Coefficient estimates of analytic endomorphisms of the unit disk fixing a point with applications to concave functions

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Dedicated to Professor Yukio Kusunoki on the occasion of his 90th birthday

Abstract In this article, we discuss the coefficient regions of analytic self-maps of the unit disk with a prescribed fixed point. As an application, we solve the Fekete–Szegő problem for normalized concave functions with a pole in the unit disk.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk in the complex plane $\mathbb{C}$. The class $\mathcal{B}_p$ for $p \in \mathbb{D}$ will denote the set of holomorphic maps $\varphi : \mathbb{D} \to \mathbb{D}$ satisfying $\varphi(p) = p$. In what follows, we will always assume without loss of generality that $0 \le p < 1$.

A function $\varphi \in \mathcal{B}_p$ can be expanded near the origin in the form

$$\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{n=0}^{\infty} c_n z^n. \tag{1.1}$$

Note that $|c_n| \le 1$ for each n. We define the coefficient body $\mathbf{X}_n(\mathcal{F})$ of order $n \ge 0$ for a class $\mathcal{F}$ of analytic functions at the origin as the set

$$\big\{(c_0, c_1, \dots, c_n) \in \mathbb{C}^{n+1} : \varphi(z) = c_0 + c_1 z + \cdots + c_n z^n + O(z^{n+1}) \text{ for some } \varphi \in \mathcal{F}\big\}.$$

Note that $\pi_{m,n}(\mathbf{X}_n(\mathcal{F})) = \mathbf{X}_m(\mathcal{F})$ for $0 \le m < n$, where $\pi_{m,n} : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ is the projection $(c_0, c_1, \dots, c_n) \mapsto (c_0, c_1, \dots, c_m)$.

Obviously, $\mathbf{X}_0(\mathcal{B}_0) = \{0\}$ and $\mathbf{X}_1(\mathcal{B}_0) = \{(0, c) : |c| \le 1\}$. In the present article, we describe $\mathbf{X}_n(\mathcal{B}_p)$ for $n = 0, 1$ and $0 < p < 1$. Note that the authors [12] describe $X_2(\mathcal{B}_p)$ to investigate the second Hankel determinant. In the following, it is convenient to put

$$P = p + \frac{1}{p} = \frac{1 + p^2}{p}.$$

Note that $P > 2$.

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THEOREM 1

Let $p \in (0,1)$.

(i) $\mathbf{X}_0(\mathcal{B}_p) = \{c_0 \in \mathbb{C} : |c_0 - P^{-1}| \le P^{-1}\}$. *For a function* $\varphi(z) = c_0 + c_1 z + \cdots$ *in* $\mathcal{B}_p$, $c_0 \in \partial \mathbf{X}_0(\mathcal{B}_p)$ *if and only if* φ *is an analytic automorphism of* $\mathbb{D}$.

(ii) $\mathbf{X}_1(\mathcal{B}_p) = \{(c_0, c_1) \in \mathbb{C}^2 : |c_1 - (1 - Pc_0 + c_0^2)| \le P[P^{-2} - |c_0 - P^{-1}|^2]\}$. *In other words, a pair* (c_0, c_1) *of complex numbers is contained in* $\mathbf{X}_1(\mathcal{B}_p)$ *if and only if*

$$
\begin{aligned}
&c_0 = P^{-1}(1 - \sigma_0) \qquad \textit{and} \\
&c_1 = P^{-2}\big[1 + (P^2 - 2)\sigma_0 + \sigma_0^2\big] + P^{-1}\big(1 - |\sigma_0|^2\big)\sigma_1
\end{aligned}
\tag{1.2}
$$

for some $\sigma_0, \sigma_1 \in \overline{\mathbb{D}}$.

Moreover, for a function $\varphi(z) = c_0 + c_1 z + \cdots$ *in* $\mathcal{B}_p$, $(c_0, c_1) \in \partial \mathbf{X}_1(\mathcal{B}_p)$ *if and only if* φ *is either an analytic automorphism of* $\mathbb{D}$ *or a Blaschke product of degree* 2.

