Log-canonical degenerations of del Pezzo surfaces in $\mathbb{Q}$-Gorenstein families

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Abstract We classify del Pezzo surfaces of Picard number 1 with log-canonical singularities admitting $\mathbb{Q}$-Gorenstein smoothings.

1. Introduction

Throughout this paper we work over the complex number field $\mathbb{C}$. A *smoothing* of a surface X is a flat family $\mathfrak{X} \to \mathfrak{D}$ over a unit disk $0 \in \mathfrak{D} \subset \mathbb{C}$ such that the fiber $\mathfrak{X}_0$ is isomorphic to X and the general fiber is smooth. In this situation X can be considered as a degeneration of a fiber $\mathfrak{X}_t$, $0 \neq t \in \mathfrak{D}$. A smoothing is said to be $\mathbb{Q}$-*Gorenstein* if the total family $\mathfrak{X}$ is. Throughout this paper a *del Pezzo surface* means a normal projective surface whose anticanonical divisor is $\mathbb{Q}$-Cartier and ample. We study $\mathbb{Q}$-Gorenstein smoothings of del Pezzo surfaces with log-canonical singularities. This is interesting for applications to birational geometry and the minimal model program (see, e.g., [21], [24]) as well as to moduli problems (see, e.g., [17], [7]). Smoothings of del Pezzo surfaces with log-terminal singularities were considered in [20], [8], and [23].

THEOREM 1.1

Let X be a del Pezzo surface with only log-canonical singularities and $\rho(X) = 1$. Assume that X admits a $\mathbb{Q}$-Gorenstein smoothing, and assume that there exists at least one non-log-terminal point $(o \in X)$. Let $\eta : Y \to X$ be the minimal resolution. Then there is a rational curve fibration $\varphi : Y \to T$ over a smooth curve T such that a component C_1 of the η-exceptional divisor dominating T is unique, it is a section of φ, and its discrepancy equals -1. Moreover, o is the only non-log-terminal singularity and singularities of X outside o are at worst Du Val of type A. *The surface X and singular fibers of φ are described in Table 1. All the cases except possibly for 3^o with $5 \le n \le 8$ and 4^o with $5 \le n \le 10$ occur.*

For a precise description of the surfaces that occur in our classification, we refer to Section 8.

Kyoto Journal of Mathematics, Vol. 59, No. 4 (2019), 1041–1073
First published online August 10, 2019.
DOI 10.1215/21562261-2019-0006,
Received April 18, 2017. Revised July 11, 2017. Accepted July 11, 2017.
2010 Mathematics Subject Classification: Primary 14J17; Secondary 14B07, 14E30.

Table 1

	Singularities $(o \in X)$	*Singularities* $X \setminus \{o\}$	$\rho(Y)$	K_X^2	*Singular fibers of* φ	*Condition on* n
1^o	Ell_n	$\varnothing$	2	n	$\varnothing$	$n \le 9$
2^o	$[n;[2]^4]$	$4\,\mathrm{A}_1$	10	$n-2$	$4(\mathrm{I}_2)$	$3 \le n \le 6$
3^o	$[n,2,2;[2]^4]$	$2\,\mathrm{A}_1$	10	$n-2$	$2(\mathrm{I}_2)(\mathrm{II})$	$3 \le n \le 8$
4^o	$[2,2,n,2,2;[2]^4]$	$\varnothing$	10	$n-2$	$2(\mathrm{II})$	$3 \le n \le 10$
5^o	$[n;[3]^3]$	$3\,\mathrm{A}_2$	11	$n-1$	$3(\mathrm{I}_3)$	2, 3, 4
6^o	$[n;[2],[4]^2]$	$\mathrm{A}_1,\ 2\,\mathrm{A}_3$	12	$n-1$	$(\mathrm{I}_2)2(\mathrm{I}_4)$	2, 3
7^o	$[2;[2],[3],[6]]$	$\mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{A}_5$	13	1	$(\mathrm{I}_2)(\mathrm{I}_3)(\mathrm{I}_6)$	

Table 2

	I	$(X \ni P)$	*Condition*	μ_P	$-\mathrm{K}^2$
$1^\bullet$	2	$[n_1,\dots,n_s;[2]^4]$	$\sum(n_i-3) \le 3$	$4-\sum(n_i-3)$	$\sum(n_i-2)$
$2^\bullet$	3	$[n;[3],[3],[3]]$	$n=2,3,4$	$4-n$	n
$3^\bullet$	4	$[n;[2],[4],[4]]$	$n=2,3$	$3-n$	$n+1$
$4^\bullet$	6	$[2;[2],[3],[6]]$		0	4

To show the existence of $\mathbb{Q}$-Gorenstein smoothings we use the unobstructedness of deformations (see Proposition 7.5) and a local investigation of the $\mathbb{Q}$-Gorenstein smoothability of log-canonical singularities.

THEOREM 1.2

Let $(X \ni P)$ be a strictly log-canonical surface singularity of index $I > 1$ admitting a $\mathbb{Q}$-Gorenstein smoothing. Then it belongs to one of types listed in Table 2, where μ_P is the Milnor fiber of the smoothing.

$\mathbb{Q}$-Gorenstein smoothings exist in cases $2^\bullet$, $3^\bullet$, $4^\bullet$, as well as in the case $1^\bullet$ for singularities of types $[n;[2]^4]$ with $n \le 6$, $[n_1,\dots,n_s;[2]^4]$ with $\sum(n_i-2) \le 2$, $[4,3;[2]^4]$, and $[3,3,3;[2]^4]$. In all other cases the existence of $\mathbb{Q}$-Gorenstein smoothings is unknown.

The smoothability of log-canonical singularities of index 1 was studied earlier (see, e.g., [19, Example 6.4], [28, Corollary 5.12]).

As a by-product we construct essentially canonical threefold singularities of index 5 and 6. We say that a canonical singularity $(\mathfrak{X} \ni o)$ is *essentially canonical* if there exists a crepant divisor with center o. V. Shokurov conjectured that essentially canonical singularities of given dimension have bounded indices. This is well known in dimension 2: canonical surface singularities are Du Val and their index equals 1. Shokurov's conjecture was proved in dimension 3 by M. Kawakita in [11]. More precisely, he proved that the index of an essentially canonical threefold singularity is at most 6. The following theorem supplements Kawakita's result.

THEOREM 1.3
For any $1 \le I \le 6$ there exists a 3-dimensional essentially canonical singularity of index I.

In fact, our result is new only for $I = 5$ and 6: [9] classified threefold canonical hyperquotient singularities, and among them, there are examples satisfying conditions of our theorem with $I \le 4$. Theorem 1.3 together with [11] gives the following.

THEOREM 1.4
Let $\mathfrak{I}$ be the set of indices of 3-dimensional essentially canonical singularities. Then

$$\mathfrak{I} = \{1, 2, 3, 4, 5, 6\}.$$

The paper is organized as follows. Section 2 is preliminary. In Section 3, we obtain necessary conditions for the $\mathbb{Q}$-Gorenstein smoothability of 2-dimensional log-canonical singularities. In Section 4 we construct examples of $\mathbb{Q}$-Gorenstein smoothings. Theorem 1.3 will be proved in Section 5. In Section 6, we collect important results on del Pezzo surfaces admitting $\mathbb{Q}$-Gorenstein smoothings. The main birational construction for the proof of Theorem 1.1 is outlined in Section 7, which will be considered in Sections 8 and 9.

2. Log-canonical singularities

For basic definitions and terminology of the minimal model program, we refer to [16] or [14].

2.1

Let $(X \ni o)$ be a log-canonical surface singularity. The *index* of $(X \ni o)$ is the smallest positive integer I such that IK_X is Cartier. We say that $(X \ni o)$ is *strictly log-canonical* if it is log-canonical but not log-terminal.

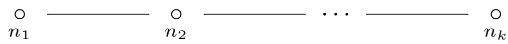
DEFINITION 2.2
A normal Gorenstein surface singularity is said to be *simple elliptic* if the exceptional divisor of the minimal resolution is a smooth elliptic curve. We say that a simple elliptic singularity is of type Ell_n if the self-intersection of the exceptional divisor equals $-n$. A normal Gorenstein surface singularity is called a *cusp* if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.

2.3

We recall some notation on weighted graphs. Let $(X \ni o)$ be a rational surface singularity, let $\eta : Y \to X$ be its minimal resolution, and let $E = \sum E_i$ be the exceptional divisor. Let $\Gamma = \Gamma(X, o)$ be the dual graph of $(X \ni o)$, that is, Γ is a weighted graph whose vertices correspond to exceptional prime divisors E_i and edges join vertices meeting each other. In the usual way we attach to each vertex E_i the number $-E_i^2$. Typically, we omit 2 if $-E_i^2 = 2$.

If $(X \ni o)$ is a cyclic quotient singularity of type $\frac{1}{r}(1,q)$, $\gcd(r,q) = 1$, then the graph Γ is a chain:

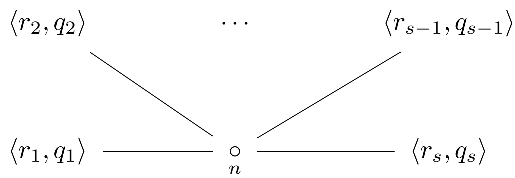
(2.3.1)



We denote it by $[n_1, \dots, n_k] = \langle r, q \rangle$. The numbers n_i are determined by the expression of r/q as a continued fraction (see [3]). For positive integers n, r_i, q_i, $\gcd(r_i, q_i) = 1$, $i = 1, \dots, s$, the symbol

$$\langle n; r_1, \dots, r_s; q_1, \dots, q_s \rangle$$

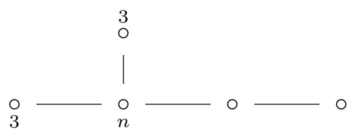
denotes the graph



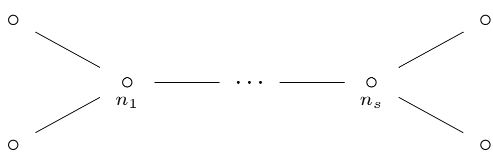
For short, we will omit q_i's: $\langle n; r_1, \dots, r_s \rangle$. If $\langle r_i, q_i \rangle = [n_{i,1}, n_{i,2}, \dots]$, then we also denote

$$\langle n; r_1, \dots, r_s; q_1, \dots, q_s \rangle = \big[n; [n_{1,1}, n_{1,2}, \dots], \dots, [n_{s,1}, n_{s,2}, \dots]\big].$$

For example, $\langle n; 3, 3, 3; 1, 1, 2 \rangle = [n; [3], [3], [2, 2]]$ is the graph



The graph



will be denoted by $[n_1, \dots, n_s; [2]^4]$.

THEOREM 2.4 ([12, Section 9])

Let $(X \ni o)$ be a strictly log-canonical surface singularity of index I. Then one of the following holds:

(i) $I = 1$ *if and only if $(X \ni o)$ is either a simple elliptic singularity or a cusp,*

(ii) $I=2$ *if and only if* $\Gamma(X,o)$ *is of type* $[n_1,\dots,n_s;[2]^4]$, $s\geq 1$,
(iii) $I=3$ *if and only if* $\Gamma(X,o)$ *is of type* $\langle n;3,3,3\rangle$,
(iv) $I=4$ *if and only if* $\Gamma(X,o)$ *is of type* $\langle n;2,4,4\rangle$,
(v) $I=6$ *if and only if* $\Gamma(X,o)$ *is of type* $\langle n;2,3,6\rangle$.

COROLLARY 2.4.1
A strictly log-canonical surface singularity is not rational if and only if it is of index 1.

2.5
Let $(X\ni o)$ be a strictly log-canonical surface singularity of index I, let $\eta: Y\to X$ be its minimal resolution, and let $E=\sum E_i$ be the exceptional divisor. Let us contract all the components of E with discrepancies greater than -1:

$$\eta: Y \xrightarrow{\tilde{\eta}} \tilde{X} \xrightarrow{\sigma} X. \tag{2.5.1}$$

Let $\tilde{C}=\sum \tilde{C}_i := \tilde{\eta}_* E$ be the σ-exceptional divisor. Then the pair $(\tilde{X},\tilde{C})$ has only divisorial log-terminal (dlt) singularities and the following relation holds:

$$K_{\tilde{X}} = \sigma^* K_X - \tilde{C}.$$

The extraction $\sigma:\tilde{X}\to X$ is called the *dlt modification* of $(X\ni o)$.

