

Local Jacquet–Langlands correspondences for simple supercuspidal representations

Naoki Imai and Takahiro Tsushima

Abstract We give a description of the local Jacquet–Langlands correspondence for simple supercuspidal representations via type theory. As a consequence, we show that the endoclasses for such representations are invariant under the local Jacquet–Langlands correspondence.

0. Introduction

Let K be a non-Archimedean local field with residue characteristic p . Let A be a central simple algebra over K . We put $n = [A : K]^{1/2}$. The *local Jacquet–Langlands correspondence* (LJLC) gives a correspondence between irreducible essentially square-integrable representations of $\mathrm{GL}_n(K)$ and A^\times . The irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ have been classified in [13] via type theory, which describes supercuspidal representations as compact inductions of representations of some open subgroups that are compact modulo center. More generally, type theory for representations of A^\times has been developed in the series of papers [24]–[27], [4], and [28]. So it is natural to seek a description of the LJLC via type theory.

In the case where A is a division algebra, such descriptions have been studied in [16], [18], [8], and [10] if n is a prime number and in [9] if p is odd, n is a power of p , and representations are totally ramified. For general A , such descriptions have been given in [29] for level zero discrete series representations and in [11] for essentially tame representations.

In this article, we give a description of the LJLC via type theory for the simple supercuspidal representations. We define the simple supercuspidal representations of A^\times in Definition 1.1 after [23], which are equivalent to the supercuspidal representations of conductor $n + 1$ as a result. Such representations appear in [19]. For $A^\times = \mathrm{GL}_n(K)$, simple supercuspidal representations are studied in [1] (cf. [22], [30]) and they are called epipelagic representations in [12]. We note that the simple supercuspidal representations of A^\times are essentially tame if n is prime to p , but they are not essentially tame if p divides n .

Kyoto Journal of Mathematics, Vol. 58, No. 3 (2018), 623–638

First published online June 19, 2018.

DOI [10.1215/21562261-2017-0032](https://doi.org/10.1215/21562261-2017-0032), © 2018 by Kyoto University

Received June 12, 2015. Revised November 18, 2016. Accepted November 30, 2016.

2010 Mathematics Subject Classification: Primary 11F70; Secondary 11L05.

As a consequence of the description of the LJLC, we show that the endoclasses are invariant under the LJLC for the simple supercuspidal representations. This verifies Broussous–Sécherre–Stevens’s [4, Conjecture 9.5] for the simple supercuspidal representations. The conjecture is verified in [21] for totally ramified representations of unit groups of division algebras, based on results in [9].

In Section 1, we give a construction and a definition of the simple supercuspidal representations of A^\times . Further, we show that they are equivalent to the supercuspidal representations of conductor $n + 1$. In Section 2, we give formulas for the characters of the simple supercuspidal representations at some elements, which are elliptic quasiregular in the sense of [11, 1.1 Remark]. The values of the characters at these elements are written in terms of variants of Gauss sums and generalized Kloosterman sums (cf. Propositions 2.6 and 2.9). We believe that these formulas are interesting in themselves. In Section 3, we give a description of the LJLC via type theory in Theorem 3.5. We determine the description by checking character relations at some elements. In Section 4, we give another proof of our main theorem using Godement–Jacquet local constants. This proof is based on a formula in [5] and [6], which calculates Godement–Jacquet local constants with respect to some kind of Gauss sums. In Section 5, we show the invariance of the endoclasses under the LJLC for the simple supercuspidal representations.

Type theory for irreducible supercuspidal representations of A^\times naturally appears in the study of the geometric realization of the local Langlands correspondence and the LJLC. The results in this article are used in [20] to show that the LJLC is realized in the cohomology of the reductions of a family of affinoids in the Lubin–Tate perfectoid space.

Notation. For a non-Archimedean local field F , let \mathcal{O}_F denote the ring of integers in F . For a field F and a positive integer l , let $\mu_l(F)$ denote the group of the l th roots of unity inside F . For an Abelian group X , we write X^\vee for its character group $\text{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. For a group G , its subgroup H , a character θ of H , and $g \in G$, we put $H^g = gHg^{-1}$ and define a character θ^g of H^g by $\theta^g(h) = \theta(g^{-1}hg)$ for $h \in H^g$.

1. Simple supercuspidal representation

In this section, we give a construction and a definition of the simple supercuspidal representations of the multiplicative group of a central simple algebra over a non-Archimedean local field. Further, we give a characterization of the simple supercuspidal representations by the conductors.

Let K be a non-Archimedean local field with residue field k . We set $q = |k|$. Let \mathfrak{p}_K be the maximal ideal of \mathcal{O}_K . Any central simple algebra over K is isomorphic to a matrix algebra over a central division algebra over K . Let m be a positive integer, and let D be a central division algebra over K . We put $A = M_m(D)$ and $G = A^\times$.

Let π be an irreducible smooth representation of G . We fix a nontrivial character $\psi \in k^\vee$. For $x \in \mathcal{O}_K$, let \bar{x} denote the image of x under the reduction map $\mathcal{O}_K \rightarrow k$. We take a character $\psi_K \in K^\vee$ such that

$$\begin{aligned} \psi_K(x) &= \psi(\bar{x}) \quad \text{for } x \in \mathcal{O}_K, \\ \psi_K(x) &= 1 \quad \text{for } x \in \mathfrak{p}_K. \end{aligned}$$

Let $\epsilon(\pi, s, \psi_K)$ be the Godement–Jacquet local constant of π with respect to ψ_K . Then there exists an integer $f(\pi, \psi_K)$ such that

$$\epsilon(\pi, s, \psi_K) = q^{-f(\pi, \psi_K)s} \epsilon(\pi, 0, \psi_K)$$

by [17, Theorem 3.3(4)]. We put $r = [D : K]^{1/2}$ and $n = mr$. We define the conductor $c(\pi)$ of π by

$$c(\pi) = f(\pi, \psi_K) + n.$$

Let $\mathcal{A}_{D,m}^{n+1}$ denote the set of isomorphism classes of the supercuspidal representations of G of conductor $n + 1$.

