\colon \widetilde{W} \to E$ for some open set $\widetilde{W} \subseteq W$ containing $W \cap L(G)$, and thus $\phi(\widetilde{W}) \to E$, $z \mapsto f(\phi^{-1}(z))$ is a $\mathbb{C}$-analytic extension of $\pi_v|_{W \cap L(G)}$. We now find $n \in \mathbb{N}$ such that $V_n \subseteq \phi(\widetilde{W})$ and $U_n \subseteq \mathrm{im}(\psi)$, using the notation from [15]. Hence $v \in \widetilde{E}_n$, and hence $v \in E_{4n}$, by [15, Lemma 3.2]. □

Proof of Lemma 4.7
For each $k \in K$, there are $g_k \in G$ and $v_k \in V_n$ such that $k = g_k v_k$. Let $P_k \subseteq V_n$ be a compact neighborhood of v_k. Then $(g_k P_k^0)_{k \in K}$ is an open cover of K, whence there exists a finite subset $F \subseteq K$ such that $K \subseteq \bigcup_{k \in F} g_k P_k$. Then $P := \bigcup_{k \in F} P_k$ is a compact subset of V_n and $GK \subseteq GP$. □

Proof of Lemma 4.8
If $z \in K$, then $z = h\ell$ for some $h \in G$ and $\ell \in L$. Then $h = z\ell^{-1} \in KL^{-1}$. For $g \in G$, we have

$$\left|\gamma(z^{-1}g)\right|e^{Nd(g)} = \left|\gamma\big(\ell^{-1}(h^{-1}g)\big)\right|e^{Nd(g)} \leq e^{Nd(h)}\left|\gamma\big(\ell^{-1}(h^{-1}g)\big)\right|e^{Nd(h^{-1}g)}$$

as $d(g) = d(h(h^{-1}g)) \leq d(h) + d(h^{-1}g)$. The assertion follows. □

Proof of Lemma 4.10
Let $K_1 \subseteq K_2 \subseteq \cdots$ be compact subsets of Y such that $Y = \bigcup_{n\in\mathbb{N}} K_n$. Then $g_n(z) := \int_{K_n} f(z,y)\,d\mu(y)$ exists for all $z \in U$ (see [29, Proposition 1.2.3]). By Lemma 1.8, the map $g_n\colon U \to E$ is C^1 with $dg_n(z,w) = \int_{K_n} d^{(1,0)}f(z,y,w)\,d\mu(y)$, which is $\mathbb{C}$-linear in $w \in Z$. As E is sequentially complete, this implies that g_n is $\mathbb{C}$-analytic (see [18, 1.4]). For each continuous seminorm q on E, we have $\int_Y q(f(z,y))\,d\mu(y) \leq \int_Y m_q(y)\,d\mu(y) < \infty$. Since $\lim_{n\to\infty}\int_{K_n} m_q(y)\,d\mu(y) = \int_Y m_q(y)\,d\mu(y)$, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\int_{Y\setminus K_n} m_q(y)\,d\mu(y) < \varepsilon$ for all $n \geq N$. Hence

$$q\big(g_\ell(z) - g_n(z)\big) \leq \int_{K_\ell\setminus K_n} m_q(y)\,d\mu(y) < \varepsilon \tag{27}$$

for all $\ell \geq n \geq N$, showing that $(g_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in E and hence convergent to some element $g(z) \in E$. For each continuous linear functional $\lambda\colon E \to \mathbb{C}$, we have $|\lambda(f(z,y))| \leq m_{|\lambda|}(y)$, whence the function $|\lambda(f(z,\cdot))|$ is μ-integrable and $\int_Y \lambda(f(z,y))\,d\mu(y) = \lim_{n\to\infty}\int_{K_n}\lambda(f(z,y))\,d\mu(y) = \lim_{n\to\infty}\lambda(g_n(z)) = \lambda(\lim_{n\to\infty} g_n(z)) = \lambda(g(z))$. Hence $g(z)$ is the weak integral $\int_Y f(z,y)\,d\mu(y)$. As $\int_Y q(f(z,y))\,d\mu(y) \leq \int_Y m_q(y)\,d\mu(y) < \infty$, the integral $\int_Y f(z,y)\,d\mu(y)$ is absolutely convergent. Letting $\ell \to \infty$ in (27), we see that $q(g(z) - g_n(z)) \leq \varepsilon$ for all $z \in U$ and $n \geq N$. Thus $g_n \to g$ uniformly. Since E is sequentially complete, the uniform limit g of $\mathbb{C}$-analytic functions is $\mathbb{C}$-analytic (see [7, Proposition 6.5]), which completes the proof. □

