Canonical Kähler metrics and arithmetics: Generalizing Faltings heights

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Abstract We extend the Faltings modular heights of Abelian varieties to general arithmetic varieties, show direct relations with the Kähler–Einstein geometry, the minimal model program, and Bost–Zhang's heights and give some applications. Along the way, we propose the "arithmetic Yau–Tian–Donaldson conjecture" (the equivalence of a purely *arithmetic* property of a variety and its *metrical* property) and partially confirm it.

1. Introduction

To metrize (schemes or bundles on them) is to "compactify"—as a motto, this expresses the philosophy of Arakelov [1], whose aim was to get a nice intersection theory for arithmetic varieties. Faltings's proof (see [36]) of the Mordell conjecture takes advantage of intersection theory. In particular, he introduced the key invariant—the *Faltings (modular) height* of Abelian varieties.

Our general aim is to show how this motto fairly compatibly fits with the recent studies of canonical Kähler metrics such as Kähler–Einstein metrics and geometric flows toward them. In particular, we discuss the so-called *Yau–Tian–Donaldson correspondence*—the equivalence of the existence of a *canonical Kähler metric* and some variant of geometric invariant theory (GIT) stability. In such studies related to the canonical Kähler metrics, we have occasionally encountered the problem of constructing compactifications of moduli spaces (*K-moduli*[1]). Thus, the author suspects that the motto expressing Arakelov's philosophy should sound familiar to experts of the recent studies of canonical Kähler metrics and related (K-)moduli theory. In other words, it also vaguely gives yet another heuristic "explanation" of why the compactification problem naturally arises in such studies of metrics.

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[1]That is, vaguely speaking, moduli of K-stable (polarized) varieties (see [41], [40], [60], [26], [72], [71], [92], [32], [78], and last year's settlement of the (smoothable) Fano case [93], [57], [76]; for the general K-moduli conjecture see [74, Section 3]; see also Section 3.3 of this article).

The basic invariant in the field is the *Donaldson–Futaki invariant* (see [98], [28]). It captures the asymptotics of the *K-energy* (see [59]) of Kähler metrics along certain variations.

In this article, we unify all the above invariants which originally appeared in different fields—Faltings height, Donaldson–Futaki invariant, and K-energy—into one, namely, the *Arakelov–Donaldson–Futaki invariant* $h_K(\mathcal{X},\mathcal{L},h)$ ($\in \mathbb{R}$) for an arithmetic polarized variety. We also call it *modular height* for brevity. If it would not sound too confusing, *K-modular height* would be a better name in some contexts. We also introduce its siblings invariants and give some applications.

We collect some basic properties of the (K-)modular height below, deliberately stated in a vague form for simpler illustration. The precise meanings are put in later sections.

THEOREM 1.1

The invariant $h_K(\mathcal{X},\mathcal{L},h) \in \mathbb{R}$ (which we introduce in Definition 2.4) for an arithmetic polarized projective scheme $(\mathcal{X},\mathcal{L})$ over $\mathcal{O}_K$, the ring of integers in a number field K,[2] attached with a Hermitian metric h on the complex line bundle $\mathcal{L}(\mathbb{C})$ over $\mathcal{X}(\mathbb{C})$ which is invariant under complex conjugation, satisfies the following statements. We denote the generic fiber of $(\mathcal{X},\mathcal{L})$ as (X,L).

(1) *If $(\mathcal{X},\mathcal{L},h)$ is a polarized Abelian scheme with the cubic metric h, then $h_K(\mathcal{X},\mathcal{L},h)$ essentially coincides with the Faltings height (see [36]) of the generic fiber X. Please see Theorem 2.11 for the details, where we allow bad reductions. We remark that, as a special case of Theorem (5) and later Theorem 2.14, we see that such $(\mathcal{X},\mathcal{L},h)$ minimizes h_K among all metrized integral models.*

(2) *With respect to change of metrics, it behaves as*

$$h_K(\mathcal{X},\mathcal{L},h\cdot e^{-2\varphi}) - h_K(\mathcal{X},\mathcal{L},h) = \frac{(L^{\dim(X)})}{[K:\mathbb{Q}]}\cdot \mu_{\omega_h}(\varphi),$$

where μ denotes the Mabuchi K-energy (see [59]).

(3) *If we birationally change the model $(\mathcal{X},\mathcal{L})$ along some finite closed fibers (while preserving h), then $h_K(\mathcal{X},\mathcal{L},h)$ behaves very similarly to the Donaldson–Futaki invariant in the equicharacteristic situation (see [28], [103], [69] for its definition and formulae). Please find the precise meaning in Section 2.*

(4) *From (2) and (3), it follows that h_K decreases along a combination of arithmetic minimal model program (MMP) with scaling and the Kähler–Ricci flow (which are compatible). Please find the precise meaning in Theorem 2.20.*

(5) *Given a polarized projective variety (X,L) defined over K which possesses Kähler–Einstein metrics (over infinite places), h_K of integral models of (X,L) minimizes at a "minimal-like" integral model over $\mathcal{O}_K$ (defined in terms of birational geometry) with the Kähler–Einstein metric attached. Please find the precise meaning in Theorem 2.14.*

[2]The subscript K of h_K comes from K-stability rather than the base field. Indeed, the quantity h_K remains the same after extension of the scalars.

The invariant h_K can also be seen as a "limited version" of some normalization of Bost–Zhang's height (see [10], [11], [108]) as we explain later.

Recall that [98] and [28] introduced the Donaldson–Futaki invariant for a test configuration, which is a flat isotrivial family (with $\mathbb{C}^*$-action) over $\mathbb{C}$ or a complex disk, and [103] and [69] independently showed that it is an intersection number on the global total space, as a simple application of the (equivariant) Riemann–Roch-type theorems. Statements (1), (2), and (3) of the above theorem explain and extend some of the known facts in the field of canonical Kähler metrics (as well as the author's study of K-stability), while some variants of statements (2), (3), and (4) (see Section 2.4) explain and refine some theorems in Arakelov geometry by [10], [11], [108], and so on.

Indeed, as we show in Section 3 via some asymptotic analysis of *Ray–Singer [82] torsion*, our modular height $h_K(\mathcal{X}, \mathcal{L})$ is exactly what controls the first nontrivial asymptotic behavior of the (Chow) heights of Bost–Zhang (see [10], [11], [108]) of $\mathcal{X}$ embedded by $|\bar{\mathcal{L}}^{\otimes m}|$ with respect to $m \to \infty$. This is yet another important feature of our modular height h_K.

Thus, in particular, via our "unification" h_K with its properties proved in the present article, direct relations among the three quantities below (which appeared in different contexts) follows indirectly:

(a) Faltings's (see [36]) height of arithmetic Abelian varieties,
(b) K-energy (see [59]) and the Donaldson–Futaki invariant (see [28]), and
(c) Bost–Zhang's (see [10], [11], [108]) heights.

That is, we have the following relations.

COROLLARY 1.2

Among the above three invariants, we have the following.

- *(a) ↔ (b) Faltings's height for Abelian varieties is essentially (a special case of) an arithmetic version of the Donaldson–Futaki invariant.*
- *(b) ↔ (c) Mabuchi's K-energy is essentially an infinite place part of a limit of (modified) Bost–Zhang's heights.*
- *(c) ↔ (a) Faltings's height for Abelian varieties is essentially a limit of (modified) Bost–Zhang's heights.*

We again deliberately gave rough statements above, rather than lengthy precise statements, as both the precise statements and proofs will be clear to the readers of Sections 2 and 3. The first relation (a) ↔ (b) follows from Theorem 2.11, combined with Theorem 2.14 and Proposition 2.17. The second relation (b) ↔ (c) follows from Theorem 3.7 (combined with Proposition 2.8). The last relation (c) ↔ (a) follows from Theorem 2.11 combined with Theorem 3.7.

For those who are not really tempted to go through the details, we make brief comments about essential points of the proofs of the above statements. The essential point of the first relation is (the coincidence of a scheme-theoretic line bundle and) the calculation of the *Weil–Petersson metric* on the base. The second relation can be seen as a sort of quantization of K-energy. The last relation essentially follows from an asymptotic analysis of *Ray–Singer torsion* combined with a refinement of the asymptotic Hilbert–Samuel formula from Proposition 3.8.

We remark that Theorem 1.1(2) gives yet another way of connecting Mabuchi K-energy and the Donaldson–Futaki invariant via our h_K (see Proposition 2.8) and roughly shows us that Mabuchi's K-energy is essentially an intersection number.

In Section 2, several other basic "height-type" invariants are introduced and studied as well, partially for future further analysis. They are connected simultaneously to, on one hand, some other functionals over the space of Kähler metrics (see, e.g., [6]) and, on the other hand, some other intersection-theoretic quantities (analyzed in [71], [24], [13], etc.).

One point of this whole set of analogies is that the two protagonists in the field, metrics and polarizations (i.e., ample line bundles), both need to be positive by their definitions (or by their nature) and the varieties or families are the "models" which "realize" the required positivity. For that realization, we need a change of models by geometric flows of metrics or, equivalently, (mostly) birational modifications, such as the minimal model program. Then such flows are supposed to arrive at nice canonical metrics or models, which give "canonical" compactifications of moduli spaces. At least, in this way, we can give yet another heuristic explanation to the Yau–Tian–Donaldson conjecture, the K-moduli conjecture (see [74, Conjecture 3.1]), and perhaps also the (arithmetic) MMP itself.

Most parts of this article have their origins in algebrogeometric versions (or Kähler geometric versions), which are already established, and thus, corresponding geometric articles are cited in each appropriate section. We recommend that interested readers review the geometric counterparts. Another recent extension of the Donaldson–Futaki invariant or the K-stability theory in a still algebrogeometric realm (such as partial resolutions of singularities) is in [75].

Finally, we would like to mention that there has recently been another interesting generalization of the Faltings heights in a different direction, that is, introduced for *motives* ("height of motives") and its studies due to Kato [48] and Koshikawa [54].

We organize our article as follows. After this introductory section, in Section 2, we introduce our modular height h_K and its related variants, mainly after [71] and [13]. Then in Section 3, we discuss applications as well as deeper results and speculations. First, we show that "asymptotic Chow semistability" does *not* admit a semistable reduction in that sense. Second, we propose an arithmetic version of the Yau–Tian–Donaldson conjecture. During the arguments, by using a result about the asymptotic behavior of Ray–Singer analytic torsion due to Bismut and Vasserot [9], we show that "quantization" of our invariants is essentially the original heights introduced by Bost [10], [11] and Zhang [108].

In this article, unless otherwise mentioned, we work under the following setting.

1.1. Notations and conventions

(1) We work with an arithmetic scheme of the form $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K) =: C$ with relative dimension n, where K is a number field, $\mathcal{O}_K$ is the ring of integers of K, $\mathcal{X}$ is flat and (relatively) projective over C, and $\mathcal{L}$ is a (relatively) ample line bundle on $\mathcal{X}$. The generic point of C is denoted by η, and the generic fiber is denoted as $(\mathcal{X}_\eta, \mathcal{L}_\eta)$ or simply (X, L).

(2) For simplicity, throughout this article, we assume that $\mathcal{X}$ is normal. We put $K_{\mathcal{X}^{\mathrm{sm}}/C} := \bigwedge^n_{\mathcal{O}_{\mathcal{X}^{\mathrm{sm}}}} \Omega_{\mathcal{X}^{\mathrm{sm}}/\mathcal{O}_K}$, where $\mathcal{X}^{\mathrm{sm}} \subset \mathcal{X}$ denotes the open dense subset of $\mathcal{X}$ where π is smooth. Then we further assume, for simplicity, the $\mathbb{Q}$-Gorenstein condition, that is, with some $m \in \mathbb{Z}_{>0}$, $(K_{\mathcal{X}^{\mathrm{sm}}/C})^{\otimes m}$ extends to an invertible sheaf $((K_{\mathcal{X}^{\mathrm{sm}}/C})^{\otimes m})\widehat{}$ on the whole $\mathcal{X}$. (We call the condition $\mathbb{Q}$-Gorenstein following the custom in birational algebraic geometry.) Then, the *discrepancy* of $\mathcal{X}$ along some exceptional divisor over it is defined similarly to that in the geometric case (see [53]) as follows.

Suppose that $f : \mathcal{Y} \to \mathcal{X}$ is a blow-up morphism of the arithmetic scheme $\mathcal{X}$ as above, where we also assume $\mathcal{Y}$ is normal. Then if we write $(K_{\mathcal{Y}})^{\otimes m} \otimes \pi^*((K_{\mathcal{X}^{\mathrm{sm}}/C})^{\otimes -m})\widehat{} = \mathcal{O}_{\mathcal{Y}}(\sum a_i m E_i)$ with exceptional prime divisors E_i of $\mathcal{Y}$ and $a_i \in \mathbb{Q}$, a_i is called the *discrepancy* of E_i over $\mathcal{X}$. Accordingly, we call $\mathcal{X}$ *log-canonical* (resp., *log-terminal*) when $a_i \geq -1$ (resp., $a_i > -1$) for all f and i. Note that, instead of using resolutions of singularities, we use all normal blowups. For the study of related classes of singularities from the discrepancy viewpoint, we recommend recent references, for example, [52] and [95].

(3) For such a polarized arithmetic variety $(\mathcal{X}, \mathcal{L})$, we associate a complex geometric generic fiber $(\mathcal{X}(\mathbb{C}) =: X_\infty, \mathcal{L}(\mathbb{C}) =: L_\infty)$. Note that X_∞ is usually *not* connected (though equidimensional), for example, when the base field K is not $\mathbb{Q}$, but this would not cause any technical problems.

(4) h is a continuous Hermitian metric of real type on L_∞ which is C^∞ at the smooth locus X_∞^{sm} of X_∞, and $c_1(L_\infty|_{X_\infty^{\mathrm{sm}}}, h)$ extends as a closed *positive* $(1,1)$-current with locally continuous potential (through singularities of X_∞). In this article, we call such a metric an *almost smooth* Hermitian metric (of real type). The curvature of h is assumed to be positive semidefinite. For $n+1$ such metrized line bundles $\bar{\mathcal{L}}_i (i = 0, \ldots, n)$, via generic resolution (see, e.g., [65, 5.1.1]), the Gillet–Soulé [44] intersection number is well defined as we explain in Section 2.1.

(5) $c_1(L, h)$ means the first Chern form (current) $c_1(L_\infty, h)$, that is, it is locally $-\frac{i}{2\pi}\partial\bar{\partial}\log(h(s,\bar{s}))$, where s is an arbitrary local nonvanishing holomorphic section of L_∞. More precisely, it is a pushforward of the smooth (usual) first Chern form at the generic resolution. We also denote it as ω_h.

(6) $\mathcal{H}(L)$ means the space of appropriate Hermitian metrics {almost smooth h on L_∞ of real type with positive $c_1(L_\infty, h)$}. In this article, it is enough to treat this only as a set.

(7) We denote the model with metric as $\pi : (\mathcal{X}, \bar{\mathcal{L}}^h := (\mathcal{L}, h)) \to \mathrm{Spec}(\mathcal{O}_K)$. We also sometimes write $\bar{\mathcal{L}}^{\omega_h}$ instead of $\bar{\mathcal{L}}^h$, where ω_h means $c_1(\mathcal{L}(\mathbb{C}), h)$ as above.

(8) Occasionally we fix a reference global model with the specified generic fiber, denoted as $\pi : (\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}}) \to \mathrm{Spec}(\mathcal{O}_K)$, and work with various $(\mathcal{X}, \mathcal{L}, h)$ with the isomorphic fiber (with possibly different metric) with specified isomorphisms.

