Homological dimensions of rigid modules

Majid Rahro Zargar, Olgur Celikbas, Mohsen Gheibi, and Arash Sadeghi

Abstract We obtain various characterizations of commutative Noetherian local rings $(R, \mathfrak{m})$ in terms of homological dimensions of certain finitely generated modules. Our argument has a series of consequences in different directions. For example, we establish that R is Gorenstein if the Gorenstein injective dimension of the maximal ideal $\mathfrak{m}$ of R is finite. Moreover, we prove that R must be regular if a single $\mathsf{Ext}^n_R(I, J)$ vanishes for some integrally closed $\mathfrak{m}$-primary ideals I and J of R and for some positive integer n.

1. Introduction

Throughout this article, R is a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and all modules over R are assumed to be finitely generated. It is well known that the projective dimension of an R-module M is determined by the vanishing of $\mathsf{Ext}^n_R(M, k)$; that is, if $\mathsf{Ext}^n_R(M, k) = 0$ for some positive integer n, then $\mathsf{pd}(M) \leq n - 1$. In fact, $\mathsf{pd}(M) = \sup\{i \in \mathbb{Z} : \mathsf{Ext}^i_R(M, k) \neq 0\}$. Furthermore, it follows from classical theorems of Auslander, Buchsbaum, and Serre that the finiteness of the projective or the injective dimension of the residue field k characterizes the ring itself: R is regular if $\mathsf{pd}(k) < \infty$ or $\mathsf{id}(k) < \infty$ (see [11, Theorem 2.2.7 and Exercise 3.1.26]).

The main task in this article is to introduce a class of modules, called *rigid-test* modules, that replace the residue field k in the aforementioned classical results (see (2.3) for the definition). A special case of our main result, Theorem 5.8, can be summarized as follows (see also Corollaries 6.1 and 6.11).

THEOREM 1.1

Let $(R, \mathfrak{m})$ be a local ring, and let M and N be nonzero R-modules. Assume that N is a rigid-test module (e.g., $N = k$).

(i) *If $\mathsf{Ext}^n_R(M, N) = 0$ for some $n \geq \mathsf{depth}(N)$, then $\mathsf{pd}(M) \leq n - 1$.*
(ii) $\mathsf{pd}(M) = \sup\{i \in \mathbb{Z} : \mathsf{Ext}^i_R(M, N) \neq 0\}$.
(iii) *If $\mathsf{pd}(N) < \infty$ or $\mathsf{id}(N) < \infty$, then R is regular.*

Kyoto Journal of Mathematics, Vol. 58, No. 3 (2018), 639–669
First published online June 19, 2018.
DOI 10.1215/21562261-2017-0033,
Received February 19, 2016. Revised December 8, 2016. Accepted December 9, 2016.
2010 Mathematics Subject Classification: Primary 13D07; Secondary 13D05, 13H10.

To motivate our approach, let us note that Corso, Huneke, Katz, and Vasconcelos [18, Corollary 3.3] established that integrally closed $\mathfrak{m}$-primary ideals are rigid-test modules (see (A.2)). As an immediate consequence of Theorem 1.1, we have the following result.

COROLLARY 1.2
Let $(R,\mathfrak{m})$ be a local ring of positive depth, and let I and J be integrally closed $\mathfrak{m}$-primary ideals of R. If $\mathsf{Ext}_R^n(I,J)=0$ for some $n\geq 1$, then R is regular.

Our argument has applications in several directions. We use Theorem 1.1 and deduce the following characterization of regularity from a beautiful result of Avramov, Hochster, Iyengar, and Yao [10, Theorem 1.1] (see (6.7) and Corollary 6.8).

COROLLARY 1.3
Let R be a complete local ring of prime characteristic p with a perfect residue field, and let M and N be nonzero R-modules with $\mathsf{Ext}_R^i({}^{\varphi^n}M,N)=0$ for some $i\geq \mathsf{depth}(N)$ and $n\geq 1$. If N is a rigid-test module, then R is regular.

Here ${}^{\varphi^n}M$ is the R-module M with the R-action given by the nth iterate of the Frobenius endomorphism φ (see (A.3)). As an example, we note that nonzero modules of infinite projective dimension are rigid-test over $R=\mathbb{F}_p[\![x,y,z]\!]/(xy-z^2)$, with p being an odd prime, and thus, Corollary 1.3 implies that $\mathsf{Ext}_R^{n+1}({}^{\varphi^e}M,{}^{\varphi^r}N)\neq 0$ for all positive integers e,n,r and for all nonzero R-modules M,N (see (A.4) and (A.6)).

Theorem 1.1 determines the Gorensteinness of R via the *Gorenstein injective dimension*, a refinement of the usual injective dimension introduced by Enochs and Jenda [23]. We prove in Corollary 7.6 that R is Gorenstein if the Gorenstein injective dimension $\mathsf{Gid}(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ of R is finite. This, combined with the results in the literature, seems to give a fairly complete picture: R is Gorenstein if and only if at least one of the dimensions $\mathsf{Gid}(\mathfrak{m})$, $\mathsf{Gid}(R)$, or $\mathsf{Gid}(k)$ is finite (see also (7.7) and Avramov's remark following Question 7.1).

A rigid-test module is, by definition, Tor-rigid (see [1]) and a test module (for projectivity) in the sense of [14] (see (2.3)). Among those already discussed, there are quite a few motivations to study test and rigid-test modules: it was established in [14, Corollary 3.7] that if the dualizing module of a Cohen–Macaulay ring is a test module, then there are no nonfree totally reflexive modules. Proposition 4.11 extends [14, Corollary 3.7] and establishes that there are no nonfree totally reflexive modules if there exists a nonzero test module over R—not necessarily maximal Cohen–Macaulay—of finite injective dimension. Another motivation for us to introduce rigid-test modules comes from the fact that the hypothesis—N is a test module—in Theorem 1.1 cannot be dropped in general. Tor-rigidity has remarkable consequences (see [1], [21]), but if N is a Tor-rigid module, which is not a test module (i.e., not a rigid-test module), then the vanishing of $\mathsf{Ext}_R^n(M,N)$, even for all $n \gg 0$, does not necessarily force M to have finite projective dimension in general (see Theorem 1.1 and Example 6.3).

We make various observations in Sections 3 and 4 and prove our main result, Theorem 5.8, in Section 5. Sections 6 and 7 are devoted to applications of our argument. We also collect some examples of test and rigid-test modules from the literature in the Appendix.

2. Definitions

2.1

An R-module M is said to be *Tor-rigid* provided that the following holds for all R-modules N (see [1]):

$$\text{if } \operatorname{Tor}_n^R(M,N)=0 \text{ for some } n\geq 1, \quad \text{then } \operatorname{Tor}_{n+1}^R(M,N)=0.$$

The notion of Tor-rigidity was initially used in the study of the Koszul complex; it was later formulated and analyzed for modules by Auslander [1]. An interesting result of Lichtenbaum [42, Theorem 3] shows that modules over regular local rings and those of finite projective dimension over hypersurfaces—quotients of power series rings over fields—are Tor-rigid.

2.2

An R-module M is said to be a *test module for projectivity* provided that the following holds for all R-modules N (see [14, Definition 1.1]):

$$\text{if } \mathsf{pd}(N)=\infty, \quad \text{then } \operatorname{Tor}_n^R(M,N)\neq 0 \text{ for infinitely many integers } n.$$

We will call a test module for projectivity simply a *test module*.

Motivated by a question of Lichtenbaum [42, page 226, question 4], we define the following.

2.3

A Tor-rigid test module is called a *rigid-test module*. More precisely, M is called a rigid-test module provided that the following holds for all R-modules N:

$$\text{if } \operatorname{Tor}_n^R(M,N)=0 \text{ for some } n\geq 1, \quad \text{then } \operatorname{Tor}_{n+1}^R(M,N)=0 \text{ and } \mathsf{pd}(N)<\infty.$$

Dao, Li, and Miller [22] defined *strong rigidity* to study the Tor-rigidity of the Frobenius endomorphism over Gorenstein rings.

2.4

An R-module M is said to be *strongly rigid* provided that the following holds for all R-modules N (see [22, Definition 2.1]):

$$\text{if } \operatorname{Tor}_n^R(M,N)=0 \text{ for some } n\geq 1, \quad \text{then } \mathsf{pd}(N)<\infty.$$

It follows from the definition that rigid-test modules are strongly rigid, but we do not know whether the converse is true in general. Most of our results work for strongly rigid modules. However, to obtain the conclusion of Theorem 1.1(i) when $n \geq \mathsf{depth}(N)$, we need Tor-rigidity (see Theorem 5.8 and Corollary 6.1). This leads us to pose the following question for further study.

QUESTION 2.5

Let R be a local ring, and let M be an R-module. If M is strongly rigid, then must M be a rigid-test module or, equivalently, must M be Tor-rigid?

We give some relations between the above definitions in diagram form:

$$
\begin{tikzcd}
\text{Tor-rigid} \arrow[r, Rightarrow, shift left, "(1)"] \arrow[d, Rightarrow, shift left, "/" marking, "(3)"] & \text{test} \arrow[l, Rightarrow, shift left, "/" marking, "(2)"] \arrow[d, Rightarrow, shift left, "/" marking, "(6)"] \\
\text{rigid-test} \arrow[u, Rightarrow, shift left, "(4)"] \arrow[r, Rightarrow, shift left, "(7)"] & \text{strongly rigid} \arrow[u, Rightarrow, shift left, "(5)"] \arrow[l, Rightarrow, shift left, "?"]
\end{tikzcd}
$$

The implications in the diagram can be justified as follows. For (1) and (3), see Example 6.3. For (2) and (6), see Example 6.4. For (4), (5), and (7), these follow from the definitions (see (2.1), (2.3), and (2.4)).

3. Projective and injective dimensions via rigid modules

Let R be a local ring. If N is a nonzero rigid-test module over R, then the vanishing of $\mathsf{Tor}_i^R(N,N)$ is not mysterious at all: it follows from the definition—unless R is regular—that $\mathsf{Tor}_i^R(N,N) \neq 0$ for all $i \geq 0$ (see (2.3)). Hence, it seems interesting to consider the vanishing of $\mathsf{Tor}_i^R(M,N)$ when N is a rigid-test module and M is an arbitrary R-module. In particular, we seek to find whether the vanishing of $\mathsf{Tor}_i^R(M,N)$ for all $i \gg 0$ yields the exact value of the projective dimension of M. Auslander remarked that if $\mathsf{depth}(N) = 0$ and $\mathsf{pd}(M) = s < \infty$, then $\mathsf{Tor}_s^R(M,N) \neq 0$ (see [1, Proposition 1.1]). Therefore, an immediate observation is the following.

3.1

If R is a local ring and N is a test module such that $\mathsf{depth}(N) = 0$, then it follows that $\mathsf{pd}(M) = \sup\{i \in \mathbb{Z} : \mathsf{Tor}_i^R(M,N) \neq 0\}$ (see (2.2)).

A rigid-test module of positive depth does not necessarily detect the exact value of the projective dimension via the vanishing of Tor in general (cf. Theorem 1.1).

EXAMPLE 3.2

Let $R = k[\![x,y,z]\!]/(xy)$, let $T = R/(x)$, and let $N = T \oplus \Omega T = R/(x) \oplus R/(y)$. Then $\mathsf{depth}(N) = 2$ and N is Tor-rigid (see [48, Corollary 1.9]). Moreover, since

$\mathsf{pd}(N) = \infty$, it follows that N is a rigid-test module (see (A.4)). Setting $M = R/(z)$, we see that $1 = \mathsf{pd}(M) \neq \sup\{i \in \mathbb{Z} : \mathsf{Tor}_i^R(M,N) \neq 0\} = 0$.

If N is a rigid-test module that is not necessarily of depth zero, Proposition 3.3 can be useful to detect the projective dimension of M (see also Remark 3.4). In the following, $\mathsf{q}^{\mathsf{syz}}(N)$ denotes the largest integer n for which N can be an nth syzygy module in a minimal free resolution of an R-module. The assumption that $\mathsf{depth}(N) = \mathsf{q}^{\mathsf{syz}}(N)$ in Proposition 3.3 holds, for example, when N is reflexive and $N_{\mathfrak{p}}$ is free for all prime ideals $\mathfrak{p}$ of R with $\mathfrak{p} \neq \mathfrak{m}$ (see [24, Corollary 3.9]).