We fix a uniformizer ϖ of K . Let $\eta = (\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$. In the following, we define a smooth representation $\pi_{D,m,\eta}$ of G . Let K_r be the unramified extension of K of degree r . We take an element $\varphi_{D,\zeta} \in D^\times$, an embedding $K_r \hookrightarrow D$, and an integer $1 \leq s \leq r - 1$ which is prime to r such that $\varphi_{D,\zeta}^r = \zeta\varpi$ and $\varphi_{D,\zeta} d \varphi_{D,\zeta}^{-1} = d^{q^s}$ for $d \in \mu_{q^r-1}(K_r)$. We put

$$\varphi_\zeta = \begin{pmatrix} \mathbf{0} & I_{m-1} \\ \varphi_{D,\zeta} & \mathbf{0} \end{pmatrix} \in A \quad \text{and} \quad L_\zeta = K(\varphi_\zeta) \subset A.$$

Since $\varphi_\zeta^n = \zeta\varpi$, the field L_ζ is a totally ramified extension over K of degree n . We have

$$(1.1) \quad \text{Nrd}_{A/K}(\varphi_\zeta) = (-1)^{n-1} \zeta\varpi, \quad \text{Trd}_{A/K}(\varphi_\zeta^{-1}) = 0.$$

Let \mathcal{O}_D denote the maximal order of D , and let \mathfrak{p}_D denote the maximal ideal of \mathcal{O}_D . For a positive integer l , let k_l be the extension of k of degree l . We identify $\mathcal{O}_D/\mathfrak{p}_D$ with k_r . Let C be the subring of $M_m(k_r)$ consisting of all upper triangular matrices. Let \mathfrak{A} denote the inverse image of C under the reduction map $M_m(\mathcal{O}_D) \rightarrow M_m(k_r)$. Then \mathfrak{A} is an order in A . Let $\mathfrak{P}_{\mathfrak{A}}$ be the Jacobson radical of \mathfrak{A} . Note that $\mathfrak{P}_{\mathfrak{A}} = \varphi_\zeta \mathfrak{A}$ and the normalizer of \mathfrak{A} in G equals $L_\zeta^\times \mathfrak{A}^\times$. For a positive integer i , we set $U_{\mathfrak{A}}^i = 1 + \mathfrak{P}_{\mathfrak{A}}^i$. Let $\theta_{D,m,\eta} : L_\zeta^\times U_{\mathfrak{A}}^1 \rightarrow \mathbb{C}^\times$ be the character defined by

$$(1.2) \quad \begin{aligned} \theta_{D,m,\eta}(x) &= \chi(\bar{x}) \quad \text{for } x \in \mu_{q-1}(K), \\ \theta_{D,m,\eta}(x) &= (\psi_K \circ \text{Trd}_{A/K})(\varphi_\zeta^{-1}(x - 1)) \quad \text{for } x \in U_{\mathfrak{A}}^1, \\ \theta_{D,m,\eta}(\varphi_\zeta) &= (-1)^{m-1} c. \end{aligned}$$

We put $\pi_{D,m,\eta} = \text{c-Ind}_{L_\zeta^\times U_{\mathfrak{A}}^1}^G \theta_{D,m,\eta}$.

DEFINITION 1.1

We say that an irreducible supercuspidal representation π of G is simple supercuspidal if $\pi \simeq \pi_{D,m,\eta}$ for some $\eta \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$.

LEMMA 1.2

The representation $\pi_{D,m,\eta}$ is a supercuspidal representation of conductor $n + 1$.

Proof

We define a chain lattice $\Lambda = \{\Lambda_i\}_{i \in \mathbb{Z}}$ in $D^{\oplus m}$ by

$$\Lambda_{mj+l} = (\mathfrak{p}_D^j)^{\oplus m-l} \oplus (\mathfrak{p}_D^{j+1})^{\oplus l}$$

for $j \in \mathbb{Z}$ and $0 \leq l \leq m - 1$. Then we have

$$\mathfrak{A} = \{g \in A \mid g\Lambda_i \subset \Lambda_i \text{ for all } i \in \mathbb{Z}\}$$

and $\varphi_\zeta \Lambda_i \subset \Lambda_{i+1}$ for $i \in \mathbb{Z}$. Hence, \mathfrak{A} is a hereditary order in A (cf. [24, Définition 1.3]). We consider a stratum $[\mathfrak{A}, 1, 0, \varphi_\zeta^{-1}]$ of A (cf. [24, Définition 2.1]). We set

$$B = \{x \in A \mid xz = zx \text{ for all } z \in L_\zeta\}.$$

Then we have $B = L_\zeta$. Using this, we see that the critical exponent of this stratum equals -1 (cf. [24, Section 2.1]). Hence, this stratum is simple (cf. [24, Définition 2.3]). Since the simple pair $[0, \varphi_\zeta^{-1}]$ over K is minimal in the sense of [24, Section 2.3.3], we have

$$(1.3) \quad H^1(\varphi_\zeta^{-1}, \mathfrak{A}) = J^1(\varphi_\zeta^{-1}, \mathfrak{A}) = U_{\mathfrak{A}}^1 \subset J(\varphi_\zeta^{-1}, \mathfrak{A}) = \mathcal{O}_{L_\zeta}^\times U_{\mathfrak{A}}^1$$

under the notation in [24, (65)]. We have

$$(1.4) \quad \mathcal{C}(\varphi_\zeta^{-1}, 0, \mathfrak{A}) = \{\theta_{D,m,\eta}|_{U_{\mathfrak{A}}^1}\}$$

by [24, Lemma 3.23] under the notation in [24, Définition 3.45]. Then, by (1.2), (1.3), and (1.4), we can check that the pair

$$(J, \lambda) = (\mathcal{O}_{L_\zeta}^\times U_{\mathfrak{A}}^1, \theta_{D,m,\eta}|_{\mathcal{O}_{L_\zeta}^\times U_{\mathfrak{A}}^1})$$

is a maximal simple type of level greater than 0 for G with respect to the simple stratum $[\mathfrak{A}, 1, 0, \varphi_\zeta^{-1}]$ in the sense of [26, Sections 4.1 and 5.1]. Hence, $\pi_{D,m,\eta}$ is a supercuspidal representation by [26, Théorème 5.2]. We see $c(\pi_{D,m,\eta}) = n + 1$ by using [6, Theorem 3.3.8] (cf. the proof of [2, Proposition 2.6]). \square

PROPOSITION 1.3

The map

$$\Phi: \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times \rightarrow \mathcal{A}_{D,m}^{n+1}, \quad \eta \mapsto \pi_{D,m,\eta}$$

is a bijection.

Proof

We show the injectivity. We take

$$\eta = (\zeta, \chi, c), \quad \eta' = (\zeta', \chi', c') \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$$

such that $\pi_{D,m,\eta} \simeq \pi_{D,m,\eta'}$. By [28, Corollary 7.3], we see that $\zeta = \zeta'$ and $\chi = \chi'$. By $\pi_{D,m,\eta} \simeq \pi_{D,m,\eta'}$, there exists $g_0 \in G$ such that

$$(1.5) \quad \text{Hom}_{L_\zeta^\times U_{\mathfrak{A}}^1 \cap (L_\zeta^\times U_{\mathfrak{A}}^1)^{g_0}}(\theta_{D,m,\eta}, \theta_{D,m,\eta'}^{g_0}) \neq 0.$$

This implies that

$$\text{Hom}_{U_{\mathfrak{A}}^1 \cap (U_{\mathfrak{A}}^1)^{g_0}}(\theta_{D,m,\eta}, \theta_{D,m,\eta}^{g_0}) \neq 0,$$

since $\theta_{D,m,\eta}$ coincides with $\theta_{D,m,\eta'}$ on $U_{\mathfrak{A}}^1$. Then we have $g_0 \in L_\zeta^\times U_{\mathfrak{A}}^1$ by [25, Proposition 2.10]. Hence, we have $\eta = \eta'$ by (1.5).