(9) From the non-Archimedean geometric perspective, it can be seen that the main body of this article discusses basically only model metrics. However, all the arguments in Sections 2.1–2.7 and some other parts can be straightforwardly extended to that for semipositive *adèlic* (metrized) line bundles $\bar{\mathcal{L}}$ without any technical difficulties. To extend each claim in Sections 2.1–2.7, where we assume the vertical ampleness (resp., vertical nefness) of some arithmetic line bundles (model metrics), we can straightforwardly extend the claim to that for vertically ample (resp., vertically nef) adèlic (metrized) line bundles. Such vertical ampleness (resp., vertical nefness) of adèlic (metrized) line bundles is simply defined as being a uniform limit[3] of vertically ample (resp., vertically nef) arithmetic line bundles (model metrics). Hence, we wish to just omit and possibly rewrite those extensions more explicitly in the future.[4]

For the above $\pi : (\mathcal{X}, \bar{\mathcal{L}} = (\mathcal{L}, h)) \to \mathrm{Spec}(\mathcal{O}_K)$, we associate a real number invariant $h_K(\mathcal{X}, \mathcal{L}, h)$, which we call the Arakelov–Donaldson–Futaki invariant or (K-)modular height. It extends the Faltings modular height of a polarized Abelian variety (as a generic fiber) and encodes the K-energy and Donaldson–Futaki invariant. For the precise meanings, please read below carefully.

2. Arakelov intersection-theoretic functionals

2.1. General preparations on intersection theory

2.1.1. Arakelov intersection numbers for singular varieties

As in the notation given above in Section 1.1, we work with an $(n+1)$-dimensional arithmetic scheme of the form $\pi : (\mathcal{X}, \mathcal{L}) \to \mathrm{Spec}(\mathcal{O}_K) =: C$. Although the Arakelov–Gillet–Soulé (see [44]) intersection theory is (usually) defined for a regular scheme, in so far as one only considers the intersection numbers of $n+1$ arithmetic line bundles, the full regularity is not required, as we see below. One way to see it is via the use of generic resolution (see, e.g., [65, 5.1.1]) as follows.

DEFINITION-PROPOSITION 2.1

Suppose that $\mathcal{X}$ is a normal scheme which is flat and projective over $C = \mathrm{Spec}(\mathcal{O}_K)$ and $n+1$ line bundles $\mathcal{L}_0, \dots, \mathcal{L}_n$ of $\mathcal{X}$ attached with almost smooth metrics h_i (see Section 1.1), which we denote by $\bar{\mathcal{L}}_i = (\mathcal{L}_i, h_i)$ $(i = 0, \dots, n)$.

[3] In the sense of [107].

[4] Indeed, the only nontrivial technically necessary change in such an extension of Sections 2.1–2.7 is to replace the Moriwaki–Hodge index theorem (see [64]) for model metrics by the recent extension by Yuan and Zhang [105, 1.3] or the continuity of intersection numbers [107, 1.4(a)].

If we take a birational proper morphism $\pi:\tilde{\mathcal{X}}\to\mathcal{X}$ from generically smooth $\tilde{\mathcal{X}}$ (it exists by Hironaka [47]; see, e.g., [65, 5.1.1]), then the definition of Gillet–Soulé's [44] intersection number $(\pi^*\bar{\mathcal{L}}_0.\cdots.\pi^*\bar{\mathcal{L}}_n)$ works and also does not depend on π. We simply denote the value by $(\bar{\mathcal{L}}_0.\cdots.\bar{\mathcal{L}}_n)$.

To see the above well-definedness, that is, that the Gillet–Soulé [44] intersection theory works, the essential point is to confirm the well-definedness of the wedge product

$$T\mapsto c_1(L,h)\wedge T$$

with any closed positive (d,d)-current T $(0\le d<n)$. (The other term of the $*$-product, i.e., $[\log(h(s,s))]|_{Z(\mathbb{C})}$ for $Z\subset X$, is well defined by the local integrability of $\log(h(s,s))$ with holomorphic section s.) This desired wedge product is standard after Bedford and Taylor [4, p. 4] (see also [23, III, Section 3]), as we are able to put $c_1(L,h)\wedge T=dd^c(\log(h(s,s))T)$ with nonvanishing local holomorphic section of L.

As the proof of the independence from π is straightforward by the use of a common generic resolution (which again exists by [47]) and a projection formula (see, e.g., [65, Proposition 5.5]), we omit the details of the proof. We denote the above intersection number simply as $(\bar{\mathcal{L}}_0.\cdots.\bar{\mathcal{L}}_n)$ and will use it throughout this work.

2.1.2. Change of metrics

We introduce the Arakelov-theoretic versions of the functionals of the space of Kähler metrics $\mathcal{H}(L)$ and show the compatibility with both the non-Archimedean analogues for test configurations in the style of [13] (which is for an equicharacteristic base) and the classical Kähler version of the functionals, such as those in [59]. In particular, our results give another explanation (in addition to [14]), with certain mathematical statements, of why the intersection number–type invariants from [13] can be seen as non-Archimedean analogues of the corresponding functionals over the space of Kähler metrics.

Arakelov intersection theory can be decomposed into "local" functionals as follows. The proposition below is essentially a calculation of the Bott–Chern secondary class (see [12], [91]) and matches the history of [25], [99], and [85].

PROPOSITION 2.2

We follow the notation in the introduction and discuss the arithmetic (relatively) projective variety $\mathcal{X}$. Suppose $\mathcal{L}_i$ $(i=0,1,\dots,n)$ are (relatively) ample arithmetic line bundles on $\mathcal{X}$. If we change only the infinite place part (i.e., the metric h), then we get a functional of the space of Kähler metrics as follows. Set

$$\mathcal{G}:\prod_{i=0}^{n}\mathcal{H}(\mathcal{L}_i(\mathbb{C}))\to\mathbb{R}\qquad \text{as } \mathcal{G}(h_0,\dots,h_n):=(\bar{\mathcal{L}}^{h_0}.\bar{\mathcal{L}}^{h_1}.\cdots.\bar{\mathcal{L}}^{h_n}),$$

simply the Arakelov–Gillet–Soulé theoretic intersection number. Then we have

$$\mathcal{G}(e^{-2\varphi}\cdot h_0,\dots,h_n)-\mathcal{G}(h_0,\dots,h_n)=\int_X \varphi\cdot c_1\big(\mathcal{L}_1(\mathbb{C}),h_1\big)\wedge\cdots\wedge c_1\big(\mathcal{L}_n(\mathbb{C}),h_n\big).$$

Proof
Take general meromorphic sections s_i of $\mathcal{L}_i(\mathbb{C})$ $(i=0,1,\dots,n)$ which do not have common components. Then, the difference of the two first arithmetic Chern classes can be expressed as

$$\hat{c_1}(\mathcal{L}_0,e^{-2\varphi}\cdot h_0)-\hat{c_1}(\mathcal{L}_0,h_0)=\overline{(0,2\varphi)}\in \hat{CH}^1(\mathcal{X}),$$

by considering the same meromorphic section s_1. Hence,

$$\begin{aligned}\mathcal{G}(e^{-2\varphi}\cdot h_0,\dots,h_n)-\mathcal{G}(h_0,\dots,h_n)&=\hat{c_1}(\mathcal{L}_n,h_n).\cdots.\hat{c_1}(\mathcal{L}_1,h_1).(0,2\varphi)\\ &=\int_X \varphi\cdot c_1\big(\mathcal{L}_1(\mathbb{C}),h_1\big)\wedge\cdots\wedge c_1\big(\mathcal{L}_n(\mathbb{C}),h_n\big),\end{aligned}$$

by the commutativity of the Gillet–Soulé intersection pairing ((higher) Weil reciprocity). □

The resemblance of some algebrogeometric intersection-theoretic invariants and functionals of the space of Kähler metrics such as the Aubin–Mabuchi (Monge–Ampère) energy and the K-energy, about which we continue discussion below, were first discussed properly by Boucksom, Hisamoto, and Jonsson [13]. A version of a part, that is, for the Monge–Ampère energy case, is also in [75] (still after the fruitful discussions with Boucksom in 2014) in a somewhat generalized setting compared with that in [13].

2.2. Modular height for general arithmetic schemes

Before the introduction of the Arakelov-theoretic (global) version, we recall the classical K-energy (see [59]) through the formula by Chen [18] and Tian [100], which we regard here as the definition. We simultaneously recall the definitions of Ricci energy, Aubin–Mabuchi energy, and entropy (see, e.g., [6]) which form parts of them.

DEFINITION 2.3 ([59], [18], [100], [6])
Keeping the above notation, we recall the following notions:

$$\mu_\omega(\varphi):=\frac{\bar{S}}{n+1}\mathcal{E}_\omega(\varphi)-\mathcal{E}^{\mathrm{Ric}(\omega)}(\varphi)+\frac{1}{V}\,\mathrm{Ent}_\omega(\omega_\varphi)\qquad\text{(K-energy)},$$

where $\bar{S}$ is the average scalar curvature, V is the volume $\int\omega_h^n$, and

$$\mathcal{E}_\omega(\varphi):=\frac{1}{V}\sum_{0\le i\le n}\int \varphi\omega^i\wedge\omega_\varphi^{n-i}\qquad\text{(Aubin–Mabuchi energy)},$$

$$\mathcal{E}^{\mathrm{Ric}(\omega)}(\varphi):=\frac{1}{V}\sum_{0\le i\le n-1}\int \varphi\,\mathrm{Ric}(\omega)\wedge\omega^i\wedge\omega_\varphi^{n-1-i}\qquad\text{(Ricci energy)},$$

$$\mathrm{Ent}_\omega(\varphi) := \int \log\Big(\frac{\omega_\varphi^n}{\omega^n}\Big)\omega_\varphi^n \qquad \text{(entropy)}.$$

2.2.1. *Definition*

We require the curvature of (L,h) to be positive. Here is the definition of our main invariant *(K-)modular height* h_K.

DEFINITION 2.4 (MODULAR HEIGHT)

Suppose that $(\mathcal{X},\mathcal{L},h)$ is an arithmetic projective scheme over $\mathrm{Spec}(\mathcal{O}_K)$, satisfying the conditions in Section 1.1. Then we define

$$\begin{aligned} h_K(\mathcal{X},\mathcal{L},h) := \frac{1}{[K:\mathbb{Q}]}\big(&-n(L^{n-1}.K_X)\big((\bar{\mathcal{L}}^h)^{n+1}\big) \\ &+ (n+1)(L^n)\big((\bar{\mathcal{L}}^h)^n.\overline{K_{\mathcal{X}/B}}^{\mathrm{Ric}(\omega_h)}\big)\big). \end{aligned}$$

Here we recall that (X,L) is the generic fiber of our $(\mathcal{X},\mathcal{L})$ as we follow the notation in Section 1.1. We remark (again) that the subscript K of h_K comes from K-stability (thus, it is the K of Kähler) and *not* our base field K. Indeed, it is easy to see that our definition does not depend on the base field K, that is, finite extension of K makes no change. We also recall here (as we defined in Section 1.1) that the metric on $K_{\mathcal{X}_\eta}$ is induced from h, that is, the determinant metric of the metric induced by $\omega_h := c_1(L_\infty,h)$. Also, the author is happy to acknowledge here that R. Berman taught the author in May of 2016 that in 2012 he had found essentially the same definition as Definition 2.4 and obtained a result closely related to Theorem 3.7, which we discuss later.

Here and throughout this article, the overbar $\bar{}$ denotes metrizations of the line bundles. The above definition can be extended to integrable adèlic metrics on L and $K_{X_{\mathrm{gen}}}$ by the extended Arakelov intersection theory (see [107]), which fits the philosophy of, for example, [30].

PROPOSITION 2.5 (ARCHIMEDEAN RIGIDITY)

For any $c\in\mathbb{R}$, we have $h_K(\mathcal{X},\mathcal{L},e^{2c}\cdot h) = h_K(\mathcal{X},\mathcal{L},h)$.

As this follows from fairly straightforward and short calculations, we omit the proof. The above is an analogue of the rigidity of [39, 4.6], and indeed, the proof follows in exactly the same way. We also have the following non-Archimedean version, which is then completely similar to [39, 4.6].

PROPOSITION 2.6 (NON-ARCHIMEDEAN RIGIDITY)

For any Cartier divisor D on C, we have $h_K(\mathcal{X},\mathcal{L}(\pi^*D),h) = h_K(\mathcal{X},\mathcal{L},h)$.

We avoid writing down the easy proof for the same reason as above.

There is a subtle issue about the definition of the Donaldson–Futaki invariant (which we inherit here) of families with nonreduced closed fibers. The author believes that idealistically one can define the Donaldson–Futaki invariant after semistable reduction, but to avoid confusion and to follow the custom of this field (cf. [28], [103], [70]), we do not make a change.

For the Kähler–Einstein case, by simply substituting terms of Definition 2.4, we get the following result.

PROPOSITION 2.7 (SPECIAL CASE OF MODULAR HEIGHT)

Suppose that $K_X \equiv aL$ with some $a \in \mathbb{R}$. Then

$$h_K(\mathcal{X},\mathcal{L},h) = \frac{(L^n)}{[K:\mathbb{Q}]}\left((\overline{\mathcal{L}}^h)^n.(n+1)\overline{(K_{\mathcal{X}/B})}^{\mathrm{Ric}(\omega_h)} - na\overline{\mathcal{L}}^h\right).$$

An important property of the above invariant is the following.

PROPOSITION 2.8

We have that $h_K(\mathcal{X},\mathcal{L},e^{-2\varphi}\cdot h) - h_K(\mathcal{X},\mathcal{L},h) = \frac{(L^n)}{[K:\mathbb{Q}]}\cdot \mu_{\omega_h}(\varphi)$, where μ denotes the Mabuchi K-energy (see [59]).

Proof

We obtain the proof as a special case of Proposition 2.2 applied to the definition. □

Proposition 2.8 refines a Bott–Chern interpretation of K-energy (see [99, p. 215]) and shows that, with a precise meaning, Mabuchi's K-energy (see [59]) is essentially an intersection number. Special cases of a variant of the arithmetic Donaldson–Futaki invariant (modular height) are treated in the following.

EXAMPLE 2.9

As an important example, we explain that, for an Abelian variety whose metrical structure corresponds to the (Ricci) flat Kähler metric, the Faltings [36] "moduli-theoretic height" is the special case of our modular height $h_K(\mathcal{X},\mathcal{L},h_{\mathrm{KE}})$. First we recall its original definition.

DEFINITION 2.10 ([36])

Suppose that $(\mathcal{X}^{oo},\mathcal{L}^{oo})$ is a semi-Abelian scheme of relative dimension n over $\mathcal{O}_K$ which has a proper generic fiber (X,L). We denote the zero section as $\epsilon: C \to \mathcal{X}$. We consider $\epsilon^* K_{\mathcal{X}^{oo}/C}$ and metrize it by $(\alpha,\bar{\alpha})_{\mathrm{Falt}} := (\frac{i}{2})^n \int_{X_\infty} \alpha\wedge\bar{\alpha}$. Then we set $h_{\mathrm{Falt}}(\mathcal{X},\mathcal{L}) := \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}(\epsilon^* K_{\mathcal{X}^{oo}/C})$ and call it the *Faltings modular height.*[5]

[5] We need to be careful *not* to be confused by the other "Faltings height" introduced in [38] (see Section 2.4).

Then we observe that this is a special case of our modular height in the following sense.

THEOREM 2.11

In the situation of Definition 2.10, we set $(\mathcal{X}, \mathcal{L})$ *as a relative compactification*[6] *of the Néron model* $(\mathcal{X}^o, \mathcal{L}^o)$ *of* (X, L) *such that* $\operatorname{codim}((\mathcal{X} \setminus \mathcal{X}^0) \subset \mathcal{X}) \geq 2$. *Then we have*

$$h_K(\mathcal{X}, \mathcal{L}, h_{\mathrm{KE}}) = (n+1)(L^n) \cdot \Big(h_{\mathrm{Falt}}(X) + \frac{1}{2}\log\Big(\frac{(L^n)}{n!}\Big)\Big),$$

where $h_{\mathrm{Falt}}(-)$ *denotes the Faltings [36] "modular-theoretic height."*

Later, as a special case of Theorem 2.14 via "birational geometry," we also find that this $(\mathcal{X}, \mathcal{L}, h_{\mathrm{KE}})$ minimizes h_K among all models, so that the canonicity of the Faltings height follows.