Our motivation for the present study comes from an intimate relation between $\mathcal{B}_p$ and the class $\mathcal{C}o_p$ of concave functions f normalized by $f(0) = f'(0) - 1 = 0$ with a pole at p. Here, a meromorphic function f on $\mathbb{D}$ is said to be *concave* if it maps $\mathbb{D}$ conformally onto a concave domain in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; in other words, f is a univalent meromorphic function on $\mathbb{D}$ such that $\mathbb{C} \setminus f(\mathbb{D})$ is convex. The class $\mathcal{C}o_p$ has been intensively studied in recent years by Avkhadiev, Bhowmik, Pommerenke, Wirths, and others (see, e.g., [1]–[6]).

The following representation of concave functions in terms of functions in $\mathcal{B}_p$ belongs to the first author [11].

THEOREM A

Let $0 < p < 1$, *and put* $P = p + 1/p$. *A meromorphic function* f *on* $\mathbb{D}$ *with* $f(0) = 0$ *is contained in the class* $\mathcal{C}o_p$ *if and only if there exists a function* $\varphi \in \mathcal{B}_p$ *such that*

$$
f'(z) = (1 - Pz + z^2)^{-2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1 - \zeta\varphi(\zeta)}\, d\zeta. \tag{1.3}
$$

For a given function $f \in \mathcal{C}o_p$ with the expansion

$$
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n=1}^{\infty} a_n z^n, \quad |z| < p, \tag{1.4}
$$

we consider the Fekete–Szegő functional

$$
\Lambda_\mu(f) = a_3 - \mu a_2^2
$$

for a real number μ. For example, $\Lambda_1(f) = a_3 - a_2^2 = S_f(0)/6$, where $S_f = (f''/f')' - (f''/f')^2/2$ is the Schwarzian derivative of f. For some background on the Fekete–Szegő functional, the reader may refer to [7] and references therein. As an application of Theorem 1, we will prove the following.

THEOREM 2

Let $0 < p < 1$ and $\mu \in \mathbb{R}$, and put $P = p + 1/p$. Then the maximum $\Phi(\mu)$ of the Fekete–Szegő functional $|\Lambda_\mu(f)|$ over $f \in \mathcal{C}o_p$ is given as follows:

$$\Phi(\mu) = \begin{cases} (1-\mu)P^2 - 1 & \text{if } \mu \le \mu_1(P), \\ -\frac{1}{3}(P^3 - 2P + 3) + \frac{(P+2)^2(2P-1)^2}{12(P+3\mu)} & \text{if } \mu_1(P) \le \mu \le \mu_2(P), \\ \Psi(P,\mu) & \text{if } \mu_2(P) \le \mu \le \mu_4(P), \\ (\mu - 1)P^2 + 1 & \text{if } \mu_4(P) \le \mu. \end{cases}$$

Here,

$$\Psi(P,\mu) = \begin{cases} P^2 - 3 - \mu(P^2 - 4 + 4P^{-2}) \\ \quad \text{if either } P_2 \le P \le P_*, \mu_3^-(P) \le \mu \le \mu_3^+(P), \\ \quad \text{or } P_* \le P, \mu_2(P) \le \mu \le \mu_3^+(P), \\ (1-\mu)P(P^2-2)\sqrt{\frac{P^2 - 4\mu}{4\mu\{(1-\mu)(P^2-1)^2 - 1\}}} & \text{otherwise,} \end{cases}$$

and

$$\mu_1(P) = \frac{1}{2} - \frac{1}{3P},$$

$$\mu_2(P) = \begin{cases} \frac{1}{72}(4 + P^2 \\ \quad + 4P^4 - \sqrt{16P^8 + 8P^6 - 543P^4 + 1160P^2 + 16}) & \text{if } P \le P_*, \\ \frac{P(3P+2)}{6(P^2-2)} & \text{if } P_* \le P, \end{cases}$$

$$\mu_3^{\pm}(P) = \frac{P^2(3P^4 - 12P^2 + 14) \pm P^2\sqrt{P^8 - 16P^6 + 84P^4 - 176P^2 + 132}}{4(P^2-1)(P^2-2)^2},$$

$$\mu_4(P) = \frac{3P^4 - 4P^2 - 2 + \sqrt{P^8 - 12P^4 + 16P^2 + 4}}{4P^2(P^2-1)},$$

where $P_* \approx 2.88965$ is the unique zero of the polynomial

$$U(P) = 6P^4 - P^3 - 38P^2 - 28P + 4$$

on the interval $2 < P < +\infty$ and $P_2 \approx 2.82343$ is the largest zero of the polynomial

$$V(P) = P^8 - 16P^6 + 84P^4 - 176P^2 + 132$$

on the positive real axis. Moreover,

$$\frac{1}{3} < \mu_1 < \frac{1}{2} < \mu_2 < \mu_4 < \frac{8}{9}$$

on the interval $2 < P$, and $\mu_2 < \mu_3^- < \mu_3^+ < \mu_4$ on $P_2 < P < P_*$, whereas $\mu_3^- < \mu_2 < \mu_3^+ < \mu_4$ on $P_* < P$.