Proof of Lemma 4.11
(See [15, Section 4.3, p. 1592] for an alternative argument.) Given $z_0 \in V_n$ and $x \in G$, let $K \subseteq V_n$ be a compact neighborhood of z_0. If q is a G-continuous seminorm on E, then there exist $C_q \geq 0$ and $m \in \mathbb{N}_0$ such that $q(\pi(y,v)) \leq q(v)C_q e^{md(y)}$ (see (14)). Choose $\ell \in \mathbb{N}_0$ such that $C := \int_G e^{-\ell d(y)}\,d\lambda_G(y) < \infty$ (see (13)). For $N \in \mathbb{N}_0$ with $N \geq m + \ell$, we obtain, using $d(y) = d(xx^{-1}y) \leq d(x) + d(x^{-1}y)$,

$$\begin{aligned} q\big(\widetilde{\gamma}(z^{-1}y)\pi(y,v)\big) &\leq \left|\widetilde{\gamma}(k^{-1}x^{-1}y)\right| q\big(\pi(y,v)\big) \\ &\leq \left|\widetilde{\gamma}(k^{-1}x^{-1}y)\right| e^{Nd(x^{-1}y)} C_q e^{(m-N)d(y)} e^{Nd(x)} q(v) \\ &\leq C_q e^{Nd(x)} \|\gamma\|_{K,N} e^{-\ell d(y)} q(v) \end{aligned} \tag{28}$$

for all $z = xk$ with $k \in K$, and all $y \in G$. Hence Lemma 4.10 shows that the integral in (17) converges absolutely for all $z \in xK^0$ and defines a $\mathbb{C}$-analytic function $xK^0 \to E$. Since $xz_0 \in GV_n$ was arbitrary, the integral in (17) exists for all $z \in GV_n$ and defines a $\mathbb{C}$-analytic function $\eta\colon GV_n \to E$. For $x \in G$,

$$\pi(x,w) = \pi(x,\cdot)\Big(\int_G \gamma(y)\pi(y,v)\,d\lambda_G(y)\Big) = \int_G \gamma(y)\pi\big(x,\pi(y,v)\big)\,d\lambda_G(y)$$
$$= \int_G \gamma(y)\pi(xy,v)\,d\lambda_G(y) = \int_G \gamma(x^{-1}y)\pi(y,v)\,d\lambda_G(y) = \eta(x)$$

by left invariance of the Haar measure. Hence η is a $\mathbb{C}$-analytic extension of π_w to GV_n, whence $w \in E_n$ and $\widetilde{\pi_w} = \eta$. □