Proof of Theorem 2.11

We write the unique Ricci flat metric g_{KE} whose normalized Kähler form ω_{KE} sits in $c_1(L_\infty)$, and we denote its determinantal metric on K_{X_∞} by $\det(g_{\mathrm{KE}})$. Then if we set $K_{\mathcal{X}/C} = \pi^* D$ with some arithmetic divisor D on C, we observe that

$$h_K(\mathcal{X}, \mathcal{L}, h_{\mathrm{KE}}) = (L^n)\widehat{\deg}\big(\mathcal{O}_C(D, h)\big) \tag{1}$$

with Hermitian metric h which satisfies $h(\alpha, \beta) = (\alpha_x, \beta_x)_{\det(g_{\mathrm{KE}})}$ for any $x \in X_\infty = \mathcal{X}(\mathbb{C})$. Note also that this corresponds to the fundamental equality of the Deligne pairing (see, e.g., [107, p. 79])

$$\langle \bar{\mathcal{L}}, \ldots, \bar{\mathcal{L}}, \overline{K_{\mathcal{X}/C}} \rangle = \mathcal{O}_C\big((L^n)D\big).$$

For $\alpha \in \Gamma(K_{X_\infty})$ and $x \in X_\infty = \mathcal{X}(\mathbb{C})$, we have

$$(\alpha_x, \alpha_x)_{\det(g_{\mathrm{KE}})} \cdot \omega^n_{g_{KE}} = (-1)^{\frac{n(n-1)}{2}} \cdot \Big(\frac{i}{2}\Big)^n \cdot n!(\alpha \wedge \bar{\alpha}).$$

This can be confirmed by an easy local calculation for the Kähler–Einstein metric; indeed, the above holds for any Kähler metric. Then integrating the above over X_∞, we have

$$\frac{(L^n)}{n!} \cdot (\alpha_x, \alpha_x)_{\det(g_{\mathrm{KE}})} = (-1)^{\frac{n(n-1)}{2}} \cdot \Big(\frac{i}{2}\Big)^n \int_{X_\infty} \alpha \wedge \bar{\alpha}, \tag{2}$$

by Ricci flatness, that is, the constancy of the quantity $(\alpha_x, \alpha_x)_{\det(g_{\mathrm{KE}})}$. Hence, combining this together with (1), we get the assertion. Also note that this essentially derives (for a geometric family, just by applying the above result fiberwise) the well-known potential description of the Weil–Petersson metric (e.g., [97, Theorem 2]). □

[6]That is, an open immersion to a *proper* scheme $\mathcal{X}$ over C. We also assume $\mathcal{L}$ is (relatively) ample over C.

EXAMPLE 2.12 ([37])

The case of an arithmetic surface $\mathcal{X} \to C$ whose generic fiber X has genus $g > 1$ with the Arakelov–Faltings metric attached is the one essentially treated since [37]. In particular, Faltings showed

$$(\overline{\omega_{\mathcal{X}/C}}^{\mathrm{Ar}}.\overline{\omega_{\mathcal{X}/C}}^{\mathrm{Ar}}) \geq 0$$

in [37, Theorem 5(a)], which is loosely related to *Arakelov K-semistability* (of a hyperbolic curve), which we introduce in the next section. Indeed, we have

$$h_K(\mathcal{X}, \mathcal{L} := K_{\mathcal{X}/C}, h^{\mathrm{Ar}}) = \frac{2g-2}{[K:\mathbb{Q}]}(\overline{\omega_{\mathcal{X}/C}}^{\mathrm{Ar}}.\overline{\omega_{\mathcal{X}/C}}^{\mathrm{Ar}}),$$

where h^{Ar} is a metric on K_{X_∞} corresponding to the Arakelov metric.

EXAMPLE 2.13 (TOWARD A LOGARITHMIC SETTING)

With [31] in mind, it is natural to think of generalization to a logarithmic setting, that is, to think of effective an $\mathbb{R}$-Cartier divisor $\mathcal{D}$ in $\mathcal{X}$ and correspondingly canonical Kähler metrics with conical singularities along $\mathcal{D}(\mathbb{C})$. Once there is an appropriate Arakelov intersection theory for such singular metrics, it would be straightforward to give the definitions of the generalizing modular height (and other invariants/functionals which we will discuss later) as in the algebrogeometric situation (see [79]). For example, Montplet [63] essentially treats such a *log-Arakelov–Donaldson–Futaki invariant* for the pointed stable curves case by applying an extended Arakelov intersection theory (see [15]). We leave the general definition of such a log extension to the future.

Morally speaking, we show the decomposition of modular heights to places of a number field

$$\text{modular height } h_K = \int_{\text{places}} \text{(local) K-energy}.$$

We extend this picture to other kinds of intersection-theoretic invariants from Section 2.4.

2.3. Modular height and birational geometry (MMP)

In the geometric setting, that is, when the base C is a complex curve, the K-stability or more general behavior of Donaldson–Futaki invariants are observed to be crucially controlled by the MMP-based birational geometry from [71] and [69] and later developed in [56], [104], [24], [13], and so on. In this section, we partially establish an arithmetic version of the phenomenon.

2.3.1. Minimizing modular heights

Our theorem below Theorem 2.14 (partially) justifies via our h_K a speculation of Manin [62, p. 76] from the 1980s shortly after [36]. “Our limited understanding of A-geometry [Arakelov geometry] suggests the special role of those A-manifolds [Arakelov variety] for which (X_v, ω_v) are Kähler–Einstein. This condition appears to be a reasonable analogue of the minimality of X_f over $\mathrm{Spec}(R)$." Here, in his notation, R is our $\mathcal{O}_K$, X_f corresponds to our $\mathcal{X}$ as it is a finite-type, proper, flat, surjective generically smooth scheme over $C=\mathrm{Spec}(R)$, v is a place of K, and ω_v is a Kähler form on the v-component of the complex variety X_∞ (in our notation). Now we come back to our notation and state a justification of the above sentences.

Roughly speaking, the below says that, for a fixed generic fiber, if we take a sort of "arithmetic minimal (or canonical) model" from the MMP perspective as a scheme and associate a canonical Kähler metric such as the Kähler–Einstein metric, then the model minimizes the modular height h_K among all possible models. In other words, such an Arakelov *minimal* model can be partially justified by the *minimality* of the modular height h_K.

THEOREM 2.14

(1) *(Calabi–Yau case) Let $(\mathcal{X},\mathcal{L})$ be a log-terminal arithmetic polarized projective flat scheme of $(n+1)$-dimension over $C:=\mathrm{Spec}(\mathcal{O}_K)$ such that $K_{\mathcal{X}/C}$ is relatively numerically trivial and $(\mathcal{X},\mathcal{X}_c=\pi^{-1}(c))$ are log-canonical (resp., purely log-terminal) for any closed point $c\in C$. We metrize $\mathcal{X}(\mathbb{C})=:X_\infty$ with the unique (singular) Kähler–Einstein metric and take its corresponding continuous Hermitian metric of $L_\infty=\mathcal{L}(\mathbb{C})$ which we denote as h_{KE} (see [34], [35]). For any other flat polarized family $(\mathcal{X}',\mathcal{L}')\to C$ whose generic fiber is isomorphic to that of $\mathcal{X}\to C$ with possibly different metric h, we have*

$$h_K(\mathcal{X},\bar{\mathcal{L}}^{h_{\mathrm{KE}}})\leq (\textit{resp.}, <)\ h_K(\mathcal{X}',\bar{\mathcal{L}}'^{h}).$$

(2) *(Canonical model case) Let $\mathcal{X}$ be a log-terminal arithmetic projective flat scheme of $(n+1)$-dimension over $C:=\mathrm{Spec}(\mathcal{O}_K)$, the ring of integers of a number field K such that $K_{\mathcal{X}/C}$ is relatively ample over C, and we set $\mathcal{L}:=\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}/C})$ with sufficiently divisible $m\in\mathbb{Z}_{>0}$.*[7] *Assume $(\mathcal{X},\mathcal{X}_c)$ are log-canonical pairs for any closed point $c\in C$. We metrize $\mathcal{X}(\mathbb{C})=:X_\infty$ with the (singular) Kähler–Einstein metric and take its corresponding continuous Hermitian metric of $L_\infty=\mathcal{L}(\mathbb{C})$, which we denote as h_{KE} (see [34], [35]). Then, for any other flat polarized family $(\mathcal{X}',\mathcal{L}')\to C$ whose generic fiber is isomorphic to that of $\mathcal{X}\to C$ with possibly different metric h, we have*

$$h_K(\mathcal{X},\bar{\mathcal{L}}^{h_{\mathrm{KE}}})< h_K(\mathcal{X}',\bar{\mathcal{L}}'^{h}).$$

(3) *(Special $\mathbb{Q}$-Fano varieties case) Let $\mathcal{X}$ be a log-terminal arithmetic projective flat scheme of $(n+1)$-dimension over $C:=\mathrm{Spec}(\mathcal{O}_K)$, the ring of integers of a number field K such that $-K_{\mathcal{X}/C}$ is relatively ample, and we set $\mathcal{L}=\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}/C})$ with some sufficiently divisible $m\in\mathbb{Z}_{>0}$. Suppose that $\mathrm{glct}((\mathcal{X},\mathcal{X}_c);-K_{\mathcal{X}})$, which is defined as*

$$\sup\bigl\{t\geq 0\,\big|\,(\mathcal{X},\mathcal{X}_c+tD)\ \textit{is log-canonical for all effective}\ D\equiv_{/C}-K_{\mathcal{X}/C}\bigr\},$$

[7] m is an unessential parameter due to the homogeneity of h_K, that is, $h_K(\bar{\mathcal{L}}^{\otimes c})=c^{2n}h_K(\bar{\mathcal{L}})$.

is at least (resp., bigger than) $\frac{n}{n+1}$ *for any closed point* $c \in C$. *From the theorem of Tian [96], we know the existence of (singular) Kähler–Einstein metric on* $\mathcal{X}(\mathbb{C})$, *and we denote its corresponding Hermitian metric of* $\mathcal{O}_X(-mK_X)$ *by* h_{KE}. *Then for any other flat polarized family* $(\mathcal{X}', \mathcal{L}') \to C$ *whose geometric generic fiber is isomorphic to that of* $\mathcal{X} \to C$ *with possibly different Hermitian metric* h, *we have*

$$h_K(\mathcal{X}, \bar{\mathcal{L}}^{h_{\mathrm{KE}}}) \leq (\textit{resp.}, <)\ h_K(\mathcal{X}', \bar{\mathcal{L}'}^h).$$

The theorem above Theorem 2.14 can also be regarded as a dequantized[8] version of Zhang's [108] Chow-stable reduction and is obtained as an arithmetic version of [104, Theorem 6] and [74, Section 4]. The log-terminality condition on general fiber/total space above is used to avoid the technical difficulty of Kähler–Einstein metrics on varieties with (semi-)log-canonicity such as in [7]. We take a somewhat complicated description via log pairs $(\mathcal{X}, \mathcal{X}_c)$ (see, e.g., [53]), as the inversion of adjunction in arithmetic setting is unfortunately not established yet.

CONJECTURE 2.15 (INVERSION OF ADJUNCTION)

We keep the notation of Section 1.1 and set $\mathcal{X}_c = \pi^{-1}(c)$ *for a closed point of* C *as above. Let* D *be an effective* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor whose support does not contain any component of* $\mathcal{X}_c$, *and let* t *be a nonnegative real number. Then each of the following equivalences holds.*

(1) $(\mathcal{X}, \mathcal{X}_c + tD)$ *is log-canonical in the neighborhood of* $\mathcal{X}_c$ *if and only if* $(\mathcal{X}_c, D|_{\mathcal{X}_c})$ *is geometrically semi-log-canonical, that is, semi-log-canonical after base change to algebraic closure of the residue field* $\kappa(c)$ *at* c.

(2) $(\mathcal{X}, \mathcal{X}_c + tD)$ *is purely log-terminal in the neighborhood of* $\mathcal{X}_c$ *if and only if* $(\mathcal{X}_c, D|_{\mathcal{X}_c})$ *is log-terminal.*

Indeed, due to the recent progress such as [21], it looks hopeful to establish Conjecture 2.15 up to $n = 2$ in the near future.

Proof of Theorem 2.14

The basic strategy of the proofs is the same as in the equicharacteristic geometric case (see [71], [77], [74, Section 4]; for Theorem 2.14(2), see also [104]). For each case (1), (2), and (3), we actually prove that the following holds:

$$h_K(\mathcal{X}, \bar{\mathcal{L}}^{h_{\mathrm{KE}}}) \leq (\text{resp.}, <)\ h_K(\mathcal{X}, \bar{\mathcal{L}}^h) \leq (\text{resp.}, <)\ h_K(\mathcal{X}', \bar{\mathcal{L}'}^h).$$

We prove these two inequalities separately and then the desired inequality will be obtained. The first inequality is a consequence of the now standard fact that K-energy is minimized at Kähler–Einstein metrics when they exist (see, e.g., [3], [19]).[9]

[8] In the sense after Donaldson [27]; in this article, quantization refers to this sense.

[9] One month after we posted this article on arXiv, the author learned about a letter from S. Zhang [106] to P. Deligne dated 3 February 1993. In that letter, he essentially found the

The proof of the second inequality is in exactly same manner as that of [71], [77], and [74]. For the reader's convenience, we describe the proofs, partially referring to the original [71], [77], and [74]. We first take a normal projective flat scheme $\mathcal{Y}$ which dominates both $\mathcal{X}$ and $\mathcal{X}'$ via birational morphisms. We denote the birational morphisms as $p : \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{Y} \to \mathcal{X}'$. As the model $\mathcal{Y}$ only needs to be normal (rather than regular), it can be easily obtained as the blowup of the indeterminacy ideal of the birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}'$ (see, e.g., [69]) or the normalization of the graph of f. Then we prove the desired inequalities as consequences of comparing $\mathcal{L}$ on $\mathcal{X}$ and $q^*\mathcal{L}'$ on $\mathcal{Y}$, which is the main nontrivial part of the proof. In the original algebrogeometric settings of [71, Section 2] and [77, Sections 3–5], our $(\mathcal{X}, \mathcal{L})$ corresponds to the trivial test configuration, so that its Donaldson–Futaki invariants were zero. But the same estimation techniques referred to above work to show that $(h_K(\mathcal{Y}, q^*\bar{\mathcal{L}'}^h) - h_K(\mathcal{X}, \bar{\mathcal{L}}^h)) > 0$.

The details of each proof are as follows. For the situation from Theorem 2.14(1) of Calabi–Yau varieties, we have

$$\big(h_K(\mathcal{Y}, q^*\bar{\mathcal{L}'}^h) - h_K(\mathcal{X}, \bar{\mathcal{L}}^h)\big) = \frac{(n+1)(L^n)}{[K:\mathbb{Q}]}\big((q^*\bar{\mathcal{L}'}^h)^n.\bar{K}_{\mathcal{Y}/\mathcal{X}}\big),$$

and this is positive (resp., nonnegative) if $K_{\mathcal{Y}/\mathcal{X}}$ is nonzero effective (resp., effective), since $q^*\bar{\mathcal{L}'}^h$ is vertically nef. This follows from the assumption of the pure log-terminality (resp., log-canonicity) of $(\mathcal{X}, \mathcal{X}_c)$. Heuristically, the above $((q^*\bar{\mathcal{L}'}^h)^n.\bar{K}_{\mathcal{Y}/\mathcal{X}})$ is the non-Archimedean version of the entropy for Monge–Ampère measures (see [13], Section 2.6), and originally we called this the discrepancy term in [71] and [69].

For the situation from Theorem 2.14(2) of canonical models, following [71, Section 2], we decompose the quantity as

$$\begin{aligned}
&\big(h_K(\mathcal{Y}, q^*\bar{\mathcal{L}'}^h) - h_K(\mathcal{X}, \bar{\mathcal{L}}^h)\big)\\
&\quad= \frac{(n+1)(L^n)}{[K:\mathbb{Q}]}\big((q^*\bar{\mathcal{L}'}^h)^n.(p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}})^{\otimes(n+1)} \otimes (q^*\bar{\mathcal{L}'}^h)^{\otimes(-n)}\big)\\
&\qquad+ \frac{(n+1)(L^n)}{[K:\mathbb{Q}]}\big((q^*\bar{\mathcal{L}'}^h)^n.\bar{K}_{\mathcal{Y}/\mathcal{X}}\big).
\end{aligned}$$

As we saw above, the latter term is nonnegative because of the log-canonicity assumption for $(\mathcal{X}, \mathcal{X}_c)$, and we rewrite the former term $((q^*\bar{\mathcal{L}'}^h)^n.(p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}})^{\otimes(n+1)} \otimes (q^*\bar{\mathcal{L}'}^h)^{\otimes(-n)})$ as in [71, p. 2280, (2)]. Heuristically, this part corresponds to the Aubin–Mabuchi energy plus the Ricci energy.