PROPOSITION 3.3

Let R be a local ring, and let M and N be nonzero R-modules. Assume that N is a rigid-test module and that $\mathsf{q}^{\mathsf{syz}}(N) = \mathsf{depth}(N) \leq \mathsf{pd}(M)$. Then

$$\sup\big\{i \in \mathbb{Z} : \mathsf{Tor}_i^R(M,N) \neq 0\big\} = \mathsf{pd}(M) - \mathsf{depth}(N).$$

Proof

We may suppose that $\mathsf{pd}(M) < \infty$ (see (2.3)). Set $\mathsf{pd}(M) = n$, $\mathsf{depth}(N) = t$, and $q = \sup\{i \in \mathbb{Z} : \mathsf{Tor}_i^R(M,N) \neq 0\}$. We proceed by induction on t and prove that $\mathsf{Tor}_i^R(M,N) \neq 0$ for all $i = 0, \dots, n-t$. Note that, since N is Tor-rigid, it is enough to show that $\mathsf{Tor}_{n-t}^R(M,N) \neq 0$.

If $t = 0$, then it follows from Auslander's remark that $\mathsf{Tor}_n^R(M,N) \neq 0$ (see (3.1)). Hence, suppose that $t \geq 1$, and pick a nonzero divisor x on N. Then one can see that there is a long exact sequence of the form

$$\cdots \to \mathsf{Tor}_{n-t+1}^R(M,N) \to \mathsf{Tor}_{n-t+1}^R(M,N/xN) \to \mathsf{Tor}_{n-t}^R(M,N) \to \cdots.$$

It is easy to see that N/xN is rigid-test over R (see [14, Proposition 2.2]). Thus, the induction hypothesis yields $\mathsf{Tor}_{n-t+1}^R(M,N/xN) \neq 0$. Therefore, $\mathsf{Tor}_{n-t}^R(M,N) \neq 0$. In particular, we have $q \geq n-t$.

Let $\mathfrak{p} \in \mathrm{Ass}(\mathsf{Tor}_q^R(M,N))$. Then the depth formula [1, Theorem 1.2] implies that

$$\text{(3.3.1)} \quad \mathsf{pd}(M_{\mathfrak{p}}) - \mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \mathsf{depth}(R_{\mathfrak{p}}) - \mathsf{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = q.$$

If $\mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq \mathsf{depth}(N)$, then it follows that

$$q = \mathsf{pd}(M_{\mathfrak{p}}) - \mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \leq \mathsf{pd}(M) - \mathsf{depth}(N) = n - t.$$

This shows that $q = n - t$ and, hence, completes the proof.

Next suppose that $\mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \mathsf{depth}(N) = t$. Since $\mathsf{depth}(N) = \mathsf{q}^{\mathsf{syz}}(N)$, we know that N is a tth syzygy module. This implies that $\mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq \min\{t, \mathsf{depth}\, R_{\mathfrak{p}}\}$ (see [11, Exercise 1.3.7]). Hence, $\mathsf{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \geq \mathsf{depth}\, R_{\mathfrak{p}}$. Therefore, by (3.3.1), we deduce that $q = 0$. Now the fact that $\mathsf{Tor}_{n-t}^R(M,N) \neq 0$ yields $n - t = 0$; that is, $q = n - t$. $\square$

REMARK 3.4

Jorgensen [34, Theorem 2.2] proved that if M is a module over a local ring

R with $\mathsf{pd}(M) < \infty$, then $q^R(M,N) = \sup\{i \in \mathbb{Z} : \mathsf{Tor}_i^R(M,N) \neq 0\}$ is equal to $\sup\{\mathsf{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) - \mathsf{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}(M \otimes_R N)\}$. Therefore, if $\mathsf{pd}(M) < \infty$, one can deduce from Jorgensen's result that $q^R(M,N) \geq \mathsf{pd}(M) - \mathsf{depth}(N)$.

In the case in which $\mathsf{q}^{\mathsf{syz}}(N) = \mathsf{depth}(N) \leq \mathsf{pd}(M) < \infty$, Proposition 3.3 establishes the equality $q^R(M,N) = \mathsf{pd}(M) - \mathsf{depth}(N)$ without appealing to [34, Theorem 2.2]. Note that, by Example 3.2, the hypothesis $\mathsf{depth}(N) \leq \mathsf{pd}(M)$ is required, but we do not know whether the condition $\mathsf{q}^{\mathsf{syz}}(N) = \mathsf{depth}(N)$ is essential.

If $(R, \mathfrak{m}, k)$ is a local ring and N is a nonzero R-module, then the k-vector spaces $\mathsf{Ext}_R^n(k,N)$ are nonzero for all n, where $\mathsf{depth}(N) \leq n \leq \mathsf{id}(N)$ (see [50, Theorem 2]). In other words, if $\mathsf{Ext}_R^n(k,N) = 0$ for some $n \geq \mathsf{depth}(N)$, then $\mathsf{id}(N) < \infty$. Since k is strongly rigid, this leads us to pose the following question (see also (2.4)).

QUESTION 3.5

Let R be a local ring, and let M and N be nonzero R-modules. Suppose that M is strongly rigid and $\mathsf{Ext}_R^n(M,N) = 0$ for some $n \geq \mathsf{depth}(N)$. Then must N have finite injective dimension?

In the case in which M is a test module (not necessarily strongly rigid) and R has a dualizing complex (i.e., R is a homomorphic image of a Gorenstein ring), it follows from [14, Theorem 3.2] that $\mathsf{id}(N) < \infty$ if and only if $\mathsf{Ext}_R^i(M,N)$ vanishes for all $i \gg 0$. Here our aim is to examine the case in which M is strongly rigid and a single $\mathsf{Ext}_R^n(M,N)$ vanishes for some $n \geq \mathsf{depth}(N)$. In Proposition 3.6 we obtain a partial affirmative answer to Question 3.5 over Cohen–Macaulay rings. This, in particular, gives an affirmative answer when R is Artinian (see Corollary 3.8).

PROPOSITION 3.6

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a dualizing module, and let M and N be nonzero R-modules. Suppose that the following hold.

(i) $\mathsf{pd}_{R_\mathfrak{p}}(M_p) < \infty$ *for all* $p \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ *(e.g., R has an isolated singularity).*

(ii) $\mathsf{Ext}_R^j(M,N) = 0$ *for some* $j \geq \dim(R) + 1$.

(iii) *M is strongly rigid.*

Then $\mathsf{id}(N) < \infty$.

Proof

As R has a dualizing module, we can consider a maximal Cohen–Macaulay approximation of N, that is, a short exact sequence of R-modules

$$0 \to Y \to C \to N \to 0, \tag{3.6.1}$$

where C is maximal Cohen–Macaulay and $\mathsf{id}(Y) < \infty$. Set $n = j - d$ where $d = \dim(R)$. Applying $\mathsf{Hom}_R(M, -)$ to (3.6.1), we get the following long exact sequence:

$$\cdots \to \mathsf{Ext}^j_R(M,Y) \to \mathsf{Ext}^j_R(M,C) \to \mathsf{Ext}^j_R(M,N) \to \cdots . \tag{3.6.2}$$

Note that $\mathsf{Ext}^j_R(M,Y) = 0$. So it follows from (3.6.2) and (ii) that $\mathsf{Ext}^j_R(M,C) = 0$. Observe, by (3.6.1), that $\mathsf{id}(C) < \infty$ if and only if $\mathsf{id}(N) < \infty$. Therefore, we may assume that N is maximal Cohen–Macaulay. Consider the following standard spectral sequence:

$$E_2^{p,q} = \mathsf{Ext}^p_R\big(\mathsf{Tor}^R_q(N^\dagger, M), \omega\big) \quad \Longrightarrow \quad H^{p+q} = \mathsf{Ext}^{p+q}_R(M,N).$$

Here $N^\dagger = \mathsf{Hom}(N, \omega)$ and ω is the dualizing module of R. Observe that $\mathsf{Tor}^R_q(M, N^\dagger)$ has finite length for all $q \geq 1$: this follows from (i) and the fact that $N^\dagger$ is maximal Cohen–Macaulay (see [58, Lemma 2.2]). Therefore, $E_2^{p,q} = 0$ if $q \geq 1$ and $p \neq d$. Furthermore,

$$\mathsf{Ext}^j_R(M,N) = H^j \cong E_2^{d,n} = \mathsf{Ext}^d_R\big(\mathsf{Tor}^R_n(M, N^\dagger), \omega\big).$$

Now, by (ii), the local duality theorem [11, Corollary 3.5.11(b)] yields that $\mathsf{Tor}^R_n(M, N^\dagger) = 0$. Thus, (iii) gives the required conclusion (see also (2.4)). ☐

We will use the following observation several times (see Corollaries 6.1 and 6.13).

3.7

Let R be a local ring, and let M and N be nonzero R-modules. Suppose that $\mathsf{Ext}^n_R(M,N) = 0$ for some $n \geq \mathsf{depth}(N)$. If $\mathsf{depth}(N) = 0$, then $\mathsf{Hom}(M,N) \neq 0$ (see, e.g., [11, Proposition 1.2.3]). Therefore, n is positive.

COROLLARY 3.8

Let R be an Artinian ring, and let M and N be nonzero R-modules. If M is strongly rigid and $\mathsf{Ext}^n_R(M,N) = 0$ for some $n \geq 0$, then N is injective.

Proof

In view of (3.7), the required result follows from Proposition 3.6. ☐

If R is an Artinian hypersurface—that is, the quotient of a power series ring over a field, M is an R-module of infinite projective dimension, and $\mathsf{Ext}^n_R(M,N) = 0$ for some $n \geq 0$—then [8, Proposition 5.12] and [12, Corollary 4.7] show that N is injective. One can recover this result from Corollary 3.8, since each module of infinite projective dimension is strongly rigid over such an Artinian hypersurface (see (A.4) and (A.6)).

COROLLARY 3.9

Let R be a d-dimensional excellent Cohen–Macaulay local ring, let N be a nonzero

R-module, and let I be an integrally closed $\mathfrak{m}$-primary ideal of R. Suppose that $\operatorname{Ext}^n_R(I,N)=0$ for some $n \geq d$. Then $\operatorname{id}(N)<\infty$.

Proof
It follows that $I \otimes_R \widehat{R}$ is an integrally closed $\mathfrak{m}\widehat{R}$-primary ideal of $\widehat{R}$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R (see, e.g., [31, Corollary 19.2.5]). So we may assume that R is complete with a dualizing module. Note that, by (3.7), we have $n \geq 1$. Moreover, I is strongly rigid, and $I_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ (see (A.2)). Thus, if $\operatorname{Ext}^n_R(I,N)=0$ for some $n \geq d$, then Proposition 3.6 implies that $\operatorname{id}(N)<\infty$. □

COROLLARY 3.10
Let R be a d-dimensional complete Cohen–Macaulay local ring. If $\operatorname{Ext}^n_R(I,R)=0$ for some $n \geq d$ and some integrally closed $\mathfrak{m}$-primary ideal I of R, then R is Gorenstein.

It was posed in [14, p. 303] whether or not the test property is preserved under completion (see (2.2)). An affirmative answer has been recently obtained in [15].

3.11
Let R be a local ring, and let M be a nonzero R-module. Then M is a test module over R if and only if $M \otimes_R \widehat{R}$ is a test module over the $\mathfrak{m}$-adic completion $\widehat{R}$ of R (see [15]).

In light of (3.11), the excellent hypothesis on R in Corollary 3.9 can be removed provided there is an affirmative answer to the following longstanding open problem.

QUESTION 3.12
Let R be a local ring. If M is a Tor-rigid module over R, then must $M \otimes_R \widehat{R}$ be Tor-rigid over $\widehat{R}$?

REMARK 3.13
It is established in Corollary 3.9 that if $\operatorname{Ext}^{n+1}_R(R/I,N)=0$, then $\operatorname{id}(N)<\infty$. Since R/I has finite length, it is worth noting that the vanishing of $\operatorname{Ext}^n_R(M,N)$ for an arbitrary R-module M of finite length does not necessarily force N to have finite injective dimension in general. To see this we can use any module M that has finite projective dimension. For example, if $R=k[\![x,y]\!]/(xy)$, $M=R/(x+y)$, and $N=k$, then $\operatorname{Ext}^i_R(M,N)=0$ for all $i \geq 2$, but $\operatorname{id}(N)=\infty$.

4. Auslander's transpose and remarks on Tor-rigidity

4.1
Let M be an R-module with a projective presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$. Then the *transpose* $\operatorname{Tr}M$ of M is the cokernel of $f^*=\operatorname{Hom}_R(f,R)$ and, hence, is given

by the exact sequence $0 \to M^* \to P_0^* \to P_1^* \to \mathsf{Tr}M \to 0$ (see [3]). Note that the transpose $\mathsf{Tr}M$ is well defined up to projective summands. If n is a positive integer, then $\mathcal{T}_n M$ denotes the transpose of the $(n-1)$st syzygy of M; that is, $\mathcal{T}_n M = \mathsf{Tr}\Omega^{n-1}M$.