COROLLARY 2.5.2 (see [12, Section 9], [14, Section 3], [16, Section 4.1], [22, Section 6.1])
In the above notation, one of the following holds:

(i) $I=1$, $\tilde{X}=Y$ *is smooth, and* $(X\ni o)$ *is either a simple elliptic or a cusp singularity;*

(ii) $I=2$, $\tilde{C}=\sum_{i=1}^{s}\tilde{C}_i$ *is a chain of smooth rational curves meeting transversely at smooth points of* $\tilde{X}$ *so that* $\tilde{C}_i\cdot\tilde{C}_{i+1}=1$, *and the singular locus of* $\tilde{X}$ *consists of two Du Val points of type* A_1 *lying on* $\tilde{C}_1$ *and two Du Val points of type* A_1 *lying on* $\tilde{C}_s$ *(the case* $s=1$ *is also possible and then* $\tilde{C}=\tilde{C}_1$ *is a smooth rational curve containing four Du Val points of type* A_1*);*

(iii) $I=3$, 4, *or* 6, $\tilde{C}$ *is a smooth rational curve, the pair* $(\tilde{X},\tilde{C})$ *has only purely log-terminal (plt) singularities, and the singular locus of* $\tilde{X}$ *consists of three cyclic quotient singularities of types* $\frac{1}{r_i}(1,q_i)$, $\gcd(r_i,q_i)=1$, *with* $\sum 1/r_i=1$. *In this case* $I=$ *least common denominator* $(\mathrm{lcm})(r_1,r_2,r_3)$.

2.6
Let $(X\ni o)$ be a log-canonical singularity of index I (of arbitrary dimension). Recall (see, e.g., [16, Definition 5.19]) that the *index* 1 *cover* of $(X\ni o)$ is a finite morphism $\pi: X^\sharp\to X$, where

$$X^\sharp := \operatorname{Spec}\Bigl(\bigoplus_{i=0}^{I-1}\mathscr{O}_X(-iK_X)\Bigr).$$

Then $X^\sharp$ is irreducible, $o^\sharp=\pi^{-1}(o)$ is one point, π is étale over $X\setminus\operatorname{Sing}(X)$, and $K_{X^\sharp}=\pi^*K_X$ is Cartier.

In this situation, $(X^\sharp \ni o^\sharp)$ is a log-canonical singularity of index 1. Moreover, if $(X \ni o)$ is log-terminal (resp., canonical, terminal), then so is the singularity $(X^\sharp \ni o^\sharp)$.

COROLLARY 2.6.1

A strictly log-canonical surface singularity of index $I > 1$ is a quotient of a simple elliptic or cusp singularity $(X^\sharp \ni o^\sharp)$ by a cyclic group $\boldsymbol{\mu}_I$ of order $I = 2$, 3, 4, or 6 whose action on $X^\sharp \setminus \{o^\sharp\}$ is free.

CONSTRUCTION 2.7 (see [12, Proof of Theorem 9.6])

Let $(X \ni o)$ be a strictly log-canonical surface singularity of index $I > 1$, let $\pi : (X^\sharp \ni o^\sharp) \to (X \ni o)$ be the index 1 cover, and let $\tilde{\sigma} : (\tilde{X}^\sharp \supset \tilde{C}^\sharp) \to (X^\sharp \ni o^\sharp)$ be the minimal resolution. The action of $\boldsymbol{\mu}_I$ lifts to $\tilde{X}^\sharp$ so that the induced action on $\mathscr{O}_{\tilde{X}^\sharp}(K_{\tilde{X}^\sharp} + \tilde{C}^\sharp) = \tilde{\sigma}^* \mathscr{O}_{X^\sharp}(K_{X^\sharp})$ and $H^0(\tilde{C}^\sharp, \mathscr{O}_{\tilde{C}^\sharp}(K_{\tilde{C}^\sharp}))$ is faithful. Let $(\tilde{X} \supset \tilde{C}) := (\tilde{X}^\sharp \supset \tilde{C}^\sharp)/\boldsymbol{\mu}_I$. Thus, we obtain the following diagram:

$$\begin{array}{ccc} \tilde{X}^\sharp & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ \Big\downarrow{\scriptstyle \tilde{\sigma}} & & \Big\downarrow{\scriptstyle \sigma} \\ X^\sharp & \xrightarrow{\pi} & X \end{array} \qquad (2.7.1)$$

Here $\sigma : (\tilde{X} \supset \tilde{C}) \to (X \ni o)$ is the dlt modification.

The following definition can be given in arbitrary dimension. For simplicity we state it only for dimension 2, which is sufficient for our needs.

2.8. Adjunction

Let X be a normal surface, and let D be an effective $\mathbb{Q}$-divisor on X. Write $D = C + B$, where C is a reduced divisor on X, B is effective, and C and B have no common component. Let $\nu : C' \to C$ be the normalization of C. One can construct an effective $\mathbb{Q}$-divisor $\operatorname{Diff}_C(B)$ on C', called the *different*, as follows (see [14, Chapter 16] or [26, Section 3] for details). Take a resolution of singularities $f : X' \to X$ such that the proper transform C' of C on X' is also smooth. Clearly, C' is nothing but the normalization of the curve C. Let B' be the proper transform of B on X'. One can find an exceptional $\mathbb{Q}$-divisor A on X' such that $K_{X'} + C' + B' \equiv_f A$. The different $\operatorname{Diff}_C(B)$ is defined as the $\mathbb{Q}$-divisor $(B' - A)|_{C'}$. Then $\operatorname{Diff}_C(B)$ is effective, and it satisfies the equality (adjunction formula)

$$K_{C'} + \operatorname{Diff}_C(B) = \nu^*(K_X + C + B)|_C. \qquad (2.8.1)$$

THEOREM 2.8.2 (Inversion of Adjunction [26], [10])

The pair $(X, C + B)$ is log-canonical (lc) (resp., plt) near C if and only if the pair $(C', \operatorname{Diff}_C(B))$ is lc (resp., Kawamata log-terminal (klt)).

PROPOSITION 2.8.3

Let $(X \ni P)$ be a surface singularity, and let $o \in C \subset X$ be an effective reduced divisor such that the pair (X, C) is plt. Then $(P \in C \subset X)$ is analytically isomorphic to

$$\big(0 \in \{x_1 - axis\} \subset \mathbb{C}^2\big)/\boldsymbol{\mu}_r(1, q), \qquad \gcd(r, q) = 1.$$

In particular, C is smooth at P and $\operatorname{Diff}_C(0) = (1 - 1/r)P$. The dual graph of the minimal resolution of $(X \ni P)$ is a chain (2.3.1), and the proper transform of C is attached to one of its ends.

3. $\mathbb{Q}$-Gorenstein smoothings of log-canonical singularities

In this section we prove the classificational part of Theorem 1.2.

NOTATION 3.1

Let $(X \ni P)$ be a normal surface singularity, let $\eta : Y \to X$ be the minimal resolution, and let $E = \sum E_i$ be the exceptional divisor. Write

$$K_Y = \eta^* K_X - \Delta, \tag{3.1.1}$$

where Δ is an effective $\mathbb{Q}$-divisor with $\operatorname{Supp}(\Delta) = \operatorname{Supp}(E)$. Thus, one can define the self-intersection $\mathrm{K}^2_{(X,P)} := \Delta^2$, which is a well-defined natural invariant. We usually write K^2 instead of $\mathrm{K}^2_{(X,P)}$ if no confusion is likely. The value K^2 is nonpositive, and it equals zero if and only if $(X \ni P)$ is a Du Val point.

- We denote by ς_P the number of exceptional divisors over P.

LEMMA 3.2

Let $(X \ni P)$ be a normal surface singularity, and let $\mathfrak{X} \to \mathfrak{D}$ be its $\mathbb{Q}$-Gorenstein smoothing. If $(X \ni P)$ is log-terminal, then the pair $(\mathfrak{X}, X)$ is plt, and the singularity $(\mathfrak{X} \ni P)$ is terminal. If $(X \ni P)$ is log-canonical, then the pair $(\mathfrak{X}, X)$ is lc, and the singularity $(\mathfrak{X} \ni P)$ is isolated canonical.

Proof

By the higher-dimensional version of the inversion of adjunction (see [16, Theorem 5.50], [10], and Theorem 2.8.2) the singularity $(X \ni P)$ is log-terminal (resp., log-canonical) if and only if the pair $(\mathfrak{X}, X)$ is plt (resp., lc) at P. Since X is a Cartier divisor on $\mathfrak{X}$, the assertion follows. □

LEMMA 3.3 ([13, Proposition 6.2.8])

Let $(X \ni P)$ be a rational surface singularity. If $(X \ni P)$ admits a $\mathbb{Q}$-Gorenstein smoothing, then K^2 is an integer.

THEOREM 3.4 ([17, Proposition 3.10], [19, Proposition 5.9])

Let $(X \ni P)$ be a log-terminal surface singularity. The following are equivalent:

(i) *$(X \ni P)$ admits a $\mathbb{Q}$-Gorenstein smoothing,*

(ii) $\mathrm{K}^2 \in \mathbb{Z}$,

(iii) $(X \ni P)$ *is either a Du Val or a cyclic quotient singularity of the form* $\frac{1}{m}(q_1, q_2)$ *with*

$$(q_1 + q_2)^2 \equiv 0 \mod m, \qquad \gcd(m, q_i) = 1.$$

A log-terminal singularity satisfying equivalent conditions above is called a T-*singularity.*

REMARK 3.4.1 (see [17])

It easily follows from (iii) that any non-Du Val singularity of type T can be written in the form

$$\frac{1}{dn^2}(1, dna - 1).$$

Below we describe log-canonical singularities with integral K^2. Note, however, that, in general, the condition $\mathrm{K}^2 \in \mathbb{Z}$ is necessary but not sufficient for the existence of $\mathbb{Q}$-Gorenstein smoothing (see Theorem 1.2 and Proposition 3.5(DV)).

PROPOSITION 3.5

Let $(X \ni P)$ *be a rational strictly log-canonical surface singularity. Then in the notation of Theorem 2.4 the invariant* K^2 *is integral if and only if* X *is either of type* $[n_1, \dots, n_s; [2]^4]$ *or of type* $\langle n; r_1, r_2, r_3; \varepsilon, \varepsilon, \varepsilon \rangle$, *where* $(r_1, r_2, r_3) = (3, 3, 3)$, $(2, 4, 4)$, *or* $(2, 3, 6)$ *and* $\varepsilon = 1$ *or* -1. *Moreover, we have:*

(DV) *if* X *is of type* $[n_1, \dots, n_s; [2]^4]$ *or* $\langle n; r_1, r_2, r_3; -1, -1, -1 \rangle$, *then*

$$-\mathrm{K}^2 = n - 2,$$

where in the case $[n_1, \dots, n_s; [2]^4]$, *we put* $n := \sum (n_i - 2) + 2$;

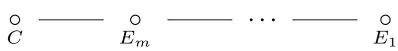
(nDV) *if* X *is of type* $\langle n; r_1, r_2, r_3; 1, 1, 1 \rangle$, *then*

$$-\mathrm{K}^2 = n - 9 + \sum r_i.$$

For the proof we need the following lemma.

LEMMA 3.5.1

Let V *be a smooth surface, and let* $C, E_1, \dots, E_m \subset V$ *be proper smooth rational curves on* V *whose configuration is a chain:*



Let $D = C + \sum \alpha_i E_i$ *be a* $\mathbb{Q}$*-divisor such that* $(K_V + D) \cdot E_j = 0$ *for all* j.

(i) *If all the* E_i*'s are* (-2)*-curves, then* $D^2 - C^2 = m/(m+1)$.

(ii) *If* $m = 1$ *and* $E_1^2 = -r$, *then* $D^2 - C^2 = (r-1)(3-r)/r$.

Proof
Assume that $E_i^2=-2$ for all i. It is easy to check that $D=C+\sum_{i=1}^m \frac{i}{m+1}E_i$. Then

$$D^2-C^2=\frac{2m}{m+1}+\Big(\sum_{i=1}^m \frac{i}{m+1}E_i\Big)^2$$
$$=\frac{2m}{m+1}+\frac{2}{(m+1)^2}\Big(-\sum_{i=1}^m i^2+\sum_{i=1}^{m-1} i(i+1)\Big)=\frac{m}{m+1}.$$

Now let $m=1$ and $E_1^2=-r$. Then $D=C+\frac{r-1}{r}E_1$. Hence,

$$D^2-C^2=\frac{2(r-1)}{r}-\frac{(r-1)^2}{r}=\frac{(r-1)(3-r)}{r}. \qquad \square$$

Proof of Proposition 3.5
Let Δ be as in (3.1.1), and let $C:=\lfloor\Delta\rfloor$. Write $\Delta=C+\sum\Delta_i$, where the Δ_i's are effective connected $\mathbb{Q}$-divisors. By Lemma 3.5.1 we have

$$\delta_i:=\big((C+\Delta_i)^2-C^2\big)=\begin{cases}1-\frac{1}{r_i} & \text{if } \Delta_i \text{ is of type } \frac{1}{r_i}(1,-1),\\ 4-r_i-\frac{3}{r_i} & \text{if } \Delta_i \text{ is of type } \frac{1}{r_i}(1,1).\end{cases}$$

Then

$$\mathrm{K}^2=\Big(C+\sum\Delta_i\Big)^2=C^2+\sum\delta_i.$$

If $(X\ni P)$ is of type $[n_1,\dots,n_s,[2],[2],[2],[2]]$, then

$$\mathrm{K}^2=C^2+2=-\sum(n_i-2).$$

Assume that C is irreducible, and assume that $(X\ni P)$ is of type $\langle n;r_1,r_2,r_3\rangle$, where $\sum 1/r_i=1$.