For that, we prepare some more notation. We can and do assume $p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}} = q^*\bar{\mathcal{L}'}^h(E)$ for some *effective* divisor E, after replacing $\mathcal{L}$ by $\mathcal{L}(\pi^*D)$ for some Cartier divisor D on C (π is the projection to C) if needed. Recall that replace-

K-energy for the curve (arithmetic surface) case in the manner of our general formula from Theorem 3.7 and showed that the Poincaré metric minimizes it among the (Archimedean) metrics. The author appreciates S. Zhang for sharing this.

ment does not change h_K by Proposition 2.6 so are our terms which consist h_K. As $-E$ is p-ample from our construction, p is a blowup along a closed subscheme Z. We set $s := \dim(Z)$. Then we have

$$\begin{aligned}&\big((q^*\bar{\mathcal{L}'}^h)^n.(p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}})^{\otimes(n+1)} \otimes (q^*\bar{\mathcal{L}'}^h)^{\otimes(-n)}\big)\\&= \Big(-E^2.\sum_{i=1}^{n}(n+1-i+\epsilon_i)(q^*\bar{\mathcal{L}'}^h)^{n-i}.(p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}})^{i-1}\Big)\\&\quad - \epsilon'\big((-E)^{n+1-s}.(p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}})^s\big),\end{aligned}$$

for $0<|\epsilon_i|\ll 1$ $(1\le i\le n)$ and $0<\epsilon'\ll 1$ as in [71, p. 2280, (2)]. Thus, this is positive by the same lemma (i.e., [71, Lemma 2.8]) as in the geometric case. Lemma 2.8 of [71] can be proved in the same way once we replace the (equicharacteristic, geometric) Hodge index theorem by the Moriwaki–Hodge index theorem [64].

For the situation Theorem 2.14(3) of special $\mathbb{Q}$-Fano varieties, we decompose the quantity as

$$\begin{aligned}&\big(h_K(\mathcal{Y}, q^*\bar{\mathcal{L}'}^h) - h_K(\mathcal{X},\bar{\mathcal{L}}^h)\big) - \big((q^*\bar{\mathcal{L}'}^h)^n.p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}}\big)\\&\quad + \big((q^*\bar{\mathcal{L}'}^h)^n.(n+1)mK_{\mathcal{Y}/\mathcal{X}} - nE\big).\end{aligned}$$

The first term $-((q^*\bar{\mathcal{L}'}^h)^n.p^*\bar{\mathcal{L}}^{h_{\mathrm{KE}}})$ is nonnegative by [77, 4.3]. The positivity (resp., nonnegativity) of the second term $((q^*\bar{\mathcal{L}'}^h)^n.(n+1)mK_{\mathcal{Y}/\mathcal{X}} - nE)$ follows from the assumption on glct. Indeed, the vertical divisor $(n+1)mK_{\mathcal{Y}/\mathcal{X}} - nE$ is nonzero effective (resp., effective) under the glct assumptions by the same argument as in [77, Section 3] (see also [73, esp., pp. 7–9]), so we omit its details. □

Note that the above does *not* use (the arithmetic version of) MMP Conjectures 2.18 and 2.19 but instead uses simple birational geometric arguments juggling with discrepancies. For Theorems 2.14(1) and 2.14(2), it naturally extends to semi-log-canonical $\mathcal{X}$, but because of technical difficulties in treating metrics of infinite diameters, we omit and do not claim it here. Also note that the left-hand sides of the above inequalities coincide with the height on the quotient variety as in Zhang [108] and Maculan [61].

In general, for stable reduction, we naturally expect the following as the arithmetic counterpart of [75, (4.3)].

CONJECTURE 2.16 (CANONICAL REDUCTION)

We fix a normal projective variety (X,L) over a number field K and consider all integral models, that is, $(\mathcal{X},\mathcal{L})$ over $\mathcal{O}_{K'}$ where K' is a finite extension of K. If $(\mathcal{X},\mathcal{L})$ takes minimal h_K among those while fixing (X,L), that is, $h_K(\mathcal{X},\mathcal{L}) \le h_K(\mathcal{X}',\mathcal{L}')$ for any other integral projective model $(\mathcal{X}',\mathcal{L}')$ over a finite extension of K, then such an h_K-minimizing model satisfies the following properties.

(1) *All the geometric fibers are reduced and semi-log-canonical.*

(2) *If the generic fiber F is a Kawamata–log-terminal $\mathbb{Q}$-Fano variety, then so are all fibers; that is, they are also klt $\mathbb{Q}$-Fano varieties.*

(3) *If $K_F \equiv a\mathcal{L}|_F$ with $a \geq 0$, then $h_K(\mathcal{X}, \mathcal{L})$ is minimum among all the models if and only if any fiber G is reduced and geometrically semi-log-canonical with $K_G \equiv a\mathcal{L}|_G$.*

By comparison with the proof of the geometric case, what is lacking in the arithmetic situation is the presence of a *log* MMP as well as the stable reduction (in the sense of [51, Chapter II]) which could possibly take more than a decade. In the light of [32] and [33], the above (esp., Conjecture 2.16(2)) can be seen as a sort of "arithmetic (pointed) Gromov–Hausdorff limit."

2.3.2. Decrease of modular heights by semistable reduction and normalization
Take an arbitrary principally polarized Abelian variety A over a number field K and its base change $A^{(s)}$ to a finite extension K'/K admitting semi-Abelian reduction, which exists due to Grothendieck and Deligne [46]. Then, it has been shown that

$$h_{\mathrm{Fal}}(A) - h_{\mathrm{Fal}}(A^{(s)}) = \frac{1}{[K:\mathbb{Q}]} \sum_{\mathfrak{p}} c(A, \mathfrak{p}) \log(N\mathfrak{p}),$$

where $\mathfrak{p}$ runs over all prime ideals of $\mathcal{O}_{K'}$ and each $c(A, \mathfrak{p})$ is a positive real number called the *base change conductor* by [17]. Indeed, it follows from the definition of the Faltings [36] height that the above formula holds once we set

$$c(A, \mathfrak{p}) := \frac{1}{e(\mathfrak{p}; K'/K)} \operatorname{length}_{\mathcal{O}_{K'}} \left(\frac{\Gamma(\mathrm{Spec}(\mathcal{O}_{K'}), \epsilon^* \omega_{\mathcal{A}_{K'}/\mathcal{O}_{K'}})}{\Gamma(\mathrm{Spec}(\mathcal{O}_K), \epsilon^* \omega_{\mathcal{A}/\mathcal{O}_K}) \otimes \mathcal{O}_{K'}} \right),$$

where $e(\mathfrak{p}; K'/K)$ is the ramifying index at $\mathfrak{p}$ and $\mathcal{A}$ (resp., $\mathcal{A}_{K'}$) means the Néron model of A (resp., $A^{(s)}$). The fact that the base change conductor above is independent of the extension K' follows easily from the fact that, after the semi-Abelian reduction at $\mathcal{O}_{K'}$, say, if we extend it further as K''/K', then $\mathcal{A}_{K'} \times_{K'} K'' \hookrightarrow \mathcal{A}_{K''}$ is an open immersion. In particular, from the above observation we have

$$h_{\mathrm{Fal}}(A) \geq h_{\mathrm{Fal}}(A^{(s)}).$$

We prove a somewhat analogous result for general arithmetic varieties as follows.

PROPOSITION 2.17
When $(\mathcal{X}, \mathcal{L}, h) \to \mathrm{Spec}(\mathcal{O}_K)$ is replaced by normalization of the finite base change $\mathcal{O}_{K'}/\mathcal{O}_K$, that is, when we consider $(\mathcal{X}_{K'}, \mathcal{L}_{K'}, h_{K'}) := (\mathcal{X}, \mathcal{L}, h) \times_{\mathcal{O}_K} \mathcal{O}_{K'}$ and denote its normalization as $(\tilde{\mathcal{X}}_{K'}, \tilde{\mathcal{L}}_{K'}, \tilde{h}_{K'})$, then we have

$$h_K(\tilde{\mathcal{X}}_{K'}, \tilde{\mathcal{L}}_{K'}, \tilde{h}_{K'}) \leq h_K(\mathcal{X}, \mathcal{L}, h).$$

Proof

This is not hard to prove and follows essentially from the simple fact that $K_{\tilde{\mathcal{X}}_{K'}/\mathcal{X}} := K_{\tilde{\mathcal{X}}_{K'}/\mathcal{O}_{K'}} - \nu^* K_{\mathcal{X}/\mathcal{O}_K}$, where $\nu : \tilde{\mathcal{X}}_{K'} \to \mathcal{X}$ denotes the finite morphism, is antieffective. Indeed, as we assume the normality of the total space, and thus of a generic fiber, Archimedean data does not affect. Thus, $h_K(\mathcal{X}, \mathcal{L}, h) - h_K(\tilde{\mathcal{X}}_{K'}, \tilde{\mathcal{L}}_{K'}, \tilde{h}_{K'}) = -\frac{(n+1)(L^n)}{[K:\mathbb{Q}]}(\overline{\tilde{\mathcal{L}}_{K'}}^n . \overline{K_{\tilde{\mathcal{X}}_{K'}/\mathcal{X}}}) \geq 0$, which completes the proof. □

The above phenomenon is essentially the one observed for geometric cases in [83, 5.1, 5.2], [69, 3.8], and [75, around 2.5, 4.3] among others. It shows that, assuming the semistable reduction conjecture, we can always replace the integral model, after some extension of scalars, by those which have reduced fibers and lower modular heights h_K.

2.3.3. Decrease of modular heights by flow

As a preparation, we consider the following natural generalization of the MMP (with scaling) (see [8]) to the arithmetic setting.

CONJECTURE 2.18 (ARITHMETIC MMP WITH SCALING)

Starting from a log-canonical arithmetic projective variety $(\mathcal{X}, \mathcal{L})$ *with* $K_X \equiv aL$ $(a \in \mathbb{R})$ *on the general fiber, we can run the semistable* $K_{\mathcal{X}/C}$*-MMP with scaling* $\mathcal{L}$ *as in [8] (see also [42]). More precisely, there is a sequence of birational modifications* $\mathcal{X} = \mathcal{X}_0 \dashrightarrow \mathcal{X}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{X}_n$ *(each step is a flip or a divisorial contraction) so that the following holds.*

(1) *There is a monotonically increasing sequence of real numbers* $0 = t_0 < t_1 < \cdots < t_l = \infty$ *such that the strict transform of (an* $\mathbb{R}$*-divisor corresponding to)* $\mathcal{L}(tK_{\mathcal{X}_i/C})$ $(t_{i-1} < t < t_i)$ *is a (relatively) ample* $\mathbb{R}$*-line bundle over the base* C*. We naturally finish with* $\mathcal{X}_n$*, which is either a relative minimal model or a relative Mori fibration (see [53], [8] for the basics).*

(2) *The above birational modifications are compatible with the Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\operatorname{Ric}(\omega_t)$$

in the sense that ω_t $(t_{i-1} \leq t \leq t_i)$ *are Kähler currents of* $\mathcal{X}_i(\mathbb{C})$.

Recent theory of the analytic MMP with scaling (see, e.g., [16], [90]) shows the compatibility written in Conjecture 2.18(2), which was pioneered in [101] and [102]. For our purpose, we use the following normalized version.

CONJECTURE 2.19 (NORMALIZED ARITHMETIC MMP WITH SCALING)

Starting from a log-canonical arithmetic projective variety $(\mathcal{X}, \mathcal{L})$ *with* $K_X \equiv aL$ $(a \in \mathbb{R})$ *on the general fiber, we can run the semistable* $K_{\mathcal{X}/C}$*-MMP with scaling* $\mathcal{L}$ *as in [8] (see also [42]). More precisely, there is a sequence of birational* *modifications* $\mathcal{X} = \mathcal{X}_0 \dashrightarrow \mathcal{X}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{X}_n$ *(each step is a flip or a divisorial contraction) so that the following holds. There is a monotonically increasing sequence of real numbers* $0 = t_0 < t_1 < \cdots < t_l = \infty$ *such that the strict transform of (an* $\mathbb{R}$*-divisor corresponding to)* $\mathcal{L}(tE)$ $(t_{i-1} < t < t_i)$ *is (relatively) ample over the base* C*. From this it naturally follows that* E *is contracted (i.e., its strict transform vanishes) in* $\mathcal{X}_n$*, which is either a relative minimal model or a relative Mori fibration (see [53], [8] for the basics).*

It is natural to simultaneously run the normalized Kähler–Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\operatorname{Ric}(\omega_t) + a\omega_t,$$

from some $\omega_0 = c_1(\mathcal{L}(\mathbb{C}), h_0)$ with some positively curved Hermitian metric h_0 of $\mathcal{L}(\mathbb{C})$. Of course, the regular arithmetic surfaces case of the above Conjecture 2.19 was settled by Lichtenbaum [58] and is now classical. Also the case of terminal threefolds with geometrically semi-log-canonical fibers[10] was settled by Kawamata [49], [50]. Recently, Tanaka [95] has confirmed the extension of [58] to the klt arithmetic surface case, in the modern MMP framework, as well.

The following "monotonically decreasing" theorem has its origin in the geometric counterpart from [56] and [75, 4.1].

THEOREM 2.20

Assume the above Conjecture 2.19. For a generically Kähler–Einstein case, h_K *decreases along the arithmetic MMP with scaling in the sense of the above Conjecture 2.19. More precisely, if* $(\mathcal{X}, \mathcal{L})$ *satisfies the conditions of Conjecture 2.19 and we define for* $t_{i-1} \le t < t_i$*,* $F(t) := h_K(\mathcal{X}_i, \mathcal{L}_i, h_i)$*, then it monotonically decreases; that is, for* $t < s$*,* $F(t) > F(s)$*.*

Proof

First we need to check that $(\mathcal{X}_i, \mathcal{L}_i, h_i)$ satisfies our conditions from Section 1.1. The only nontrivial parts for that are the preservation of the normal $\mathbb{Q}$-Gorenstein property and the almost smoothness (in the sense of Section 1.1) of the Hermitian metrics. For the former half, the same proof from the geometric case (see, e.g., [53]) applies. The latter is proven in [90]. In the direction of $\bar{E} := \overline{K_{\mathcal{X}/B}}^{\operatorname{Ric}(\omega_h)} - a\bar{\mathcal{L}}^h$, the derivation of h_K is $\frac{dF(t)}{dt} = ((\overline{\mathcal{L}}^h)^{n-1}.\bar{E}^2)$.

The proof of decrease of the *finite*, that is, non-Archimedean part, of the modular heights h_K along time development is nearly the same as in the geometric case (see [56], [74], [75]), and we only need to replace the use of the usual Hodge index theorem by the arithmetic Hodge index theorem [64, Theorem B] (after Faltings and Hriljac). If we deal with integrable adèlic metrics, we use the generalized arithmetic Hodge index theorem for those metrics due to [105, Theorem 1.3].

[10] Accurately, when $(\mathcal{X}, \mathcal{X}_c)$ is dlt for any closed point $c \in C$.