There are exact sequences of functors (see [3, Theorem 2.8])

(i) $\mathsf{Tor}_2^R(\mathcal{T}_{n+1}M, -) \to \big(\mathsf{Ext}_R^n(M,R) \otimes_R -\big) \to \mathsf{Ext}_R^n(M,-) \twoheadrightarrow \mathsf{Tor}_1^R(\mathcal{T}_{n+1}M,-)$,

(ii) $\mathsf{Ext}_R^1(\mathcal{T}_{n+1}M,-) \hookrightarrow \mathsf{Tor}_n^R(M,-) \to \mathsf{Hom}\big(\mathsf{Ext}_R^n(M,R),-\big) \to \mathsf{Ext}_R^2(\mathcal{T}_{n+1}M,-)$.

The next useful fact was initially addressed by Auslander [2]. Auslander's original remark is for regular local rings, but in the case in which Tor-rigidity holds, it also holds over arbitrary local rings (see (2.1)).

4.2

Let M and N be nonzero R-modules. Suppose that N is Tor-rigid. Assume further that $\mathsf{Ext}_R^n(M,N) = 0$ for some nonnegative integer n. It follows from (4.1)(i) that $\mathsf{Tor}_1^R(\mathcal{T}_{n+1}M,N) = 0$. This implies, since N is Tor-rigid, that $\mathsf{Tor}_i^R(\mathcal{T}_{n+1}M,N) = 0$ for all $i \geq 1$. We can now use (4.1)(i) once more and conclude that $\mathsf{Ext}_R^n(M,R) \otimes_R N = 0$. Therefore, $\mathsf{Ext}_R^n(M,R) = 0$. In particular, if $M = N$, then $\mathsf{Tor}_1^R(\mathsf{Tr}\Omega^n M, \Omega^n M) = 0$ so that [59, Lemma 3.9] implies $\Omega^n M$ is free; that is, $\mathsf{pd}(M) \leq n-1$ (see [2, Corollary 6]; see also [37]).

As ${}^{\varphi}R$ is Tor-rigid over complete intersection rings, we deduce the following.

4.3

Suppose that R is an F-finite local complete intersection ring with prime characteristic p. If $\mathsf{Ext}_R^i({}^{\varphi^n}R, {}^{\varphi^n}R) = 0$ for some positive integers i and n, then it follows from (A.3)(iii) and (4.2) that R is regular (cf. Corollary 1.3).

One can find remarkable applications of (4.2) in the literature. For example, Jorgensen [35, Proposition 2.5] proved that if R is a complete intersection ring and M is an R-module such that $\mathsf{Ext}_R^2(M,M) = 0$, then $\mathsf{pd}(M) \leq 1$; (4.2) plays an important role in Jorgensen's proof. On the other hand, Dao [20] exploited (4.2) and obtained new results on the noncommutative crepant resolutions.

We give two applications of Auslander's rigidity result recorded in (4.2). The first one, (4.5), is an immediate observation, although it will be quite useful later (see the proof of Proposition 4.11). Our second application is given in Proposition 4.6: it yields a characterization of Cohen–Macaulay rings in terms of Tor-rigidity. We proceed by recalling a remarkable result of Foxby.

4.4

R is Gorenstein if and only if there exists a nonzero finitely generated R-module M such that $\mathsf{pd}(M) < \infty$ and $\mathsf{id}(M) < \infty$ (see [11, Exercise 3.1.25]).

4.5

Let M be a nonzero Tor-rigid R-module. If $\mathsf{id}(M) < \infty$, then it follows from (4.2) that $\mathsf{pd}(M) < \infty$ and hence, by (4.4), R is Gorenstein.

The Tor-rigidity hypothesis in (4.5) cannot be replaced with the test property: a local ring admitting a nonzero test module of finite injective dimension is not necessarily Gorenstein; however, such a ring R is G-regular (see [53]); that is, $\mathsf{G\text{-}dim}(M) = \mathsf{pd}(M)$ for all R-modules M (see (2.2), Example 6.4, and Corollary 4.12).

The *grade* of a pair of nonzero modules (M,N), denoted by $\operatorname{grade}(M,N)$, is defined as $\inf\{i \in \mathbb{N} \cup \{0\} : \mathsf{Ext}_R^i(M,N) \neq 0\}$. Setting $\operatorname{grade}(M) = \operatorname{grade}(M,R)$, we see that $\operatorname{grade}(M) < \infty$.

PROPOSITION 4.6

Let R be a local ring, and let M and N be nonzero R-modules. Suppose that N is Tor-rigid. Set $\operatorname{grade}(M) = n$ and $\operatorname{grade}(M,N) = s$. Then $s \leq n$, and $\mathsf{Ext}_R^i(M,N) \neq 0$ for all $i = s, \dots, n$.

Proof

We have, by definition, that $\mathsf{Ext}_R^n(M,R) \neq 0$. If $\mathsf{Ext}_R^n(M,N) = 0$, then it follows from (4.2) that $\mathsf{Ext}_R^n(M,R) = 0$, which is a contradiction. Therefore, $\mathsf{Ext}_R^n(M,N) \neq 0$ and hence $s \leq n$. Now suppose that $\mathsf{Ext}_R^i(M,N) = 0$ for some $s < i < n$. Set $r = \min\{j \in \mathbb{Z} : \mathsf{Ext}_R^j(M,N) = 0 \text{ with } s < j < n\}$. We know, since $r < n$, that $\mathsf{Ext}_R^r(M,R) = 0$. Therefore, $\mathcal{T}_r M$ is stably isomorphic to $\Omega \mathcal{T}_{r+1} M$. Moreover, it follows from (4.2) that $\mathsf{Tor}_i^R(\mathcal{T}_{r+1}M, N) = 0$ for all $i \geq 1$. This implies that $\mathsf{Tor}_i^R(\mathcal{T}_r M, N) = 0$ for all $i \geq 1$. Now we use (4.1)(i) and deduce that $\mathsf{Ext}_R^{r-1}(M,N) = 0$. This contradicts the choice of r and finishes the proof. $\square$

Auslander [1, Theorem 4.3] proved that if R is a local ring and N is a nonzero finitely generated Tor-rigid R-module, then every N-regular sequence is R-regular. This shows that a local ring admitting a Tor-rigid maximal Cohen–Macaulay module is Cohen–Macaulay. A consequence of Proposition 4.6 recovers this fact and gives a characterization of the Cohen–Macaulay property of local rings.

COROLLARY 4.7

Let R be a local ring, and let N be a nonzero Tor-rigid R-module.

(i) $\mathsf{depth}(N) \leq \dim(M) + \operatorname{grade}(M)$ *for all nonzero R-modules M.*
(ii) $\mathsf{depth}(N) \leq \mathsf{depth}(R)$.
(iii) *If N is maximal Cohen–Macaulay, then R is Cohen–Macaulay.*

Proof

It follows from Proposition 4.6 that $\operatorname{grade}(M,N) \leq \operatorname{grade}(M)$. On the other hand $\mathsf{depth}(N) - \dim(M) \leq \operatorname{grade}(M,N)$ (see [57, Theorem 2.1(a)]). This implies $\mathsf{depth}(N) \leq \dim(M) + \operatorname{grade}(M)$ and justifies (i). In particular, if $M = k$, we deduce that $\mathsf{depth}(N) \leq \dim(k) + \operatorname{grade}(k) = \mathsf{depth}(R)$ and hence (ii) follows. Note that (iii) is an immediate consequence of (ii). □

It is an open question whether or not every complete Noetherian local ring has a maximal Cohen–Macaulay module: this is known as the small Cohen–Macaulay conjecture. However, in lower dimensions, we can use Corollary 4.7 and observe the following.

4.8
Suppose that R is a local ring that is not Cohen–Macaulay.

(i) If R is one-dimensional and $\mathfrak{p}$ is a minimal prime ideal of R, then $R/\mathfrak{p}$ is a maximal Cohen–Macaulay R-module that is not Tor-rigid. For example, if we put $R = k[\![x,y]\!]/(x^2, xy)$, then $R/(x)$ is not a Tor-rigid R-module. In fact, $\mathsf{Tor}_1^R(R/(x), R/(y)) = 0 \neq \mathsf{Tor}_2^R(R/(x), R/(y))$ (see [42, Question 3]).

(ii) If R is a two-dimensional complete domain, then the integral closure $\overline{R}$ of R in its field of fractions is a (finitely generated) maximal Cohen–Macaulay R-module that is not Tor-rigid. For example, if $R = k[\![x^4, x^3y, xy^3, y^4]\!]$, then $\overline{R} = R[\![x^2y^2]\!]$ is not a Tor-rigid R-module.

One can also find examples of three-dimensional non–Cohen–Macaulay local rings that admit maximal Cohen–Macaulay modules (see, e.g., Hochster [25, 5.4, 5.6, and 5.9]). These modules are not Tor-rigid, for example, by (4.7).

Recall that an R-module C is called *semidualizing* if the natural map $R \to \mathsf{Hom}(C, C)$ is bijective and $\mathsf{Ext}_R^i(C, C) = 0$ for all $i \geq 1$. If C is a semidualizing module such that $\mathsf{id}(C) < \infty$, then R is Cohen–Macaulay and C is dualizing.

4.9
Let C be a semidualizing module over R. Then the *C-projective dimension* $\mathsf{C\text{-}pd}(M)$ of a nonzero R-module M is defined as the infimum of the integers n such that there exists an exact sequence

$$0 \to C^{b_n} \to C^{b_{n-1}} \to \cdots \to C^{b_1} \to C^{b_0} \to M \to 0,$$

where each b_i is a positive integer. It follows that $\mathsf{C\text{-}pd}(M) = \mathsf{pd}(\mathsf{Hom}_R(C, M))$, and the *$C$-injective dimension* $\mathsf{C\text{-}id}(M)$ of M is defined similarly: $\mathsf{C\text{-}id}(M) = \mathsf{id}(C \otimes_R M)$ (see [54, 1.6, 2.8, 2.9, and 2.11]). Note that if $\mathsf{pd}(C) < \infty$, then $C \cong R$ and hence $\mathsf{C\text{-}pd}(N) = \mathsf{pd}(N)$ and $\mathsf{C\text{-}id}(N) = \mathsf{id}(N)$.

4.10
Let M be a nonzero R-module, and let C be a semidualizing R-module. If $\mathsf{C\text{-}pd}(M) < \infty$ (resp., $\mathsf{C\text{-}id}(M) < \infty$), then $\mathsf{Ext}_R^i(C, M) = 0$ for all $i \geq 1$ (resp., $\mathsf{Tor}_i^R(C, M) = 0$ for all $i \geq 1$) (see [54, Corollary 2.9]). In particular, if M is a nonzero test module and $\mathsf{C\text{-}id}(M) < \infty$ for some semidualizing R-module C, then $C \cong R$ and hence $\mathsf{id}(M) < \infty$ (see (2.1) and (4.9)).

PROPOSITION 4.11

Let R be a local ring, and let C be a semidualizing R-module. Suppose that M is a nonzero test module over R. Assume further that $\mathsf{C}\text{-}\mathsf{id}(M) < \infty$. If X is an R-module such that $\mathsf{Ext}^i_R(X,R) = 0$ for all $i \gg 0$, then $\mathsf{pd}(X) < \infty$.

Proof

Suppose that X is an R-module with $\mathsf{Ext}^i_R(X,R) = 0$ for all $i \gg 0$. Note that, by (4.10), we have $\mathsf{id}(M) < \infty$. This yields $\mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(X,R),M) \simeq X \otimes^{\mathbf{L}}_R M$ (see [16, A.4.24]). Therefore, $\mathsf{Tor}^R_i(M,X) = 0$ for all $i \gg 0$ so that $\mathsf{pd}(X) < \infty$. □

It was proved in [14, Corollary 3.7] that if the dualizing module of a Cohen–Macaulay local ring R is a test module, then R is G-regular; that is, $\mathsf{pd}(M) = \mathsf{G}\text{-}\mathsf{dim}(M)$ for all R-modules M (see [53]). A straightforward application of Proposition 4.11 extends this.

COROLLARY 4.12

A local ring admitting a nonzero test module of finite injective dimension is G-regular.

Before we proceed to prove our main result, we give an overview of what has been established so far in terms of the injective dimension of test and rigid modules.