We see numerically that $p_* \in (0,1)$ satisfying $P_* = p_* + 1/p_*$ is approximately 0.401984. Also, we have $p_2 \approx 0.415252$ for $p_2 \in (0,1)$ with $P_2 = p_2 + 1/p_2$. The behavior of $\mu_1(P)$, $\mu_2(P)$, $\mu_3^{\pm}(P)$, and $\mu_4(P)$ can be observed in Figure 1.

The Fekete–Szegő problem was solved by Bhowmik, Ponnusamy, and Wirths [6] for the different but related classes $\mathcal{C}o(\alpha)$ for $1 < \alpha \le 2$. Here, the class $\mathcal{C}o(\alpha)$

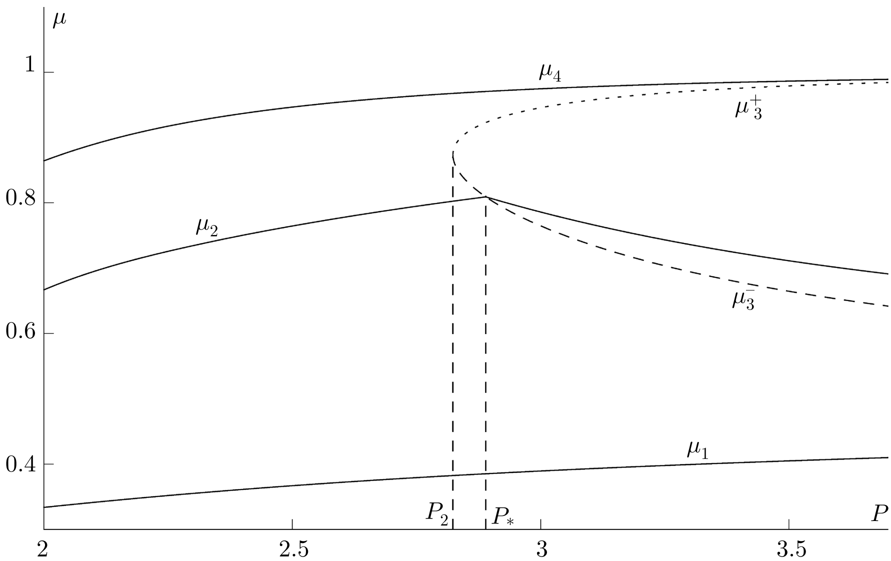

Figure 1. The graphs of $\mu_1(P)$, $\mu_2(P)$, $\mu_3^\pm(P)$, and $\mu_4(P)$ in the $P\mu$-plane.

consists of functions f analytic and univalent on $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $f(1) = \infty$ such that $\mathbb{C} \setminus f(\mathbb{D})$ is convex and such that the opening angle of the image $f(\mathbb{D})$ at ∞ is at most $\pi\alpha$. It is interesting to observe that the case $\alpha = 2$ of their main theorem in [6, p. 438] agrees with the limiting case of our Theorem 2 as $p \to 1^-$ (equivalently, $P \to 2^+$).

With the special choice $\mu = 0$, we have the following known fact.

COROLLARY 3

Let $f(z) = z + a_1 z + a_2 z^2 + \cdots$ be a function in $\mathcal{C}o_p$. Then the following sharp inequality holds:

$$|a_3| \le P^2 - 1 = p^2 + 1 + \frac{1}{p^2}.$$

Indeed, the above inequality is still valid as long as f is a univalent meromorphic function on $\mathbb{D}$ with a pole at p (see Jenkins [10]). Avkhadiev, Pommerenke, and Wirths [1] (see also [5]) proved the even stronger result that the variability region of a_3 over $f \in \mathcal{C}o_p$ is given as $|a_3 - P^2 + 2| \le 1$. (This can also be proved by our method given below.)

Since $\Phi(1) = 1$ by Theorem 2, we get another corollary.