If all the $\mathrm{Supp}(\Delta_i)$'s are Du Val chains, then

$$\mathrm{K}^2=C^2+\sum\Big(1-\frac{1}{r_i}\Big)=-n+2.$$

If $(X\ni P)$ is of type $\langle n;r_1,r_2,r_3;1,1,1\rangle$, then

$$\mathrm{K}^2=C^2+\sum\Big(4-r_i-\frac{3}{r_i}\Big)=-n+9-\sum r_i.$$

It remains to consider the "mixed" case. Assume, for example, that $(X\ni P)$ is of type $\langle n;3,3,3\rangle$. Then $\delta_i\in\{0,2/3\}$. Since $\sum\delta_i$ is an integer, the only possibility is $\delta_1=\delta_2=\delta_3$, that is, all the chains Δ_i are of the same type. The cases $\langle n;2,4,4\rangle$ and $\langle n;2,3,6\rangle$ are considered similarly. $\square$

COROLLARY 3.5.2
Let $(X\ni P)$ be a strictly log-canonical surface singularity of index $I\ge 2$ admitting a $\mathbb{Q}$-Gorenstein smoothing. Let $(X^\sharp\ni P^\sharp)\to(X\ni P)$ be the index 1 cover. Then

$$-\mathrm{K}^2_{(X^\sharp\ni P^\sharp)}=\begin{cases}I(n-2) & \textit{in the case } (\mathrm{DV}),\\ I(n-1) & \textit{in the case } (\mathrm{nDV}).\end{cases}$$

Proof
Let us consider the (nDV) case. We use the notation of (2.5.1) and (2.7.1). Let E_1, E_2, E_3 be the $\tilde{\eta}$-exceptional divisors. Then

$$K_{\tilde{X}} = \sigma^* K_X - \tilde{C}, \qquad K_Y = \eta^* K_X - \Delta = \tilde{\eta}^* K_{\tilde{X}} - \sum \frac{r_i - 2}{r_i} E_i.$$

Therefore,

$$\Delta = \tilde{\eta}^* \tilde{C} + \sum \frac{r_i - 2}{r_i} E_i, \qquad \Delta^2 = \tilde{C}^2 - \sum \Big(r_i - 4 + \frac{4}{r_i} \Big),$$

$$-\tilde{C}^2 = n + 3 - \sum \frac{4}{r_i} = n - 1, \qquad -\mathrm{K}^2_{(X^\sharp \ni P^\sharp)} = -I\tilde{C}^2 = I(n-1). \qquad \square$$

REMARK 3.5.3
In the above notation we have (see, e.g., [16, Theorem 4.57])

$$\operatorname{mult}(X^\sharp \ni P^\sharp) = \max(2, -\mathrm{K}^2_{(X^\sharp \ni P^\sharp)}),$$
$$\operatorname{emb\,dim}(X^\sharp \ni P^\sharp) = \max(3, -\mathrm{K}^2_{(X^\sharp \ni P^\sharp)}).$$

The following proposition is the key point in the proof of Theorem 1.2.

PROPOSITION 3.6
Let $(X \ni P)$ be a strictly log-canonical rational surface singularity of index $I \geq 3$ admitting a $\mathbb{Q}$-Gorenstein smoothing. Then $(X \ni P)$ is of type $[n; [r_1], [r_2], [r_3]]$.

Proof
By Lemma 3.3 the number K^2 is integral, and by Proposition 3.5 $(X \ni P)$ is either of type (nDV) or of type (DV). Assume that $(X \ni P)$ is of type DV.

3.7
Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{D}$ be a $\mathbb{Q}$-Gorenstein smoothing. By Lemma 3.2 the pair $(\mathfrak{X}, X)$ is log-canonical, and $(P \in \mathfrak{X})$ is an isolated canonical singularity. Let

$$\pi : (\mathfrak{X}^\sharp \ni P^\sharp) \to (\mathfrak{X} \ni P)$$

be the index 1 cover (see Section 2.6), and let $X^\sharp := \pi^* X$. Then $X^\sharp$ is a Cartier divisor on $\mathfrak{X}^\sharp$, the singularity $(\mathfrak{X}^\sharp \ni P^\sharp)$ is canonical (of index 1), and the pair $(\mathfrak{X}^\sharp, X^\sharp)$ is lc. Moreover, $\mathfrak{X}^\sharp$ is Cohen–Macaulay (CM), $X^\sharp$ is hence normal, and the canonical divisor $K_{X^\sharp}$ is Cartier. Therefore, π induces the index 1 cover $\pi_X : (X^\sharp \ni P^\sharp) \to (X \ni P)$. In particular, the index of $(P \in \mathfrak{X})$ equals I. Since $I \geq 3$, the singularity $(X^\sharp \ni P^\sharp)$ is simple elliptic, and the dlt modification coincides with the minimal resolution.

3.8
First we consider the case where $(P \in \mathfrak{X})$ is *terminal*. Below we essentially use the classification of terminal singularities (see, e.g., [25]). In our case, $(\mathfrak{X}^\sharp \ni P^\sharp)$ is either smooth or an isolated compound Du Val (cDV) singularity. In particular,

$$\operatorname{emb}\dim(X^\sharp \ni P^\sharp) \le \operatorname{emb}\dim(\mathfrak{X}^\sharp \ni P^\sharp) \le 4.$$

By our assumption $(X \ni P)$ is of type DV. So, by Corollary 3.5.2 and Remark 3.5.3

$$\operatorname{emb}\dim(X^\sharp \ni P^\sharp) = I(n-2). \tag{3.8.1}$$

If $\operatorname{emb}\dim(\mathfrak{X}^\sharp \ni P^\sharp) = 3$, that is, $(\mathfrak{X}^\sharp \ni P^\sharp)$ is smooth, then $\operatorname{emb}\dim(X^\sharp \ni P^\sharp) = 3$, $\operatorname{mult}(X^\sharp \ni P^\sharp) = 3$, and $I = n = 3$. In this case $(\mathfrak{X} \ni P)$ is a cyclic quotient singularity of type $\frac{1}{3}(1,1,-1)$ (see [25]). We may assume that $(\mathfrak{X}^\sharp, P^\sharp) = (\mathbb{C}^3, 0)$ and $X^\sharp$ is given by an invariant equation $\psi(x_1,x_2,x_3) = 0$ with $\operatorname{mult}_0 \psi = 3$. Since $(X^\sharp \ni P^\sharp)$ is a simple elliptic singularity, the cubic part ψ_3 of ψ defines a smooth elliptic curve on $\mathbb{P}^2$. Hence, we can write $\psi_3 = x_3^3 + \tau(x_1,x_2)$, where $\tau(x_1,x_2)$ is a cubic homogeneous polynomial without multiple factors. The minimal resolution $\tilde{X}^\sharp \to X^\sharp$ is the blowup of the origin. In the affine chart $\{x_2 \ne 0\}$ the surface $\tilde{X}^\sharp$ is given by the equation $\tau(x_1',1) + x_3'^3 + x_2'(\cdots) = 0$ and the action of $\boldsymbol{\mu}_3$ is given by the weights $(0,1,1)$. Then it is easy to see that $\tilde{X}$ has three singular points of type $\frac{1}{3}(1,1)$. This contradicts our assumption.

Thus, we may assume that $\operatorname{emb}\dim(\mathfrak{X}^\sharp \ni P^\sharp) = 4$, that is, $(\mathfrak{X}^\sharp \ni P^\sharp)$ is a hypersurface singularity. Then $I = 4$ by (3.8.1). We may assume that $(\mathfrak{X}^\sharp \ni P^\sharp) \subset (\mathbb{C}^4 \ni 0)$ is a hypersurface given by an equation $\phi(x_1,\ldots,x_4) = 0$ with $\operatorname{mult}_0 \phi = 2$ and $X^\sharp$ is cut out by an invariant equation $\psi(x_1,\ldots,x_4) = 0$. Furthermore, we may assume that $x_1,\ldots,x_4$ are semi-invariants with $\boldsymbol{\mu}_4$-weights $(1,1,-1,b)$, where $b = 0$ or 2 (see [25]).

Consider the case $\operatorname{mult}_0 \psi = 1$. Since ψ is invariant, we have $\psi = x_4 +$ (higher degree terms) and $b = 0$. In this case the only quadratic invariants are x_1x_3, x_2x_3, and x_4^2. Thus, ϕ_2 is a linear combination of x_1x_3, x_2x_3, x_4^2. Since $I = 4$ and $b = 0$, by the classification of terminal singularities ϕ contains either x_1x_3 or x_2x_3 (see [25]). Then by eliminating x_4 we see that $(X^\sharp \ni P^\sharp)$ is a hypersurface singularity whose equation has quadratic part of rank at least 2. In this case, $(X^\sharp \ni P^\sharp)$ is a Du Val singularity of type A_n, a contradiction.

Now let $\operatorname{mult}_0 \psi > 1$. Then (see Remark 3.5.3)

$$\operatorname{emb}\dim(X^\sharp \ni P^\sharp) = -K_{(X^\sharp \ni P^\sharp)} = \operatorname{mult}(X^\sharp \ni P^\sharp) = 4 = I.$$

According to [16, Theorem 4.57] the curve given by quadratic parts of ϕ and ψ in the projectivization $\mathbb{P}(T_{P^\sharp,\mathfrak{X}^\sharp})$ of the tangent space is a smooth elliptic curve. According to the classification from [25], there are two cases.

Case: $b = 0$ and ϕ is an invariant

In this case, as above, ϕ_2 and ψ_2 are linear combinations of x_1x_3, x_2x_3, and x_4^2, so $\{\phi_2 = \psi_2 = 0\}$ cannot be smooth, a contradiction.

Case: $b = 2$ and ϕ is a semi-invariant of weight 2

Then, up to linear coordinate change of x_1 and x_2, we can write

$$\phi_2 = a_1x_1x_2 + a_2x_1^2 + a_3x_2^2 + a_4x_3^2, \qquad \psi_2 = b_1x_1x_3 + b_2x_2x_3 + b_3x_4^2.$$

Since $\phi_2 = \psi_2 = 0$ defines a smooth curve, $a_1x_1x_2 + a_2x_1^2 + a_3x_2^2$ has no multiple factors, so up to a linear coordinate change of x_1 and x_2 we may assume that $\phi_2 = x_1x_2 + x_3^2$. Similarly, $b_1, b_2, b_3 \neq 0$. Then easy computations (see, e.g., [15, (7.7.1)]) show that $(X^\sharp \ni P^\sharp)$ is a singularity of type $[2;[2],[4]^2]$. This contradicts our assumption.

3.9

Now we assume that $(P \in \mathfrak{X})$ is *strictly canonical*. Let $\gamma : \tilde{\mathfrak{X}} \to \mathfrak{X}$ be the *crepant blowup* of $(P \in \mathfrak{X})$. By definition, $\tilde{\mathfrak{X}}$ has only $\mathbb{Q}$-factorial terminal singularities and $K_{\tilde{\mathfrak{X}}} = \gamma^* K_{\mathfrak{X}}$. Let $E = \sum E_i$ be the exceptional divisor, and let $\tilde{X}$ be the proper transform of X. Since the pair $(\mathfrak{X}, X)$ is log-canonical, we can write

$$K_{\tilde{\mathfrak{X}}} + \tilde{X} + E = \gamma^*(K_{\mathfrak{X}} + X), \qquad \gamma^* X = \tilde{X} + E. \tag{3.9.1}$$

The pair $(\tilde{\mathfrak{X}}, \tilde{X} + E)$ is log-canonical and $\tilde{\mathfrak{X}}$ has isolated singularities, so $\tilde{X} + E$ has generically normal crossings along $\tilde{X} \cap E$. Hence, $C := \tilde{X} \cap E$ is a reduced curve. By the adjunction we have

$$K_{\tilde{X}} + C = (K_{\tilde{\mathfrak{X}}} + \tilde{X} + E)|_{\tilde{X}} = \gamma^*(K_{\mathfrak{X}} + X)|_{\tilde{X}} = \gamma_{\tilde{X}}^* K_X.$$

Thus, $\gamma_{\tilde{X}} : \tilde{X} \to X$ is a dlt modification of $(X \ni P)$. Since $I \geq 3$, there is only one divisor over $P \in X$ with discrepancy -1. Hence, this divisor coincides with C, and so C is irreducible and smooth. In particular, $\tilde{X}$ meets only one component of E.