The infinite place part is nothing but the known fact that the K-energy decreases along the (normalized) Kähler–Ricci flow (see, e.g., [20]), which can be proved as follows: if we set φ_t as $\mathrm{Ric}(\omega_t)+a\omega_t=i\partial\bar\partial\varphi$, then we have

$$\begin{aligned}\frac{d\mu_{\omega_0}(\omega_t)}{dt}&=\int_{X_\infty}\Big(\frac{d\varphi_t}{dt}\Big|_{t=0}\Big)\frac{i\partial\bar\partial}{2}\Big(\frac{d\varphi_t}{dt}\Big|_{t=0}\Big)\omega_0^{n-1}\\&=-\int_{X_\infty}\frac{i}{2}\Big(\partial\frac{d\varphi_t}{dt}\Big|_{t=0}\Big)\overline{\Big(\partial\frac{d\varphi_t}{dt}\Big|_{t=0}\Big)}\omega_0^{n-1}\leq 0,\end{aligned}$$

as $-\frac{i}{2}\partial\psi\overline{\partial\psi}$ is the semipositive $(1,1)$-form for arbitrary real function ψ. □

2.4. Arakelov energy

We keep the same notation. The invariant we introduce in this section is essentially just a self-intersection number of Gillet and Soulé and has repeatedly appeared in various contexts before (see, e.g., [38], [91], [86], [5]).

DEFINITION 2.21

The *Arakelov (Aubin–Mabuchi) energy* is (simply) defined as

$$\mathcal{E}^{\mathrm{Ar}}\big(\mathcal{X},\bar{\mathcal{L}}(=(\mathcal{L},h))\big):=\frac{1}{[K:\mathbb{Q}]}(\bar{\mathcal{L}}^h)^{n+1},$$

an Arakelov–Gillet–Soulé intersection theory (see [91]).

As the name leads us to expect, the following holds.

PROPOSITION 2.22

We have that $\mathcal{E}^{\mathrm{Ar}}(\mathcal{X},\mathcal{L},e^{-2\varphi}\cdot h)-\mathcal{E}^{\mathrm{Ar}}(\mathcal{X},\mathcal{L},h)=\frac{(L^n)}{[K:\mathbb{Q}]}\cdot\mathcal{E}_{\omega_h}(\varphi)$, the Monge–Ampère energy.

Proof

This follows is a special case of Proposition 2.2. □

In this case, the corresponding limiting analysis of [14] was also recently discussed in [84], especially for the low-dimensional case.

Experts in Arakelov geometry should notice right away that if $\mathcal{L}=\mathcal{O}(1)$ for an embedding $\mathcal{X}\subset\mathbb{P}(\mathcal{E})$ with arithmetic metrized bundle $\mathcal{E}$, then this is essentially the *(cycle's) height* treated in [38],[11] [91], and so on.

Hence, in our language, the Cornalba–Harris–Bost–Zhang inequality gives a lower bound of this "Arakelov Monge–Ampère energy" of Chow (semi)stable varieties by some "quantized" invariant. Here, we allow some twist for the projective bundle under consideration.

[11] Hence, this is also sometimes called *Faltings height* but we distinguish this from our h_K crucially.

2.5. Arakelov–Ricci energy

We use a reference model $\pi_{\mathrm{ref}} : (\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}}) \to \mathrm{Spec}(\mathcal{O}_K)$.

DEFINITION 2.23

We define the *Arakelov–Ricci energy* as follows. For $(\mathcal{X}, \mathcal{L}, h)$, we construct a common generic resolution $\tilde{\mathcal{X}}$ which is normal and dominates both models; that is, there are birational proper morphisms $p : \tilde{\mathcal{X}} \to \mathcal{X}$ and $q : \tilde{\mathcal{X}} \to \mathcal{X}_{\mathrm{ref}}$. Then we set

$$\mathcal{E}^{\mathrm{Ar.Ric}}_{(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}})}(\mathcal{X}, \mathcal{L}, h) := \frac{1}{[K : \mathbb{Q}]} \big(\big((p^* \bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^n - q^* \bar{\mathcal{L}}^h \big)^n \big) . \overline{K_{\tilde{\mathcal{X}}/C}}^{\mathrm{Ric}(\omega_h)} \big)$$

as an Arakelov intersection number on $\tilde{\mathcal{X}}$. It is easy to see that this does not depend on the choice of $\tilde{\mathcal{X}}$.

PROPOSITION 2.24

We have $\mathcal{E}^{\mathrm{Ar.Ric}}_{(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}})}(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, e^{-2\varphi} \cdot h_{\mathrm{ref}}) = \frac{(L^n)}{[K:\mathbb{Q}]} \mathcal{E}^{\mathrm{Ric}}_{\omega_{h_{\mathrm{ref}}}}(\varphi)$, *where* $\mathcal{E}^{\mathrm{Ric}}$ *denotes the Ricci energy (see Definition 2.3).*

Proof

Again, this follows as a special case of Proposition 2.2. □

2.6. Entropy

Arakelov entropy is defined as follows. Again, we use the (same) reference model $\pi_{\mathrm{ref}} : (\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}}) \to \mathrm{Spec}(\mathcal{O}_K)$.

DEFINITION 2.25

For $(\mathcal{X}, \mathcal{L}, h)$, we construct a model $\tilde{\mathcal{X}}$ which dominates both models; that is, there are birational proper morphisms $p : \tilde{\mathcal{X}} \to \mathcal{X}$ and $q : \tilde{\mathcal{X}} \to \mathcal{X}_{\mathrm{ref}}$. Then we set $\mathrm{Ent}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}})}(\mathcal{X}, \mathcal{L}, h)$ equal to

$$\frac{1}{[K : \mathbb{Q}]} \big((p^* \bar{\mathcal{L}}^h)^n . p^* \overline{K_{\mathcal{X}/C}}^{\mathrm{Ric}(\omega_h)} - q^* \overline{K_{\mathcal{X}_{\mathrm{ref}}/C}}^{\mathrm{Ric}(\omega_{h_{\mathrm{ref}}})} \big),$$

as the Gillet–Soulé intersection number (see [91]). It is easy to see that this does not depend on the common resolution $\tilde{\mathcal{X}}$.

PROPOSITION 2.26

We have $\mathrm{Ent}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}})}(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, e^{-2\varphi} \cdot h_{\mathrm{ref}}) = \frac{(L^n)}{[K:\mathbb{Q}]} \mathrm{Ent}_{\omega_h}((\omega_h + dd^c\varphi)^n)$, *where* Ent *means the (usual) entropy (see Definition 2.3).*

Proof

Again, this follows as a special case of Proposition 2.2. □

The sum of the above three is the Arakelov–Donaldson–Futaki invariant as follows from an analogue of Definition 2.3 and the Donaldson–Futaki invariants formula from [103] and [70].

PROPOSITION 2.27 (DECOMPOSING THE MODULAR HEIGHT)

We have that

$$h_K(\mathcal{X},\mathcal{L},h) - h_K(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})$$
$$= \frac{\bar{S}}{n+1}\mathcal{E}^{\mathrm{Ar}}_{\omega}(\mathcal{X},\mathcal{L},h) - \mathcal{E}^{\mathrm{Ar.Ric}(\omega)}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})}(\mathcal{X},\mathcal{L},h) + \frac{\mathrm{Ent}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})}(\mathcal{X},\mathcal{L},h)}{[K:\mathbb{Q}]},$$

where $\bar{S}$ is the average scalar curvature of ω_h of a geometric generic fiber and V is the volume of ω_h.

The proof is straightforward from the definitions.

2.7. Arakelov–Aubin functionals

We recall the original Aubin functionals.

DEFINITION 2.28 ([2])

For an ω_h-plurisubharmonic (PSH) smooth function φ, we set

(1) $I_{\omega_h}(\varphi) := \frac{1}{V}\int_{X_\infty}\varphi(\omega_h^n - \omega_\varphi^n)$,
(2) $J_{\omega_h}(\varphi) := \frac{1}{V}\int_{X_\infty}\varphi\omega_h^n - \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \varphi(\omega_\varphi^j \wedge \omega_h^{n-j})$.

Now we define the arithmetic (Arakelov) version of the Aubin functionals $\mathcal{I}^{\mathrm{Ar}}$, $\mathcal{J}^{\mathrm{Ar}}$ as follows. Again, we use the (same) reference model $\pi : (\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}}) \to \mathrm{Spec}(\mathcal{O}_K)$ and keep the notation of Sections 2.5 and 2.6.

DEFINITION 2.29

We have that

$$\mathcal{I}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})}(\mathcal{X},\mathcal{L},h)$$
$$:= \frac{1}{[K:\mathbb{Q}]}$$
$$\times\Big(-(p^*\bar{\mathcal{L}}^h)^{n+1} - (q^*\bar{\mathcal{L}}^{h_{\mathrm{ref}}}_{\mathrm{ref}})^{n+1} + \big(p^*\bar{\mathcal{L}}^h.(q^*\bar{\mathcal{L}}^{h_{\mathrm{ref}}}_{\mathrm{ref}})^n\big) + \big(q^*\bar{\mathcal{L}}^{h_{\mathrm{ref}}}_{\mathrm{ref}}.(p^*\bar{\mathcal{L}}^h)^n\big)\Big),$$

$$\mathcal{J}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})}(\mathcal{X},\mathcal{L},h)$$
$$:= \frac{1}{[K:\mathbb{Q}]}\Big(\big(p^*\overline{\mathcal{L}}^h.(q^*\bar{\mathcal{L}}^{h_{\mathrm{ref}}}_{\mathrm{ref}})^n\big) - \frac{1}{n+1}\big(q^*(\bar{\mathcal{L}}^{h_{\mathrm{ref}}}_{\mathrm{ref}})^{n+1}\big)\Big) + \frac{n}{n+1}\big(p^*(\bar{\mathcal{L}}^h)^{n+1}\big)\big).$$

PROPOSITION 2.30

For an $\omega_{h_{\mathrm{ref}}}$-PSH smooth function φ, we have

$$\mathcal{I}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})}(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},e^{-2\varphi}\cdot h_{\mathrm{ref}}) = \frac{(L^n)}{[K:\mathbb{Q}]}\mathcal{I}_{\omega_{h_{\mathrm{ref}}}}(\varphi),$$

the (usual) Aubin functional.

Proof

Again, this follows as a special case of Proposition 2.2. □

One can hope to use the above Arakelov–Aubin functional as a certain "norm" when estimating our modular height h_K, as in the original algebrogeometric setting. For example, the recent theory of uniform K-stability from [24] and [13] makes use of the Aubin functional from Definition 2.28. Similarly to the original Kähler situation (see [2, pp. 146–147]), we have the following fundamental inequality for our arithmetic situation.

PROPOSITION 2.31

Keeping the notation used previously, we have

$$0 \leq \frac{1}{n+1}\mathcal{I}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})} \leq \mathcal{J}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})} \leq \frac{n}{n+1}\mathcal{I}^{\mathrm{Ar}}_{(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})}.$$

Proof

Although the essential techniques are completely the same as in the known classical case, we hope the following gives a simpler explanation for readers. Indeed, for example, the corresponding estimates for test configurations are done in [13, Propositions 7, 8].

In our proof, we only use the Hodge index theorem in the Arakelov-geometric setting due to [64] and [105, Theorem 1.3]. We take the value at $(\mathcal{X},\mathcal{L},h)$ of the functionals. We set $\mathcal{O}_{\tilde{\mathcal{X}}}(\bar{E}) := (p^*\bar{\mathcal{L}}^h) \otimes (q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^{\otimes(-1)}$, with an Arakelov divisor $\bar{E}$. Then the desired inequalities can be rewritten after some simple calculations as

(1) $(-\bar{E}^2.(p^*\bar{\mathcal{L}}^h)^{n-1} + (p^*\bar{\mathcal{L}}^h)^{n-2}.(q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^1 + \cdots + (q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^{n-1}) \geq 0$,
(2) $(-\bar{E}^2.\sum_{k=0}^{n-1}(n-k)(p^*\bar{\mathcal{L}}^h)^{n-1-k}.(q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^k) \geq 0$,
(3) $(-\bar{E}^2.\sum_{k=0}^{n-1}(n-k)(q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^{n-1-k}.(p^*\bar{\mathcal{L}}^h)^k) \geq 0$.

Indeed, the first inequality (1) gives $\mathcal{I}^{\mathrm{Ar}} \geq 0$, the second inequality gives $\frac{1}{n+1}\mathcal{I}^{\mathrm{Ar}} \leq \mathcal{J}^{\mathrm{Ar}}$, and the last inequality gives $\mathcal{J}^{\mathrm{Ar}} \leq \frac{n}{n+1}\mathcal{I}^{\mathrm{Ar}}$. □

We end this section with a remark that the arguments of [71, (2.6), (2.7), and (2.8)] are via similar techniques and indeed give us the following similar inequalities under the same notation as above:

$$(n+1)\big((p^*\bar{\mathcal{L}}^h)^n.q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}}\big) \geq n(p^*\bar{\mathcal{L}}^h)^{n+1} + (q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^{n+1}, \tag{3}$$

$$(n+1)\big((q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^n.p^*\bar{\mathcal{L}}^h\big) \geq n(q^*\bar{\mathcal{L}}_{\mathrm{ref}}^{h_{\mathrm{ref}}})^{n+1} + (p^*\bar{\mathcal{L}}^h)^{n+1}. \tag{4}$$

As we partially show later in Section 3.3, all the above functionals defined so far can be encoded as metrized line bundles over a higher-dimensional arithmetic base and partially on arithmetic moduli spaces (*Arakelov K-moduli*). It will be done simply and straightforwardly by replacing the Gillet–Soulé intersection number we use here in our definitions by the Deligne pairings.

2.8. Non-Archimedean scalar curvature and Calabi energy

To discuss the general case of a constant scalar curvature Kähler (cscK) metric or more broadly extremal metrics in the sense of E. Calabi, we certainly need discussions of scalar curvature and related functionals such as the Calabi energy. Regarding this kind of energy, even the non-Archimedean analogues have not yet been introduced as far as the author knows; hence, we would like to start with that. In this section, our base is a *smooth projective curve C over a field k* and $\pi : (\mathcal{X}, \mathcal{L}) \to C$ is a projective flat family of relative dimension n. For simplicity, we suppose that $\mathcal{X}$ is normal and $\mathbb{Q}$-factorial in this section. Temporarily, we do not discuss extension to adèlic metrics, and from here to the end of our article, we do not mean that such extension is automatic anymore.

2.8.1. Equicharacteristic situation

For $\pi : (\mathcal{X}, \mathcal{L}) \to C$ with $\mathcal{X}_0 = \bigcup_i E_i$, the *non-Archimedean scalar curvature* S^{nA} is a function from the set of irreducible components of $\mathcal{X}_0$ defined as

$$S^{nA} : E_i \mapsto \frac{-n(\mathcal{L}|_{E_i}^{n-1}.K_{\mathcal{X}/C}|_{E_i})}{(\mathcal{L}|_{E_i}^{n})}.$$

We make a brief review of the minimization of the (normalized) Donaldson–Futaki invariant from [75, Section 4]. Fixing a general fiber (X, L) over $\eta \in C$, we consider all polarized models $\pi : (\mathcal{X}, \mathcal{L}) \to \tilde{C} \to C$, where $\mathcal{X} \to \tilde{C}$ is projective and $\tilde{C} \to C$ is a finite covering. We call

$$\frac{\mathrm{DF}(\mathcal{X}, \mathcal{L})}{\deg(\tilde{C} \to C)} := \frac{-n(L^{n-1}.K_X)(\mathcal{L}^{n+1}) + (n+1)(L^n)(\mathcal{L}^n.K_{\mathcal{X}/\tilde{C}})}{\deg(\tilde{C} \to C)}$$

the *normalized Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})(\to \tilde{C} \to C)$ and denote it by $\mathrm{nDF}(\mathcal{X}, \mathcal{L})$.

Among all models $(\mathcal{X}, \mathcal{L})$ over finite coverings $\tilde{C}$ of C, if π is minimizing the above normalized Donaldson–Futaki invariant, we [75, (4.3)] proved that $\mathcal{X}_0$ is reduced and only admits semi-log-canonical singularities. In some situations, we proved more (see [75, Section 4] for more details). We add one more property of this family in our context.

PROPOSITION 2.32

If $(\mathcal{X}, \mathcal{L}) \to \tilde{C}$ takes the minimal normalized Donaldson–Futaki invariant among the models of (X, L) over finite coverings $\tilde{C}$ of C, then the non-Archimedean scalar curvature S^{nA} is constant; that is, $S^{nA}(E_i)$ does not depend on i.