COROLLARY 4

Let $0 < p < 1$, and suppose that $f(z) = z + a_1 z + a_2 z^2 + \cdots$ is a function in $\mathcal{C}o_p$. Then the following sharp inequality holds:

$$|a_3 - a_2^2| \le 1.$$

Recall that $6(a_3 - a_2^2) = S_f(0)$ is the Schwarzian derivative of f evaluated at $z = 0$. The inequality $|a_3 - a_2^2| \le 1$ is valid for a univalent holomorphic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ on $\mathbb{D}$ (see, e.g., [9, Example 1, p. 70]). Indeed, it is obtained by a simple application of Gronwall's area theorem for the function $1/f(1/w)$. Since the Schwarzian derivative S_f is unchanged under the postcomposition with Möbius transformations, the above corollary can also be obtained from this classical result.

In the final section, we will focus on the variability region of $\Lambda_1(f) = a_3 - a_2^2$ over $f \in \mathcal{C}o_p$. Section 2 will be devoted to the proof of Theorem 1. To apply Theorem 1 to concave functions, in Section 3 we consider a maximum value problem for a quadratic polynomial over the closed unit disk. The proof of Theorem 2 will be given in Section 4.

2. Proof of Theorem 1

For the proof of Theorem 1, we recall a useful lemma due to Dieudonné [8, pp. 351–352]. The following form is due to Duren [9, p. 198]. To clarify the equality case in the lemma, we will give an outline of the proof. Throughout this section, it is helpful to use the special automorphism

$$T_a(z) = \frac{a - z}{1 - \bar{a}z} \tag{2.1}$$

of $\mathbb{D}$ for $a \in \mathbb{D}$. This is indeed an analytic involution of $\mathbb{D}$ and interchanges 0 and a. Moreover,

$$T_a'(z) = \frac{|a|^2 - 1}{(1 - \bar{a}z)^2}.$$

In particular,

$$T_a'(0) = |a|^2 - 1 \qquad \text{and} \qquad T_a'(a) = \frac{1}{|a|^2 - 1}.$$

LEMMA 5 (DIEUDONNÉ'S LEMMA)

Let $z_0, w_0 \in \mathbb{D}$ with $|w_0| < |z_0|$. Then the region of values of $w = g'(z_0)$ for holomorphic functions $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$ and $g(z_0) = w_0$ is given as the closed disk

$$\left| w - \frac{w_0}{z_0} \right| \le \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |z_0|^2)}. \tag{2.2}$$

Equality holds if and only if g is a Blaschke product of degree 2 fixing 0.

Proof

The function $h(z) = g(z)/z$ is an analytic endomorphism of $\mathbb{D}$ which sends z_0 to $\omega_0 = w_0/z_0 \in \mathbb{D}$. Thus, $H = T_{\omega_0} \circ h \circ T_{z_0}$ belongs to $\mathcal{B}_0$. The Schwarz lemma now gives $|H'(0)| \le 1$, which turns out to be equivalent to (2.2) with $w = g'(z_0)$. Moreover, equality holds if and only if $H(z) = \zeta z$ for some $\zeta \in \partial\mathbb{D}$, which means that h is an analytic automorphism of $\mathbb{D}$. Then,

$$g(z) = zT_{\omega_0}\big(\zeta T_{z_0}(z)\big)$$

is certainly a Blaschke product of degree 2 fixing 0. Allowing $\zeta \in \overline{\mathbb{D}}$ in this form of $g(z)$, we see that the disk in the assertion is filled with the values of the derivative $g'(z_0)$ at z_0. □

We are now ready to prove Theorem 1.

Proof of Theorem 1
For a function $\varphi \in \mathcal{B}_p$, we consider $\psi = T_p \circ \varphi \circ T_p : \mathbb{D} \to \mathbb{D}$. Then $\psi \in \mathcal{B}_0$. The Schwarz lemma implies $|\psi(p)| \le p$. Namely,

$$\big|T_p(c_0)\big| = \left|\frac{p - c_0}{1 - p\,c_0}\right| \le p,$$

which is equivalent to

$$0 \le |1 - pc_0|^2 - \left|1 - \frac{c_0}{p}\right|^2 = \frac{1 - p^4}{p^2}\left[\left(\frac{p}{1+p^2}\right)^2 - \left|c_0 - \frac{p}{1+p^2}\right|^2\right]. \tag{2.3}$$

The range is optimal because the function φ corresponding to $\psi(z) = T_p(c_0)z/p$ belongs to $\mathcal{B}_p$. Suppose now that $c_0 \in \partial\mathbf{X}_0(\mathcal{B}_p)$. Then, by the above argument, we have $\psi(z) = \zeta z$, where $\zeta = T_p(c_0)/p \in \partial\mathbb{D}$. Thus, $\varphi(z) = T_p(\zeta T_p(z))$ is an analytic automorphism of $\mathbb{D}$ fixing p. Hence, the first assertion follows.