CLAIM

Let $Q \in \tilde{\mathfrak{X}}$ be a point at which E is not Cartier. Then in a neighborhood of Q we have $\tilde{X} \sim K_{\tilde{\mathfrak{X}}}$. In particular, $Q \in C$.

Proof

We are going to apply the results of [11]. The extraction $\gamma : \tilde{\mathfrak{X}} \to \mathfrak{X}$ can be decomposed in a sequence of elementary crepant blowups

$$\gamma_i : \mathfrak{X}_{i+1} \longrightarrow \mathfrak{X}_i, \quad i = 0, \dots, N,$$

where $\mathfrak{X}_0 = \mathfrak{X}$, $\mathfrak{X}_N = \tilde{\mathfrak{X}}$, for $i = 1, \dots, N$ each $\mathfrak{X}_i$ has only $\mathbb{Q}$-factorial canonical singularities, and the γ_i-exceptional divisor $E_{i+1,i}$ is irreducible. Kawakita [11] defined a divisor F with $\operatorname{Supp}(F) = E$ on $\mathfrak{X}_N = \tilde{\mathfrak{X}}$ inductively: $F_1 = E_{1,0}$ on $\mathfrak{X}_1$ and $F_{i+1} = \lceil \gamma_i^* F_i \rceil$. In our case, by (3.9.1) the divisor F is reduced, that is, $F = E$. Then by [11, Theorem 4.2] we have $E \sim -K_{\tilde{\mathfrak{X}}}$ near Q. Since $\tilde{X} + E$ is Cartier, $\tilde{X} \sim K_{\tilde{\mathfrak{X}}}$ near Q. □

CLAIM

The singular locus of $\tilde{\mathfrak{X}}$ near C consists of three cyclic quotient singularities P_1, P_2, P_3 of types $\frac{1}{r_i}(1, -1, b_i)$, where $\gcd(b_i, r_i) = 1$ and $(r_1, r_2, r_3) = (3,3,3)$, $(2,4,4)$, and $(2,3,6)$ in cases $I = 3$, 4, 6, respectively.

Proof
Let $P_1, P_2, P_3 \in C$ be singular points of $\tilde{X}$. Since $C = \tilde{X} \cap E$ is smooth, E is not Cartier at the P_i's. Hence, $P_1, P_2, P_3 \in \tilde{\mathfrak{X}}$ are (terminal) non-Gorenstein points. Now the assertion follows by [11, Theorem 4.2]. □

Therefore, $P_i \in \tilde{X}$ is a point of index $r_i/\gcd(2, r_i)$. Hence, the singularities of $\tilde{X}$ are of types $\frac{1}{r_i}(1,1)$. This proves Proposition 3.6. □

3.10
Let $(X \ni P)$ be a normal surface singularity admitting a $\mathbb{Q}$-Gorenstein smoothing $\mathfrak{f}: \mathfrak{X} \to \mathfrak{D}$. Let M_P be the Milnor fiber of $\mathfrak{f}$. Thus, $(M_P, \partial M_P)$ is a smooth 4-manifold with boundary. Denote by $\upmu_P = \mathrm{b}_2(M_P)$ the Milnor number of the smoothing. In our case we have (see [6])

$$\mathrm{b}_1(M_P) = 0, \qquad \mathrm{Eu}(M_P) = 1 + \upmu_P. \tag{3.10.1}$$

PROPOSITION 3.10.2 ([8, Section 2.3])
Let $(X \ni P)$ be a rational surface singularity. Assume that $(X \ni P)$ admits a $\mathbb{Q}$-Gorenstein smoothing. Then for the Milnor number $\upmu_P$ we have

$$\upmu_P = \mathrm{K}^2_{(X,P)} + \varsigma_P. \tag{3.10.3}$$

Proof
Obviously, $\mathrm{K}^2_{(X,P)} + \varsigma_P$ depends only on the analytic type of the singularity $(X \ni P)$. According to [18, Appendix], for $(X \ni P)$ there exist a projective surface Z with a unique singularity isomorphic to $(X \ni P)$ and a $\mathbb{Q}$-Gorenstein smoothing $\mathfrak{Z}/(\mathfrak{T} \ni 0)$. Let $\eta: Y \to Z$ be the minimal resolution. Write

$$K_Y = \eta^* K_Z - \Delta, \qquad K_Y^2 = K_Z^2 + \Delta^2.$$

Let Z' be the general fiber. Since

$$\mathrm{Eu}(Y) = \mathrm{Eu}(Z) + \varsigma_P, \qquad \chi(\mathscr{O}_Y) = \chi(\mathscr{O}_Z),$$

by Noether's formula we have

$$\begin{aligned} 0 &= K_Y^2 + \mathrm{Eu}(Y) - 12\chi(\mathscr{O}_Z) \\ &= K_Z^2 + \Delta^2 + \mathrm{Eu}(Z) + \varsigma_P - 12\chi(\mathscr{O}_{Z'}) \\ &= \Delta^2 + \varsigma_P + \mathrm{Eu}(Z) + K_{Z'}^2 - 12\chi(\mathscr{O}_{Z'}) \\ &= \Delta^2 + \varsigma_P + \mathrm{Eu}(Z) - \mathrm{Eu}(Z'). \end{aligned}$$

By (3.10.1) we have $\upmu_P = \Delta_P^2 + \varsigma_P$. □

COROLLARY 3.10.4 (see [20, Proposition 13])
If $(X \ni P)$ is a T*-singularity of type $\frac{1}{dm^2}(1, dma-1)$, then*

$$\upmu_P = d - 1, \qquad -\mathrm{K}^2 = \varsigma_P - d + 1. \tag{3.10.5}$$

Proposition 3.10.2 implies the following.

COROLLARY 3.10.6

Let $(X \ni P)$ be a strictly log-canonical surface singularity of index $I > 1$ admitting a $\mathbb{Q}$-Gorenstein smoothing. Then

$$\mu_P = \begin{cases} 4 - \sum(n_i - 3) & \text{in the case (DV) with } I = 2, \\ 13 - n - \sum r_i & \text{in the case (nDV).} \end{cases} \tag{3.10.7}$$

Proof of the classificational part of Theorem 1.2

Let

$$\pi : (X^\sharp \ni P^\sharp) \to (X \ni P)$$

be the index 1 cover. A $\mathbb{Q}$-Gorenstein smoothing $(X \ni P)$ is induced by an equivariant smoothing of $(X^\sharp \ni P^\sharp)$ (see Section 3.7). In particular, $(X^\sharp \ni P^\sharp)$ is smoothable. Assume that $(X \ni P)$ is of type $[n_1, \dots, n_s; [2]^4]$ with $s > 1$. Then $(X^\sharp \ni P^\sharp)$ is a cusp singularity. By [29, Theorem 5.6] its smoothability implies that

$$\operatorname{mult}(X^\sharp \ni P^\sharp) \le \varsigma_{P^\sharp} + 9.$$

Since $\varsigma_{P^\sharp} = 2\varsigma_P - 10$, by Corollary 3.5.2 and Remark 3.5.3 we have

$$2\sum(n_i - 2) \le 2\varsigma_P - 1, \qquad \sum(n_i - 3) \le 3.$$

In the case where $(X \ni P)$ is of type $[n; [2]^4]$, the singularity $(X^\sharp \ni P^\sharp)$ is simple elliptic. Then $\operatorname{mult}(X^\sharp \ni P^\sharp) \le 9$ (see, e.g., [19, Example 6.4]). Hence, $n \le 6$. In the case where $(X^\sharp \ni P^\sharp)$ is of type $[n, [r_1], [r_2], [r_3]]$, the assertion follows from Corollary 3.10.6 because $\mu_P \ge 0$. □

The existence of $\mathbb{Q}$-Gorenstein smoothings follows from examples and discussions in the next two sections.

4. Examples of $\mathbb{Q}$-Gorenstein smoothings

PROPOSITION 4.1 ([27, Corollary 19])

A rational surface singularity of index 2 and multiplicity 4 admits a $\mathbb{Q}$-Gorenstein smoothing.

Recall that for any rational surface singularity $(X \ni P)$ one has

$$\operatorname{mult}(X \ni P) = -\mathcal{Z}^2,$$

where $\mathcal{Z}$ is the fundamental cycle on the minimal resolution (see [1, Corollary 6]).

LEMMA 4.1.1

Let $(X \ni P)$ be a log-canonical surface singularity of type $[n_1, \dots, n_s; [2]^4]$. Then

$$-\mathcal{Z}^2 = \max\Big(4, 2 + \sum(n_i - 2)\Big) = \max(4, 2 - \mathrm{K}^2).$$

Proof
If either $s \geq 2$ and $n_1, n_s \geq 3$ or $s = 1$ and $n_1 \geq 4$, then $\mathcal{Z} = \lceil \Delta \rceil$ and so $\mathcal{Z}^2 = \Delta^2 - 2 = -n$ by Proposition 3.5. If $\sum(n_i - 2) = 1$, then $\mathcal{Z} = 2\Delta$ and so $\mathcal{Z}^2 = 4\Delta^2 = -4$. □

COROLLARY 4.1.2
A log-canonical singularity of type $[n_1, \ldots, n_s; [2]^4]$ *with* $\sum(n_i - 2) \leq 2$ *admits a* $\mathbb{Q}$*-Gorenstein smoothing.*

Let us consider explicit examples.

EXAMPLE 4.1.3
Let $\mathfrak{X} = \mathbb{C}^3/\boldsymbol{\mu}_2(1,1,1)$, and let

$$\mathfrak{f}: \mathfrak{X} \to \mathbb{C}, \qquad (x_1, x_2, x_3) \mapsto x_1^2 + (x_2^2 + c_1 x_3^{2k})(x_3^2 + c_2 x_2^{2m}),$$

where $k, m \geq 1$ and c_1, c_2 are constants. The central fiber $X = \mathfrak{X}_0$ is a log-canonical singularity of type

$$[\underbrace{2, \ldots, 2}_{k-1}, 3, \underbrace{2, \ldots, 2}_{m-1}; [2]^4].$$

Indeed, the $\frac{1}{2}(1,1,1)$-blowup of $X' \to X \ni 0$ has irreducible exceptional divisor. If $k, m \geq 3$, then the singular locus of X' consists of two Du Val singularities of types D_{k+1} and D_{m+1}. Other cases are similar.

EXAMPLE 4.1.4
Let $\boldsymbol{\mu}_2$ act on $\mathbb{C}^4_{x_1,\ldots,x_4}$ diagonally with weights $(1,1,1,0)$, and let $\phi(x_1, \ldots, x_4)$ and $\psi(x_1, \ldots, x_4)$ be invariants such that $\operatorname{mult}_0 \phi = \operatorname{mult}_0 \psi = 2$ and the quadratic parts $\phi_{(2)}$, $\psi_{(2)}$ define a smooth elliptic curve in $\mathbb{P}^3$. Let $\mathfrak{X} := \{\phi = 0\}/\boldsymbol{\mu}_2(1,1,1,0)$. Consider the family

$$\mathfrak{f}: \mathfrak{X} \longrightarrow \mathbb{C}, \qquad (x_1, \ldots, x_4) \longmapsto \psi.$$

The central fiber $X = \mathfrak{X}_0$ is a log-canonical singularity of type $[4; [2]^4]$.

PROPOSITION 4.1.5 ([4, Example 4.2])
Singularities of types $[5; [2]^4]$, $[4, 3; [2]^4]$, *and* $[3, 3, 3; [2]^4]$ *admit* $\mathbb{Q}$*-Gorenstein smoothings.*

Now consider singularities of index greater than 2.

EXAMPLE 4.2 ([15, (6.7.1)])
Let $\mathfrak{X} = \mathbb{C}^3/\boldsymbol{\mu}_3(1,1,2)$, and let

$$\mathfrak{f}: (x_1, x_2, x_3) \longmapsto x_1^3 + x_2^3 + x_3^3.$$

The central fiber $X = \mathfrak{X}_0$ is a log-canonical singularity of type $[2; [3]^3]$.

EXAMPLE 4.3
Let $\mathfrak{X} = \mathbb{C}^3/\boldsymbol{\mu}_9(1,4,7)$, and let

$$\mathfrak{f}: (x_1, x_2, x_3) \longmapsto x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2.$$

The central fiber $X = \mathfrak{X}_0$ is a log-canonical singularity of type $[4;[3]^3]$. The total space has a canonical singularity at the origin.

EXAMPLE 4.4 ([15, (7.7.1)])
Let

$$\mathfrak{X} = \{x_1 x_2 + x_3^2 + x_4^{2k+1} = 0\}/\boldsymbol{\mu}_4(1,1,-1,2), \quad k \geq 1.$$

Consider the family

$$\mathfrak{f}: \mathfrak{X} \longrightarrow \mathbb{C}, \qquad (x_1, \dots, x_4) \longmapsto x_4^2 + x_3(x_1 + x_2) + \psi_{\geq 3}(x_1, \dots, x_4),$$

where $\psi_{\geq 3}$ is an invariant with $\operatorname{mult}(\psi_{\geq 3}) \geq 3$. The central fiber $X = \mathfrak{X}_0$ is a log-canonical singularity of type $[2;[2],[4]^2]$. The singularity of the total space is terminal of type cAx/4.