Proof

Suppose the contrary; then either $(\mathcal{X}, \mathcal{L}(\epsilon E_i))$ with $0 < \epsilon \ll 1$ or $(\mathcal{X}, \mathcal{L}(-\epsilon E_i))$ with $0 < \epsilon \ll 1$ has a lesser (normalized) Donaldson–Futaki invariant than that of $(\mathcal{X}, \mathcal{L})$. Thus, it contradicts the fact that $(\mathcal{X}, \mathcal{L})$ minimizes the (normalizing) Donaldson–Futaki invariant. $\square$

The above Proposition 2.32 also extends an observation made for the Calabi–Yau case (see [74, 4.2(i)]).

We give a new way of interpreting the K-stability via scalar curvature of the $(n+1)$-dimensional total space of test configurations.

DEFINITION-PROPOSITION 2.33

A polarized projective variety (X, L) is K-polystable if and only if the following $(*)$ holds.

$(*)$ *For any test configuration with $\mathbb{Q}$-line bundle of exponent 1, if we think of a natural compactification*[12] *$(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$ by attaching (X, L) at $\infty \in \mathbb{P}^1$, then by supposing $\mathcal{L}$ is (absolutely) ample (we replace $\mathcal{L}$ by $\mathcal{L}(mF)$ with $m \gg 0$ to make it ample if not),*

$$\frac{-(n+1)(\mathcal{L}^n.K_{\mathcal{X}/C})}{(\mathcal{L}^{n+1})} \leq \frac{-n(L^{n-1}.K_X)}{(L^n)},$$

with equality holding exactly when $(\mathcal{X}, \mathcal{L})$ is a (naturally compactified) product test configuration, that is, (X, L)-fiber bundle. Note that the above inequality remains equivalent even when change m.

The above new way of paraphrasing K-stability is analogous to slope theories (see Mumford [66], Takemoto [94], Ross and Thomas [83]) and follows straightforwardly from the general formula for the Donaldson–Futaki invariant (see [103], [69]). Note that the left-hand side is a sort of scalar curvature average of $(\mathcal{X}, \mathcal{L})$, and the right-hand side is precisely the scalar curvature average of (X, L). The above reinterpretation of K-stability "via average scalar curvatures" was found during discussions with R. Thomas in 2013.

2.8.2. Non-Archimedean Calabi functional

Inspired by the observation in the above arguments, let us propose our working definition of a non-Archimedean Calabi functional and an Arakelov–Calabi functional as follows. We will come back to further study of these in the future.

DEFINITION 2.34 (NON-ARCHIMEDEAN CALABI FUNCTIONAL)

For a projective flat family $\mathcal{X}$ over a smooth proper curve C with a relatively ample line bundle $\mathcal{L}$ and finite closed points set $S \subset C(k)$, we set

$$\mathrm{Ca}_S(\mathcal{X}, \mathcal{L}) := \sum_i \Big(\frac{(\mathcal{L}|_{E_i}^{n-1}.K_{\mathcal{X}/C})}{(\mathcal{L}|_{E_i})^n} \Big)^2 \ (>0),$$

where $\bigcup_i E_i = \mathrm{Supp}(\bigcup_{s \in S} \mathcal{X}_s)$ is the irreducible decomposition of the support of the union of the special fibers $\mathcal{X}_s$.

DEFINITION 2.35 (ARAKELOV–CALABI FUNCTIONAL)

Suppose $\mathcal{X}$ is a regular variety which is projective over $C := \mathrm{Spec}(\mathcal{O}_K)$, the ring of integers of K, with a relatively ample line bundle $\mathcal{L}$ with a Hermitian metric h. For a finite set S of places of K, we decompose it into the finite places and infinite

[12] See [70] for precise details.

places as $S = S^{\text{fin}} \cup S^{\infty}$. Then we set $\mathrm{Ca}^{\mathrm{Ar}}(\mathcal{X}, \bar{\mathcal{L}} := (\mathcal{L}, h))$ as

$$\sum_{\bigcup_i E_i = \bigcup_{s\in S^{\text{fin}}} \mathcal{X}_s} \Big(\frac{(\bar{\mathcal{L}}|_{E_i}^{n-1} . (\overline{K_{\mathcal{X}/C}}^{\mathrm{Ric}(\omega_h)}))}{(\bar{\mathcal{L}}|_{E_i})^n} \Big)^2 + \frac{1}{n^2} \sum_{\sigma \in S^{\infty}} \int_{\mathcal{X}(\sigma)} S(\omega_h)^2 \omega_h^n \ (>0).$$

Here, as above, $\bigcup_i E_i = \mathrm{Supp}(\mathcal{X}_0)$ is the support of the special fiber over a closed point $0 \in C$.

3. Further discussions

In this section, we argue closely related issues as well as give applications.

3.1. Failure of asymptotic semistable reduction

First, let us recall that, as long as we consider the Chow stability of embedded projective varieties, we have a stable reduction theorem which we recall in the following general form.

PROPOSITION 3.1 (GIT (POLY)STABLE REDUCTION ([68], [87]))
Suppose that R is a discrete valuation ring which is a Nagata ring[13] *as well, let K be its fractional field, and let k be its residue field. Let G be a reductive group scheme over R (i.e., all of its geometric fibers are reductive algebraic groups) acting on a projective scheme $(H, \mathcal{O}_H(1))$ over R. If $x \in H^{\mathrm{ss}}(K)$, a semistable point, then for a finite extension of K' and the integral closure R' of R in K', x extends to a morphism $\tilde{x} : \mathrm{Spec}(R') \to H^{\mathrm{ss}}$ such that $x(k) \in H^{\mathrm{ss}}(k)$ is a polystable*[14] *point.*

Proof
As the proof for the geometric case ($R = k[[t]]$) is written in [68, Lemma 5.3] and basically our general case follows similarly, we only briefly note the differences of which we need to take care. The facts used in the geometric case's proof of Mumford which are not proven for arithmetic case in the original GIT [67] nor [68] are, first, the existence of a quasiprojective GIT quotient (with its compatibility with base change to fibers) and second the existence of a group-invariant homogeneous polynomial separating arbitrary given group-invariant closed subsets. The former is established as [87, p. 269, Theorem 4, note (v)], and the latter is established as [87, p. 254, Proposition 7(3)]. □

However, if we consider the abstract polarized variety (X, L) and consider *asymptotic Chow (semi)stability*, that is, the Chow (semi)stability for $X \subset$

[13]Noetherian ring which is "universally Japanese" (i.e., all finitely generated integral domain extensions $(R \subset) R'$ are "Japanese," i.e., satisfy the finiteness of integral closure in finite extensions L'/K' of the fractional field $K' = \mathrm{Frac}(R')$).
[14]That is, semistable with minimal (closed) orbit.

$\mathbb{P}(H^0(X, L^{\otimes m}))$ for $m \gg 0$, then the desired stable reduction fails. We give counterexamples below, but the key proposition is the following observation after the local stability theory of Eisenbud and Mumford.

PROPOSITION 3.2 ([68], [88])[15]

Suppose (X, L) is an n-dimensional projective variety over a field. If there is a closed point $x \in X$ such that $\mathrm{mult}_x(X) > (n+1)!$, then (X, L) is asymptotically Chow unstable, that is, for $l \gg 0$, $X \subset \mathbb{P}(H^0(X, L^{\otimes l}))$, embedded by the complete linear system, is Chow unstable.

Proposition 3.2 has been used repeatedly, as the key, in Shepherd-Barron [89], [71], Wang–Xu [104], and others, to show that "classical GIT does *not* work for compactifying moduli of higher-dimensional varieties." Although the Kollár's surface example (see [104]) essentially works in our situation as well, we give a simpler series of examples in arbitrary dimensions as follows.

EXAMPLE 3.3

For each prime number p, we consider the integer parameters $a_0, \dots, a_n$ which are all coprime to p and coprime to each other. Then we consider an integral model of (weighted) Brieskorn–Pham type

$$(\mathcal{X}, \mathcal{O}(1)) := \Big[\sum_{0 \le i < n} x_i^{d_i} + p x_n^{a_0 \cdots a_{n-1}} = 0 \subset \mathbb{P}_{\mathbb{Z}}(a_0, \dots, a_n)\Big],$$

where $d_i := \prod_{j \neq i} a_j$ and $\min_i\{a_i\} \gg n$. The reduction $\mathcal{X}_p$ at p is

$$\Big[\sum_{0 \le i < n} x_i^{d_i} = 0 \subset \mathbb{P}(a_0, \dots, a_n)\Big].$$

Obviously, $\mathcal{X}$ is regular scheme and $\mathcal{X}_p$ has only one singular point $x = [0 : \cdots : 0 : 1]$. The singularity is a quotient of (we put $x_i = X_i^{a_i}$)

$$(0, \dots, 0) \in \Big[\sum_{0 \le i \le n-1} X_i^{\prod_{0 \le j \le n} a_j}\Big] \subset \mathbb{A}^1_{X_0, \dots, X_{n-1}}$$

by $\prod_{0 \le i \le n} \boldsymbol{\mu}_{a_i}$ and, thus, log-canonical. In the meantime, the multiplicity of $x \in \mathcal{X}_p$ is at least that of the cyclic quotient singularity $\frac{1}{a_n}(a_0, \dots, a_{n-1})$. If we choose $a_0, \dots, a_n$ carefully, then it is easy to make the multiplicity bigger than $(n+1)!$. From the way we take the a_i's, all the fibers of $\mathcal{X}$ are normal with ample canonical class.

To show some pathological properties of the above examples, we need some preparations. First we recall the arithmetic (twisted) variant of Chow weight defined

[15] Also compare [70], which gives a modern version of them via discrepancy.

by [11]. Also [10] and [108] contain very closely related variants,[16] and we propose a common generalization a while later in Definition 3.15.

DEFINITION 3.4 ([11])

Keeping the notation, we suppose further that $\mathcal{X}$ is generically smooth. Then the *Chow height* $h_C(\mathcal{X}, \bar{\mathcal{L}} = (\mathcal{L}, h))$ is defined as

$$\frac{(\bar{\mathcal{L}})^{n+1}}{(\dim(X)+1)(\mathcal{L}_\eta)^n[K:\mathbb{Q}]} - \frac{\hat{\deg}(\pi_*\bar{\mathcal{L}})}{\operatorname{rank}(\pi_*\mathcal{L})[K:\mathbb{Q}]},$$

where the direct image sheaf $(\pi_*\bar{\mathcal{L}})$ is with the natural L^2-metric h_{L^2}, which we obtain via h and the Kähler metric corresponding to $c_1(L,h)$:

$$h_{L^2}(s,\bar{t}) := \int_{\mathcal{X}(\mathbb{C})} \langle s(x), \overline{t(x)} \rangle_h \omega_h^n.$$

Note that the definition of the L^2-metric above is slightly different from the original [11], which normalizes the volume form to be a probability measure. This adjustment is for the compatibility with our Definition 3.15 and Theorem 3.16.

DEFINITION 3.5

We fix a reference integral model $\pi : (\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}}) \to \operatorname{Spec}(\mathcal{O}_K)$ of a generically smooth projective variety over the ring of integers of a number field K. Then for another integral model $(\mathcal{X}, \mathcal{L}, h_{\mathrm{ref}})$, we set

$$h_{K,(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}})}(\mathcal{X},\mathcal{L}) := h_K(\mathcal{X},\mathcal{L},h_{\mathrm{ref}}) - h_K(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})$$

and call it the *relative (scheme-theoretic) modular height* of $(\mathcal{X}, \mathcal{L})$ with respect to $(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}})$. Note that it easily follows from Proposition 2.8 that it is independent of the choice of the reference metric h_{ref} so that is well defined and is a quantity of a purely arithmetic nature.

Similarly, we set

$$h_{C,(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}})}(\mathcal{X},\mathcal{L}) := h_C(\mathcal{X},\mathcal{L},h_{\mathrm{ref}}) - h_C(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}},h_{\mathrm{ref}})$$

and call it the *relative Chow height* of $(\mathcal{X}, \mathcal{L})$ with respect to $(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}})$. Note that it is again easy to prove that it is independent of the choice of the reference metric h_{ref} so that is well defined and is also a quantity of a purely arithmetic nature.

From the definition it is easy to see the cocycle condition.

PROPOSITION 3.6

For any three integral models $(\mathcal{X}_i, \mathcal{L}_i)$ $(1 \le i \le 3)$ of a common polarized variety

[16]But please be a little careful as, the author supposes, there are some (unessential) typos which do not cause any troubles in the definitions of their papers: [10, Theorems I and III] have presumably wrong signs on the right-hand sides, and in [108], the place where "$[K:\mathbb{Q}]$" is put seems to be wrong.

(X, L) *over a number field* K*, we have*

$$h_{K,(\mathcal{X}_1,\mathcal{L}_1)}(\mathcal{X}_3,\mathcal{L}_3) = h_{K,(\mathcal{X}_1,\mathcal{L}_1)}(\mathcal{X}_2,\mathcal{L}_2) + h_{K,(\mathcal{X}_2,\mathcal{L}_2)}(\mathcal{X}_3,\mathcal{L}_3)$$

and

$$h_{C,(\mathcal{X}_1,\mathcal{L}_1)}(\mathcal{X}_3,\mathcal{L}_3) = h_{C,(\mathcal{X}_1,\mathcal{L}_1)}(\mathcal{X}_2,\mathcal{L}_2) + h_{C,(\mathcal{X}_2,\mathcal{L}_2)}(\mathcal{X}_3,\mathcal{L}_3).$$

The following Theorem 3.7 is an Arakelov-theoretic analogue of the fact that the "Donaldson–Futaki invariant is a limit of Chow weights" (see the original definition from [28], which is for isotrivial geometric families with $\mathbb{G}_m$-action) but the proof cannot be obtained as simple imitation and we use the asymptotic analysis of the *Ray–Singer [82] analytic torsion* (as well as a simple "anomaly" formula) in addition to the Gillet–Soulé [45] arithmetic Riemann–Roch theorem.

THEOREM 3.7 ((DE-)QUANTIZATION)
Keeping the notation, we still suppose that $\mathcal{X}$ *is generically smooth over* $C = \operatorname{Spec}(\mathcal{O}_K)$*. Then the following asymptotic behavior of Chow heights holds:*

$$h_C(\mathcal{X}, \mathcal{L}^{\otimes m}, h^m) = 2(n+1)(L^n)^2 h_K(\mathcal{X}, \mathcal{L}, h) + \frac{n}{4}\log(m) + o(1)$$

for $m \to \infty$*. Hence, in particular, the (K-)modular height is essentially a limit of slightly modified Chow heights:*

$$h_K(\mathcal{X}, \mathcal{L}, h) = \frac{1}{2(n+1)(L^n)^2}\Big(h_C(\mathcal{X}, \mathcal{L}^{\otimes m}, h^m) - \frac{n}{4}\log(m)\Big) + o(1)$$

for $m \to \infty$*.*

Due to the presence of the logarithmic term ($\frac{n}{4}\log(m)$), it also shows that Zhang's height positivity conjecture [108, p. 78] is always "asymptotically true" with respect to the twist of line bundle, even without the (Chow) semistability assumption. The author is happy to acknowledge that Robert Berman told me in May of 2016 that he had a closely related result to Theorem 3.7 in 2012.