For the second assertion, we use Dieudonné's lemma. Note that

$$\psi'(p) = T_p'(c_0) \cdot \varphi'(0) \cdot T_p'(p) = \frac{c_1}{(1 - pc_0)^2}.$$

Applying Dieudonné's lemma to the function ψ with the choices $z_0 = p$ and $w_0 = \psi(p) = T_p(c_0)$, we get

$$\left|\frac{c_1}{(1 - p\,c_0)^2} - \frac{p - c_0}{p\,(1 - p\,c_0)}\right| \le \frac{p^2 - |\frac{p - c_0}{1 - p\,c_0}|^2}{p\,(1 - p^2)}.$$

Here, if $|w_0| = p = |z_0|$, then the above inequality (in fact, equality) obviously holds. Note that the above range of c_1 for a fixed c_0 is optimal by Dieudonné's lemma. Using the identity in (2.3), we obtain the first description of the set $\mathbf{X}_1(\mathcal{B}_p)$. The second description of $\mathbf{X}_1(\mathcal{B}_p)$ is obtained by letting $\sigma_0 = P(P^{-1} - c_0) = 1 - Pc_0$ and $\sigma_1 = (c_1 - (1 - Pc_0 + c_0^2))/(P^{-1} - P|c_0 - P^{-1}|^2) = P(c_1 - P^{-2}(1 + (P^2 - 2)\sigma_0 + \sigma_0^2))/(1 - |\sigma_0|^2)$.

We now prove the final assertion. Suppose that $(c_0, c_1) \in \partial\mathbf{X}_1(\mathcal{B}_p)$ for a function $\varphi(z) = c_0 + c_1 z + \cdots$ in $\mathcal{B}_p$. By Theorem 1(i), we know that $c_0 \in \partial\mathbf{X}_0(\mathcal{B}_p)$ if and only if φ is an analytic automorphism of $\mathbb{D}$ fixing p. Thus, we may assume that c_0 is an interior point of $\mathbf{X}_0(\mathcal{B}_p)$; namely, $|T_p(c_0)| < p$. Then, by the equality case in Dieudonné's lemma, $\psi = T_p \circ \varphi \circ T_p$ is a Blaschke product of degree 2 fixing 0. Therefore, we have proved the "only if" part. The "if" part is easy to check. □

3. Maximum value problem for a quadratic polynomial

To apply Theorem 1 for concave functions, we consider the following problem: What is the value of the quantity

$$Y(a,b,c) = \max_{z\in\overline{\mathbb{D}}}\left(|a+bz+cz^2|+1-|z|^2\right) \tag{3.1}$$

for real numbers a, b, c?

In fact, a more general and symmetric problem was considered in [7]. Let

$$\Omega(A,B,K,L,M) = \max_{u,v\in\overline{\mathbb{D}}}\left\{|A|\left(1-|u|^2\right)+|B|\left(1-|v|^2\right)+|Ku^2+2Muv+Lv^2|\right\}$$

for $A,B,K,L,M\in\mathbb{C}$. When K, L, M are all real numbers, the value of $\Omega(A,B,K,L,M)$ was computed in [7, Theorem 3.1]. By virtue of the maximum modulus principle, one can see that

$$\Omega(1,0,c,a,b/2) = \max_{u\in\overline{\mathbb{D}},v\in\partial\mathbb{D}}\left\{\left(1-|u|^2\right)+|cu^2+buv+av^2|\right\} = Y(a,b,c).$$

As an immediate consequence of [7, Theorem 3.1], we obtain the following result. (Note that, under the notation adopted in [7], $\max\{\Phi_1,\Phi_2\}\geq 0$ because of $B=0$ so that $S\geq |A|+|B|=1$ in [7, Theorem 3.1(3c)].)