4.13

Let R be a local ring, and let N be a nonzero R-module such that $\mathsf{id}(N) < \infty$.

(i) If N is Tor-rigid, then R is Gorenstein (see (4.5)).

(ii) If N is a test module over R, then R is G-regular (see Corollary 4.12).

(iii) If N is a rigid-test module over R, then it follows from (i) and (ii) that R is regular (see also Question 2.5 and Corollary 6.11).

5. Main theorem

This section is dedicated to a proof of our main result, Theorem 5.8. In the following, $\mathsf{H}\text{-}\mathsf{dim}$ denotes a homological dimension of finitely generated modules (see (5.2) and, e.g., [7, Theorems 8.6–8.8] for details). The special case—where H is the projective dimension pd—is what we really need for the proof of Theorem 1.1, stated in the introduction. However, one can follow our argument word for word by replacing $\mathsf{H}\text{-}\mathsf{dim}$ with projective dimension pd so there is no extra penalty for this generality. Furthermore, such a generality is useful to examine the *Gorenstein dimension* $\mathsf{G}\text{-}\mathsf{dim}$ of Tor-rigid modules (see Corollary 6.13).

5.1

A finitely generated module M over a commutative Noetherian ring R is said to be *totally reflexive* if the canonical map $M \to \mathsf{Hom}(\mathsf{Hom}(M,R),R)$ is bijective and $\mathsf{Ext}^i_R(M,R) = 0 = \mathsf{Ext}^i_R(\mathsf{Hom}_R(M,R),R)$ for all $i \geq 1$ (see [3]).

The infimum of nonnegative integers n for which there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$ such that each X_i is totally reflexive is called the Gorenstein dimension of M. If M has Gorenstein dimension n, we write $\mathsf{G\text{-}dim}(M) = n$. Therefore, M is totally reflexive if and only if $\mathsf{G\text{-}dim}(M) \leq 0$, where it follows by convention that $\mathsf{G\text{-}dim}(0) = -\infty$.

5.2

Throughout we suppose that $\mathsf{H\text{-}dim}$ satisfies the following conditions.

(i) $\mathsf{G\text{-}dim}(M) \leq \mathsf{H\text{-}dim}(M) \leq \mathsf{pd}(M)$ for all R-modules M.

(ii) If $\mathsf{H\text{-}dim}(M) = 0$, then $\mathsf{Tr}M = 0$ or $\mathsf{H\text{-}dim}(\mathsf{Tr}M) = 0$ for all R-modules M.

Although we will not use it, we note that the *complete intersection dimension* (see [9]) is an example of a homological dimension—in general, distinct from the Gorenstein and projective dimensions—that satisfies the conditions in (5.2).

5.3

For our purposes we recall a few properties of $\mathsf{H\text{-}dim}$ (see [7, Theorems 3.1.2, 8.7, and 8.8]).

(i) If one of the dimensions in (5.2)(i) is finite, then it equals the one on its left.

(ii) If $\mathsf{H\text{-}dim}(M) < \infty$, then $\mathsf{H\text{-}dim}(M) = \sup\{i \in \mathbb{Z} : \mathsf{Ext}^i_R(M,R) \neq 0\}$.

(iii) If $\mathsf{H\text{-}dim}(M) < \infty$, then $\mathsf{H\text{-}dim}(M) \leq \mathsf{depth}(R)$.

If X and Y are nonzero R-modules and $\mathsf{H\text{-}dim}$ is a homological dimension of modules, we consider the following condition for (X, Y, H).

5.4

If $\mathsf{Tor}^R_1(X,Y) = 0$, then $\mathsf{H\text{-}dim}(X) = \mathsf{depth}(Y) - \mathsf{depth}(X \otimes_R Y)$.

The condition in (5.4) is not restrictive for rigid modules. For example, Auslander [1, Theorem 1.2] proved that if R is a local ring and X and Y are nonzero R-modules where $\mathsf{pd}(X) < \infty$ and Y is Tor-rigid, then (5.4) holds for (X, Y, pd) (see (2.1)).

Recently, Christensen and Jorgensen [17, Corollary 5.3] established a similar result over AB rings: if R is AB and X and Y are nonzero R-modules either of which is Tor-rigid, then $(X, Y, \mathsf{G\text{-}dim})$ satisfies the condition in (5.4). Recall that a Gorenstein local ring R is said to be AB (see [30]) if, for all R-modules M and N, $\mathsf{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ implies that $\mathsf{Ext}^i_R(M,N) = 0$ for all $i > \dim(R)$. The class of AB rings strictly contains that of complete intersections (see [30]).

Next we summarize the two aforementioned results.

5.5
Let R be a local ring, and let X and Y be nonzero R-modules. Then (X,Y,H) satisfies the condition in (5.4) if at least one of the following conditions holds.

(i) Y is a rigid-test module, and $\mathsf{H\text{-}dim} = \mathsf{pd}$ (see (2.3) and [1, Theorem 1.2]).

(ii) R is AB, X or Y is Tor-rigid, and $\mathsf{H\text{-}dim} = \mathsf{G\text{-}dim}$ (see (2.1) and [17, Corollary 5.3]).

Next is the key result we use for our proof of Theorem 5.8 (see also (4.1)).

5.6 (Auslander and Bridger; see [3, Chapter 2])
Let R be a local ring, let M be a nonzero R-module, and let n be a positive integer. If M is n-torsion-free, then it is n-reflexive (i.e., if $\mathsf{Ext}^i_R(\mathsf{Tr}M,R)=0$ for all $i=1,\dots,n$), then $M \approx \Omega^n_R(\mathcal{T}_{n+1}(\mathsf{Tr}_R M))$; that is, M is isomorphic to $\Omega^n_R(\mathcal{T}_{n+1}(\mathsf{Tr}_R M))$ up to a projective summand.

Recall that $\mathsf{depth}(0)=\infty$.

5.7 (see [3])
Let R be a local ring, and let M be an R-module. Then M is said to satisfy $(\widetilde{\mathsf{S}}_n)$ if $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{n, \mathsf{depth}(R_\mathfrak{p})\}$ for all $\mathfrak{p} \in \operatorname{Supp}(M)$.

If the ring is Cohen–Macaulay, then $(\widetilde{\mathsf{S}}_n)$ coincides with Serre's condition (S_n) (see [24]), but in general $(\widetilde{\mathsf{S}}_n)$ is a weaker condition. For example, if $R = k[\![x,y]\!]/(x^2,xy)$, then, by definition, R satisfies $(\widetilde{\mathsf{S}}_n)$ for all nonnegative integers n but fails to satisfy (S_1) since $\mathsf{depth}(R)=0$ (see also the discussion following [44, Definition 10]).

Theorem 5.8 is a generalization of a result of Jothilingam [38, Corollary 1]. We give the argument in two steps as the conclusion of part (1) may be of independent interest. It is already known that the vanishing of $\mathsf{Ext}^i_R(\mathsf{Tr}M,R)$ for all $i=1,\dots,r$ forces M to be an rth syzygy module and that forces M to satisfy $(\widetilde{\mathsf{S}}_r)$ (see (5.7) and [44, Propositions 11 and 40]).

THEOREM 5.8
Let R be a local ring, and let M and N be nonzero R-modules. Suppose that M satisfies $(\widetilde{\mathsf{S}}_r)$ for some nonnegative integer r.

(1) *If $\mathsf{G\text{-}dim}(\mathsf{Tr}M)<\infty$, then $\mathsf{Ext}^i_R(\mathsf{Tr}M,R)=0$ for all $i=1,\dots,r$.*

(2) *Suppose that $\mathsf{Ext}^n_R(M,N)=0$ for some positive integer n.*

(i) *If N is strongly rigid and $\mathsf{depth}(R) \leq n+r$, then $\mathsf{pd}(M) \leq n-1$.*

(ii) *If N is Tor-rigid, $\mathsf{depth}(N) \leq n+r$, and $(\mathcal{T}_{n+1}M,N,\mathsf{H})$ satisfies the condition in (5.4), then $\mathsf{H\text{-}dim}(M) \leq n-1$.*

Proof
We proceed to prove (1). Suppose that $\mathsf{G\text{-}dim}(\mathsf{Tr}M) < \infty$ and $r \geq 1$. If $\mathsf{G\text{-}dim}(\mathsf{Tr}M) = 0$, then $\mathsf{Ext}^i_R(\mathsf{Tr}M, R) = 0$ for all $i \geq 1$ so that there is nothing to prove. So we suppose that $\mathsf{G\text{-}dim}(\mathsf{Tr}M) \geq 1$ and set $s = \inf\{i \geq 1 : \mathsf{Ext}^i_R(\mathsf{Tr}M, R) \neq 0\}$.

Let $\mathfrak{p} \in \mathrm{Ass}_R(\mathsf{Ext}^s_R(\mathsf{Tr}M, R))$. Then we have

$$\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}\left(\mathsf{Ext}^s_{R_\mathfrak{p}}(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}, R_\mathfrak{p})\right)$$

and that $s = \inf\{i \geq 1 : \mathsf{Ext}^i_{R_\mathfrak{p}}(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}, R_\mathfrak{p}) \neq 0\}$. It follows from the definition of the transpose (see (4.1)) that there is an injection

$$0 \to \mathsf{Ext}^s_{R_\mathfrak{p}}(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}, R_\mathfrak{p}) \hookrightarrow \mathcal{T}_s(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}),$$

which shows that $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}(\mathcal{T}_s(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}))$. Hence, $\mathsf{depth}_{R_\mathfrak{p}}(\mathcal{T}_s(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p})) = 0$. Since $\mathsf{Ext}^i_{R_\mathfrak{p}}(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}, R_\mathfrak{p}) = 0$ for all $i = 1, \dots, s-1$, we conclude from (5.6) that

$$M_\mathfrak{p} \approx \Omega^{s-1}_{R_\mathfrak{p}}\big(\mathcal{T}_s(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p})\big).$$

Note that $s \leq \mathsf{G\text{-}dim}_{R_\mathfrak{p}}(\mathsf{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}) \leq \mathsf{depth}\, R_\mathfrak{p}$. Therefore, $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = s - 1$. Furthermore, since M satisfies $(\widetilde{S}_r)$, it follows that $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{r, \mathsf{depth}\, R_\mathfrak{p}\}$. Hence, $\mathsf{depth}\, R_\mathfrak{p} \geq r + 1$ and $r \leq s - 1$. Consequently, $\mathsf{Ext}^i_R(\mathsf{Tr}M, R) = 0$ for all $i = 1, \dots, r$.

We now proceed to prove (2). If $\mathcal{T}_{n+1}M = 0$, then $\Omega^n M$ is free and hence $\mathsf{pd}(M) \leq n-1$ (see (4.1)). In particular, this implies that $\mathsf{H\text{-}dim}(M) \leq n-1$ (see (5.3)(i)). Therefore, we may suppose that $\mathcal{T}_{n+1}M \neq 0$. Since M satisfies $(\widetilde{S}_r)$, it follows that $\Omega^n M$ satisfies $(\widetilde{S}_{n+r})$ (see (5.7)). Therefore, by the first part of the theorem, we conclude that

$$\text{(5.8.1)} \qquad \mathsf{Ext}^i_R(\mathcal{T}_{n+1}M, R) = 0 \quad \text{for all } i = 1, \dots, n+r.$$

Note that, since $\mathsf{Ext}^n_R(M, N) = 0$, we have by (4.1)(i)

$$\text{(5.8.2)} \qquad \mathsf{Tor}^R_1(\mathcal{T}_{n+1}M, N) = 0.$$

If (i) holds, then it follows from (2.4) and (5.8.2) that $\mathsf{pd}(\mathcal{T}_{n+1}M) < \infty$. Therefore, since $\mathsf{depth}(R) \leq n + r$, we use (5.3)(i) and deduce

$$\text{(5.8.3)} \qquad \mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) = \mathsf{pd}(\mathcal{T}_{n+1}M) \leq \mathsf{depth}(R) \leq n + r.$$

On the other hand, if (ii) holds, then it follows from our assumption that $\mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) = \mathsf{depth}(N) - \mathsf{depth}(\mathcal{T}_{n+1}M \otimes_R N)$. Since $\mathsf{depth}(N) \leq n + r$, we obtain

$$\text{(5.8.4)} \qquad \mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) \leq n + r.$$

Consequently, if either (i) or (ii) holds, then $\mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) \leq n + r$, where for part (i), $\mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) = \mathsf{pd}(\mathcal{T}_{n+1}M)$ (see (5.8.3) and (5.8.4)). Recall that $\mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) = \sup\{i : \mathsf{Ext}^i_R(\mathcal{T}_{n+1}M, R) \neq 0\} < \infty$ (see (5.3)(ii)). Since $\mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) \leq n + r$, it follows from (5.8.1) that $\mathsf{H\text{-}dim}(\mathcal{T}_{n+1}M) = 0$; that is, $\mathsf{H\text{-}dim}(\mathsf{Tr}\Omega^n M) = 0$ (see (4.1)). Thus, by (5.2)(ii), we have $\mathsf{H\text{-}dim}(\Omega^n M) = 0$. This yields that $\mathsf{H\text{-}dim}(M) \leq n$.