We show the surjectivity. Let $\pi \in \mathcal{A}_{D,m}^{n+1}$. By [27, Théorème 5.21 and Corollaire 5.22], we have $\pi \simeq c\text{-Ind}_J^G \theta$ for a maximal simple type (J, λ) with a simple stratum $[\mathfrak{A}_0, l, 0, \beta]$ and an extension θ of λ to

$$\bar{J} = \{g \in G \mid J^g = J, \lambda^g = \lambda\}.$$

Let $\mathfrak{P}_{\mathfrak{A}_0}$ be the Jacobson radical of \mathfrak{A}_0 . By [2, Proposition 2.6], we have $l = 1$ and $\mathfrak{p}_K \mathfrak{A}_0 = \mathfrak{P}_{\mathfrak{A}_0}$. Replacing \mathfrak{A}_0 by its conjugate, we may assume that $\mathfrak{A}_0 = \mathfrak{A}$ (cf. [6, (1.5.2) Proposition (ii)]). We put $U_K^1 = 1 + \mathfrak{p}_K$. We take $\zeta \in \mu_{q-1}(K)$ such that

$$(1.6) \quad \text{Nrd}_{A/K}(\beta^{-1}) \equiv (-1)^{n-1} \zeta \varpi \pmod{U_K^1}.$$

We write $\beta^{-1} = \varphi_\zeta a u$, where $u \in U_{\mathfrak{A}}^1$ and $a = (a_{ij})_{1 \leq i,j \leq m} \in \mathfrak{A}^\times$ is a diagonal matrix such that $a_{ii} \in \mu_{q^r-1}(K_r)$ for $1 \leq i \leq m$. We put $b = \prod_{1 \leq i \leq m} a_{ii}$. Then we see that $\varphi_\zeta a$ is conjugate to

$$\begin{pmatrix} \mathbf{0} & I_{m-1} \\ \varphi_{D,\zeta} b & \mathbf{0} \end{pmatrix} \in A$$

by an element of \mathfrak{A}^\times . Further, this is conjugate to φ_ζ by an element of \mathfrak{A}^\times , since we have $\text{Nrd}_{D/K}(b) = 1$ by (1.6). Hence, we may assume that $\beta = \varphi_\zeta^{-1}$.

We see that θ is a character by the definition of maximal simple type (cf. [26, Sections 4.1 and 5.1], [25, Section 2.2], and (1.3)). We define $\chi \in (k^\times)^\vee$ by $\chi(\bar{x}) = \theta(x)$ for $x \in \mu_{q-1}(K)$ and put $c = (-1)^{m-1} \theta(\varphi_\zeta)$. Then we have $\pi = \Phi(\eta)$ for $\eta = (\zeta, \chi, c)$. \square

REMARK 1.4

If $m = n$, then Proposition 1.3 follows from [12, Sections 2.1 and 2.2]. Actually, the proof of the injectivity of Φ is logically unnecessary in this article. We need only the surjectivity of Φ in the case $m = n$ for the proof of Theorem 3.5, and the injectivity of Φ follows from Theorem 3.5 and the injectivity of Φ in the case $m = n$.

2. Formula for character

In this section, we give formulas for characters of the simple supercuspidal representations of G at some elements.

DEFINITION 2.1 (cf. [11, 1.1 Remark])

Let $g \in G$. Let $f_g(x) \in K[x]$ be the reduced characteristic polynomial of g over K .

- (1) We say that g is regular if $f_g(x)$ is separable.
- (2) We say that g is quasiregular if $f_g(x)$ has no repeated irreducible factor over K .
- (3) We say that g is elliptic if the minimal polynomial of g is irreducible over K .

We write G_{reg} , G_{qr} , and $G_{\text{qr}}^{\text{ell}}$ for the set of the regular elements of G , the set of the quasiregular elements of G , and the set of the elliptic quasiregular elements of G , respectively. Let $\mathcal{H}(G)$ be the space of locally constant compactly supported functions $G \rightarrow \mathbb{C}$. We take a Haar measure $d\mu_G$ on G . We recall the following fact due to Bushnell–Henniart.

PROPOSITION 2.2

For an irreducible smooth representation π of G , there is a locally constant function $\text{tr } \pi: G_{\text{qr}} \rightarrow \mathbb{C}$ characterized by

$$\text{tr } \pi(f) = \int_G \text{tr } \pi(g) f(g) d\mu_G$$

for all $f \in \mathcal{H}(G)$ with support contained in G_{qr} .

Proof

This is proved in [7, (A.11) Corollary] in the case where $G = \text{GL}_n(K)$. The same arguments work also in our situation (cf. [11, 1.1 Remark and Proposition]). \square

For an irreducible smooth representation π of G , let $\text{tr } \pi$ be the function in Proposition 2.2, which we call the character of π . Let $\eta = (\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$.

LEMMA 2.3

We have

$$\text{tr } \pi_{D,m,\eta}(g) = \sum_{x \in G/L_\zeta^\times U_{\mathfrak{A}}^1, x^{-1}gx \in L_\zeta^\times U_{\mathfrak{A}}^1} \theta_{D,m,\eta}(x^{-1}gx)$$

for $g \in G_{\text{qr}}^{\text{ell}}$.

Proof

This follows from [11, 1.1 Remark and (1.2.2)] (cf. [7, (A.14)]). \square

We put $U_K^1 = 1 + \mathfrak{p}_K \subset \mathcal{O}_K^\times$. We set $g_u = \varphi_\zeta(1 + \varphi_\zeta u) \in \mathfrak{A}$ for $u \in \mathfrak{A}$.

LEMMA 2.4

Let $u \in \mathfrak{A}$. We set

$$f_{g_u}(x) = x^n + \sum_{i=0}^{n-1} a_i(u)x^i \in K[x].$$

Then, we have

$$a_i(u) \in \mathfrak{p}_K \quad \text{for } 1 \leq i \leq n-1, \quad -\frac{a_0(u)}{\zeta\varpi} \in U_K^1.$$

Furthermore, g_u is an elliptic quasiregular element.