Proof of Lemma 4.12
Since $L(G)$ is a compact Lie algebra, there exists a positive definite bilinear form $\langle\cdot,\cdot\rangle\colon L(G)\times L(G)\to\mathbb{R}$ making $e^{\mathrm{ad}(x)} = \mathrm{Ad}(\exp_G(x))$ an isometry for each $x \in L(G)$. Since G is generated by the exponential image, it follows that $\mathrm{Ad}(g)$ is an isometry for each $g \in G$. Now use the same symbol, $\langle\cdot,\cdot\rangle$, for the unique extension to a Hermitian form $L(G)_{\mathbb{C}} \times L(G)_{\mathbb{C}} \to \mathbb{C}$. Write $B_r \subseteq L(G)_{\mathbb{C}}$ for the open ball of radius r around zero. After replacing the form by a positive multiple if necessary, we may assume that $\exp_{G_{\mathbb{C}}}$ restricts to a homeomorphism ϕ from B_1 onto a relatively compact, open subset of $G_{\mathbb{C}}$. Then the sets $V_n := \exp_{G_{\mathbb{C}}}(B_{1/n})$ form a basis of relatively compact, connected open identity neighborhoods, such that $\overline{V_{n+1}} \subseteq V_n$ and $gV_ng^{-1} = \exp_{G_{\mathbb{C}}}(\mathrm{Ad}(g)(B_{1/n})) = \exp_{G_{\mathbb{C}}}(B_{1/n}) = V_n$. If $K \subseteq V_n$ is compact, then $A := \phi^{-1}(K)$ is a compact subset of $B_{1/n}$, and thus $r := \max\{\sqrt{\langle x,x\rangle}\colon x\in A\} < 1/n$. Then $\exp_{G_{\mathbb{C}}}(\overline{B_r})$ is a compact, conjugation-invariant subset of G which contains K, and thus $\overline{\{gxg^{-1}\colon g\in G, x\in K\}} \subseteq \exp_{G_{\mathbb{C}}}(\overline{B_r}) \subseteq V_n$. □

Proof of Lemma 4.13
Let $x_0 \in G$, $z_0 \in V_n$ and $K \subseteq V_n$ be a compact neighborhood of y_0. Then $K_1 := \overline{\{gzg^{-1}\colon g\in G, z\in K\}} \subseteq V_n$ is compact, by choice of V_n. If q is a G-continuous seminorm on E, then there exist $C_q \geq 0$ and $m \in \mathbb{N}_0$ such that $q(\pi(y,v)) \leq q(v)C_q e^{md(y)}$ (see (14)). Then $\|v\|_{K_1,q} := \sup q(\widetilde{\pi}_v(K_1)) < \infty$. Choose $\ell \in \mathbb{N}_0$ such that $C := \int_G e^{-\ell d(y)}\,d\lambda_G(y) < \infty$ (see (13)). Note that $\widetilde{\pi}_v(xzy) = \widetilde{\pi}_v(xyy^{-1}zy) = \pi(xy, \widetilde{\pi}_v(y^{-1}zy))$ for all $x\in G$, $z\in K$, and $y \in G$, where $y^{-1}zy \in K_1$. Thus

$$\begin{aligned}
q\big(\gamma(y)\widetilde{\pi}_v(xzy)\big) &= \big|\gamma(y)\big|q\big(\pi\big(xy, \widetilde{\pi}_v(y^{-1}zy)\big)\big)\\
&= \big|\gamma(y)\big|C_q e^{md(xy)} q\big(\widetilde{\pi}_v(y^{-1}zy)\big)\\
&\leq C_q\|v\|_{K_1,q}\big|\gamma(y)\big|e^{md(y)}e^{md(x)}\\
&\leq C_q\|v\|_{K_1,q}e^{md(x)}\|\gamma\|_{m+\ell}e^{-\ell d(y)},
\end{aligned} \tag{29}$$

using the notation from (6). Hence Lemma 4.10 shows that the integral in (18) converges absolutely for all $z \in xK^0$ and defines a $\mathbb{C}$-analytic function $xK^0 \to E$. Notably, this holds for $x = x_0$. Since $x_0z_0 \in GV_n$ was arbitrary, the integral in (18) exists for all $z \in GV_n$ and defines a $\mathbb{C}$-analytic map $\eta\colon GV_n \to E$. For $x\in G$, we have $\pi(x,w) = \int_G \gamma(y)\pi(x,\pi(y,v))\,d\lambda_G(y) = \eta(x)$. Hence η is a $\mathbb{C}$-analytic extension of π_w to GV_n, and thus $w \in E_n$ and $\widetilde{\pi}_w = \eta$. □

Proof of Lemma 4.14
We need only show that Π is separately continuous. In fact, $\mathcal{A}(G)$ is barreled, being a locally convex direct limit of the Fréchet spaces $\mathcal{A}_n(G)$ (see [40, II.7.1, II.7.2]). Hence, if Π is separately continuous, it automatically is hypocontinuous in its second argument (see [40, II5.2]) and hence sequentially continuous (see [33, p. 157, Remark following §40.1(5)]).