EXAMPLE 4.5
Let $\mathfrak{X} := \{x_1 x_2 + x_3^2 + x_4^2 = 0\}/\boldsymbol{\mu}_8(1,5,3,7)$. Consider the family

$$\mathfrak{f}: \mathfrak{X} \longrightarrow \mathbb{C}, \qquad (x_1, \dots, x_4) \longmapsto x_1 x_4 + x_2 x_3.$$

The central fiber $X = \mathfrak{X}_0$ is a log-canonical singularity of type $[3;[2],[4]^2]$. The singularity of the total space $\mathfrak{X}$ is canonical (see [9]).

More examples of $\mathbb{Q}$-Gorenstein smoothings will be given in the next section.

5. Indices of canonical singularities

NOTATION 5.1
Let $S = S_d \subset \mathbb{P}^d$ be a smooth del Pezzo surface of degree $d \geq 3$. Let Z be the affine cone over S, and let $z \in Z$ be its vertex. Let $\delta: \tilde{Z} \to Z$ be the blowup along the maximal ideal of z, and let $\tilde{S} \subset \tilde{Z}$ be the exceptional divisor. The affine variety Z can be viewed as the spectrum of the anticanonical graded algebra

$$Z = \operatorname{Spec} R(-K_S), \qquad R(-K_S) := \bigoplus_{n \geq 0} H^0\big(S, \mathscr{O}_S(-nK_S)\big),$$

and the variety $\tilde{Z}$ can be viewed as the total space $\operatorname{Tot}(\mathscr{L})$ of the line bundle $\mathscr{L} := \mathscr{O}_S(K_S)$. Here $\tilde{S}$ is the negative section. Denote by $\gamma: \tilde{Z} \to S$ the natural projection.

LEMMA 5.2
The map δ is a crepant morphism, and $(Z \ni z)$ is a canonical singularity.

Proof
Write $K_{\tilde{Z}} = \delta^* K_Z + a\tilde{S}$. Then

$$K_{\tilde{S}} = (K_{\tilde{Z}} + \tilde{S})|_{\tilde{S}} = (a+1)\tilde{S}|_{\tilde{S}}.$$

Under the natural identification $S = \tilde{S}$ one has $\mathscr{O}_{\tilde{S}}(K_{\tilde{S}}) \simeq \mathscr{O}_S(-1) \simeq \mathscr{O}_{\tilde{S}}(\tilde{S})$. Hence, $a = 0$. □

CONSTRUCTION 5.3
Assume that S admits an action $\varsigma : G \to \mathrm{Aut}(S)$ of a finite group G. The action naturally extends to an action on the algebra $R(-K_S)$, the cone Z, and its blowup $\tilde{Z}$. We assume that

(A) $G \simeq \boldsymbol{\mu}_I$ is a cyclic group of order I,
(B) the action G on S is free in codimension 1, and
(C) the quotient S/G has only Du Val singularities.

Let G_P be the stabilizer of a point $P \in S$. Since $\mathscr{L} = \mathscr{O}_S(K_S)$, the fiber $\mathscr{L}_P$ of $\gamma : \tilde{Z} = \mathrm{Tot}(\mathscr{L}) \to S$ is naturally identified with $\bigwedge^2 T^\vee_{P,S}$, where $T_{P,S}$ is the tangent space to S at P. By our assumptions (B) and (C), in suitable analytic coordinates x_1, x_2 near P, the action of G_P is given by

$$(x_1, x_2) \longmapsto (\zeta_{I_P}^{b_P} \cdot x_1, \zeta_{I_P}^{-b_P} \cdot x_2), \tag{5.3.1}$$

where ζ_{I_P} is a primitive I_Pth root of unity, $\gcd(I_P, b_P) = 1$, and I_P is the order of G_P. Therefore, the action of G_P on $\mathscr{L}_P \simeq \wedge^2 T^\vee_{P,S}$ is trivial. Let $\tilde{P} := \mathscr{L}_P \cap \tilde{S}$.

The algebra $R(-K_S)$ also admits a natural $\mathbb{C}^*$-action compatible with the grading. Thus, $\gamma : \tilde{Z} \to S$ is a $\mathbb{C}^*$-equivariant $\mathbb{A}^1$-bundle, where $\mathbb{C}^*$-action on S is trivial and the induced action $\lambda : \mathbb{C}^* \to \mathrm{Aut}(\tilde{Z})$ is just multiplication in fibers. Fix an embedding $G = \boldsymbol{\mu}_I \subset \mathbb{C}^*$. Then two actions ς and λ commute, and so we can define a new action of G on $\tilde{Z}$ by

$$\varsigma'(\alpha) = \lambda(\alpha)\varsigma(\alpha), \quad \alpha \in G. \tag{5.3.2}$$

Take local coordinates x_1, x_2, x_3 in a neighborhood of $\tilde{P} \in \tilde{Z}$ compatible with the decomposition $T_{\tilde{P},\tilde{Z}} = T_{\tilde{P},\tilde{S}} \oplus T_{\tilde{P},\mathscr{L}_P}$ of the tangent space and (5.3.1). Then the action of G_P is given by

$$(x_1, x_2, x_3) \longmapsto (\zeta_{I_P}^{b_P} \cdot x_1, \zeta_{I_P}^{-b_P} \cdot x_2, \zeta_{I_P}^{a_P} \cdot x_3), \qquad \gcd(a_P, I_P) = 1. \tag{5.3.3}$$

CLAIM 5.4
The quotient $\tilde{\mathfrak{X}} := \tilde{Z}/\varsigma'(G)$ has only terminal singularities.

Proof
All the points of $\tilde{Z}$ with nontrivial stabilizers lie on the negative section $\tilde{S}$. The image of such a point $\tilde{P}$ on $\tilde{\mathfrak{X}}$ is a cyclic quotient singularity of type $\frac{1}{I_P}(b_P, -b_P, a_P)$ by (5.3.3). □

By the universal property of quotients, there is a contraction $\varphi : \tilde{\mathfrak{X}} \to \mathfrak{X}$ contracting E to a point, say, o, where $\mathfrak{X} := Z/G$ and $E := \tilde{S}/G$. Thus, we have the following diagram:

$$
\begin{array}{ccccccccc}
 & \tilde{S} & \subset & \tilde{Z} & \longrightarrow & \tilde{\mathfrak{X}} & \supset & E & \\
 & \big\downarrow & & \big\downarrow{\scriptstyle \delta} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow & \\
 & z & \in & Z & \xrightarrow{\ \pi\ } & \mathfrak{X} & \ni & o & \\
\end{array}
\qquad \gamma : \tilde{S} \to S
\tag{5.4.1}
$$

PROPOSITION 5.5

The germ $(\mathfrak{X} \ni o)$ *is an isolated canonical nonterminal singularity of index* $|G|$.

Proof

Since the action ς' is free in codimension 1, the contraction φ is crepant by Lemma 5.2. The index of $(\mathfrak{X} \ni o)$ is equal to the least common multiple of $|G_P|$ for $P \in S$. On the other hand, by the holomorphic Lefschetz fixed point formula G has a fixed point on S. Hence, $G = G_P$ for some P. □

5.6

Now we construct explicit examples of del Pezzo surfaces with cyclic group actions satisfying the conditions (A)–(C).

EXAMPLE 5.6.1

Recall that a del Pezzo surface S of degree 6 is unique up to isomorphism and can be given in $\mathbb{P}^1_{u_0:u_1} \times \mathbb{P}^1_{v_0:v_1} \times \mathbb{P}^1_{w_0:w_1}$ by the equation

$$u_1 v_1 w_1 = u_0 v_0 w_0.$$

Let $\alpha \in \operatorname{Aut}(S)$ be the following element of order 6:

$$\alpha : (u_0 : u_1; v_0 : v_1; w_0 : w_1) \longmapsto (v_1 : v_0; w_1 : w_0; u_1 : u_0).$$

Points with nontrivial stabilizers belong to one of three orbits, and representatives are the following:

- $P = (1:1;1:1;1:1)$, $|G_P| = 6$,
- $Q = (1:\zeta_3;1:\zeta_3;1:\zeta_3)$, $|G_Q| = 3$,
- $R = (1:1;1:-1;1:-1)$, $|G_R| = 2$.

It is easy to check that they give us Du Val points of type A_5, A_2, A_1, respectively.

EXAMPLE 5.6.2

A del Pezzo surface S of degree 5 is obtained by blowing up four points P_1, P_2, P_3, P_4 on $\mathbb{P}^2$ in general position. We may assume that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$, $P_4 = (1:1:1)$. Consider the following Cremona transformation:

$$\alpha : (u_0 : u_1 : u_2) \longmapsto \big(u_0(u_2 - u_1) : u_2(u_0 - u_1) : u_0 u_2\big).$$

It is easy to check that $\alpha^5 = \mathrm{id}$ and the indeterminacy points are exactly P_1, P_2, P_3. Thus, α lifts to an element $\alpha \in \mathrm{Aut}(S)$ of order 5.

CLAIM

Let $\alpha \in \mathrm{Aut}(S)$ be any element of order 5. Then α has only isolated fixed points, and the singular locus of the quotient $S/\langle a\rangle$ consists of two Du Val points of type A_4.

Proof

For the characteristic polynomial of α on $\mathrm{Pic}(S)$ there is only one possibility: $t^5 - 1$. Therefore, the eigenvalues of α are $1, \zeta_5, \ldots, \zeta_5^4$. This implies that every invariant curve is linearly proportional (in $\mathrm{Pic}(S)$) to $-K_S$. In particular, this curve must be an ample divisor.

Assume that there is a curve of fixed points. By the above it meets any line. Since on S there are at most two lines passing through a fixed point, all the lines must be invariant. In this case α acts on S identically, a contradiction.

Thus, the action of α on S is free in codimension 1. By the topological Lefschetz fixed point formula, α has exactly two fixed points, say, Q_1 and Q_2. We may assume that actions of α in local coordinates near Q_1 and Q_2 are diagonal:

$$(x_1, x_2) \longmapsto (\zeta_5^r x_1, \zeta_5^k x_2), \qquad (y_1, y_2) \longmapsto (\zeta_5^l y_1, \zeta_5^m y_2),$$

where r, k, l, m are not divisible by 5. Then by the holomorphic Lefschetz fixed point formula

$$1 = (1 - \zeta_5^r)^{-1}(1 - \zeta_5^k)^{-1} + (1 - \zeta_5^l)^{-1}(1 - \zeta_5^m)^{-1}.$$

Easy computations with cyclotomics show that up to permutations and modulo 5 there is only one possibility: $r = 1$, $k = 4$, $l = 2$, $m = 3$. This means that the quotient has only Du Val singularities of type A_4. □

EXAMPLE 5.6.3

Let $\boldsymbol{\mu}_3$ act on $S = \mathbb{P}^2$ diagonally with weights $(0, 1, 2)$. The quotient has three Du Val singularities of type A_2.

EXAMPLE 5.6.4

Let $\boldsymbol{\mu}_4$ act on $S = \mathbb{P}^1_{u_0:u_1} \times \mathbb{P}^1_{v_0:v_1}$ by

$$(u_0 : u_1; v_0 : v_1) \longmapsto (v_0 : v_1; u_1 : u_0).$$

The quotient has three Du Val singularities of types A_1, A_3, A_3.

Note that in all the examples above, the group generated by α^n also satisfies the conditions (A)–(C). We summarize the above information in Table 3. Together with Proposition 5.5 this proves Theorem 1.3. Note that our table agrees with the corresponding one in [11].

Table 3

No.	K_S^2	*Example*	G	I	$\mathrm{Sing}(\tilde{\mathfrak{X}})$
$1^{\#}$	6	5.6.1	$\langle\alpha\rangle$	6	$\frac{1}{6}(1,-1,1)$, $\frac{1}{3}(1,-1,1)$, $\frac{1}{2}(1,1,1)$
$2^{\#}$	5	5.6.2	$\langle\alpha\rangle$	5	$\frac{1}{5}(1,-1,1)$, $\frac{1}{5}(2,-3,1)$
$3^{\#}$	8	5.6.4	$\langle\alpha\rangle$	4	$2\times\frac{1}{4}(1,-1,1)$, $\frac{1}{2}(1,1,1)$
$4^{\#}$	6	5.6.1	$\langle\alpha^2\rangle$	3	$3\times\frac{1}{3}(1,-1,1)$
$5^{\#}$	9	5.6.3	$\langle\alpha\rangle$	3	$3\times\frac{1}{3}(1,-1,1)$
$6^{\#}$	6	5.6.1	$\langle\alpha^3\rangle$	2	$4\times\frac{1}{2}(1,1,1)$
$7^{\#}$	8	5.6.4	$\langle\alpha^2\rangle$	2	$4\times\frac{1}{2}(1,1,1)$

Now we apply the above technique to construct examples of $\mathbb{Q}$-Gorenstein smoothings.