First proof
By clearing the denominators of the Chow heights $h_C(\mathcal{X}, \mathcal{L}^{\otimes m}, h^m)$, what we are to analyze is the asymptotic behavior of

$$\begin{aligned}
&(\bar{\mathcal{L}}^{n+1})h^0(L^{\otimes m})m - (n+1)(L^n)\hat{\deg}(\pi_*\bar{\mathcal{L}}^{\otimes m}) \\
&= m(\bar{\mathcal{L}}^{n+1})\Big(\frac{(L^n)}{n!}m^n - \frac{(L^{n-1}.K_X)}{2(n-1)!}m^{n-1} + \cdots\Big) \\
&\quad - (n+1)(L^n)\hat{\deg}(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2})
\end{aligned}$$

with respect to $m \gg 0$. To clarify that the L^2-metric above is induced by h^m on $L^{\otimes m}$ and mg_h on T_X corresponding to $c_1(L^{\otimes m}, h^m)$, it should be read as $h_{L^2}(h^m, mg_h)$. Note that this is *not* $h_{L^2}(h^m, g_h)$. So we make the asymptotic analysis of $\hat{\deg}(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h))$. We decompose it as

$$\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, g_h)\big) + \big(\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h)\big) - \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, g_h)\big)\big)$$

and then

$$= \big(\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h)\big) - \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, g_h)\big)\big) \tag{5}$$

$$+ \big(\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, g_h)\big) - \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_Q(h^m, g_h)\big)\big) \tag{6}$$

$$+ \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_Q(h^m, g_h)\big), \tag{7}$$

where h_Q of (6) and (7) stand for the Quillen metrics. It follows directly from the definition that (5) coincides with $\frac{[K:\mathbb{Q}]}{2}$ times

$$\mathrm{rank}(\pi_*\mathcal{L}^{\otimes m}) \cdot \log(m^n) = \frac{(L^n)}{(n-1)!} m^n \log(m) + O\big(m^{n-1}\log(m)\big).$$

Note that $\log(m^n)$ appears as an entropy. From the definition of the Quillen metric, the second part (6) is half of the Ray–Singer analytic torsion, which we denote as

$$\frac{d\zeta_m^{\mathrm{sp},g}(s)}{ds}\Big|_{s=0} = \sum_{q=0}^{n}(-1)^{q+1} q \frac{d\zeta_{m,q}^{\mathrm{sp},g}(s)}{ds}\Big|_{s=0}.$$

We also denote the above as $T(X_\infty, g, L_\infty^{\otimes m}, h^m)$.[17] Here, note that $\zeta_{m,q}^{\mathrm{sp},g}(s)$ denotes the spectral zeta function $\sum_n \lambda_{m,q,n}(g)^{-s}$, the sum of $(-s)$-powers of all positive eigenvalues $\lambda_{m,q,n}(g)$ of the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial},m}(g)$ on $\mathcal{A}^{0,q}(L^{\otimes m})$, the space of $(0,q)$-forms of C^∞-class with coefficients in $L^{\otimes m}$. We set $\zeta_m^{\mathrm{sp},g}(s) := \sum_q (-1)^{q+1} q \zeta_{m,q}^{\mathrm{sp},g}(s)$. In our situation, $g = g_h$. By Bismut and Vasserot [9, Theorem 8],[18] we conclude that (6) is

$$\frac{1}{2} T(X_\infty, g, L_\infty^{\otimes m}, h^m) = \frac{[K:\mathbb{Q}]}{4}\Big\{\frac{(L^n)}{(n-1)!} m^n \log(m) + o(m^n)\Big\}.$$

Finally we have that (7) is

$$\begin{aligned}
&\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_Q(h^m, g_h)\big) \\
&= \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_Q(h^m, mg_h)\big) \\
&= \bar{\deg}\Big(\Big(m^{n+1}\frac{\hat{c_1}(\bar{\mathcal{L}})^{n+1}}{(n+1)!} + m^n \frac{\hat{c_1}(\bar{\mathcal{L}})^n}{n!} + o(m^n)\Big) \\
&\quad \cdot \hat{td}(T_{\mathcal{X}/C}, \omega_h) \cdot \big(1 - a\big(R(T_{X(\mathbb{C})}, \omega_h)\big)\big)\Big) \\
&= \frac{\hat{c_1}(\bar{\mathcal{L}})^{n+1}}{(n+1)!} m^{n+1} + \frac{(\hat{c_1}(\bar{\mathcal{L}})^n . \hat{c_1}(T_{\mathcal{X}/C}, \omega_h))}{2(n!)} m^n + o(m^n),
\end{aligned}$$

[17] Different notation from [11] up to an additive constant due to the normalization of the Kähler form and the difference of the L^2-metric.

[18] Note that $r^\circ/2\pi$ of that paper is the identity matrix Id in our situation.

by the Gillet–Soulé [45] arithmetic Riemann–Roch theorem, where $a(R(-,-))$ is the R-genus.

Combining this with the above analysis, we have the following fundamental refinement of the asymptotic Hilbert–Samuel formula (originally Gillet–Soulé [43]). The reason the author gives its proof and statement here is that he unfortunately could not find any literature for it, but he presumes it is well known to experts.

PROPOSITION 3.8 (ASYMPTOTIC HILBERT–SAMUEL FORMULA)

Suppose that $(\mathcal{X},\mathcal{L})$ is a normal polarized projective variety over $\operatorname{Spec}(\mathcal{O}_K)$, which is furthermore generically smooth, and a Hermitian metric h on L_∞ of real type. Then we have, for $m \to \infty$,

$$\begin{aligned}&\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h)\big)\\&\quad= \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)!}m^{n+1} - \frac{(L^n)}{4((n-1)!)}m^n\log(m) - \frac{(\bar{\mathcal{L}}^n.\overline{K_{\mathcal{X}/C}})}{2(n!)}m^n + o(m^n).\end{aligned}$$

From Proposition 3.8, we have

$$\begin{aligned}&(\bar{\mathcal{L}}^{n+1})h^0(L^{\otimes m})m^{n+1} - (n+1)(L^n)\Big(\hat{\deg}(\pi_*\bar{\mathcal{L}}^{\otimes m}) - \frac{(L^n)}{4((n-1)!)}m^n\log(m)\Big)\\&= \frac{1}{2(n!)}h_K(\mathcal{X},\mathcal{L},h)m^{n+1} + O(m^n),\end{aligned}$$

which completes the first proof of Theorem 3.7. □

More direct analytic proof of Theorem 3.7

We can prove the above theorem without (really) considering metrics of type $h_{**}(h^m, g_h)$. The asymptotic analysis of $\hat{\deg}(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h))$ can be replaced as

$$\begin{aligned}&\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h)\big)\\&= \Big(\hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_{L^2}(h^m, mg_h)\big) - \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_Q(h^m, mg_h)\big)\Big) \quad (8)\\&\quad + \hat{\deg}\big(\pi_*\bar{\mathcal{L}}^{\otimes m}, h_Q(h^m, mg_h)\big)\big). \quad (9)\end{aligned}$$

Note that (8) is half of the Ray–Singer torsion $T(X, mg, L^{\otimes m}, h^m)$. From the anomaly formula ([11, Proposition 4.4]) combined with asymptotics of the Ray–Singer torsion [9, Theorem 8], we have the following result.

LEMMA 3.9

We have that $T(X_\infty, mg_h, \bar{L_\infty}^{\otimes m}) = o(m^n)$.

Lemma 3.9 extends [11, Proposition 4.2] for Abelian varieties in a weak form (but note the slight difference with [11] due to normalizations in the definition). On the other hand, (9) can be calculated by the Gillet–Soulé [45] arithmetic

Riemann–Roch theorem again, as (9) equals

$$\hat{\deg}\Big(\Big(m^{n+1}\frac{\hat{c}_1(\bar{\mathcal{L}})^{n+1}}{(n+1)!}+m^n\frac{\hat{c}_1(\bar{\mathcal{L}})^n}{n!}+o(m^n)\Big)$$
$$\cdot\,\hat{td}(T_{\mathcal{X}/C},m\omega_h)\cdot\big(1-a\big(R(T_{X(\mathbb{C})},\omega_h)\big)\big),$$

where $a(R(-,-))$ is the R-genus again. If we use the description of secondary class [11, (4.2.9)], we have

$$\hat{td}(T_{\mathcal{X}/C},m\omega_h)=\hat{td}(T_{\mathcal{X}/C},\omega_h)-\log(m)\big(Td'(T_{X_\infty},\omega_{g_h})\big),$$

where Td' stands for the characteristic form defined by the derivative of formal series which corresponds to the Todd class (see [11, 4.2.2]). As being similar to the first proof, simple additions of the above gives us the assertion again. □

We have the following application of the above dequantization process (Theorem 3.7). This is an arithmetic version of [104], which in turn builds upon [68, 3.12] and [71, 1.1 and its proof] (see also [74, Section 3]).

THEOREM 3.10

The generic fiber $(\mathcal{X}_\eta,K_{\mathcal{X}_\eta})$ *of Example 3.3 does not have weakly asymptotic Chow semistable reduction at the prime* p, *that is, there is no integral model* $(\mathcal{X}',\mathcal{L}')$ *which satisfies that it extends* $(\mathcal{X}_\eta,K_{\mathcal{X}_\eta})$, *and for infinitely many* $l\gg 0$, $(\mathcal{X}'_p,\mathcal{L}'_p)$ *embedded by* $|(\mathcal{L}'_p)^{\otimes l}|$ *are Chow semistable.*

Proof

Suppose the contrary, and let $\pi':(\mathcal{X}',\mathcal{L}')\to C$ be such an integral model with weakly asymptotically Chow semistable reduction $(\mathcal{X}'_p,\mathcal{L}'_p)$ at p. We also fix a reference integral model $\pi:(\mathcal{X}_{\rm ref},\mathcal{L}_{\rm ref},h_{\rm ref})\to \mathrm{Spec}(\mathcal{O}_K)$ as above. We follow the strategy of [104] and deduce a contradiction by using Proposition 3.2.

From the assumption of the contrary, there are infinitely many $m\gg 0$ such that $[\mathcal{X}'\subset\mathbb{P}(\pi'_*\mathcal{L}'^{\otimes m})]$ is minimizing the relative Chow height $h_{C,(\mathcal{X}_{\rm ref},\mathcal{L}_{\rm ref}^{\otimes m})}$ (Definitions 3.4, 3.5) among all integral models. Thus, from Theorem 3.7, it also minimizes $h_{K,(\mathcal{X}_{\rm ref},\mathcal{L}_{\rm ref})}$ (Definition 3.5) or equivalently minimizes the modular height h_K (if we fix a reference metric). Thus, Theorem 2.14 tells us that $(\mathcal{X}',\mathcal{L}')\simeq(\mathcal{X},\mathcal{L})$.

On the other hand, Proposition 3.2 shows that the multiplicity of any closed point in $\mathcal{X}_p$ is at most $(n+1)!$. This contradicts the fact that Example 3.3 violates the condition. □

REMARK 3.11

In Example 3.3, if we replace $\mathrm{Spec}(\mathbb{Z})$ and p with $\mathbb{A}^1_k=\mathrm{Spec}(k[t])$ and t for an algebraically closed field k, then they give yet another but simpler variant to the examples of [89], [71], and [104].

3.2. Arithmetic Yau–Tian–Donaldson conjecture

We propose the following conjectures, which speculate relations between purely metrical properties of (arithmetic) varieties and their purely arithmetic properties.

CONJECTURE 3.12 (ARITHMETIC YAU–TIAN–DONALDSON CONJECTURE)
For an arbitrary smooth projective variety (X, L) *with finite automorphism group* $\mathrm{Aut}_K(X, L)$ *over a number field* K*, the following conditions are equivalent.*

(1) *(Differential geometric side)* $X(\mathbb{C})$ *admits a cscK metric* $\omega \in c_1(L(\mathbb{C}))$.

(2) *(Arithmetic side) There is an integral model* $(\mathcal{X}, \mathcal{L})$ *of* (X, L) *possibly after finite extension of* K *such that, for each prime ideal* $\mathfrak{p}$ *of* $\mathcal{O}_K$*, the reduction* $(\mathcal{X}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$ *is K-semistable. Furthermore, for almost all (other than finite exceptions)* $\mathfrak{p}$*, the reduction is K-stable.*

Note that this property is purely arithmetic. (The notion does not depend on the reference model a posteriori.) (X, L) *is said to be arithmetically K-stable if* $\mathrm{Aut}_K(X, L)$ *is finite and the above conditions (the previous paragraph of (2)) hold.*

We believe that the above equivalence also holds for singular varieties (see, e.g., [34]) and log pairs (see, e.g., [31], [79]) as well. For example, if the (log-)canonical class is ample or numerically trivial, then it is natural to admit geometrically *semi-log-canonical* singularities, and if the (log-)canonical class is antiample, then it is natural to admit *(Kawamata) log-terminal* singularities (see [71], [70], [7], [79]).

We call the lower boundedness of modular height h_K among all (metrized) integral models $(\mathcal{X}, \mathcal{L}, h)$ over finite extensions of K the Arakelov K-semistability of (X, L). Then the semistability version of the above conjecture is as follows.

CONJECTURE 3.13 (SEMISTABLE ARITHMETIC YAU–TIAN–DONALDSON CONJECTURE)
For an arbitrary smooth projective variety (X, L) *over a number field* K*, the following are equivalent.*

(1) *(Differential geometric side)* $(X(\mathbb{C}), L(\mathbb{C}))$ *is K-semistable (in the sense of [28]).*

(2) *(Mixed)* (X, L) *is Arakelov K-semistable, that is,* $h_K(\mathcal{X}, \mathcal{L}, h)$*, where* $(\mathcal{X}, \mathcal{L})$ *runs through all the models of* (X, L) *and all* h *whose curvature is positive, are (uniformly) lower bounded.*

(3) *(Arithmetic side) For an integral model* $(\mathcal{X}, \mathcal{L})$ *over* $\mathcal{O}_K$ *and a maximal ideal* $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$*, the reduction* $(\mathcal{X}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$ *is K-semistable.*[19]

(4) *(Arithmetic side) For any integral model* $(\mathcal{X}, \mathcal{L})$ *over* $\mathcal{O}_K$ *and for almost all (i.e., outside finite exceptions) maximal ideals* $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$*, the reduction*

[19]Note that the definition of K-stability from [28] works over any field, although in that paper the base is assumed to be $\mathbb{C}$.

$(\mathcal{X}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$ *is K-semistable. We call this condition of* (X, L) *arithmetic K-semistability.*

The above expected equivalence between Conjecture 3.13(1) and Conjectures 3.13(3) and 3.13(4), when it is true, gives a partial geometric meaning to the K-(semi)stability of polarized varieties over a *positive* characteristic field via lifting.

We partially confirm the above conjecture as follows. First, it straightforwardly follows from Proposition 2.8 that Conjecture 3.13(2) implies the lower boundedness of Mabuchi's K-energy (see [55] for the (geometric) Fano case). The geometric version of the above conjecture is (partially) discussed in [75, Sections 3 and 4.2]), where the Arakelov K-semistability corresponds to the generic K-semistability introduced in [75, Definition 3.1].

A possibly interesting remark would be that the above two conjectures also give us an insight that existence or nonexistence of Kähler–Einstein metrics and other canonical Kähler metrics remains the same once we replace the coefficients of the defining equations by some conjugates (i.e., by an element of $\mathrm{Gal}(\mathbb{Q})$).

We only partially confirm the above conjectures as follows. As in Theorem 2.14, the statements involve the language of log pairs $(\mathcal{X}, \mathcal{X}_{\mathfrak{p}})$ and their log discrepancies (see, e.g., [53]), but it is due to the lack of inversion of adjunction, which is natural to expect. Each of the conditions on $(\mathcal{X}, \mathcal{X}_{\mathfrak{p}})$ below means the mildness of singularities of fibers $\mathcal{X}_{\mathfrak{p}}$ (or their anticanonical divisors in Conjecture 3.13(3)), as was precisely so in the geometric case. We refer the interested readers to [104, Theorem 6] and [74, Section 4] corresponding to the following.