If (i) holds, then since $\mathsf{Ext}^n(M,N)=0$, we conclude that $\mathsf{H}\text{-}\mathsf{dim}(M)\leq n-1$ (see, e.g., [45, Chapter 19, Lemma 1(iii)]). On the other hand, if (ii) holds, then since N is Tor-rigid, it follows from (5.8.2) and (4.1)(i) that $\mathsf{Ext}^n(M,R)=0$. Therefore, we see that $\mathsf{H}\text{-}\mathsf{dim}(M)\leq n-1$. □

6. Corollaries of the main theorem

In this section we give various applications of Theorem 5.8 and examine homological dimensions of test and rigid-test modules. Corollary 6.1, a reformulation of Theorem 5.8, is fundamental to our work: it shows that one can use an arbitrary nonzero strongly rigid, or a rigid-test module N, just like the residue field k, to determine the exact value of the projective dimension of M via the vanishing of $\mathsf{Ext}^i_R(M,N)$. Besides this, Corollary 6.1 yields a series of related results. Among those is Corollary 7.6, which proves, in particular, that R is Gorenstein if the Gorenstein injective dimension (see [23]) of the maximal ideal $\mathfrak{m}$ is finite.

Corollary 6.1 is well known for the special case in which $N=k$. Recall that a rigid-test module is, by definition, strongly rigid, but we do not know whether or not all strongly rigid modules are rigid-test (see Question 2.5).

COROLLARY 6.1

Let R be a local ring, and let M and N be nonzero R-modules.

(i) *Suppose that $\mathsf{Ext}^n_R(M,N)=0$ for some $n\geq \mathsf{depth}(R)$. Assume further that N is strongly rigid. Then $\mathsf{pd}(M)=\sup\{i\in\mathbb{Z}:\mathsf{Ext}^i_R(M,N)\neq 0\}\leq n-1$.*

(ii) *Suppose that $\mathsf{Ext}^n_R(M,N)=0$ for some integer $n\geq \mathsf{depth}(N)$. Assume further that N is a rigid-test module. Then $\mathsf{pd}(M)=\sup\{i\in\mathbb{Z}:\mathsf{Ext}^i_R(M,N)\neq 0\}\leq n-1$.*

Proof

Note that, for parts (i) and (ii), it suffices to prove that $\mathsf{pd}(M)$ cannot exceed $n-1$ (see, e.g., [45, Chapter 19, Lemma 1(iii)]).

Suppose (i). Suppose that $n=0$. Then $\mathsf{depth}(R)=0$, and so by Corollary 4.7, we have $\mathsf{depth}(N)=0$. However, this contradicts the fact that $\mathsf{Hom}(M,N)=0$. Thus, $n\geq 1$. Now setting $r=0$ in Theorem 5.8(2)(i), we conclude that $\mathsf{pd}(M)\leq n-1$.

Next assume (ii). Note that, by (3.7), n is a positive integer. Moreover, since N is a rigid-test module, (5.4) holds for $(\mathcal{T}_{n+1}M,N,\mathsf{pd})$ (see (5.5)(i)). Therefore, we obtain the required conclusion by setting $r=0$ in Theorem 5.8(2)(ii). □

The conclusion of Corollary 6.1 is sharp: Example 6.2 shows that the condition on n cannot be removed. Examples 6.3 and 6.4, respectively, highlight the fact that the assumption "N is Tor-rigid" or "N is a test module" is not merely enough to deduce that M has finite projective dimension (see (2.1), (2.2), and (2.3)).

EXAMPLE 6.2

Let k be a field, $R=k[\![x,y,z]\!]/(xy-z^2)$, $M=k$, and $N=\Omega^2 k$. Then N is a rigid-test module, $\mathsf{Ext}^1_R(M,N)=0$, $\mathsf{depth}(N)=2$, and $\mathsf{pd}(M)=\infty$.

EXAMPLE 6.3

Let k be a field, $R=k[\![x,y]\!]/(xy)$, $M=k$, and $N=R/(x+y)$. Then, since $\mathsf{pd}(N)=1$, N is Tor-rigid. Furthermore, since R is not regular, N is not a test module. Consequently, N is not a rigid-test or a strongly rigid module. Note that $\mathsf{pd}(M)=\infty$ and $\mathsf{Ext}^i_R(M,N)=0$ for all $i\geq 2$.

EXAMPLE 6.4

Let k be a field, and put $R=k[\![x,y,z]\!]/(y^2-xz,x^2y-z^2,x^3-yz)$. Let $N=\omega$, the canonical module of R. As R is a one-dimensional domain with minimal multiplicity, it is Golod (see [6, Example 5.2.8]). Hence, since $\mathsf{pd}(N)=\infty$, it follows from (A.4) that N is a test-module. Huneke and Wiegand [32, Example 4.8] proved that there exists an R-module M such that $M\otimes_R N$ is torsion-free and M has torsion. We now follow the proof of [32, Lemma 1.1].

Let $\overline{M}$ be the torsion-free part of M. Then, since R is a domain, there is an exact sequence $0\to\overline{M}\to F\to C\to 0$, where F is a free R-module. Tensoring this short exact sequence with N, we obtain an injection $\mathsf{Tor}_1^R(C,N)\hookrightarrow \overline{M}\otimes_R N$. Since $\overline{M}\otimes_R N\cong M\otimes_R N$ and $\mathsf{Tor}_1^R(C,N)$ is torsion, we see that $\mathsf{Tor}_1^R(C,N)=0$. Note that $\mathsf{pd}(C)=\infty$: otherwise $\overline{M}$ is free, and this would force M to be free. Therefore, N is not a strongly rigid module. For completeness, we also remark that N is not Tor-rigid (see (4.5)). Note that $\mathsf{Ext}^i_R(M,N)=0$ for all $i\geq 2$ and $\mathsf{pd}(M)=\infty$.

Corollary 6.1 can be useful to determine the depth of $\mathsf{Hom}(M,N)$.

COROLLARY 6.5

Let R be a Cohen–Macaulay local ring, and let M and N be nonzero R-modules. Assume that the following conditions hold:

(i) *R has an isolated singularity; that is, $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p}\in\operatorname{Spec}(R)-\{\mathfrak{m}\}$;*
(ii) *M is nonfree and maximal Cohen–Macaulay;*
(iii) *N is strongly rigid with $\mathsf{depth}(N)\geq 2$.*

Then $\mathsf{depth}(\mathsf{Hom}_R(M,N))=2$.

Proof

Note that $\mathsf{depth}(\mathsf{Hom}_R(M,N))\geq\min\{2,\mathsf{depth}(N)\}=2$ (see [11, Exercise 1.4.19]). Thus, it suffices to prove that $\mathsf{depth}(\mathsf{Hom}_R(M,N))\leq 2$. Suppose that this is not so; that is, assume that $\mathsf{depth}(\mathsf{Hom}_R(M,N))\geq 3$. Then, by [33, Lemma 1.1], we see either $\mathsf{Ext}^1_R(M,N)=0$ or $1\leq\mathsf{depth}(\mathsf{Ext}^1_R(M,N))<\infty$. Since $\mathsf{Ext}^1_R(M,N)$ has finite length, it follows that $\mathsf{Ext}^1_R(M,N)=0$. Set $d=\dim(R)$.

Then $M = \Omega^d(X)$ for some finitely generated R-module X (see [40, A.15]). This yields $\mathsf{Ext}_R^{d+1}(X, N) = 0$. Now Corollary 6.1 shows that $\mathsf{pd}(X) < \infty$; that is, M is free. So we conclude that $\mathsf{depth}(\mathsf{Hom}_R(M, N)) = 2$. □

COROLLARY 6.6

Let $A = Q/(f)$, where $Q = k[x_1, \ldots, x_{2s+1}]$, with $s \geq 1$, is a polynomial ring over a perfect field k, and f is a nonconstant polynomial in Q. Set $R = A_{\mathfrak{m}}$, where $\mathfrak{m} = (x_1, \ldots, x_{2s+1})A$, and assume that $f \in \mathfrak{m}$ and $A_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(A) - \{\mathfrak{m}\}$. If M and N are nonfree maximal Cohen–Macaulay R-modules, then $\mathsf{depth}(\mathsf{Hom}_R(M, N)) = 2$.

Proof

In light of (A.4) and (A.8), the required conclusion follows immediately from Corollary 6.5 (cf. [19, Theorem 3.4]). □

Corollary 1.3, stated in the introduction, is now a direct consequence of our argument and the following special case of [10, Theorem 1.1] (see also (A.3)(i)).

6.7

Let R be a local ring of prime characteristic p, and let M be a nonzero finitely generated R-module (see [10, Theorem 1.1]). If ${}^{\varphi^e}M$ has finite flat dimension for some positive integer e, then R is regular.

COROLLARY 6.8

Let R be an F-finite local ring of prime characteristic p, and let N be a nonzero rigid-test module over R. If $\mathsf{Ext}_R^j({}^{\varphi^n}M, N) = 0$ for some nonzero R-module M and for some integers $n \geq 1$ and $j \geq \mathsf{depth}(N)$, then R is regular.

Proof

Note that, as R is F-finite, ${}^{\varphi^e}M$ is a finitely generated R-module. Thus, it follows from Corollary 6.1(ii) that $\mathsf{pd}({}^{\varphi^e}M) < \infty$. Now, by (6.7), R is regular. □

6.9

Proof of Corollary 1.3. As R is complete and k is perfect, it follows that R is F-finite (see, e.g., [11, page 398]). So the result follows from Corollary 6.8.

A special case of Corollaries 6.1 and 6.8 has been established in [49, Theorem B]: if R is a complete intersection ring of prime characteristic p and $\mathsf{Ext}_R^n(M, {}^{\varphi^i}R) = 0$ for some $n \geq \mathsf{depth}(R)$, then $\mathsf{pd}(M) < \infty$ (see (A.3)(iii)). We should note that [49, Theorem B] does not require an F-finite ring and relies on methods different from ours. As discussed in the introduction, our argument is not specific to rings of characteristic p and gives useful information regarding the Frobenius endomorphism even if the ring considered is not a complete intersection. For example, the next result, in view of (A.3)(ii), is immediate from Corollary 6.1 (cf. [49, Theorem A]).

COROLLARY 6.10

Let R be a one-dimensional F-finite Cohen–Macaulay local ring of prime characteristic p, and let M be an R-module. Then $\mathsf{Ext}^n_R(M, {}^{\varphi^i}R) = 0$ for some $n \geq 1$ and some $i \gg 0$ if and only if $\mathsf{pd}(M) < \infty$.

Corollary 6.1 yields a characterization of regularity in terms of C-pd and C-id dimensions of strongly rigid modules (see also (2.4) and (4.9)).

COROLLARY 6.11

Let R be a local ring, let C be a semidualizing R-module, and let M be a nonzero strongly rigid R-module. Assume that either C-$\mathsf{pd}(M) < \infty$ or C-$\mathsf{id}(M) < \infty$. Then R is regular.

Proof

We start by noting that M is a test module (see (2.2)). Assume first that C-$\mathsf{id}(M) < \infty$. Then it follows from (4.10) that $\mathsf{id}(M) < \infty$; that is, $\mathsf{Ext}^i_R(k, M) = 0$ for all $i \gg 0$. Now Corollary 6.1(i) implies that $\mathsf{pd}(k) < \infty$ so that R is regular. Next assume that C-$\mathsf{pd}(M) < \infty$. Then it follows from (4.10) that $\mathsf{Ext}^i_R(C, M) = 0$ for all $i \geq 1$. Hence, we can use Corollary 6.1(i) once more and deduce that $\mathsf{pd}(M) < \infty$. This implies that R is regular. □

A special case of Corollary 6.11 is a characterization of regularity in terms of integrally closed $\mathfrak{m}$-primary ideals (see (A.2)).