PROPOSITION 6

Let $Y(a,b,c)$ be the quantity defined in (3.1) for real numbers a, b, c. When $ac\geq 0$,

$$Y(a,b,c) = \begin{cases} |a|+|b|+|c| & \text{if } |b|\geq 2(1-|c|),\\ 1+|a|+\frac{b^2}{4(1-|c|)} & \text{if } |b|< 2(1-|c|).\end{cases}$$

When $ac<0$,

$$Y(a,b,c) = \begin{cases} 1-|a|+\frac{b^2}{4(1-|c|)} & \text{if } -4ac(c^{-2}-1)\leq b^2 \text{ and } |b|<2(1-|c|),\\ 1+|a|+\frac{b^2}{4(1+|c|)} & \text{if } b^2<\min\{4(1+|c|)^2, -4ac(c^{-2}-1)\},\\ R(a,b,c) & \text{otherwise,}\end{cases} \tag{3.2}$$

where

$$R(a,b,c) = \begin{cases} |a|+|b|-|c| & \text{if } |c|(|b|+4|a|)\leq |ab|,\\ -|a|+|b|+|c| & \text{if } |ab|\leq |c|(|b|-4|a|),\\ (|c|+|a|)\sqrt{1-\frac{b^2}{4ac}} & \text{otherwise.}\end{cases} \tag{3.3}$$

4. Proof of Theorem 2

Let $p\in(0,1)$, and put $P=p+1/p$ as before. For a given function $f\in\mathcal{C}o_p$ with expansion (1.4), there is a unique function $\varphi\in\mathcal{B}_p$ with expansion (1.1) such that the representation formula (1.3) holds. A straightforward computation yields

$$a_2 = P-c_0 \qquad \text{and} \qquad a_3 = P^2-\frac{1}{3}(c_1-c_0^2+4Pc_0+2).$$

For $\mu \in \mathbb{R}$, by substituting the expressions in (1.2), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{3}\big[(1-3\mu)c_0^2 + 2(3\mu-2)Pc_0 - c_1 + (3-\mu)P^2 - 2\big] \\ &= P^2 - 2 - \mu(P-P^{-1})^2 \\ &\quad + \big(1 - 2\mu(1-P^{-2})\big)\sigma_0 - \mu P^{-2}\sigma_0^2 - \frac{(1-|\sigma_0|^2)\sigma_1}{3P}. \end{aligned} \tag{4.1}$$

Since σ_1 is an arbitrary point in $\overline{\mathbb{D}}$, we get the sharp inequality

$$|a_3 - \mu a_2^2| \le \frac{1}{3P}\{|a + b\sigma_0 + c\sigma_0^2| + 1 - |\sigma_0|^2\},$$

where

$$\begin{aligned} &a = 3P\big[P^2 - 2 - \mu(P-P^{-1})^2\big], \\ &b = 3P - 6\mu(P-P^{-1}), \qquad \text{and} \qquad c = -3\mu P^{-1}. \end{aligned}$$

Therefore, in terms of the quantity introduced in the last section, we can express $\Phi(\mu)$ by

$$\Phi(\mu) = \sup_{f\in\mathcal{C}o_p} \Lambda_\mu(f) = \frac{1}{3P} Y(a,b,c).$$

Observe that a changes its sign at $\mu = \mu_a := (P^2-2)/(P-P^{-1})^2 > 0$, whereas c changes its sign at $\mu = 0$. It is easy to verify

$$\frac{8}{9} < \mu_a < 1.$$

Furthermore, b changes its sign at $\mu = \mu_b := P/2(P-P^{-1}) \in (1/2, 2/3)$.

Case when $\mu \le 0$. In this case, $a \ge 0$, $c \ge 0$, and $b \ge 0$. Since $2(1-|c|) - |b| = 2 - 3P + 6P\mu < 0$, Proposition 6 leads to

$$\Phi(\mu) = \frac{1}{3P}(a+b+c) = (1-\mu)P - 1.$$

Case when $\mu \ge \mu_a$. In this case, $a \le 0$, $b \le 0$, and $c \le 0$, and thus $ac \ge 0$. Since $2(1-|c|) - |b| = 2 + 3P - 6P\mu < 0$ for $\mu \ge \mu_a > 1/2 + 1/3P$, by Proposition 6 we have

$$\Phi(\mu) = \frac{1}{3P}(-a-b-c) = (\mu-1)P + 1.$$

Case when $0 < \mu < \mu_a$. In this case, $a > 0$, $c < 0$, and thus $ac < 0$. We compute $b^2 + 4ac(c^{-2}-1) = H(\mu)/\mu$, where H is a quadratic polynomial in μ given by

$$H(\mu) = -36\mu^2 + (4 + P^2 + 4P^4)\mu - 4P^2(P^2-2).$$

The zeros of $H(\mu)$ are given by

$$\mu_0^{\pm} = \frac{1}{72}(4 + P^2 + 4P^4 \pm \sqrt{16P^8 + 8P^6 - 543P^4 + 1160P^2 + 16}).$$