Proof

We see that $a_i(u) \in \mathfrak{p}_K$ for $0 \leq i \leq n-1$, since the left multiplication on $\mathcal{O}_D/\varpi\mathcal{O}_D$ by g_u acts nilpotently. We have

$$a_0(u) = (-1)^n \text{Nrd}_{A/K}(g_u) \in \mathfrak{p}_K \setminus \mathfrak{p}_K^2.$$

Hence, the former assertion follows from (1.1) and $\text{Nrd}_{A/K}(1 + \varphi_\zeta u) \in U_K^1$. By Eisenstein’s irreducibility criterion, $f_{g_u}(x)$ is irreducible over K . Therefore, the latter assertion follows. \square

LEMMA 2.5

Let $g \in G$ and $u \in \mathfrak{A}$. Assume that $g^{-1}g_u g \in \mathfrak{P}_{\mathfrak{A}}$. Then we have $g \in L_\zeta^\times \mathfrak{A}^\times$.

Proof

We take $l \in \mathbb{Z}$ and $s = (s_{i,j})_{1 \leq i,j \leq m} \in \mathfrak{A} \setminus \mathfrak{P}_{\mathfrak{A}}$ such that $g = \varphi_\zeta^l s$. We put $u' = \varphi_\zeta^{-l} u \varphi_\zeta^l \in \mathfrak{A}$. Then we have

$$g^{-1}g_u g = s^{-1}g_{u'}s \in \mathfrak{P}_{\mathfrak{A}}.$$

Hence, we have $g_{u'}s\mathfrak{A} \subset s\varphi_\zeta\mathfrak{A}$. This implies that $g_{u'}s\mathfrak{A} = s\varphi_\zeta\mathfrak{A}$, since we have

$$\text{Nrd}_{A/K}(\varphi_\zeta^{-1}s^{-1}g_{u'}s) \in \mathcal{O}_K^\times.$$

Therefore, we have $\varphi_\zeta s\mathfrak{A} = s\varphi_\zeta\mathfrak{A}$ in $\mathfrak{P}_{\mathfrak{A}}/\mathfrak{P}_{\mathfrak{A}}^2$. This implies that $s_{i,i} \in \mathcal{O}_D^\times$ if and only if $s_{i+1,i+1} \in \mathcal{O}_D^\times$ for each $1 \leq i \leq m-1$. Then we have $s_{i,i} \in \mathcal{O}_D^\times$ for all $1 \leq i \leq m$, since $s \notin \mathfrak{P}_{\mathfrak{A}}$. Hence, we have $s \in \mathfrak{A}^\times$. This shows the claim. \square

We put $n_q = (n, q-1)$. For $a \in k$, we put

$$G_n(\chi, \psi, a) = \sum_{x \in \mu_{n_q}(k)} \chi(x)\psi(ax).$$

Note that $G_{q-1}(\chi, \psi, 1)$ is a usual Gauss sum, for which we write $G(\chi, \psi)$.

PROPOSITION 2.6

For $u \in \mathfrak{A}$, we have

$$\text{tr } \pi_{D,m,\eta}(g_u) = (-1)^{m-1} c G_n(\chi, \psi, \overline{\text{Trd}_{A/K}(u)}).$$

Proof

For $\lambda \in \mu_{q-1}(K)$, let

$$g(\lambda) = \text{diag}(1, \lambda, \dots, \lambda^{m-1}) \in G.$$

We put

$$H = \{dg(\lambda) \in G \mid d \in \mu_{q^r-1}(K_r), \lambda \in \mu_{q-1}(K), d^{q^s-1} = \lambda^m\}.$$

Note that λ in the definition of H automatically belongs to $\mu_{n_q}(K)$. Then, we can check that H is equal to

$$\{x = \text{diag}(d_1, \dots, d_m) \in G \mid x^{-1}g_u x \in L_\zeta^\times U_{\mathfrak{A}}^1, d_i \in \mu_{q^r-1}(K_r) \text{ for all } i\}.$$

Hence, we see that

$$\{x \in G \mid x^{-1}g_u x \in L_\zeta^\times U_{\mathfrak{A}}^1\} = HL_\zeta^\times U_{\mathfrak{A}}^1$$

by Lemma 2.5. We have the isomorphism

$$HL_\zeta^\times U_{\mathfrak{A}}^1 / L_\zeta^\times U_{\mathfrak{A}}^1 \xrightarrow{\sim} \mu_{n_q}(k), \quad dg(\lambda) \mapsto \bar{\lambda}.$$

Hence, the claim follows from Lemmas 2.3 and 2.4. \square

LEMMA 2.7

The element $1 + \varphi_\zeta$ is elliptic quasiregular.

Proof

The element φ_ζ is elliptic quasiregular by Lemma 2.4. Hence, the claim follows. \square

LEMMA 2.8

Let $g \in G$ and $\lambda \in \mu_{q-1}(K)$. Assume that $g^{-1}(1 + \varphi_{\zeta\lambda})g \in L_\zeta^\times U_{\mathfrak{A}}^1$. Then we have $g \in L_\zeta^\times \mathfrak{A}^\times$.

Proof

By the assumption, we have $g^{-1}(1 + \varphi_{\zeta\lambda})g = \lambda_0(1 + v)$ with $\lambda_0 \in \mu_{q-1}(K)$ and $v \in \mathfrak{P}_{\mathfrak{A}}$. Then we have

$$\lambda_0 = \lim_{l \rightarrow \infty} (\lambda_0(1 + v))^{q^l} = \lim_{l \rightarrow \infty} (g^{-1}(1 + \varphi_{\zeta\lambda})g)^{q^l} = 1.$$

Hence, we have $g^{-1}\varphi_{\zeta\lambda}g = v \in \mathfrak{P}_{\mathfrak{A}}$. Therefore, the claim follows from Lemma 2.5 and $L_\zeta^\times \mathfrak{A}^\times = L_{\zeta\lambda}^\times \mathfrak{A}^\times$. \square

For a finite field extension k' of k , an additive character $\psi' \in k'^\vee$, a positive integer l , and $a \in k'^\times$, we put

$$K_{l,a}(\psi') = \sum_{\zeta_1 \cdots \zeta_l = a, \zeta_i \in k'^\times} \psi' \left(\sum_{i=1}^l \zeta_i \right).$$

This is a generalized Kloosterman sum (cf. [14, Sommes trig. 7.1]).