COROLLARY 6.12

Let $(R, \mathfrak{m})$ be a local ring, and let I be an integrally closed $\mathfrak{m}$-primary ideal of R. Then R is regular if and only if there exists a semidualizing R-module C such that C-$\mathsf{id}(\Omega^n I) < \infty$ or C-$\mathsf{pd}(\Omega^n I) < \infty$ for some nonnegative integer n. In particular, R is regular if and only if $\mathsf{id}(I) < \infty$.

The conclusions of Corollaries 6.13 and 6.14 are known over complete intersection rings (see [51, Theorem 3.6]). Here we are able to show that these results carry over to AB rings. Recall that every complete intersection ring is AB, but not vice versa (see the paragraph preceding (5.5)). Furthermore, in Corollary 6.15, we obtain a nonvanishing result for Ext over hypersurfaces that are in the form of (A.8).

COROLLARY 6.13

Let R be a local AB ring, and let M and N be nonzero R-modules. Assume that N is Tor-rigid (e.g., $N = k$) and that $\mathsf{Ext}^n_R(M, N) = 0$ for some $n \geq \mathsf{depth}(N)$. Then

$\sup\{i \in \mathbb{Z} : \mathsf{Ext}^i_R(M,N) \neq 0\} = \mathsf{G}\text{-}\mathsf{dim}(M) = \mathsf{depth}(R) - \mathsf{depth}(M) \leq n-1$.

Proof
The bound on $\mathsf{G}\text{-}\mathsf{dim}(M)$ follows from Theorem 5.8(ii): the conditions in (5.4) hold for $(\mathcal{T}_{n+1}M, N, \mathsf{G}\text{-}\mathsf{dim})$ (see (5.5)(ii)). Hence, since R is an AB ring, it suffices to prove by [17, Proposition 3.2 and Theorem 6.1] that $\mathsf{Ext}^i_R(M,N) = 0$ for all $i \gg 0$.

As $\mathsf{Ext}^n_R(M,N) = 0$ and N is a Tor-rigid module, it follows from the exact sequence (4.1)(i) that $\mathsf{Tor}^R_i(\mathcal{T}_{n+1}M,N) = 0$ for all $i \geq 1$. This shows, by [17, Proposition 3.2], that Tate Tor groups $\widehat{\mathsf{Tor}}^R_i(\mathcal{T}_{n+1}M,N)$ vanish for all $i \in \mathbb{Z}$. Note that $\mathsf{G}\text{-}\mathsf{dim}(M) \leq n-1$ and $\mathsf{G}\text{-}\mathsf{dim}(\Omega^n M) = 0$ (see [3, Theorem 3.13]). We now use [8, Section 4.4.7] and conclude, for all $i \geq n+1$, that

$$\begin{aligned}\mathsf{Ext}^i_R(M,N) &\cong \mathsf{Ext}^{i-n}_R(\Omega^n M,N) \cong \widehat{\mathsf{Ext}}^{i-n}_R(\Omega^n M,N)\\ &\cong \widehat{\mathsf{Tor}}^R_{-i+n-1}\big(\mathsf{Hom}(\Omega^n M,R),N\big)\\ &\cong \widehat{\mathsf{Tor}}^R_{-i+n+1}(\mathcal{T}_{n+1}M,N).\end{aligned}$$

Therefore, we have $\mathsf{Ext}^i_R(M,N) = 0$ for all $i \geq n$, and this proves our claim. $\square$

COROLLARY 6.14
Let R be a local AB ring, and let M and N be nonzero R-modules such that N is Tor-rigid. Assume that $\mathsf{depth}(N) \leq \mathsf{G}\text{-}\mathsf{dim}(M)$. Then $\mathsf{Ext}^i_R(M,N) \neq 0$ for all i, where $\mathsf{depth}(N) \leq i \leq \mathsf{G}\text{-}\mathsf{dim}(M)$. In particular, if $\mathsf{depth}(M) = 0$, then $\mathsf{Ext}^i_R(M,N) \neq 0$ for all i, where $\mathsf{depth}(N) \leq i \leq \dim(R)$.

Proof
The first part is clear from Corollary 6.13. If $\mathsf{depth}(M) = 0$, then $\mathsf{G}\text{-}\mathsf{dim}(M) = \dim(R)$ so that the second part follows. $\square$

If R is a hypersurface (the quotient of an equicharacteristic regular local ring) and N is an R-module such that $\mathsf{id}(N) < \infty$, then $\mathsf{pd}(N) < \infty$ and hence it follows from a result of Lichtenabum [42, Theorem 3] that N is Tor-rigid. Consequently, when R is such a hypersurface and M and N are nonzero R-modules such that $\mathsf{pd}(M) < \infty$, $\mathsf{depth}(M) = 0$, and $\mathsf{id}(N) < \infty$, Corollary 6.14 implies that $\mathsf{Ext}^i_R(M,N) \neq 0$ for all i, where $\mathsf{depth}(N) \leq i \leq \dim(R)$. Over certain hypersurfaces, we know that all modules are Tor-rigid so that the nonvanishing of $\mathsf{Ext}^i_R(M,N)$ occurs without any restriction on N. For example, Corollary 6.14 and (A.8) yield the following result.

COROLLARY 6.15
Let $A = Q/(f)$, where $Q = k[x_1, \ldots, x_{2s+1}]$, with $s \geq 1$, is a polynomial ring over a perfect field k, and f is a nonconstant polynomial in Q. Set $R = A_{\mathfrak{m}}$,

where $\mathfrak{m} = (x_1, \ldots, x_{2s+1})A$, *and assume that* $f \in \mathfrak{m}$ *and* $A_{\mathfrak{p}}$ *is regular for all* $\mathfrak{p} \in \operatorname{Spec}(A) - \{\mathfrak{m}\}$. *If* M *and* N *are nonzero* R*-modules such that* $\mathsf{depth}(M) = 0$, *then* $\mathsf{Ext}_R^i(M, N) \neq 0$ *for all* i, *where* $\mathsf{depth}(N) \leq i \leq \dim(R)$.

7. Gorenstein injective dimension of strongly rigid modules

Let $(R, \mathfrak{m}, k)$ be a local ring. If R is Gorenstein and the injective dimension $\mathsf{id}(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ is finite, so is the injective dimension of k, and hence, the Auslander–Buchsbaum formula implies that R is regular (see [11, Exercise 3.1.26]). Our results in this section originated in an attempt to answer the following question.

QUESTION 7.1
Let $(R, \mathfrak{m})$ be a local ring. Assume that $\mathsf{id}(\mathfrak{m}) < \infty$. Then must R be Gorenstein or, equivalently, must R be regular?

Levin and Vasconcelos [41, Theorem 1.1] proved that R is regular if $\mathsf{pd}(\mathfrak{m}M) < \infty$ for an R-module M with $\mathfrak{m}M \neq 0$. They also remarked that an argument analogous to that of [41, Theorem 1.1] would work just as well for finite injective dimension.

Avramov [4] pointed out that an affirmative answer to Question 7.1 came out in a discussion with himself and H.-B. Foxby in the summer of 1983. He also referred us to Lescot's explicit computation of the Bass series of $\mathfrak{m}$ for an example of a published treatment of this fact (see [39]). Avramov [5, Theorem 4] proved that any submodule L of a finitely generated R-module M satisfying $L \supseteq \mathfrak{m}M \supsetneq \mathfrak{m}L$ has the same injective complexity and curvature as the residue field k. It follows, for example, if $\mathfrak{m}^n M \neq 0$ and $\mathsf{id}(\mathfrak{m}^n M) < \infty$, then R is regular (see also [5, Corollary 5] and the remark following it). A very special case of this result—the case where $n = 1$ and $M = R$; that is, the case where $\mathsf{id}(\mathfrak{m}) < \infty$—also follows from (4.5).

The Gorenstein injective dimension, introduced by Enochs and Jenda [23], is a refinement of the classical injective dimension. We use Corollary 6.1 and prove that R is Gorenstein if the Gorenstein injective dimension of an integrally closed $\mathfrak{m}$-primary ideal of R is finite (see Corollary 7.6). This, in particular, refines Question 7.1 and establishes that R is Gorenstein if and only if the Gorenstein injective dimension of the maximal ideal $\mathfrak{m}$ is finite. We proceed by recalling some definitions.

7.2
An R-module M is said to be Gorenstein injective if there is an exact sequence $I_\bullet = \cdots \to I_1 \xrightarrow{\partial_1} I_0 \xrightarrow{\partial_0} I_{-1} \to \cdots$ of injective R-modules such that $M \cong \ker(\partial_0)$ and $\mathsf{Hom}_R(E, I_\bullet)$ is exact for any injective R-module E (see [23] and [16, Definition 6.2.2]). The Gorenstein injective dimension of M, $\mathsf{Gid}(M)$, is defined as the infimum of n for which there exists an exact sequence $0 \to M \to J_0 \to \cdots \to J_{-n} \to 0$, where each J_i is Gorenstein injective.

The Gorenstein injective dimension is a refinement of the classical injective dimension: $\mathsf{Gid}(M) \leq \mathsf{id}(M)$, with equality if $\mathsf{id}(M) < \infty$ (see [16, Proposition 6.2.6]). It follows that every module over a Gorenstein ring has finite Gorenstein injective dimension. Hence, if $(R, \mathfrak{m})$ is Gorenstein but not regular, then $\mathsf{Gid}(k) < \infty = \mathsf{id}(k)$.

Cohen–Macaulay local rings that admit a nonzero strongly rigid module of finite Gorenstein injective dimension are Gorenstein.

PROPOSITION 7.3

Let R be a Cohen–Macaulay local ring. If $\mathsf{Gid}(M) < \infty$ for some nonzero strongly rigid R-module M, then R is Gorenstein.

Proof

Assume that $\mathsf{Gid}(M) < \infty$ for some nonzero strongly rigid R-module M. Then, since R is Cohen–Macaulay, there exists a nonzero finitely generated R-module N such that $\mathsf{id}(N) < \infty$ and $\mathsf{Ext}^i_R(N, M) = 0$ for all $i \gg 0$ (see [26, Theorem 2.22]). Therefore, Corollary 6.1(i) shows that $\mathsf{pd}(N) < \infty$; that is, R is Gorenstein (see (4.4)). $\square$

In general, it is not known whether or not a local ring admitting a nonzero module of finite Gorenstein injective dimension must be Cohen–Macaulay (see [16]). Hence, in view of the foregoing result, it seems worth raising the following question.

QUESTION 7.4

Let R be a local ring, and let M be a nonzero strongly rigid module over R. If $\mathsf{Gid}(M) < \infty$, then must R be Gorenstein?

We are able to give an affirmative answer to Question 7.4 when M is an integrally closed $\mathfrak{m}$-primary ideal. For that we need the following result of Yassemi.

7.5 (Yassemi [56, Theorem 1.3])

Let R be a local ring, and let M be a nonzero R-module. Assume that $\mathsf{Gid}(M) < \infty$. Assume further that $\dim(M) = \dim(R)$. Then R is Cohen–Macaulay.

An integrally closed $\mathfrak{m}$-primary ideal I of $(R, \mathfrak{m})$ is a strongly rigid module with $\dim(I) = \dim(R)$ (see (A.2)). So we deduce from Proposition 7.3 and (7.5) that the following result holds.

COROLLARY 7.6

A local ring $(R, \mathfrak{m})$ is Gorenstein if and only if $\mathsf{Gid}(I) < \infty$ for some integrally closed $\mathfrak{m}$-primary ideal I of R. In particular, $(R, \mathfrak{m})$ is Gorenstein if and only if $\mathsf{Gid}(\mathfrak{m}) < \infty$.

REMARK 7.7

It is known that if $\mathsf{Gid}(k) < \infty$ or $\mathsf{Gid}(R) < \infty$, then R is Gorenstein (see [16, Theorem 6.2.7] and [27, Theorem 2.1]). However, we do not know whether the finiteness of $\mathsf{Gid}(\mathfrak{m})$ directly implies the finiteness of $\mathsf{Gid}(k)$ or $\mathsf{Gid}(R)$ via the short exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$. Thus, as far as we know, Corollary 7.6, even for the special case where $I = \mathfrak{m}$, is new.

Our final aim is to show in Proposition 7.13 that the finiteness of $\mathsf{C}\text{-}\mathsf{Gid}(\mathfrak{m})$ for a semidualizing module C detects the dualizing module, that is, forces C to be dualizing. We first record a few preliminary results.

7.8

Let C be a semidualizing R-module (see (4.9)). Then the *C-Gorenstein injective dimension* $\mathsf{C}\text{-}\mathsf{Gid}(M)$ of a nonzero R-module M can be defined as $\mathsf{Gid}_{R\ltimes C}(M)$, where $R \ltimes C$ is the trivial extension of R by C (see Holm and Jørgensen [28, Theorem 2.16]). In particular, if $C = R$, then $\mathsf{C}\text{-}\mathsf{Gid}(M) = \mathsf{Gid}(M)$ (see (7.2)).