THEOREM 5.7

Let $(X\ni o)$ be a surface log-canonical singularity of one of the following types:

$$[2;[2,3,6]],\quad [3;[2,4,4]],\quad [n;[3,3,3]],\quad n=3,4,\qquad [n;[2,2,2,2]],\quad n=5,6.$$

Then $(X\ni o)$ admits a $\mathbb{Q}$-Gorenstein smoothing.

LEMMA 5.7.1

In the notation of (5.4.1), let $C\subset S$ be a smooth elliptic G-invariant curve such that $C\sim -K_S$. Assume that C passes through all the points with nontrivial stabilizers. Let $\tilde{X}^\sharp:=\gamma^{-1}(C)$, $X^\sharp:=\delta(\tilde{X}^\sharp)$, and $X:=\pi(X^\sharp)$. Then the singularity $(X\ni o)$ is log-canonical of index $|G|$. Moreover, replacing λ with λ^{-1} if necessary we may assume that X is a Cartier divisor on $\mathfrak{X}$.

Proof

Put $\tilde{X}:=\tilde{X}^\sharp/G$. Since the divisor $\tilde{X}^\sharp+\tilde{S}$ is trivial on $\tilde{S}$, the contraction δ is log crepant with respect to $K_{\tilde{Z}}+\tilde{X}^\sharp+\tilde{S}$ and so is φ with respect to $K_{\tilde{\mathfrak{X}}}+\tilde{X}+E$. By construction, $X^\sharp$ is a cone over the elliptic curve C and $X=X^\sharp/G$. Therefore, $(X\ni o)$ is a log-canonical singularity. Comparing this with Construction 2.7 we see that the index of $(X\ni o)$ equals $|G|$. We claim that $\tilde{X}+E$ is a Cartier divisor on $\tilde{\mathfrak{X}}$. Identify C with $\tilde{C}:=\gamma^{-1}(C)\cap\tilde{S}=\tilde{S}\cap\tilde{X}^\sharp$.

Let $\omega\in H^0(C,\mathscr{O}_C(K_C))$ be a nowhere-vanishing holomorphic 1-form on C, and let α be a generator of G. Since $\dim H^0(C,\mathscr{O}_C(K_C))=1$ and G has a fixed point on C, the action of G on $H^0(C,\mathscr{O}_C(K_C))$ is faithful and we can write $\alpha^*\omega=\zeta_I\omega$, where ζ_I is a suitable primitive Ith root of unity.

Pick a point $\tilde{P}\in\tilde{Z}$ with nontrivial stabilizer G_P of order I_P. By our assumptions $\tilde{P}\in\tilde{C}$. Take semi-invariant local coordinates x_1, x_2, x_3 as in (5.3.3). Moreover, we can take them so that x_1 is a local coordinate along C. Then we can write $\omega=\varpi\,dx_1$, where ϖ is an invertible holomorphic function in a neighborhood of P. Hence, ϖ is an invariant and $\alpha^*x_1=\zeta_I^{I/I_P}x_1$. Thus, by (5.3.3), the action near $\tilde{P}$ has the form $\frac{1}{I_P}(1,-1,a_P)$. Since G faithfully acts on C with a fixed point, $I_P=2$, 3, 4, or 6. Since $\gcd(a_P,I_P)=1$, we have $a_P\in\{\pm1\}$. Then by (5.3.2), replacing λ with λ^{-1}, we may assume that $a_P=1$. In our coordinates the local equation of $\tilde{S}$ is $x_3 = 0$, and the local equation of $\tilde{X}^\sharp$ is $x_2 = 0$. Now it is easy to see that the local equation $x_2x_3 = 0$ of $\tilde{S} + \tilde{X}^\sharp$ is G_P-invariant. Therefore, $\tilde{X} + E$ is Cartier. Since it is φ-trivial, the divisor $X = \varphi_*(\tilde{X} + E)$ on $\mathfrak{X}$ is Cartier as well. □

Proof of Theorem 5.7
It is sufficient to embed X to a canonical threefold singularity $(\mathfrak{X} \ni o)$ as a Cartier divisor. Let $(X^\sharp \ni o^\sharp) \to (X \ni o)$ be the index 1 cover. Then $(X^\sharp \ni o^\sharp)$ is a simple elliptic singularity (see Section 2.6). In the notation of the examples in Section 5.6 consider the following $\boldsymbol{\mu}_I$-invariant elliptic curve $C \subset S$:

- $1^\#4^\#$ $\zeta_3(u_0w_1 - u_1w_0)(v_0 + v_1) + (u_0v_1 - u_1v_0)(w_0 + w_1)$,
- $3^\#$ $(u_1^2 - u_0^2)v_0v_1 + \zeta_4 u_0u_1(v_1^2 - v_0^2)$,
- $5^\#$ $u_0^2u_1 + u_1^2u_2 + u_2^2u_0$,
- $6^\#$ $c_1(u_0w_1 - u_1w_0)(v_0 + v_1) + c_2(u_0v_1 - u_1v_0)(w_0 + w_1)$,
- $7^\#$ $c_1(u_0^2v_0^2 - u_1^2v_1^2) + c_2v_0v_1(u_0^2 - u_1^2) + c_3(u_0^2v_1^2 - u_1^2v_0^2) + c_5u_0u_1(v_0^2 - v_1^2)$,

where c_i's are constants and ζ_n is a primitive nth root of unity. Then we apply Lemma 5.7.1. □

6. Noether's formula

PROPOSITION 6.1 ([8])
Let X be a projective rational surface with only rational singularities. Assume that every singularity of X admits a $\mathbb{Q}$-Gorenstein smoothing. Then

$$K_X^2 + \rho(X) + \sum_{P \in X} \mu_P = 10. \tag{6.1.1}$$

Proof
Let $\eta : Y \to X$ be the minimal resolution. Since X has only rational singularities, we have

$$\mathrm{Eu}(Y) = \mathrm{Eu}(X) + \sum_P \varsigma_P, \qquad \chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X).$$

Further, we can write

$$K_Y = \eta^* K_X - \sum_P \Delta_P, \qquad K_Y^2 = K_X^2 + \sum_P \Delta_P^2.$$

By the usual Noether formula for smooth surfaces

$$12\chi(\mathscr{O}_X) = K_Y^2 + \mathrm{Eu}(Y) = K_X^2 + \mathrm{Eu}(X) + \sum_P (\Delta_P^2 + \varsigma_P).$$

Now the assertion follows from (3.10.3). □

6.2

Let X be an arbitrary normal projective surface, let $\eta : Y \to X$ be the minimal resolution, and let D be a Weil divisor on X. Write $\eta^* D = D_Y + D^\bullet$, where D_Y

is the proper transform of D and $D^\bullet$ is the exceptional part of η^*D. Define the number

$$c_X(D) = -\frac{1}{2}\langle D^\bullet\rangle \cdot \bigl(\lfloor \eta^* D\rfloor - K_Y\bigr). \tag{6.2.1}$$

PROPOSITION 6.2.2 ([2, Section 1])

In the above notation we have

$$\chi\bigl(\mathscr{O}_X(D)\bigr) = \frac{1}{2}D\cdot(D-K_X) + \chi(\mathscr{O}_X) + c_X(D) + c'_X(D), \tag{6.2.3}$$

where

$$c'_X(D) := h^0\bigl(R^1\eta_*\mathscr{O}_Y\bigl(\lfloor \eta^* D\rfloor\bigr)\bigr) - h^0(R^1\eta_*\mathscr{O}_Y).$$

REMARK 6.2.4

Note that $c_X(D)$ can be computed locally as

$$c_X(D) = \sum_{P\in X} c_{P,X}(D),$$

where $c_{P,X}(D)$ is defined by the formula (6.2.1) for each germ $(X \ni P)$.

LEMMA 6.2.5

Let $(X \ni P)$ be a rational log-canonical surface singularity. Then

$$c_{P,X}(-K_X) = \Delta^2 - \lceil \Delta\rceil^2 - 3,$$

where, as usual, Δ is defined by $K_Y = \eta^*K_X - \Delta$.

Proof

Put $D := -K_X$, and write

$$\eta^* D = -K_Y - \Delta, \qquad \langle D^\bullet\rangle = \langle -\Delta\rangle = \lceil\Delta\rceil - \Delta,$$

$$\lfloor \eta^* D\rfloor - K_Y = -2K_Y - \lceil\Delta\rceil = -2\eta^*K_X + 2\Delta - \lceil\Delta\rceil.$$

Therefore,

$$c_{P,X}(D) = \frac{1}{2}\bigl(\Delta - \lceil\Delta\rceil\bigr)\cdot\bigl(-2\eta^*K_X + 2\Delta - \lceil\Delta\rceil\bigr) = \frac{1}{2}\bigl(\lceil\Delta\rceil - \Delta\bigr)\cdot\bigl(\lceil\Delta\rceil - 2\Delta\bigr).$$

Since $(X \ni P)$ is a rational singularity, we have

$$-2 = 2p_a\bigl(\lceil\Delta\rceil\bigr) - 2 = \bigl(\lceil\Delta\rceil - \Delta\bigr)\cdot\lceil\Delta\rceil, \qquad \lceil\Delta\rceil^2 + 2 = \Delta\cdot\lceil\Delta\rceil,$$

and the equality follows. □

COROLLARY 6.2.6

Let $(X \ni P)$ be a rational log-canonical surface singularity such that K^2 is integral. Then

$$c_{P,X}(-K_X) = \begin{cases} -1 & \text{in the case (DV)},\\ 0 & \text{if } (X\ni P) \text{ is log-terminal}\\ & \text{or in the case (nDV)}.\end{cases} \tag{6.2.7}$$

Proof
Let us consider the (nDV) case. (Other cases are similar.) By Proposition 3.5 we have $-\Delta^2 = n - 9 + \sum r_i$. On the other hand, $\lceil \Delta \rceil^2 = -n + 6 - \sum r_i$. Hence, $c_{P,X}(-K_X) = 0$ as claimed. □

COROLLARY 6.2.8
Let X be a del Pezzo surface with log-canonical rational singularities and $\rho(X) = 1$. Assume that for any singularity of X the invariant K^2 is integral. Then $H^i(X, \mathscr{O}_X) = 0$ for $i > 0$ and $\dim |-K_X| \geq K_X^2 - 1$.

Proof
By the Serre duality $H^2(X, \mathscr{O}_X) = H^0(X, K_X) = 0$. If the singularities of X are rational, then the Albanese map is a well-defined morphism $\mathrm{alb}: X \to \mathrm{Alb}(X)$. Since $\rho(X) = 1$, we have $\dim \mathrm{Alb}(X) = 0$ and so $H^1(X, \mathscr{O}_X) = 0$. The last inequality follows from (6.2.3) because $c'_X(-K_X) \geq 0$ and $c_X(-K_X) \geq -1$ (see (6.2.7)). □

7. Del Pezzo surfaces

ASSUMPTION 7.1
From now on let X be a del Pezzo surface satisfying the following conditions:

(i) the singularities of X are log-canonical and X has at least one non-log-terminal point $o \in X$,
(ii) X admits a $\mathbb{Q}$-Gorenstein smoothing,
(iii) $\rho(X) = 1$.

LEMMA 7.2
In the above assumptions, the following hold:

(i) $\dim |-K_X| > 0$,
(ii) *X has exactly one non-log-terminal point.*

Proof
Part (i) is implied by semicontinuity (see [20, Theorem 4]). Part (ii) follows from Shokurov's connectedness theorem (see [26, Lemma 5.7], [14, Theorem 17.4]). □

CONSTRUCTION 7.3
Let $\sigma: \tilde{X} \to X$ be a dlt modification, and let

$$\tilde{C} = \sum_{i=1}^{s} \tilde{C}_i = \sigma^{-1}(o)$$

be the exceptional divisor. Thus, $\rho(\tilde{X}) = s + 1$.

For some large k the divisor $-kK_X$ is very ample. Let $H \in |-kK_X|$ be a general member, and let $\Theta := \frac{1}{k}H$. Then $K_X + \Theta \equiv 0$ and the pair (X, Θ) is lc at o and klt outside o. We can write

$$K_{\tilde{X}} + \tilde{C} = \sigma^* K_X, \qquad K_{\tilde{X}} + \tilde{\Theta} + \tilde{C} = \sigma^*(K_X + \Theta), \tag{7.3.1}$$

where $\tilde{\Theta}$ is the proper transform of Θ on $\tilde{X}$. Clearly $\tilde{C} \cap \operatorname{Supp}(\tilde{\Theta}) = \varnothing$ and $\tilde{\Theta}$ is nef and big. Note also that $K_{\tilde{X}}$ is σ-nef.