PROPOSITION 2.9

For $\lambda \in \mu_{q-1}(K)$, we have

$$(2.1) \quad \text{tr } \pi_{D,m,\eta}(1 + \varphi_{\zeta\lambda}) = (-1)^{n-m} K_{n,\bar{\lambda}}(\psi).$$

Proof

Let $\lambda \in \mu_{q-1}(K)$. We have

$$\text{tr } \pi_{D,m,\eta}(1 + \varphi_{\zeta\lambda}) = \sum_{g \in L_{\zeta}^{\times} \mathfrak{A}^{\times} / L_{\zeta}^{\times} U_{\mathfrak{A}}^1} \theta_{D,m,\eta}(g^{-1}(1 + \varphi_{\zeta\lambda})g)$$

by Lemmas 2.3, 2.7, and 2.8. We can check that

$$\text{tr}_{A/D}(\varphi_{\zeta}^{-1}g^{-1}\varphi_{\zeta\lambda}g) \equiv \varphi_{D,\zeta}^{-1}a_{m,m}^{-1}\varphi_{D,\zeta\lambda}a_{1,1} + \sum_{i=1}^{m-1} a_{i,i}^{-1}a_{i+1,i+1} \pmod{\mathfrak{p}_D}$$

for $g = (a_{i,j})_{1 \leq i,j \leq m} \in \mathfrak{A}^{\times}$. For $d \in \mathcal{O}_D$, let \bar{d} denote the image of d by $\mathcal{O}_D \rightarrow k_r$. We have the bijection

$$L_{\zeta}^{\times} \mathfrak{A}^{\times} / L_{\zeta}^{\times} U_{\mathfrak{A}}^1 \xrightarrow{\sim} \{(\zeta_1, \dots, \zeta_m) \in (k_r^{\times})^{\oplus m} \mid \text{Nr}_{k_r/k}(\zeta_1 \cdots \zeta_m) = \bar{\lambda}\},$$

$$(a_{i,j})_{1 \leq i,j \leq m} \mapsto ((\bar{a}_{i,i}^{-1}\bar{a}_{i+1,i+1})_{1 \leq i \leq m-1}, \overline{\varphi_{D,\zeta}^{-1}a_{m,m}^{-1}\varphi_{D,\zeta\lambda}a_{1,1}}),$$

where $(a_{i,j})_{1 \leq i,j \leq m} \in \mathfrak{A}^{\times}$. Hence, we have

$$\text{tr } \pi_{D,m,\eta}(1 + \varphi_{\zeta\lambda}) = \sum_{\text{Nr}_{k_r/k}(x) = \bar{\lambda}} K_{m,x}(\psi \circ \text{Tr}_{k_r/k}).$$

Further, we have

$$\begin{aligned} \sum_{\text{Nr}_{k_r/k}(x) = \bar{\lambda}} K_{m,x}(\psi \circ \text{Tr}_{k_r/k}) &= (-1)^{m-1} \sum_{\text{Nr}_{k_n/k}(y) = \bar{\lambda}} \psi \circ \text{Tr}_{k_n/k}(y) \\ &= (-1)^{n-m} K_{n,\bar{\lambda}}(\psi) \end{aligned}$$

by [14, (7.2.5)]. Thus, we have proved the claim. □

3. Description of LJLC

In this section, we give an explicit description of the LJLC. First, we recall the statement of the LJLC. Let $\mathcal{A}_{D,m}^{\square}$ denote the set of equivalent classes of irreducible essentially square-integrable smooth representations of $\text{GL}_m(D)$. We put $G' = \text{GL}_n(K)$. We say that $g \in G_{\text{qr}}$ and $g' \in G'_{\text{qr}}$ correspond if $f_g(x) = f_{g'}(x)$.

THEOREM 3.1 ([3, Théorème 5.1] and [15, Théorème principal])

There exists a bijection

$$\text{JL}_{D,m} : \mathcal{A}_{D,m}^{\square} \rightarrow \mathcal{A}_{K,n}^{\square},$$

which is uniquely characterized by the character relation

$$\text{tr } \pi(g) = (-1)^{n-m} \text{tr } \text{JL}_{D,m}(\pi)(g')$$

for $\pi \in \mathcal{A}_{D,m}^\square$ and any corresponding $g \in G_{\text{reg}}$ and $g' \in G'_{\text{reg}}$.

The bijection $\text{JL}_{D,m}$ in Theorem 3.1 is called the LJLC.

COROLLARY 3.2

The bijection $\text{JL}_{D,m}$ satisfies the character relation

$$\text{tr } \pi(g) = (-1)^{n-m} \text{tr } \text{JL}_{D,m}(\pi)(g')$$

for $\pi \in \mathcal{A}_{D,m}^\square$ and any corresponding $g \in G_{\text{qr}}$ and $g' \in G'_{\text{qr}}$.

Proof

The complement of G^{reg} in G is a Zariski-closed subset of G with strictly smaller dimension than G . This implies that G^{reg} is dense in G in the ϖ -adic topology; hence, it is also dense in G^{qr} . Therefore, the claim follows from Theorem 3.1, since $\text{tr } \pi$ and $\text{tr } \text{JL}_{D,m}(\pi)$ for $\pi \in \mathcal{A}_{D,m}^\square$ are locally constant functions on G_{qr} and G'_{qr} , respectively. □

LEMMA 3.3

The function

$$G_{n,\chi,\psi} : k \rightarrow \mathbb{C}, \quad a \mapsto G_n(\chi, \psi, a)$$

is not equal to the zero function on k .

Proof

Let $F_{n,\chi}$ be the function on k defined by

$$F_{n,\chi}(x) = \begin{cases} \chi(x) & \text{if } x \in \mu_{n_q}(k), \\ 0 & \text{if } x \notin \mu_{n_q}(k). \end{cases}$$

Then $G_{n,\chi,\psi}$ is regarded as the Fourier transformation of $F_{n,\chi}$ with respect to ψ . Hence, the claim follows from the Fourier inversion formula. □

LEMMA 3.4

For any $a' \in k^\times \setminus \{1\}$, there is $a \in k^\times$ such that $K_{n,a}(\psi) \neq K_{n,aa'}(\psi)$.

Proof

Let $a' \in k^\times \setminus \{1\}$. We take $\chi' \in (k^\times)^\vee$ such that $\chi'(a') \neq 1$. Then we have

$$\begin{aligned} \sum_{a \in k^\times} \chi'(aa')(K_{n,a}(\psi) - K_{n,aa'}(\psi)) &= (\chi'(a') - 1) \sum_{a \in k^\times} \chi'(a)K_{n,a}(\psi) \\ &= (\chi'(a') - 1)G(\chi', \psi)^n \neq 0 \end{aligned}$$

by [14, (7.1.6)]. Hence, the claim follows. □

THEOREM 3.5

For $\eta = (\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$, we have $\mathbf{JL}_{D,m}(\pi_{D,m,\eta}) = \pi_{K,n,\eta}$.

Proof

Let $\mathbf{LJ}_{D,m}$ denote the inverse of $\mathbf{JL}_{D,m}$. The map $\mathbf{LJ}_{D,m}$ preserves conductors and supercuspidality by [2, Theorems 2.2(e), 2.2(i)]. Let $\eta' = (\zeta', \chi', c') \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$. By Proposition 1.3, there exists $\eta = (\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$ such that $\mathbf{LJ}_{D,m}(\pi_{K,n,\eta'}) = \pi_{D,m,\eta}$.