Proposition 7.9 is used for our proof of Proposition 7.13 for the case where R is Artinian (see also (2.4)).

PROPOSITION 7.9

Let R be a Cohen–Macaulay local ring with a dualizing module, and let C be a semidualizing R-module. Assume that M is a nonzero strongly rigid module over R. Assume further that $\mathsf{C}\text{-}\mathsf{Gid}(M) < \infty$. Then C is dualizing.

Proof

Note that $\mathsf{Ext}^i_R(C,\omega) = 0$ for all $i \geq 1$, where ω is the dualizing module. Hence, $X = \mathbf{R}\operatorname{Hom}_R(C,\omega) \simeq \mathsf{Hom}_R(C,\omega)$ is a maximal Cohen–Macaulay R-module. It follows from [28, Theorem 4.6] that M is in the Bass class of R with respect to X. In particular, $\mathsf{Ext}^i_R(X,M) = 0$ for all $i \gg 0$ (see [28, Remark 4.1]). Therefore, Corollary 6.1(i) implies that $C \cong \omega$. $\square$

7.10

Let M be a nonzero R-module, and let C be a semidualizing R-module. Assume that $\mathsf{C}\text{-}\mathsf{Gid}(M) < \infty$. If $\dim(M) = \dim(R)$, then $\dim_{R\ltimes C}(M) = \dim(R \ltimes C)$, so that, by (7.5), R is Cohen–Macaulay. Therefore, if M is a strongly rigid module, $\dim(M) = \dim(R)$, and $\mathsf{depth}(R) = 0$, then R is Artinian and it follows from Proposition 7.9 that C is dualizing.

For the rest of our arguments, $\overline{X}$ denotes X/xX, where X is an R-module and x is a nonzero divisor on R.

7.11

Let $(R, \mathfrak{m})$ be a local ring, and let $x \in \mathfrak{m} - \mathfrak{m}^2$ be a nonzero divisor on R. Then the surjective R-linear map $f : \mathfrak{m}/x\mathfrak{m} \twoheadrightarrow \mathfrak{m}/xR$, given by $f(y + x\mathfrak{m}) = y + xR$ for all $y \in \mathfrak{m}$, splits (see, e.g., the proof of [45, Theorem 19.2]). Therefore, there exists an $\overline{R}$-module N such that $\overline{\mathfrak{m}} \cong N \oplus \mathfrak{m}/xR$.

7.12

Let R be a local ring, and let $x \in R$. Assume that x is a nonzero divisor on R. Then x is also a nonzero divisor on $R \ltimes C$, and hence, the following holds:

$$\overline{R} \ltimes \overline{C} \cong \overline{R \ltimes C}. \tag{7.12.1}$$

Let M be an R-module. Assume that x is also a nonzero divisor on M. Assume further that $\mathsf{C}\text{-}\mathsf{Gid}(M) < \infty$ for some semidualizing R-module C. Then, in view of (7.12.1), we deduce from [52, Lemma 2] that

$$\mathsf{Gid}_{\overline{R \ltimes C}}(\overline{M}) = \mathsf{Gid}_{\overline{R} \ltimes \overline{C}}(\overline{M}) = \overline{\mathsf{C}}\text{-}\mathsf{Gid}_{\overline{R}}(\overline{M}) < \infty. \tag{7.12.2}$$

PROPOSITION 7.13

Let R be a local ring. Assume that at least one of the following conditions holds:

(i) $\mathsf{C}\text{-}\mathsf{Gid}(\mathfrak{m}) < \infty$ *for some semidualizing R-module C,*
(ii) $\mathsf{C}\text{-}\mathsf{Gid}(M) < \infty$ *for some maximal Cohen–Macaulay strongly rigid module M.*

Then C is dualizing.

REMARK 7.14

We already know from (7.10) that R must be Cohen–Macaulay in the case in which (i) or (ii) holds in Proposition 7.13.

Proof of Proposition 7.13

We proceed by induction on $d = \mathsf{depth}\, R$. If $d = 0$, then (7.10) gives the required conclusion for both cases (i) and (ii). Hence, we assume that $d \geq 1$ and pick a nonzero divisor x on R such that $x \in \mathfrak{m} - \mathfrak{m}^2$.

First suppose that (i) holds; that is, $\mathsf{C}\text{-}\mathsf{Gid}(\mathfrak{m}) < \infty$. It follows from (7.11) that there exists an $\overline{R}$-module N such that $\overline{\mathfrak{m}} \cong N \oplus \mathfrak{m}/xR$. Therefore, we conclude from (7.12.2) that $\mathsf{Gid}_{\overline{R \ltimes C}}(N \oplus \mathfrak{m}/xR) = \mathsf{Gid}_{\overline{R \ltimes C}}(\overline{\mathfrak{m}}) < \infty$. Then [26, Theorem 2.6] implies that $\mathsf{Gid}_{\overline{R \ltimes C}}(\mathfrak{m}/xR) < \infty$. Hence, we obtain the following (see also (7.10) and (7.12.1)):

$$\mathsf{Gid}_{\overline{R \ltimes C}}(\mathfrak{m}/xR) = \mathsf{Gid}_{\overline{R} \ltimes \overline{C}}(\mathfrak{m}/xR) = \overline{\mathsf{C}}\text{-}\mathsf{Gid}_{\overline{R}}(\mathfrak{m}/xR) < \infty.$$

Now the induction hypothesis forces $\overline{C}$ to be dualizing over $\overline{R}$; that is, $\mathsf{id}_{\overline{R}}(\overline{C}) < \infty$. Consequently, $\mathsf{id}_R(C) < \infty$ and, hence, C is dualizing over R.

Next assume (ii). Then it follows from (7.12.2) that $\overline{\mathsf{C}}\text{-}\mathsf{Gid}_{\overline{R}}(\overline{M}) < \infty$. Moreover, since M is maximal Cohen–Macaulay, x is a nonzero divisor on M. We can easily observe, similar to [14, Proposition 2.2], that $\overline{M}$ is strongly rigid over $\overline{R}$:

here we include an argument since [14] deals with only test modules (see also (2.2) and (2.4)).

Suppose that $\mathsf{Tor}_n^{\overline{R}}(\overline{M}, L) = 0$ for some $\overline{R}$-module L and for some positive integer n. Then, since M is a strongly rigid module over R and $\mathsf{Tor}_i^{\overline{R}}(\overline{M}, L) \cong \mathsf{Tor}_i^{R}(M, L)$ for all $i \geq 0$, we see that $\mathsf{pd}_R(L) < \infty$. Now the fact that $x \notin \mathfrak{m}^2$ implies that $\mathsf{pd}_{\overline{R}}(L) < \infty$ (see [6, Proposition 3.3.5(1)]). This proves that $\overline{M}$ is strongly rigid over $\overline{R}$. Thus, the induction hypothesis implies that $\overline{C}$ is dualizing over $\overline{R}$, and so C is dualizing over R. □

Proposition 7.13 naturally raises the following question (see (2.4)).

QUESTION 7.15

Let $(R, \mathfrak{m})$ be a local ring, and let C be a semidualizing R-module. If $\mathsf{C\text{-}Gid}(M) < \infty$ for some nonzero strongly rigid R-module M, then must R be Cohen–Macaulay?

The *Cohen–Macaulay injective dimension* $\mathsf{CMid}(M)$ (see [29, Definition 2.3]) of an R-module M is defined as $\inf\{\mathsf{C\text{-}Gid}(M) : C$ is a semidualizing R-module$\}$ (see also (7.8)). In view of this notation, Proposition 7.13 characterizes Cohen–Macaulay rings in terms of the finiteness of the Cohen–Macaulay injective dimension of the maximal ideal $\mathfrak{m}$; that is, if $\mathsf{CMid}(\mathfrak{m}) < \infty$, then R is Cohen–Macaulay with a dualizing module C.

Appendix. Some examples of test and rigid-test modules

There are quite a few examples of test and rigid-test modules in the literature. In this section we catalog a few of them (see also [14, Theorem 1.4] for a characterization of test modules over complete intersection rings). We start by pointing out that test and Tor-rigid modules are distinct in general (see (2.1) and (2.2)).

A.1

Let $S = k[\![x, y, z]\!]$ be the formal power series over a field k, and let R be the subring of S generated by monomials of degree 2; that is, the second *Veronese* subring S. Then $S = R \oplus_R L$, where L generates the class group of R (see, e.g., [13, Remark 3.16]). Since S is a finite extension of R, [14, Proposition 2.4] shows that L is a test module over R. On the other hand, by Dao's remark [20, Remark 2.6], L is not Tor-rigid.

Recall that all rigid-test modules are strongly rigid (see (2.3) and (2.4)).

A.2

If I is an integrally closed $\mathfrak{m}$-primary ideal of R and $\mathsf{Tor}_n^R(R/I, N) = 0$, then $\mathsf{pd}(N) \leq n - 1$; that is, $\Omega^i(R/I)$ is a rigid-test module for all $i \geq 0$ (see [18, Corollary 3.3]).

A.3

Let R be an F-finite local ring of prime characteristic p, and let $\varphi^n : R \to R$ be the nth iterate of the Frobenius endomorphism defined by $r \mapsto r^{p^n}$ for $r \in R$. If M is a finitely generated R-module, then ${}^{\varphi^n}M$ denotes the (finitely generated) R-module M with the R-action given by $r \cdot m = \varphi^n(r)m$.

(i) ${}^{\varphi^n}R$ is a test module over R for all $n \gg 0$ (see (2.2) and [46, Theorem 2.2.8]).

(ii) If R is a one-dimensional Cohen–Macaulay local ring, then ${}^{\varphi^n}R$ is a rigid-test module over R for all $n \gg 0$ (see (2.3) and [46, Theorem 2.1.3 and Corollary 2.2.12]).

(iii) If $\mathfrak{m}^{[p]} = 0$ or R is a complete intersection ring, then ${}^{\varphi^n}R$ is a rigid-test module over R for all $n \geq 1$ (see (2.3) and [46, Corollary 2.2.9, Remark 2.2.10, and Theorem 5.1.1]).

A.4

Let R be a Golod ring (e.g., R is a hypersurface). If M is an R-module with $\mathsf{pd}(M) = \infty$, then M is a test module over R (see [6, Section 5] and [34, Theorem 3.1]).

A.5

Let $(R, \mathfrak{m}, k)$ be a two-dimensional complete normal local domain with an algebraically closed residue field k. Assume that R has a rational singularity; that is, there exists a resolution of singularities $X \to \operatorname{Spec}(R)$, a proper birational morphism where X is a regular scheme, such that $\mathsf{H}^1(X, \mathcal{O}_X) = 0$ (see [40, Definition 6.32] or [43]). It follows that R is a Cohen–Macaulay ring with minimal multiplicity (see, e.g., [40, Corollary 6.36]). Thus, R is Golod [6, Example 5.2.8]. Hence, each R-module M with $\mathsf{pd}(M) = \infty$ is a test module over R (see (A.4)). In particular, if R is not Gorenstein, then the dualizing module of R is a test module.

A.6

Let $R = k[\![x_1, \dots, x_n]\!]/(f)$, where k is a field and $n \geq 3$. Assume that R has an isolated singularity; that is, $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Let M be an R-module. If $n = 3$ (i.e., $\dim(R) = 2$) or $\dim(M) \leq 1$ (e.g., M has finite length), then M is Tor-rigid (see [21, Lemma 2.10, Theorem 3.4, and Corollary 3.6]).

A technical but rather important point for us is that a rigid-test module, unlike the residue field k, may have arbitrary depth, and even if its depth is zero, it does not have to have finite length in general. Here is such an example.

A.7

Let k be a field, $R = k[\![x, y, z]\!]/(xy - z^2)$, and $M = \mathfrak{m}/x\mathfrak{m}$. Then M is a rigid-test module with $\mathsf{depth}(M) = 0$ and $\dim(M) = 1$ (see (A.4) and (A.6)).

The next result, in view of Dao [21, Lemma 2.10], is already known when f is homogeneous. Recently, Walker [55] removed the homogeneity assumption (see [47] and also the paragraph preceding Theorem 1.2 in [55]).