Let $\Pi_n\colon \mathcal{A}_n(G)\times E\to E^\omega$ be the restriction of Π to $\mathcal{A}_n(G)\times E$. Then Π_n is continuous (see (8)). For $\gamma\in\mathcal{A}(G)$, there exists $n\in\mathbb{N}$ such that $\gamma\in\mathcal{A}_n(G)$. Thus $\Pi(\gamma,\cdot)=\Pi_n(\gamma,\cdot)\colon E\to E^\omega$ is continuous. If $v\in E$, then the linear map $\Pi(\cdot,v)=\lim_{\longrightarrow}\Pi_n(\cdot,v)\colon \mathcal{A}(G)\to E^\omega$ is continuous. □

Proof of Lemma 4.15
If K were unbounded, we could find $x_1,x_2,\ldots$ in K and a continuous seminorm q on E such that $q(x_n)\to\infty$ as $n\to\infty$. Then $(x_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence, a contradiction. □

Proof of Lemma 4.16
Since $\widetilde{\mathcal{A}}_n(G)$ is a Fréchet space and hence barreled, it only remains to show that each bounded subset $M\subseteq\widetilde{\mathcal{A}}_n(G)$ is relatively compact. Because $\widetilde{\mathcal{A}}_n(G)$ is complete, we need only show that M is precompact. Thus, for each compact set $K\subseteq V_n$, $N\in\mathbb{N}_0$, and $\varepsilon>0$, we have to find a finite subset $\Gamma\subseteq M$ such that

$$M\subseteq\bigcup_{\gamma\in\Gamma}\big\{\eta\in\widetilde{\mathcal{A}}_n(G)\colon \|\eta-\gamma\|_{K,N}\le\varepsilon\big\}. \tag{30}$$

Since M is bounded, $C:=\sup\{\|\gamma\|_{K,N+1}\colon \gamma\in M\}<\infty$. Choose $\rho>0$ with $2Ce^{-\rho}<\varepsilon$. Then $K_1:=\{g\in G\colon d(g)\le\rho\}$ is a compact subset of G (see [13, p. 74]), and hence $L:=K^{-1}K_1$ is compact in $G_{\mathbb{C}}$. The inclusion map $\widetilde{\mathcal{A}}_n(G)\to\mathcal{O}(V_nG)$ being continuous, M is bounded also in the space $\mathcal{O}(V_nG)$ of $\mathbb{C}$-analytic functions on the finite-dimensional complex manifold $\mathcal{O}(V_nG)$, equipped with the compact open topology, which is a prime example of a Montel space. Hence, we find a finite subset $\Gamma\subseteq M$ such that

$$(\forall\eta\in M)(\exists\gamma\in\Gamma)\quad \big\|(\eta-\gamma)|_L\big\|_\infty<e^{-N\rho}\varepsilon. \tag{31}$$

Given $\eta\in M$, pick $\gamma\in\Gamma$ as in (31). Let $z\in K$, $g\in G$. If $d(g)\ge\rho$, then

$$\begin{aligned}\big|\eta(z^{-1}g)-\gamma(z^{-1}g)\big|e^{Nd(g)}&\le\big(\big|\eta(z^{-1}g)\big|+\big|\gamma(z^{-1}g)\big|\big)e^{(N+1)d(g)}e^{-d(g)}\\&\le 2Ce^{-\rho}<\varepsilon.\end{aligned}$$

If $d(g)<\rho$, then $z^{-1}g\in L$, and thus

$$\big|\eta(z^{-1}g)-\gamma(z^{-1}g)\big|e^{Nd(g)}\le e^{-N\rho}\varepsilon e^{Nd(g)}<\varepsilon,$$

by (31). Hence $\|\eta-\gamma\|_{K,N}\le\varepsilon$, showing that (30) holds for Γ. □

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