We have to show $\eta = \eta'$. We have $\chi = \chi'$, because $\mathbf{JL}_{D,m}$ preserves central characters. We write \mathfrak{A}' and φ'_ζ for \mathfrak{A} and φ_ζ , respectively, in the case $m = n$. By Proposition 2.9 and Corollary 3.2, we have

$$K_{n,\lambda}(\psi) = (-1)^{n-m} \operatorname{tr} \pi_{D,m,\eta}(1 + \varphi_{\zeta\lambda}) = \operatorname{tr} \pi_{K,n,\eta'}(1 + \varphi'_{\zeta\lambda}) = K_{n,\lambda\zeta\zeta'^{-1}}(\psi)$$

for $\lambda \in \mu_{q-1}(K)$. Hence, we have $\zeta = \zeta'$ by Lemma 3.4.

Finally, we show $c = c'$. By Lemma 3.3, we can take $a \in k$ such that $G_n(\chi, \psi, a) \neq 0$. We take $\xi_0 \in k_r$ such that $\operatorname{Tr}_{k_r/k}(\xi_0) = a$. Let $\xi \in \mathcal{O}_D$ be a lifting of ξ_0 . We set

$$u_\xi = \operatorname{diag}(\xi, 0, \dots, 0) \in \mathfrak{A}.$$

We simply write g_ξ for g_{u_ξ} . Let $\{a_i(u_\xi)\}_{0 \leq i \leq n-1}$ be as in Lemma 2.4. By Lemma 2.4, the elements

$$\alpha_i = -\frac{a_i(u_\xi)}{\zeta\varpi} \quad \text{for } 1 \leq i \leq n-1, \quad \alpha_n = -\frac{a_0(u_\xi)}{\zeta\varpi} + 1$$

are contained in \mathcal{O}_K . We set

$$u_\alpha = \sum_{i=1}^n \varphi_\zeta^{i-1} \operatorname{diag}(\alpha_i, 0, \dots, 0) \in \mathfrak{A}', \quad g_\alpha = \varphi'_\zeta(1 + \varphi'_\zeta u_\alpha) \in G'.$$

Then we have

$$f_{g_\alpha}(x) = x^n - \sum_{i=1}^{n-1} \zeta\varpi\alpha_i x^i - \zeta\varpi(1 + \alpha_n\zeta\varpi) = f_{g_\xi}(x).$$

Hence, $g_\xi \in G_{\text{qr}}$ and $g_\alpha \in G'_{\text{qr}}$ correspond. We have $\overline{\operatorname{tr}(u_\alpha)} = \overline{\operatorname{Tr}_{A/K}(u_\xi)} = a$, since

$$\operatorname{tr}(u_\alpha) \equiv \alpha_1 = (-1)^n \frac{\operatorname{Tr}_{A/K}(g_\xi^{-1}) \operatorname{Nrd}_{A/K}(g_\xi)}{\zeta\varpi} \equiv \operatorname{Tr}_{D/K}(\xi) \pmod{\mathfrak{p}_K}.$$

Therefore, we have

$$c = (-1)^{m-1} \frac{\operatorname{tr} \pi_{D,m,\eta}(g_\xi)}{G_n(\chi, \psi, a)} = (-1)^{n-1} \frac{\operatorname{tr} \pi_{K,n,\eta'}(g_\alpha)}{G_n(\chi, \psi, a)} = c'$$

by Proposition 2.6 and Corollary 3.2. □

4. Another proof of Theorem 3.5

Let $\eta = (\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$. We simply write $\epsilon(\pi_{D,m,\eta}, \psi_K)$ for the value of $\epsilon(\pi_{D,m,\eta}, s, \psi_K)$ at $1/2$. For any smooth character $\xi: K^\times \rightarrow \mathbb{C}^\times$, let ξ_A denote the composite $\xi \circ \text{Nrd}_{A/K}$.

The following lemma is a special case of a formula which represents Godement–Jacquet local constants with respect to non-Abelian congruence Gauss sums. Such a formula is studied in [5] and [6]. If $m = n$, then this lemma is just [12, 2.2 Lemma].

LEMMA 4.1

1. We have

$$\epsilon(\pi_{D,m,\eta}, \psi_K) = (-1)^{n-1}c.$$

2. For any tamely ramified character ξ of K^\times , we have

$$\epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K) = \xi((-1)^{n-1}\zeta\varpi)\epsilon(\pi_{D,m,\eta}, \psi_K).$$

Proof

Let ξ be a tamely ramified character of K^\times . Assume that the characteristic of K equals 0. We set

$$\rho = \text{Ind}_{L_\zeta^\times U_{\mathfrak{q}}^1}^{L_\zeta^\times \mathfrak{A}^\times}(\theta_{D,m,\eta} \otimes \xi_A).$$

We see that ρ is an irreducible admissible representation of $L_\zeta^\times \mathfrak{A}^\times$ by Lemma 1.2 and is nondegenerate in the sense of [6, p. 228] by (1.2).

Let ψ_K^0 be the standard continuous additive character of K defined in [6, 2.1]. In [6, (2.3.6)], the non-Abelian congruence Gauss sum $\tau(\rho)$ is defined for ρ and ψ_K^0 . However, it can be defined for any nontrivial additive character ψ'_K of K as in [5, (3.4)] (cf. [5, Remarks after Theorem (3.7)]), for which we write $\tau(\rho, \psi'_K)$.

We take an element $a \in K^\times$ such that $\psi_K(x) = \psi_K^0(ax)$ for $x \in K$. Let ω_{ρ^\vee} denote the central character of ρ^\vee . By [5, (3.4)] and [17, (3.3.5)], we have formulas

$$(4.1) \quad \begin{aligned} \tau(\rho^\vee, \psi_K) &= \omega_{\rho^\vee}(a^{-1})\tau(\rho^\vee, \psi_K^0) = (\theta_{D,m,\eta} \otimes \xi_A)(a)\tau(\rho^\vee, \psi_K^0), \\ \epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K) &= (\theta_{D,m,\eta} \otimes \xi_A)(a)\epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K^0), \end{aligned}$$

respectively. By these formulas and [6, Theorem (3.3.8)(iv)], we obtain

$$(4.2) \quad \epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K) = (-1)^{n-m}\tau(\rho^\vee, \psi_K)\text{Nf}(\rho)^{-1/2}.$$