THEOREM 3.14

The following types of projective varieties (X, L) *over a number field* K *are arithmetically K-stable (resp., arithmetically K-semistable and Arakelov K-semistable), and all the base changes to complex places admit possibly singular Kähler–Einstein metrics (resp., the K-energy of a geometric generic fiber is bounded). In particular, the arithmetic Yau–Tian–Donaldson Conjectures 3.12 and 3.13 hold for the following cases.*

(1) *A log-terminal polarized variety* (X, L) *over* K *with numerically trivial* K_X *which admits, possibly after finite extension of* K*, an integral model* $(\mathcal{X}, \mathcal{L})$ *(relatively minimal model) that satisfies the following:* $K_{\mathcal{X}}$ *is* $\mathbb{Q}$*-Cartier, and for any prime* $\mathfrak{p}$ *of* $\mathcal{O}_K$*, the reduction* $\mathcal{X}_{\mathfrak{p}}$ *is reduced with* $\mathbb{Q}$*-linearly trivial canonical divisor and* $(\mathcal{X}, \mathcal{X}_{\mathfrak{p}})$ *is log canonical in an open neighborhood of* $\mathcal{X}_{\mathfrak{p}}$*.*

(2) *A log-terminal variety*[20] X *over* K *with ample (pluri-)canonical polarization* $L = \mathcal{O}_X(mK_X)$ *with* $m \in \mathbb{Z}_{>0}$*, which has a nice (relative log-canonical model) integral model* $\mathcal{X}$*, possibly after finite extension of* K*, which satisfies the*

[20] In this case and case (1), the only technical reason why we do not lose this mildness assumption of singularities to (semi-)log-canonicity, which is more natural, is due to the difficulty of the metric (see [7]). The same holds for (3).

following: $\mathcal{X}$ is $\mathbb{Q}$-Cartier, each reduction $\mathcal{X}_{\mathfrak{p}}$ is reduced with ample canonical $\mathbb{Q}$-Cartier divisor, and $(\mathcal{X}, \mathcal{X}_{\mathfrak{p}})$ is log-canonical in a neighborhood of $\mathcal{X}_{\mathfrak{p}}$.

(3) *A log-terminal (anticanonically polarized) $\mathbb{Q}$-Fano variety (X, L) over K which has, after some finite extension of K if necessary, an integral model $\mathcal{X}$ which satisfies the following: for any prime $\mathfrak{p}$ of $\mathcal{O}_K$, $\mathcal{X}_{\mathfrak{p}}$ is a (normal and) log-terminal $\mathbb{Q}$-Fano variety whose alpha invariant (see [96], [77])*

$$\sup\bigl\{t \geq 0 \,\big|\, (\mathcal{X}, \mathcal{X}_{\mathfrak{p}} + tD) \text{ is log-canonical around } \mathcal{X}_{\mathfrak{p}} \\ \text{for all effective } D \equiv_{/C} -K_{\mathcal{X}/C}\bigr\}$$

is more than $\frac{n}{n+1}$ (resp., at least $\frac{n}{n+1}$).

Proof
It simply follows from our height minimization Theorem 2.14, when we combine that with known analytic results that Kähler–Einstein metrics exist on each geometric generic fiber for the above situations (see [34], [35], and [96]; resp., the lower boundedness of K-energy for the case of (3) with the alpha invariant $\frac{n}{n+1}$; see, e.g., [55]). □

Note that the desired integral models from (1) and (2) are naturally expected to exist unconditionally as consequences of the conjectural arithmetic log MMP.

We now review the original quantized version, that is, the version for the Chow stability of (essentially) *embedded* varieties with slight improvement. That is, we slightly extend the definition of Chow height (see [10], [11], [108]) after an idea of Donaldson [29].

DEFINITION 3.15 (EXTENDED CHOW HEIGHT)
We keep the notation as above (although $\mathcal{X}$ only needs to be normal and $\mathbb{Q}$-Gorenstein). As extra data, for a Hermitian metric of real type[21] h on $L = \mathcal{L}(\mathbb{C})$ and a Hermitian metric of real type H on $\pi_*\mathcal{L}$, we associate the *extended Chow height* $\tilde{h}_C$ as

$$\tilde{h}_C(\mathcal{X}, \mathcal{L}, h, H) := \frac{1}{[K:\mathbb{Q}]}\biggl\{\frac{(\bar{\mathcal{L}})^{n+1}}{(\dim(X)+1)(\mathcal{L}_\eta)^n} - \frac{\hat{\deg}(\pi_*\bar{\mathcal{L}})}{\operatorname{rank}(\pi_*\mathcal{L})} \\ + \log\Bigl(\int_{X_\infty} \sum_\alpha |s_\alpha|_h^2 \frac{c_1(L,h)^n}{(L^n)}\Bigr)\biggr\},$$

where s_α is the orthonormal basis of $H^0(L)$ with respect to H. Note that the integrand $\sum_\alpha |s_\alpha|_h^2$ is the ratio $\frac{h}{FS(H)}$. The difference with the original Chow height (h_C in this article) introduced in [11] and [108] is the last logarithmic term.

[21] That is, complex conjugate invariant.

The following equivalence can nowadays be regarded as a quantized version of the Yau–Tian–Donaldson correspondence, slightly refined after [29]. One direction (implication from semistability) was found by Bost [10], [11] and Zhang [108], while the other direction was also found by Zhang [108]. We slightly refine the statement partially after [29].

THEOREM 3.16 (QUANTIZED YAU–TIAN–DONALDSON CORRESPONDENCE)
If we fix a smooth polarized projective variety (X, L) over a number field K, then its Chow semistability (resp., Chow polystability) is equivalent to the lower boundedness (resp., existence of a minimum) of $\tilde{h}_C(\mathcal{X}, \mathcal{L}, h, H)$, where $(\mathcal{X}, \mathcal{L})$ is an integral model of (X, L) with a real-type Hermitian metric h on $L(\mathbb{C})$ and a real-type Hermitian metric (inner product) H on $H^0(L(\mathbb{C}))$. Indeed, if (X, L) is Chow polystable, then it minimizes at the model with the balanced metric h, its corresponding L^2-metric H, and Chow polystable reductions at all primes.

Proof
Note that the last term $\log(\int_{X_\infty} \sum_\alpha |s_\alpha|_h^2 c_1(L,h)^n)$ of $\tilde{h}_C$ only depends on the Archimedean data. So what we first want to know is that, once we fix a reference model $(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}}, h_{\mathrm{ref}})$ (with some fixed H), the relative Chow height $h_{C,(\mathcal{X}_{\mathrm{ref}},\mathcal{L}_{\mathrm{ref}})}(\mathcal{X}, \mathcal{L})$ (Definition 3.5) minimizes exactly when $(\mathcal{X}_{\mathfrak{p}}, \mathcal{L}_{\mathfrak{p}})$ is Chow polystable for any maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$. This is proved by Zhang [108]. A semistable version of the statement is also established by [10] and [11] for one direction and [108] for both directions.

Thus, what remains to show is that, when we fix the scheme-theoretic data $(\mathcal{X}, \mathcal{L})$, $\tilde{h}_C(\mathcal{X}, \mathcal{L}, h, H)$ minimizes at a balanced[22] Fubini–Study metric h and its corresponding H, that is, its L^2-metric. This result is essentially proved in Donaldson [29, Theorem 2 and Lemmas 4 and 5], where he wrote $\tilde{P}$ for the corresponding Archimedean invariant. Note that [29, Lemmas 4 and 5] also are essentially the same as [11, Propositions 2.1 and 2.2], while the definition of the L^2-metric in [11, (1.2.3)] does not have the multiplication by $\mathrm{rank}(\pi_*\mathcal{L})$. We refer the reader to [11], [108], and [29] for the details. □

Going back to our dequantized version, we show some examples of Arakelov K-unstable (i.e., not Arakelov K-semistable) arithmetic varieties.

EXAMPLE 3.17
For $C = \mathrm{Spec}(\mathbb{Z})$ and the projective plane $\mathcal{Y} := \mathbb{P}^2_{\mathbb{Z}}$ over C, polarized by $\mathcal{M} := \mathcal{O}_{\mathcal{X}}(1)$, we can take a section $S \subset \mathcal{X}$. Then we set $\mathcal{X} := Bl_S(\mathcal{Y})$, $\mathcal{L} := \mathcal{O}_{\mathcal{X}}(-K_{\mathcal{X}/C})$. If we denote by E the exceptional divisor in $\mathcal{X}$, for an arbitrary set of prime numbers $p_1 < \cdots < p_m$, we can construct an integral model $\mathcal{X}(p_1, \ldots, p_m)$ as the blowup of $\mathcal{X}$ along $\bigcup_i (E|_{\mathcal{X}_{p_i}})$ (with its reduced struc-

[22] Originally called the "critical metric" in Zhang [108].

ture) and set $\mathcal{L}(p_1,\ldots,p_m) := \pi^*\mathcal{L}(-\sum_i F_i)$, where $\pi : \mathcal{X}(p_1,\ldots,p_m) \to \mathcal{X}$ is the blowup and F_i is the π-exceptional divisor over the prime p_i.

Then we have

$$h_K\big(\mathcal{X}(p_1,\ldots,p_m),\mathcal{L}(p_1,\ldots,p_m),h\big) = h_K(\mathcal{X},\mathcal{L},h) - c\sum_i \log(p_i)$$

with some positive constant c; thus, the Arakelov K-instability of the generic fiber of $(\mathcal{X},\mathcal{L})$ follows. Its arithmetic K-instability also follows by the same arguments as above.

We end this section by supplementarily giving a definition of an intrinsic version after the idea of [11] and the proposal of a problem about the effects of arithmetic structures.

DEFINITION 3.18 (INTRINSIC (K-)MODULAR HEIGHT)

For an arithmetically K-semistable arithmetic variety (X,L) over a number field K, we set

$$h_K(X,L) := \inf_{(\mathcal{X},\mathcal{L},h,K'/K)} h_K(\mathcal{X},\mathcal{L},h),$$

where $(\mathcal{X},\mathcal{L},h,K')$ run over the set of all (vertically positive) metrized models of (X,L) over finite extensions K' of K, as in Section 1.1 (especially see (4)). We call it the *intrinsic (K-)modular height.*

Recall that the K of K-modular and that of the subscript of h_K come from K-stability and, hence, from Kähler after all, but not from the original base field. Indeed, we take all metrized models over all finite extensions of K for the definitions of our modular heights.

This formulation also follows the definition of the Faltings [36] height for Abelian varieties (recall Section 2, especially Theorem 2.11). The name is after [11, 1.2.3], a quantized (embedded varieties) original version of the above, which the author would like to distinguish by calling it the *intrinsic Chow height* (in our articles) to avoid confusion from now on.

REMARK 3.19 (ARITHMETIC DANCE)

We can rather fix a complex variety structure as follows. Starting with polarized *complex* projective variety (X,L), that is, with only its $\mathbb{C}$-structure, we can consider the set

$$\big\{\big(K,(\mathcal{X},\mathcal{L},h)\big) \,\big|\, c_1(L,h) > 0, (X,L) \text{ is a component of } \big(\mathcal{X}(\mathbb{C}),\mathcal{L}(\mathbb{C})\big)\big\},$$

which we denote by $\mathcal{D}(X,L)$. The point is that we change the K-structure (arithmetic structure). A possible question one can ask is "what is the precise relation among

$$\inf_{(\mathcal{X},\mathcal{L},h)\in\mathcal{D}(X,L)} h_K(\mathcal{X},\mathcal{L},h),$$

the Donaldson–Futaki invariant, and K-energy?"

3.3. Arithmetic moduli

We conclude this article with a brief introduction to our ongoing attempt to partially unify treatments of (arithmetic) moduli, which we hope to discuss the details of in the near future. First we introduce the following key arithmetic line bundle(s).

DEFINITION 3.20 (ARITHMETIC LINE BUNDLES)

Suppose $\pi : \mathcal{X} \to S$ is a smooth projective morphism between quasiprojective normal $\mathbb{Q}$-Gorenstein schemes over $\mathbb{Z}$. Furthermore, $\mathcal{L}$ is an arithmetic line bundle on $\mathcal{X}$ which is π-ample with a family of Hermitian metrics $h = \{h_{\bar{s}}\}_{\bar{s}}$ on $\mathcal{L}|_{\mathcal{X}_{\bar{s}}}$ over any geometric points $\bar{s}$ of S that are smooth of positive curvature (see Section 1.1 and [41]).

Then we define the following arithmetic line bundles.

(1) (Arithmetic CM line bundle) We denote the following arithmetic line bundle on S as $\bar{\lambda}_{\mathrm{CM}}(\mathcal{X}, \bar{\mathcal{L}})$ and call it the *arithmetic CM line bundle*

$$\langle \bar{\mathcal{L}}^h, \dots, \bar{\mathcal{L}}^h \rangle^{\otimes(-(n-1)(L|_{X_s}^{n-1}.K_{X_s}))} \otimes \langle \bar{\mathcal{L}}^h, \dots, \bar{\mathcal{L}}^h, \overline{K_{\mathcal{X}/S}}^{\det(\omega_h)} \rangle^{\otimes((n+1)(L|_{X_s}^{n}))},$$

where $\langle \cdot \rangle$ denotes the Deligne pairing with the Deligne metric (see [22], also [108], [81]), X_s is a π-fiber over $s \in S$, and $\det(\omega_h)$ above means the natural family of Hermitian metrics $\{\det(g_{h_{\bar{s}}})\}_{\bar{s}}$, the determinant metric on $K_{\mathcal{X}_{\bar{s}}}$ of the Kähler metric g_h whose Kähler form is $c_1(\mathcal{L}_{\bar{s}}, h_{\bar{s}})$. If $c_1(\mathcal{L}_{\bar{s}}, h_{\bar{s}})$ are normalized Kähler forms of cscK metrics, then $\bar{\lambda}_{\mathrm{CM}}$ encodes the Weil–Petersson potential essentially by [41, Sections 7 and 10]. This definition is after its geometric version, the CM line bundle, which was introduced in [41, Sections 10 and 11] for the smooth case and was later extended to the singular case (see [80]).

(2) (Arithmetic Aubin–Mabuchi line bundle) We denote the following arithmetic line bundle on S as $\bar{\lambda}_{\mathrm{AM}}(\mathcal{X}, \bar{\mathcal{L}})$ and call it the *arithmetic Aubin–Mabuchi line bundle*

$$\bar{\lambda}_{\mathrm{AM}}(\mathcal{X}, \bar{\mathcal{L}}) := \langle \bar{\mathcal{L}}^h, \dots, \bar{\mathcal{L}}^h \rangle,$$

where $\langle \cdot \rangle$ denotes, as above, the Deligne pairing with the Deligne metric (see [22], also [108], [81]). Here, "AM" stands for Aubin–Mabuchi (energy).

The above $\bar{\lambda}_{\mathrm{CM}}$ can be regarded as an extension of the modular height h_K. Although in the above definitions we suppose the smoothness of π for simplicity, we expect that the same construction works when the geometric fibers are semi-log-canonical, that is, mildly singular (with regular enough metrics),[23] and such mildness of singularities and regularity should automatically follow from the

[23] This at least works as a (scheme-theoretic) line bundle, and the inductive definition of associated metrics (see [22], [108]) also at least works for *almost smooth metrics* in our sense from Section 1.1.

Arakelov/arithmetic K-semistability as in [70] and [7]. By assuming so, our conjecture is roughly as follows (Conjecture 3.21), which we will make more precise in the near future.

If the infimum of h_K of all integral models (*model* metrics) is attained by an *adèlic* vertically semipositive metric model of (X, L), we call such an "integral" model *globally Arakelov K-semistable.*

"CONJECTURE" 3.21 (ARAKELOV K-MODULI)

For each class of (liftable) polarized varieties, a moduli algebraic stack $\mathcal{M}$ of globally Arakelov K-semistable polarized models exists, and it has the coarse moduli projective scheme M over $\mathbb{Z}$. The K-moduli stack (resp., or its coarse moduli scheme) of the class of polarized varieties over a fixed field k is $\mathcal{M} \times_{\mathbb{Z}} k$ (resp., $M \times_{\mathbb{Z}} k$). Furthermore, the arithmetic CM line bundle $\bar{\lambda}_{\mathrm{CM}}$ (Definition 3.20 above) on $\mathcal{M}$ descends on M as a vertically ample[24] *arithmetic line bundle on M with a natural Hermitian metric*[25] *whose curvature current is the Weil–Petersson current (see [41]).*

We wish to call the above arithmetic moduli *Arakelov K-moduli* as the (generalized) Weil–Petersson metric gives a canonical Arakelov compactification of the arithmetic moduli scheme M (see also [62, p. 77]). For references to the original geometric versions of the above conjecture, please review Section 1.

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[24] See, for example, [65, 5.38] for the definition.

[25] It is obtained as the Deligne metric in a modified version of Definition 3.20, applied to the family of cscK metrics of the parameterized varieties. A more precise meaning will be given in the future.

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