A.8

Let $A = Q/(f)$, where $Q = k[x_1, \ldots, x_{2s+1}]$, with $s \geq 1$, is a polynomial ring (with standard grading) over a perfect field k, and f is a nonconstant polynomial in Q. Set $R = A_{\mathfrak{m}}$, where $\mathfrak{m} = (x_1, \ldots, x_{2s+1})A$, and assume that $f \in \mathfrak{m}$ and $A_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(A) - \{\mathfrak{m}\}$. Then all R-modules are Tor-rigid (see [21, Lemma 2.10] and [55, Theorem 1.2]).

Acknowledgments. We are grateful to Luchezar Avramov, David Buchsbaum, Kamran Divaani-Aazar, Henrik Holm, Li Liang, Saeed Nasseh, Greg Piepmeyer, William Sanders, Alex Tchernev, Sean Sather-Wagstaff, Yang Zheng, Ryo Takahashi, and Mark E. Walker for valuable information and discussions related to this work during its preparation. We also thank the referees for suggestions that have greatly improved the manuscript.

Sadeghi's research was partially supported by grant 93130026 from the Institute for Research in Fundamental Sciences (IPM).

References

[1] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631–647. MR 0179211.

[2] ———, "Modules over unramified regular local rings" in *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, Inst. Mittag-Leffler, Djursholm, 1963, 230–233. MR 0175930.

[3] M. Auslander and M. Bridger, *Stable Module Theory*, Mem. Amer. Math. Soc. **94** (1969), no. 94. MR 0269685.

[4] L. L. Avramov, personal communication.

[5] ———, *Modules with extremal resolutions*, Math. Res. Lett. **3** (1996), 319–328. MR 1397681. DOI 10.4310/MRL.1996.v3.n3.a3.

[6] ———, "Infinite free resolutions" in *Six Lectures on Commutative Algebra (Bellaterra, 1996)*, Progr. Math. **166**, Birkhäuser, Basel, 1998, 1–118. MR 1648664.

[7] ———, "Homological dimensions and related invariants of modules over local rings" in *Representations of Algebra, Vol. I, II*, Beijing Norm. Univ. Press, Beijing, 2002, 1–39. MR 2067368.

[8] L. L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), 285–318. MR 1794064. DOI 10.1007/s002220000090.

[9] L. L. Avramov, V. Gasharov, and I. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. **86** (1998), 67–114. MR 1608565.

[10] L. L. Avramov, M. Hochster, S. B. Iyengar, and Y. Yao, *Homological invariants of modules over contracting endomorphisms*, Math. Ann. **353** (2012), 275–291. MR 2915536. DOI 10.1007/s00208-011-0682-z.

[11] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Stud. Adv. Math. **39**, Cambridge Univ. Press, Cambridge, 1993. MR 1251956.

[12] O. Celikbas and H. Dao, *Asymptotic behavior of Ext functors for modules of finite complete intersection dimension*, Math. Z. **269** (2011), 1005–1020. MR 2860275. DOI 10.1007/s00209-010-0771-9.

[13] ———, *Necessary conditions for the depth formula over Cohen-Macaulay local rings*, J. Pure Appl. Algebra **218** (2014), 522–530. MR 3124216. DOI 10.1016/j.jpaa.2013.07.002.

[14] O. Celikbas, H. Dao, and R. Takahashi, *Modules that detect finite homological dimensions*, Kyoto J. Math. **54** (2014), 295–310. MR 3215569. DOI 10.1215/21562261-2642404.

[15] O. Celikbas and S. Sather-Wagstaff, *Testing for the Gorenstein property*, Collect. Math. **67** (2016), 555–568. MR 3536062. DOI 10.1007/s13348-015-0163-x.

[16] L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Math. **1747**, Springer, Berlin, 2000. MR 1799866. DOI 10.1007/BFb0103980.

[17] L. W. Christensen and D. A. Jorgensen, *Vanishing of Tate homology and depth formulas over local rings*, J. Pure Appl. Algebra **219** (2015), 464–481. MR 3279366. DOI 10.1016/j.jpaa.2014.05.005.

[18] A. Corso, C. Huneke, D. Katz, and W. V. Vasconcelos, "Integral closure of ideals and annihilators of homology" in *Commutative Algebra*, Lect. Notes Pure Appl. Math. **244**, Chapman & Hall/CRC, Boca Raton, FL, 2006, 33–48. MR 2184788. DOI 10.1201/9781420028324.ch4.

[19] H. Dao, *Some observations on local and projective hypersurfaces*, Math. Res. Lett. **15** (2008), 207–219. MR 2385635. DOI 10.4310/MRL.2008.v15.n2.a1.

[20] ———, *Remarks on non-commutative crepant resolutions of complete intersections*, Adv. Math. **224** (2010), 1021–1030. MR 2628801. DOI 10.1016/j.aim.2009.12.016.

[21] ———, *Decent intersection and Tor-rigidity for modules over local hypersurfaces*, Trans. Amer. Math. Soc. **365** (2013), no. 6, 2803–2821. MR 3034448. DOI 10.1090/S0002-9947-2012-05574-7.

[22] H. Dao, J. Li, and C. Miller, *On the (non)rigidity of the Frobenius endomorphism over Gorenstein rings*, Algebra Number Theory **4** (2010), 1039–1053. MR 2832633. DOI 10.2140/ant.2010.4.1039.

[23] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), 611–633. MR 1363858. DOI 10.1007/BF02572634.

[24] E. G. Evans and P. Griffith, *Syzygies*, London Math. Soc. Lecture Note Ser. **106**, Cambridge Univ. Press, Cambridge, 1985. MR 0811636. DOI 10.1017/CBO9781107325661.

[25] M. Hochster, "Cohen–Macaulay modules" in *Conference on Commutative Algebra (Lawrence, 1972)*, Lecture Notes in Math. **311**, Springer, Berlin, 1973, 120–152. MR 0340251.

[26] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), 167–193. MR 2038564. DOI 10.1016/j.jpaa.2003.11.007.

[27] ———, *Rings with finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **132** (2004), 1279–1283. MR 2053331. DOI 10.1090/S0002-9939-03-07466-5.

[28] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), 423–445. MR 2203625. DOI 10.1016/j.jpaa.2005.07.010.

[29] ———, *Cohen–Macaulay homological dimensions*, Rend. Semin. Mat. Univ. Padova **117** (2007), 87–112. MR 2351787.

[30] C. Huneke and D. A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, Math. Scand. **93** (2003), 161–184. MR 2009580. DOI 10.7146/math.scand.a-14418.

[31] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Math. Soc. Lecture Note Ser. **336**, Cambridge Univ. Press, Cambridge, 2006. MR 2266432.

[32] C. Huneke and R. Wiegand, *Tensor products of modules and the rigidity of Tor*, Math. Ann. **299** (1994), 449–476. MR 1282227. DOI 10.1007/BF01459794.

[33] M. I. Jinnah, *Reflexive modules over regular local rings*, Arch. Math. (Basel) **26** (1975), 367–371. MR 0376662. DOI 10.1007/BF01229753.

[34] D. A. Jorgensen, *A generalization of the Auslander–Buchsbaum formula*, J. Pure Appl. Algebra **144** (1999), 145–155. MR 1732626. DOI 10.1016/S0022-4049(98)00057-7.

[35] ———, *Finite projective dimension and the vanishing of* $\mathrm{Ext}_R(M,M)$, Comm. Algebra **36** (2008), 4461–4471. MR 2473341. DOI 10.1080/00927870802179560.

[36] D. A. Jorgensen and L. M. Şega, *Nonvanishing cohomology and classes of Gorenstein rings*, Adv. Math. **188** (2004), 470–490. MR 2087235. DOI 10.1016/j.aim.2003.11.003.

[37] P. Jothilingam, *A note on grade*, Nagoya Math. J. **59** (1975), 149–152. MR 0392979.

[38] ———, *Syzygies and Ext*, Math. Z. **188** (1985), 278–282. MR 0772356. DOI 10.1007/BF01304215.

[39] J. Lescot, "La série de Bass d'un produit fibré d'anneaux locaux" in *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 35th Year (Paris, 1982)*,

Lecture Notes in Math. **1029**, Springer, Berlin, 1983, 218–239. MR 0732477. DOI 10.1007/BFb0098933.

[40] G. J. Leuschke and R. Wiegand, *Cohen–Macaulay Representations*, Math. Surveys Monogr. **181**, Amer. Math. Soc., Providence, 2012. MR 2919145. DOI 10.1090/surv/181.

[41] G. Levin and W. V. Vasconcelos, *Homological dimensions and Macaulay rings*, Pacific J. Math. **25** (1968), 315–323. MR 0230715.

[42] S. Lichtenbaum, *On the vanishing of Tor in regular local rings*, Illinois J. Math. **10** (1966), 220–226. MR 0188249.

[43] J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 195–279. MR 0276239.

[44] V. Maşek, *Gorenstein dimension and torsion of modules over commutative Noetherian rings*, Comm. Algebra **28** (2000), 5783–5811. MR 1808604. DOI 10.1080/00927870008827189.

[45] H. Matsumura, *Commutative Ring Theory*, 2nd ed., Cambridge Stud. Adv. Math. **8**, Cambridge Univ. Press, Cambridge, 1989. MR 1011461.

[46] C. Miller, "The Frobenius endomorphism and homological dimensions" in *Commutative Algebra (Grenoble/Lyon, 2001)*, Contemp. Math. **331**, Amer. Math. Soc., Providence, 2003, 207–234. MR 2013168. DOI 10.1090/conm/331/05912.

[47] W. F. Moore, G. Piepmeyer, S. Spiroff, and M. E. Walker, *Hochster's theta invariant and the Hodge–Riemann bilinear relations*, Adv. Math. **2** (2011), 1692–1714. MR 2737797. DOI 10.1016/j.aim.2010.09.005.

[48] M. P. Murthy, *Modules over regular local rings*, Illinois J. Math. **7** (1963), 558–565. MR 0156883.

[49] S. Nasseh, M. Tousi, and S. Yassemi, *Characterization of modules of finite projective dimension via Frobenius functors*, Manuscripta Math. **130** (2009), 425–431. MR 2563144. DOI 10.1007/s00229-009-0296-x.

[50] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. Éc. Norm. Supér. (4) **9** (1976), 103–106. MR 0399075.

[51] A. Sadeghi, *Vanishing of cohomology over complete intersection rings*, Glasg. Math. J. **57** (2015), 445–455. MR 3333952. DOI 10.1017/S0017089514000408.

[52] S. Salarian, S. Sather-Wagstaff, and S. Yassemi, *Characterizing local rings via homological dimensions and regular sequences*, J. Pure Appl. Algebra **207** (2006), 99–108. MR 2244383. DOI 10.1016/j.jpaa.2005.09.015.

[53] R. Takahashi, *On G-regular local rings*, Comm. Algebra **36** (2008), 4472–4491. MR 2473342. DOI 10.1080/00927870802179602.

[54] R. Takahashi and D. White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), 5–22. MR 2603458. DOI 10.7146/math.scand.a-15121.

[55] M. E. Walker, *On the vanishing of Hochster's θ invariant*, Ann. K-Theory **2** (2017), 131–174. MR 3590343. DOI 10.2140/akt.2017.2.131.

[56] S. Yassemi, *A generalization of a theorem of Bass*, Comm. Algebra **35** (2007), 249–251. MR 2287566. DOI 10.1080/00927870601041672.

[57] S. Yassemi, L. Khatami, and T. Sharif, *Grade and Gorenstein dimension*, Comm. Algebra **29** (2001), 5085–5094. MR 1856932. DOI 10.1081/AGB-100106803.

[58] K. Yoshida, *Tensor products of perfect modules and maximal surjective Buchsbaum modules*, J. Pure Appl. Algebra **123** (1998), 313–326. MR 1492906. DOI 10.1016/S0022-4049(96)00088-6.

[59] Y. Yoshino, *Cohen–Macaulay Modules over Cohen–Macaulay Rings*, London Math. Soc. Lecture Note Ser. **146**, Cambridge Univ. Press, Cambridge, 1990. MR 1079937. DOI 10.1017/CBO9780511600685.

Zargar: Department of Engineering Sciences, Faculty of Advanced Technologies, University of Mohaghegh Ardabili, Namin, Iran, and Department of Engineering Sciences, Faculty of Advanced Technologies, Sabalan University of Advanced Technologies (SUAT), Namin, Iran; zargar9077@gmail.com

Celikbas: Department of Mathematics, West Virginia University, Morgantown, West Virginia, USA; olgur.celikbas@math.wvu.edu

Gheibi: Department of Mathematics, University of Texas at Arlington, Arlington, Texas, USA; mohsen.gheibi@gmail.com

Sadeghi: School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran; sadeghiarash61@gmail.com