We put $\psi_A = \psi_K \circ \text{Trd}_{A/K}$. Then we have

$$(4.3) \quad \tau(\rho^\vee, \psi_K) = \text{Nf}(\rho)^{1/2}(\theta_{D,m,\eta} \otimes \xi_A)^\vee(\varphi_\zeta^{-1})\psi_A(\varphi_\zeta^{-1})$$

by (4.1) and [6, (2.7.4)]. We obtain

$$\begin{aligned} \epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K) &= (-1)^{n-m}(\theta_{D,m,\eta} \otimes \xi_A)^\vee(\varphi_\zeta^{-1})\psi_A(\varphi_\zeta^{-1}) \\ &= (-1)^{n-1}\xi((-1)^{n-1}\zeta\varpi)c \end{aligned}$$

by (1.1), (1.2), (4.2), and (4.3). The same arguments work also in the equal characteristic case (cf. [5, Remarks after Theorem (3.7)]). Hence, we obtain the claims. \square

Another proof of Theorem 3.5

Let $\eta' = (\zeta', \chi', c') \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$. As in the first proof of Theorem 3.5, we see that there exists a triple $\eta = (\zeta, \chi', c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$ such that $\pi_{D,m,\eta} \simeq \text{LJ}_{D,m}(\pi_{K,n,\eta'})$. We prove $\zeta = \zeta'$. Let ξ be any tamely ramified character of K^\times . We write A' for A in the case $m = n$. Since $\text{JL}_{D,m}$ preserves character twists, we have

$$(4.4) \quad \epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K) = \epsilon(\pi_{K,n,\eta'} \otimes \xi_{A'}, \psi_K)$$

by [2, Theorem 2.2(i)] (cf. [15, Section B.j.1]). We have $\epsilon(\pi_{K,n,\eta'}, \psi_K) \neq 0$ by Lemma 4.1.1. By Lemma 4.1.2 and (4.4), we have

$$\xi((-1)^{n-1} \zeta \varpi) = \frac{\epsilon(\pi_{D,m,\eta} \otimes \xi_A, \psi_K)}{\epsilon(\pi_{D,m,\eta}, \psi_K)} = \frac{\epsilon(\pi_{K,n,\eta'} \otimes \xi_{A'}, \psi_K)}{\epsilon(\pi_{K,n,\eta'}, \psi_K)} = \xi((-1)^{n-1} \zeta' \varpi).$$

Hence, we obtain $\xi(\zeta \zeta'^{-1}) = 1$. This implies that $\zeta = \zeta'$. By Lemma 4.1.1 and (4.4), we have

$$c = (-1)^{n-1} \epsilon(\pi_{D,m,\eta}, \psi_K) = (-1)^{n-1} \epsilon(\pi_{K,n,\eta'}, \psi_K) = c'.$$

Therefore, we have $\eta = \eta'$. \square

5. Invariance of endoclass

We show that endoclasses for the simple supercuspidal representations are invariant under the LJLC. Let $\mathcal{E}(K)$ be the set of endoclasses of ps-characters over K in the sense of [4, Definition 1.10]. Then we have a map

$$\Theta_G : \mathcal{A}_{D,m}^\square \rightarrow \mathcal{E}(K)$$

that was constructed in [4, Section 9.3]. Broussous–Sécherre–Stevens [4, Conjecture 9.5] conjecture that

$$\Theta_G(\pi) = \Theta_{G'}(\text{JL}_{D,m}(\pi))$$

for $\pi \in \mathcal{A}_{D,m}^\square$. The following proposition verifies the conjecture for the simple supercuspidal representations.

PROPOSITION 5.1

For any simple supercuspidal representation π of G , we have

$$\Theta_G(\pi) = \Theta_{G'}(\text{JL}_{D,m}(\pi)).$$

Proof

Let π be a simple supercuspidal representation of G . We take $\eta = (\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$ such that $\pi = \pi_{D,m,\eta}$. Then $\Theta_G(\pi_{D,m,\eta})$ is the endoclass

of the ps-character defined by

$$([\mathfrak{A}, 1, 0, \varphi_{\zeta}^{-1}], \theta_{D,m,\eta}|_{U_{\mathfrak{A}}^1}).$$

We have $\text{JL}_{D,m}(\pi_{D,m,\eta}) = \pi_{K,n,\eta}$ by Theorem 3.5. We define \mathfrak{A}' and φ'_{ζ} for $M_n(K)$ similarly as \mathfrak{A} and φ_{ζ} for $M_m(D)$. Then $\Theta_G(\pi_{K,n,\eta})$ is the endoclass of the ps-character defined by

$$([\mathfrak{A}', 1, 0, \varphi'^{-1}_{\zeta}], \theta_{K,n,\eta}|_{U_{\mathfrak{A}'}^1}).$$

Let $L_{\zeta,0}$ be a finite extension of K generated by an element $\varphi_{\zeta,0}$ such that $\varphi_{\zeta,0}^n = \zeta\varpi$. Let $\mathcal{C}_K(\varphi_{\zeta,0}^{-1}, 0)$ be the set of simple characters corresponding to the simple pair $[\varphi_{\zeta,0}^{-1}, 0]$ over K defined in [24, p. 371]. Then we have bijective transfer maps

$$\tau_{\mathfrak{A},0,\varphi_{\zeta,0}^{-1}}: \mathcal{C}_K(\varphi_{\zeta,0}^{-1}, 0) \rightarrow \mathcal{C}(\varphi_{\zeta}^{-1}, 0, \mathfrak{A}),$$

$$\tau_{\mathfrak{A}',0,\varphi_{\zeta,0}^{-1}}: \mathcal{C}_K(\varphi_{\zeta,0}^{-1}, 0) \rightarrow \mathcal{C}(\varphi'^{-1}_{\zeta}, 0, \mathfrak{A}')$$

by [24, Théorème 3.53]. We have

$$\tau_{\mathfrak{A},0,\varphi_{\zeta,0}^{-1}}^{-1}(\theta_{D,m,\eta}|_{U_{\mathfrak{A}}^1}) = \tau_{\mathfrak{A}',0,\varphi_{\zeta,0}^{-1}}^{-1}(\theta_{K,n,\eta}|_{U_{\mathfrak{A}'}^1})$$

by (1.4). Hence, we have the claim. \square

Acknowledgments. The authors thank the referees for suggestions for improvement. This work was supported by Japan Society for the Promotion of Science KAKENHI grants 26707003 (Imai) and 15K17506 (Tsushima).