7.3.2

Let $D \in |-K_X|$ be a member such that $o \in \operatorname{Supp}(D)$. This holds automatically for any member $D \in |-K_X|$ if $I > 1$ because $-K_X$ is not Cartier at o in this case. In general, such a member exists by Lemma 7.2(i). We have

$$K_{\tilde{X}} + \sum m_i \tilde{C}_i + \tilde{D} \sim 0, \quad m_i \geq 2 \ \forall i. \tag{7.3.3}$$

7.4

We distinguish two cases that will be treated in Sections 8 and 9, respectively:

(A) there exists a fibration $\tilde{X} \to T$ over a smooth curve,
(B) $\tilde{X}$ has no dominant morphism to a curve.

Note that the divisor $-(K_{\tilde{X}} + \tilde{C})$ is nef and big. Therefore, in the case (A) the generic fiber of the fibration $\tilde{X} \to T$ is a smooth rational curve.

To show the existence of $\mathbb{Q}$-Gorenstein smoothings we use the unobstructedness of deformations.

PROPOSITION 7.5 ([8, Proposition 3.1])

Let Y be a projective surface with log-canonical singularities such that $-K_Y$ is big. Then there are no local-to-global obstructions to deformations of Y. In particular, if the singularities of Y admit $\mathbb{Q}$-Gorenstein smoothings, then the surface Y admits a $\mathbb{Q}$-Gorenstein smoothing.

However, in some cases the corresponding smoothings can be constructed explicitly.

EXAMPLE 7.5.1

Consider the hypersurface $X \subset \mathbb{P}(1,1,2,3)$ given by $z^2 = y\phi_4(x_1, x_2)$. Then X is a del Pezzo surface with $K_X^2 = 1$. The singular locus of X consists of the point $(0:0:1:0)$ of type $[3;[2]^4]$ and four points $\{z = y = \phi_4(x_1,x_2) = 0\}$ of type A_1. Therefore, X is of type 2^o with $n = 3$.

EXAMPLE 7.5.2

Consider the hypersurface $X \subset \mathbb{P}(1,1,2,3)$ given by $(x_1^3 - x_2^3)z + y^3 = 0$. Then X is a del Pezzo surface with $K_X^2 = 1$. The singular locus of X consists of the point $(0:0:0:1)$ of type $[2;[3]^3]$ and three points $(1:\zeta_3^k:0:0)$, $k = 0,1,2$, of type A_2. Therefore, X is of type 5^o with $n = 2$.

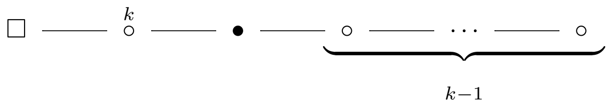
8. Proof of Theorem 1.1: Fibrations

In this section we consider the case (A) of Construction 7.3. First we describe quickly the singular fibers that occur in our classification.

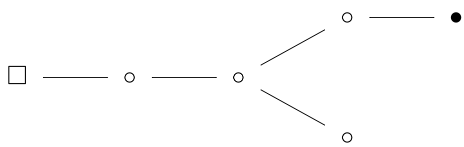
8.1

Let Y be a smooth surface, and let $Y \to T$ be a rational curve fibration. Let $\Sigma \subset Y$ be a section, and let F be a singular fiber. We say that F is of type ($\mathrm{I_k}$) or (II) if its dual graph has the following form, where $\square$ corresponds to Σ and $\bullet$ corresponds to a (-1)-curve:

($\mathrm{I_k}$)



(II)



Assume that Y has only fibers of these types ($\mathrm{I_k}$) or (II). Let $Y \to \bar{X}$ be the contraction of all curves in fibers having self-intersections less than -1, that is, corresponding to white vertices. Then $\rho(\bar{X}) = 2$ and $\bar{X}$ has a contraction $\theta : \bar{X} \to T$.

REMARK 8.1.1

Let $\bar{C} \subset \bar{X}$ be the image of Σ. Assume that $\bar{X}$ is projective, $\bar{C}^2 < 0$, that is, $\bar{C}$ is contractible, and $(K_{\bar{X}} + \bar{C}) \cdot \bar{C} = 0$. For a general fiber F of θ we have $(K_{\bar{X}} + \bar{C}) \cdot F = -1$. Therefore, $-(K_{\bar{X}} + \bar{C})$ is nef. Now let $\bar{X} \to X$ be the contraction of $\bar{C}$. Then X is a del Pezzo surface with $\rho(X) = 1$.

8.2

Recall that we use the notation of Assumption 7.1 and Construction 7.3. In this section, we assume that $\tilde{X}$ has a rational curve fibration $\tilde{X} \to T$, where T is a smooth curve (the case (A)). Since $\rho(X) = 1$, the curve $\tilde{C}$ is not contained in the fibers. A general fiber $\tilde{F} \subset \tilde{X}$ is a smooth rational curve. By the adjunction formula $K_{\tilde{X}} \cdot \tilde{F} = -2$. By (7.3.3) we have $\tilde{F} \cdot \sum m_i \tilde{C}_i = 2$ and so $\tilde{F} \cdot \tilde{D} = 0$. Hence, there exists exactly one component of $\tilde{C}$, say, $\tilde{C}_1$, such that $\tilde{F} \cdot \tilde{C}_1 = 1$, $m_1 = 2$, and for $i \neq 1$ we have $\tilde{F} \cdot \tilde{C}_i = 0$. This means that the divisor $\tilde{D}$ and the components $\tilde{C}_i$ with $i \neq 1$ are contained in the fibers, and $\tilde{C}_1$ is a section of the fibration $\tilde{X} \to T$.

Let us contract all the vertical components of $\tilde{C}$, that is, the components $\tilde{C}_i$ with $i \neq 1$. We get the following diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \nu\ } & \bar{X} \\ {\scriptstyle\sigma}\downarrow & \swarrow & \downarrow{\scriptstyle\theta} \\ X & & T \end{array}$$

Let $\bar{C} := \nu_*\tilde{C} = \nu_*\tilde{C}_1$, $\bar{\Theta} = \nu_*\tilde{\Theta}$, and $\bar{D} = \nu_*\tilde{D}$. By (7.3.1) and (7.3.3) we have

(8.2.1) $$K_{\bar{X}} + \bar{C} + \bar{\Theta} \equiv 0, \qquad K_{\bar{X}} + 2\bar{C} + \bar{D} \sim 0.$$

Moreover, the pair $(\bar{X}, \bar{C} + \bar{\Theta})$ is lc, and if $I > 1$, then $\dim |\bar{D}| > 0$.

LEMMA 8.3 ([5])

If the singularity $(X \ni o)$ is not rational, then T is an elliptic curve, $\tilde{X} \simeq \bar{X}$ is smooth, and X is a generalized cone over T.

Proof

By Theorem 2.4(i) the surface $\tilde{X}$ is smooth along $\tilde{C}$. Since $\tilde{C}_1$ is a section, we have $\tilde{C}_1 \simeq T$ and $\tilde{C}$ cannot be a combinatorial cycle of smooth rational curves. Hence, both $\tilde{C}_1$ and T are smooth elliptic curves. Then $\tilde{C} = \tilde{C}_1$ and $\rho(\tilde{X}) = \rho(X) + 1 = 2$. Hence, any fiber $\tilde{F}$ of the fibration $\tilde{X} \to T$ is irreducible. Since $\tilde{F} \cdot \tilde{C}_1 = 1$, any fiber is not multiple. This means that $\tilde{X} \to T$ is a smooth morphism. Therefore, $\tilde{X}$ is a geometrically ruled surface over an elliptic curve. □

From now on we assume that the singularities of X are rational. In this case, $T \simeq \mathbb{P}^1$ and $\dim |\bar{D}| \ge \dim |-K_X| > 0$ (see Section 7.3.2 and Lemma 7.2).

LEMMA 8.4

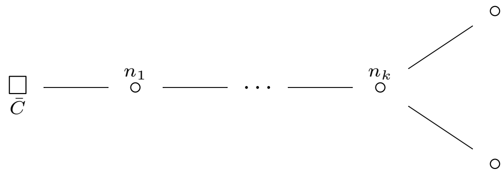
Let $\bar{F}$ be a degenerate fiber (with reduced structure). Then the dual graph of $\bar{F}$ has one of the forms described in Section 8.1:

($\mathrm{I_k}$) *with* $k = 2$, 3, 4 *or* 6, *or* (II).

Proof

Let $\bar{P} := \bar{C} \cap \bar{F}$. Since $-(K_{\bar{X}} + \bar{C} + \bar{F})$ is θ-ample, the pair $(\bar{X}, \bar{C} + \bar{F})$ is plt outside $\bar{C}$ by Shokurov's connectedness theorem. Let m be the multiplicity of $\bar{F}$. Since $\bar{C}$ is a section of θ, we have $\bar{C} \cdot \bar{F} = 1/m < 1$, and so the point $\bar{P} \in \bar{X}$ is singular.

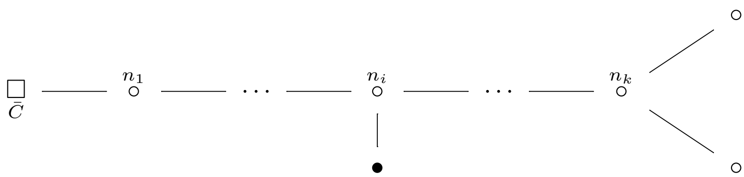
If the pair $(\bar{X}, \bar{F})$ is plt at $\bar{P}$, then $\bar{X}$ has on $\bar{F}$ two singular points and these points are of types $\frac{1}{n}(1,q)$ and $\frac{1}{n}(1,-q)$ (see, e.g., [22, Theorem 7.1.12]). We may assume that $\bar{P} \in \bar{X}$ is of type $\frac{1}{n}(1,q)$. In this case, $m = n$ and the pair $(\bar{X}, \bar{C} + \bar{F})$ is lc at $\bar{P}$ because $\bar{C} \cdot \bar{F} = 1/n$. By Theorem 1.2 we have $n = 2$, 3, 4, or 6 and $q = 1$. We get the case ($\mathrm{I_k}$). From now on we assume that $(\bar{X}, \bar{F})$ is not plt at $\bar{P}$. In particular, $(\bar{X} \ni \bar{P})$ is not of type $\frac{1}{n}(1,1)$. Then again by Theorem 1.2 the singularity $(o \in X)$ is of type $[n_1, \ldots, n_s; [2]^4]$. Hence, the part of the dual graph of F attached to C_1 has the form



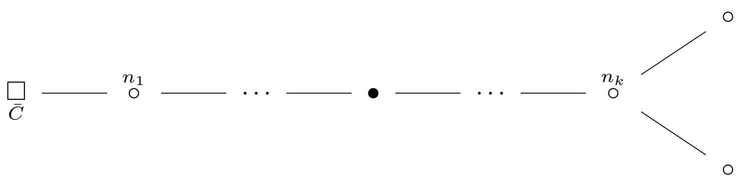
where $k \geq 1$. Then $K_{\bar{X}} + \bar{C}$ is of index 2 at $\bar{P}$ (see [14, Proposition 16.6]). Since $(K_{\bar{X}} + \bar{C}) \cdot m\bar{F} = -1$, the number $2(K_{\bar{X}} + \bar{C}) \cdot \bar{F} = -2/m$ must be an integer. Therefore, $m = 2$. Assume that $\bar{X}$ has a singular point $\bar{Q}$ on $\bar{F} \setminus \{\bar{P}\}$. We can write $\operatorname{Diff}_{\bar{F}}(0) = \alpha_1 \bar{P} + \alpha_2 \bar{Q}$, where $\alpha_1 \geq 1$ (by the inversion of adjunction) and $\alpha_2 \geq 1/2$. Then $\operatorname{Diff}_{\bar{F}}(\bar{C}) = \alpha_1' \bar{P} + \alpha_2 \bar{Q}$, where $\alpha_1' = \alpha_1 + \bar{F} \cdot \bar{C} \geq 3/2$. On the other hand, the divisor

$$-\big(K_{\bar{F}} + \operatorname{Diff}_{\bar{F}}(\bar{C})\big) = -(K_{\bar{X}} + \bar{F} + \bar{C})|_{\bar{F}}$$

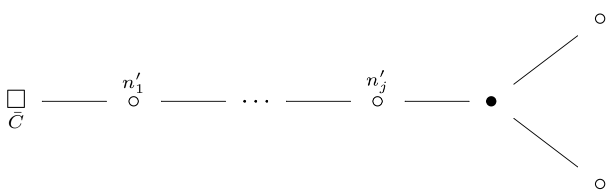
is ample. Hence, $\deg \operatorname{Diff}_{\bar{F}}(\bar{C}) < 2$, a contradiction. Thus, $\bar{P}$ is the only singular point of $\bar{X}$ on $\bar{F}$. We claim that $\bullet$ is attached to one of the (-2)-curves at the end of the graph. Indeed, assume that the dual graph of F has the form



where $1 \leq i \leq k$. Clearly, $n_i = 2$. Contracting the (-1)-curve $\bullet$ we obtain the following graph:



Continuing the process, on each step we have a configuration of the same type and finally we get the dual graph



where $j \geq 0$. Then the next contraction gives us a configuration which is not a simple normal crossing divisor. The contradiction proves our claim. Similar arguments show that $n_k = n_{k-1} = 2$ and $k = 2$, that is, we get the case (II). □

Proof of Theorem 1.1 in the case 7.3(A)
If all the fibers are smooth, then by Lemma 8.3 we have the case 1^o. If there exists a fiber of type $(\mathrm{I_k})$ with $k > 2$, then $I > 2$ and by Theorem 1.2 we have cases 5^o, 6^o, 7^o. If all the fibers are of types (I_2) or (II), then $I = 2$ and we have cases 2^o, 3^o, 4^o. The computation of $K_{\bar{X}}^2$ follows from (6.1.1) and (3.10.7). □

9. Proof of Theorem 1.1: Birational contractions

9.1

In this section we assume that $\tilde{X}$ has no dominant morphism to a curve (Construction 7.3(B)). It will be shown that this case does not occur.