References

- [1] M. Adrian and B. Liu, *Some results on simple supercuspidal representations of $\text{GL}_n(F)$* , J. Number Theory **160** (2016), 117–147. MR 3425202. DOI 10.1016/j.jnt.2015.08.002.
- [2] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld, “Depth and the local Langlands correspondence” in *Arbeitstagung Bonn 2013*, Progr. Math. **319**, Birkhäuser/Springer, Cham, 17–41. MR 3618046.
- [3] A. I. Badulescu, *Correspondance de Jacquet–Langlands pour les corps locaux de caractéristique non nulle*, Ann. Sci. Éc. Norm. Supér. (4) **35** (2002), 695–747. MR 1951441. DOI 10.1016/S0012-9593(02)01106-0.
- [4] P. Broussous, V. Sécherre, and S. Stevens, *Smooth representations of $\text{GL}_m(D)$, V: Endo-classes*, Doc. Math. **17** (2012), 23–77. MR 2889743.
- [5] C. J. Bushnell, “Gauss sums and local constants for $\text{GL}(N)$ ” in *L-functions and Arithmetic (Durham, 1989)*, London Math. Soc. Lecture Note Ser. **153**, Cambridge Univ. Press, Cambridge, 1991, 61–73. MR 1110390. DOI 10.1017/CBO9780511526053.004.

- [6] C. J. Bushnell and A. Fröhlich, *Non-abelian congruence Gauss sums and p -adic simple algebras*, Proc. Lond. Math. Soc. (3) **50** (1985), 207–264. MR 0772712. DOI 10.1112/plms/s3-50.2.207.
- [7] C. J. Bushnell and G. Henniart, *Local tame lifting for $GL(N)$, I: Simple characters*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 105–233. MR 1423022.
- [8] ———, *Correspondance de Jacquet–Langlands explicite, II: Le cas de degré égal à la caractéristique résiduelle*, Manuscripta Math. **102** (2000), 211–225. MR 1771941. DOI 10.1007/s002291020211.
- [9] ———, *Local tame lifting for $GL(n)$, III: Explicit base change and Jacquet–Langlands correspondence*, J. Reine Angew. Math. **580** (2005), 39–100. MR 2130587. DOI 10.1515/crll.2005.2005.580.39.
- [10] ———, *The Local Langlands Conjecture for $GL(2)$* , Grundlehren Math. Wiss. **335**, Springer, Berlin, 2006. MR 2234120. DOI 10.1007/3-540-31511-X.
- [11] ———, *The essentially tame Jacquet–Langlands correspondence for inner forms of $GL(n)$* , Pure Appl. Math. Q. **7** (2011), 469–538. MR 2848585. DOI 10.4310/PAMQ.2011.v7.n3.a2.
- [12] ———, *Langlands parameters for epipelagic representations of GL_n* , Math. Ann. **358** (2014), 433–463. MR 3158004. DOI 10.1007/s00208-013-0962-x.
- [13] C. J. Bushnell and P. C. Kutzko, *The Admissible Dual of $GL(N)$ via Compact Open Subgroups*, Ann. of Math. Stud. **129**, Princeton Univ. Press, Princeton, 1993. MR 1204652. DOI 10.1515/9781400882496.
- [14] P. Deligne, *Cohomologie étale*, Lecture Notes in Math. **569**, Springer, New York, 1977. MR 0463174.
- [15] P. Deligne, D. Kazhdan, and M.-F. Vigneras, “Représentations des algèbres centrales simples p -adiques” in *Representations of Reductive Groups over a Local Field*, Hermann, Paris, 1984, 33–117. MR 0771672.
- [16] P. Gérardin, “Facteurs locaux des algèbres simples de rang 4, I” in *Reductive Groups and Automorphic Forms, I (Paris, 1976/1977)*, Publ. Math. Univ. Paris VII **1**, Univ. Paris VII, Paris, 1978, 37–77. MR 0680785.
- [17] R. Godement and H. Jacquet, *Zeta Functions of Simple Algebras*, Lecture Notes in Math. **260**, Springer, New York, 1970. MR 0342495.
- [18] G. Henniart, “Correspondance de Jacquet–Langlands explicite, I: Le cas modéré de degré premier” in *Séminaire de Théorie des Nombres, Paris, 1990–91*, Progr. Math. **108**, Birkhäuser Boston, Boston, 1993, 85–114. MR 1263525.
- [19] N. Imai and T. Tsushima, *Affinoids in the Lubin–Tate perfectoid space and simple supercuspidal representations, I: Tame case*, preprint, arXiv:1308.1276v4 [math.NT].
- [20] ———, *Affinoids in the Lubin–Tate perfectoid space and simple supercuspidal representations, II: Wild case*, preprint, arXiv:1603.04693v2 [math.NT].
- [21] K. Kariyama, *Endo-class and the Jacquet–Langlands correspondence*, Kyoto. J. Math. **55** (2015), 299–320. MR 3356075. DOI 10.1215/21562261-2871767.

- [22] A. Knightly and C. Li, *Simple supercuspidal representations of $GL(n)$* , *Taiwanese J. Math.* **19** (2015), 995–1029. MR 3384676. DOI 10.11650/tjm.19.2015.3853.
- [23] M. Reeder and J.-K. Yu, *Epipelagic representations and invariant theory*, *J. Amer. Math. Soc.* **27** (2014), 437–477. MR 3164986. DOI 10.1090/S0894-0347-2013-00780-8.
- [24] V. Sécherre, *Représentations lisses de $GL(m, D)$, I: Caractères simples*, *Bull. Soc. Math. France* **132** (2004), 327–396. MR 2081220.
- [25] ———, *Représentations lisses de $GL(m, D)$, II: β -extensions*, *Compos. Math.* **141** (2005), 1531–1550. MR 2188448. DOI 10.1112/S0010437X05001429.
- [26] ———, *Représentations lisses de $GL_m(D)$, III: Types simples*, *Ann. Sci. Éc. Norm. Supér. (4)* **38** (2005), 951–977. MR 2216835. DOI 10.1016/j.ansens.2005.10.003.
- [27] V. Sécherre and S. Stevens, *Représentations lisses de $GL_m(D)$, IV: Représentations supercuspidales*, *J. Inst. Math. Jussieu* **7** (2008), 527–574. MR 2427423. DOI 10.1017/S1474748008000078.
- [28] ———, *Smooth representations of $GL_m(D)$, VI: Semisimple types*, *Int. Math. Res. Not. IMRN* **2012**, no. 13, 2994–3039. MR 2946230.
- [29] A. J. Silberger and E.-W. Zink, *An explicit matching theorem for level zero discrete series of unit groups of p -adic simple algebras*, *J. Reine Angew. Math.* **585** (2005), 173–235. MR 2164626. DOI 10.1515/crll.2005.2005.585.173.
- [30] P. Xu, *A remark on the simple cuspidal representations of $GL(n, F)$* , preprint, arXiv:1310.3519v2 [math.NT].

Imai: Graduate School of Mathematical Sciences, University of Tokyo, Tokyo, 153-8914, Japan; naoki@ms.u-tokyo.ac.jp

Tsushima: Department of Mathematics and Informatics, Faculty of Science, Chiba University, Chiba, 263-8522, Japan; tsushima@math.s.chiba-u.ac.jp