Run the $K_{\tilde{X}}$-Minimal Model Program on $\tilde{X}$. Since $-K_{\tilde{X}}$ is big, in the last step we get a Mori fiber space $\bar{X} \to T$, and by our assumption T cannot be a curve. Hence, T is a point, and $\bar{X}$ is a del Pezzo surface with $\rho(\bar{X}) = 1$. Moreover, the singularities of $\bar{X}$ are log-terminal and so $\bar{X} \not\simeq X$. Thus, we get the following diagram:

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \overset{\sigma}{\swarrow} & & \overset{\nu}{\searrow} & \\ X & - & - & \dashrightarrow & \bar{X} \end{array}$$

Put $\bar{C} := \nu_* \tilde{C}$ and $\bar{C}_i := \nu_* \tilde{C}_i$. By (7.3.3) we have

$$K_{\bar{X}} + \sum m_i \bar{C}_i + \bar{D} \sim 0, \quad m_i \geq 2. \tag{9.1.1}$$

Since $\rho(X) = \rho(\bar{X})$ and $\tilde{C}$ is the σ-exceptional divisor, the whole $\tilde{C}$ cannot be contracted by ν.

LEMMA 9.2
Any fiber $\nu^{-1}(\bar{P})$ of positive dimension meets $\tilde{C}$.

Proof
Since $\bar{X}$ is normal, $\nu^{-1}(\bar{P})$ is a connected contractible effective divisor. Since all the components of $\tilde{C}$ are $K_{\tilde{X}}$-nonnegative, $\nu^{-1}(\bar{P}) \not\subset \tilde{C}$. Since $\rho(X) = 1$, we have $\nu^{-1}(\bar{P}) \cap \tilde{C} \neq \varnothing$. □

LEMMA 9.3
If ν is not an isomorphism over $\bar{P}$, then $(\bar{X}, \bar{C})$ is plt at $\bar{P}$. In particular, $\bar{C}$ is smooth at $\bar{P}$.

Proof
Since $K_{\tilde{X}} + \tilde{C} + \tilde{\Theta} \equiv 0$, the pair $(\bar{X}, \bar{C} + \bar{\Theta})$ is lc. By the above lemma there exists a component $\tilde{E}$ of $\nu^{-1}(\bar{P})$ meeting $\tilde{C}$. By Kodaira's lemma the divisor $\tilde{\Theta} - \sum \alpha_i \tilde{C}_i$ is ample for some $\alpha_i > 0$. Hence, $\tilde{E}$ meets $\tilde{\Theta}$ and so $\mathrm{Supp}(\bar{\Theta})$ contains $\bar{P}$. Therefore, $(\bar{X}, \bar{C})$ is plt at $\bar{P}$. □

COROLLARY 9.3.1

The pair $(\bar{X}, \bar{C})$ is dlt.

LEMMA 9.4

The following hold:

(i) *$\bar{C}$ is an irreducible smooth rational curve;*
(ii) *$\bar{X}$ has at most two singular points on $\bar{C}$;*
(iii) *the singularities of X are rational* (see also [5, Corollary 1.9]).

Proof

(i) Let $\bar{C}_1 \subset \bar{C}$ be any component meeting $\bar{D}$, and let $\bar{C}' := \bar{C} - \bar{C}_1$. Assume that $\bar{C}' \neq 0$. By Corollary 9.3.1, any point $\bar{P} \in \bar{C}_1 \cap \bar{C}'$ is a smooth point of $\bar{X}$. Hence, $\operatorname{Diff}_{\bar{C}_1}(\bar{C}')$ contains $\bar{P}$ with positive integral coefficient and $\deg \operatorname{Diff}_{\bar{C}_1}(\bar{D} + \bar{C}') \geq 2$ because $\operatorname{Supp}(\bar{D}) \cap \bar{C} \neq \varnothing$. On the other hand, $-(K_{\bar{X}} + \bar{C} + \bar{D})$ is ample by (9.1.1). This contradicts the adjunction formula. Thus, $\bar{C}$ is irreducible. Again by the adjunction

$$\deg K_{\bar{C}} + \deg \operatorname{Diff}_{\bar{C}}(0) < 0.$$

Hence, $p_a(\bar{C}) = 0$.

(ii) Assume that $\bar{X}$ is singular at $\bar{P}_1, \ldots, \bar{P}_N \in \bar{C}$. Write

$$\operatorname{Diff}_{\bar{C}}(0) = \sum_{i=1}^{N} \Big(1 - \frac{1}{b_i}\Big) \bar{P}_i$$

for some $b_i \geq 2$. The coefficient of $\operatorname{Diff}_{\bar{C}}(\bar{D})$ at points of the intersection $\operatorname{Supp}(\bar{D}) \cap \bar{C}$ is at least 1. Since $\operatorname{Supp}(\bar{D}) \cap \bar{C} \neq \varnothing$, we have $N \leq 2$.

(iii) If $(X \ni o)$ is a nonrational singularity, then $p_a(\tilde{C}) = 1$ and $\tilde{X}$ is smooth along $\tilde{C}$. Hence, $p_a(\bar{C}) \geq 1$. This contradicts (i). □

LEMMA 9.5

Let $\varphi : S \to S'$ be a birational Mori contraction of surfaces with log-terminal singularities, and let $E \subset S$ be the exceptional divisor. Then $-K_S \cdot E \leq 1$, and the equality holds if and only if the singularities of S along E are at worst Du Val.

Proof

Let $\psi : S^{\min} \to S$ be the minimal resolution, and let $\tilde{E} \subset S^{\min}$ be the proper transform of E. Write $K_{S^{\min}} = \psi^* K_S - \Delta$. Since $K_{S^{\min}} \cdot \psi^* E < 0$, the divisor $K_{S^{\min}}$ is not nef over Z. Hence, $K_{S^{\min}} \cdot \tilde{E} = -1$ and so $-K_S \cdot E + \tilde{E} \cdot \Delta = 1$. □

LEMMA 9.6

Let $\nu' : \tilde{X} \to X'$ be the first extremal contraction in ν, and let $\tilde{E}$ be its exceptional divisor. Then $\tilde{E} \not\subset \tilde{C}$. Moreover, $\tilde{E} \cap \tilde{C}$ is a singular point of $\tilde{X}$ and smooth point of $\tilde{C}$.

Proof
Since $\rho(X)=1$, $\tilde{E}\cap\tilde{C}\neq\varnothing$. Since $K_{\tilde{X}}$ is σ-nef, $\tilde{E}\not\subset\tilde{C}$. Since $\bar{C}$ is a smooth rational curve, $\tilde{E}$ meets $\tilde{C}$ at a single point, say, $\tilde{P}$. Further, $\sigma(\tilde{E})$ meets $\mathrm{Supp}(\Theta)$ outside o. Hence, $\tilde{\Theta}\cdot\tilde{E}>0$. By Lemma 9.5 $K_{\tilde{X}}\cdot\tilde{E}\geq -1$. Since $K_{\tilde{X}}+\tilde{C}+\tilde{\Theta}\equiv 0$, we have $\tilde{C}\cdot\tilde{E}<1$. Hence, $\tilde{C}\cap\tilde{E}$ is a singular point of $\tilde{X}$. Since $(\tilde{X},\tilde{C})$ is dlt, $\tilde{C}\cap\tilde{E}$ is a smooth point of $\tilde{C}$ (see, e.g., [14, Proposition 16.6]). □

PROPOSITION 9.7
We have that $\rho(\tilde{X})=2$ *and* $\tilde{C}$ *is irreducible. Moreover,* $\bar{X}$ *has exactly two singular points on* $\bar{C}$ *and* $I>2$.

Proof
Assume the converse, that is, $\tilde{C}$ is reducible. By Lemma 9.4 the curve $\bar{C}$ is irreducible. Let s be the number of components of $\tilde{C}$. So, $\rho(\tilde{X})=s+1$. Hence, ν contracts $s-1$ components of $\tilde{C}$ and exactly one divisor, say, $\tilde{E}$ such that $\tilde{E}\not\subset\tilde{C}$. By Lemma 9.6 the curve $\tilde{E}$ is contracted on the first step. Note that $\tilde{C}$ is a chain $\tilde{C}_1+\cdots+\tilde{C}_s$, where both $\tilde{C}_1$ and $\tilde{C}_s$ contain two points of type A_1 and the middle curves $\tilde{C}_2,\dots,\tilde{C}_{s-1}$ are contained in the smooth locus. By Lemma 9.6 we may assume that $\tilde{E}$ meets $\tilde{C}_1$. Then ν contracts $\tilde{C}_1,\dots,\tilde{C}_{s-1}$. However, $\tilde{C}_s$ contains two points of type A_1, and it is not contracted. Thus, $\bar{X}$ has two singular points of type A_1 on $\bar{C}$. Again by Lemma 9.4 the surface $\bar{X}$ has no other singular points on $\bar{C}$. In particular, $2\bar{C}$ is Cartier, $\bar{X}$ has only singularities of type T, and $K_{\bar{X}}^2$ is an integer. On the other hand, we have $-K_{\bar{X}}=m\bar{C}+\bar{D}$, $m\geq 2$. By the adjunction formula

$$-1=\deg\big(K_{\bar{C}}+\mathrm{Diff}_{\bar{C}}(0)\big)=(K_{\bar{X}}+\bar{C})\cdot\bar{C}=-\bar{D}\cdot\bar{C}-(m-1)\bar{C}^2.$$

This gives us $\bar{D}\cdot\bar{C}=\bar{C}^2=1/2$, $m=2$, and $K_{\bar{X}}^2=9/2$, a contradiction.

Finally, by Lemmas 9.4 and 9.6 the surface $\tilde{X}$ (resp., $\bar{X}$) has exactly three (resp., two) singular points on $\tilde{C}$. □

By Theorem 1.2 the surface $\bar{X}$ has at least one non-Du Val singularity lying on $\bar{C}$. Thus, Theorem 1.1 is implied by the following.

PROPOSITION 9.8
We have that $\bar{X}$ *has only Du Val singularities on* $\bar{C}$.

Proof
Assume that the singularities of $\bar{X}$ at points lying on $\bar{C}$ are of types $\frac{1}{n_1}(1,1)$ and $\frac{1}{n_2}(1,1)$ with $n_1\geq n_2$ and $n_1>2$. In this case near $\bar{C}$ the divisor $H:=-(K_{\bar{X}}+2\bar{C})$ is Cartier. By the adjunction formula

$$K_{\bar{C}}+\mathrm{Diff}_{\bar{C}}(0)=(K_{\bar{X}}+\bar{C})|_{\bar{C}}=-(H+\bar{C})|_{\bar{C}}.$$

Hence,

$$\deg\mathrm{Diff}_{\bar{C}}(0)<2-H\cdot\bar{C}\leq 1.$$

In particular, $\bar{X}$ has at most one singular point on $\bar{C}$, a contradiction. □

Acknowledgments. I thank Brendan Hassett whose questions encouraged me to write up my computations. The questions were asked during Simons Symposia "Geometry Over Nonclosed Fields, 2016." I am grateful to the organizers of this activity for the invitation and creative atmosphere. I also would like to thank the referee for the careful reading and numerous helpful comments and suggestions.

The author's work was partially supported by the Russian Foundation for Basic Research grants 15-01-02164 and 15-01-02158, by the Russian Academic Excellence Project '5-100', and by the Russian Science Foundation grant project 14-21-00053 dated 11.08.14.

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