Since $H(2/3) = -2(P^2-4)(2P^2-5)/3 < 0$, $H(\mu_a) = 9P^4(P^2-3)^2(P^2-2)/(P^2-1)^4 > 0$, and $H(4/3) = 4(P^2-4)(P^2+11)/3 > 0$, the zeros are real and satisfy $2/3 < \mu_0^- < \mu_a < 4/3 < \mu_0^+$. Note that $H(\mu) < 0$ for $\mu \in (-\infty, \mu_0^-) \cup (\mu_0^+, +\infty)$, and note that $H(\mu) \ge 0$ for $\mu \in [\mu_0^-, \mu_0^+]$. Since $2(1-|c|)-|b| = 2(1+c)+b = 2+(3-6\mu)P < 2-P < 0$ for $\mu \ge \mu_0^- (> 2/3)$, the first case in (3.2) does not occur.

We now analyze the condition $b^2 < 4(1+|c|)^2$, which is equivalent to $|b| < 2(1+|c|) = 2(1-c)$ in the present case, and we observe that $b < 2(1-c)$ precisely when $\mu > \mu_1 = 1/2 - 1/3P$, whereas $-b < 2(1-c)$ precisely when $\mu < \mu_1' := P(3P+2)/6(P^2-2)$. Note here that $1/3 < \mu_1 < 1/2 < \mu_1' < 4/3$. Hence, for $\mu \in (0, \mu_a)$, we see that $b^2 < 4(1+|c|)^2$ if and only if $\mu_1 < \mu < \mu_1'$. Hence, by the second case of (3.2), we obtain

$$\Phi(\mu) = \frac{1}{3P}\Big(1 + a + \frac{b^2}{4(1-c)}\Big)$$

for $\mu_1 < \mu < \mu_2 = \min\{\mu_0^-, \mu_1'\}$. Substituting the explicit forms of a, b, c, we obtain the expression in the theorem. Here, keeping $\mu_1' < 4/3$ in mind, we see that $\mu_1' > \mu_0^-$ if and only if

$$H(\mu_1') = -\frac{P(2P-1)(P^2-4)U(P)}{6(P^2-2)^2} > 0,$$

where $U(P)$ is the quartic polynomial given in Theorem 2. One can check that the polynomial $U(P)$ has a unique zero $P_* \approx 2.88965$ in the interval $2 < P < +\infty$. Thus, $\mu_2 = \mu_0^-$ if $2 < P \le P_*$ and $\mu_2 = \mu_1'$ if $P_* \le P < +\infty$.

When either $0 < \mu \le \mu_1$ or $\mu_2 \le \mu < \mu_a$, we have $Y(a,b,c) = R(a,b,c)$ in (3.2). We shall take a closer look at these cases.

Subcase when $0 < \mu < \mu_1$. Since $\mu_1 < 1/2 < \mu_b$, we have $b > 0$ in this case. We compute

$$\begin{aligned} |ab| - |c|\big(|b| + 4|a|\big) &= ab + c(b+4a) \\ &= 9\big[2P^2(P^2-1)\mu^2 - (3P^4-4P^2-2)\mu + P^2(P^2-2)\big]. \end{aligned}$$

Note that the above quadratic polynomial in μ is convex and has the axis of symmetry at $\mu = (3P^4-4P^2-2)/4P^2(P^2-1) > 1/2 > \mu_1$. Therefore, it is decreasing in $0 < \mu < \mu_1$, and thus,

$$\begin{aligned} |ab| - |c|\big(|b| + 4|a|\big) &\ge 9\big[2P^2(P^2-1)\mu_1^2 - (3P^4-4P^2-2)\mu_1 + P^2(P^2-2)\big] \\ &= \frac{9}{2P}(6P^4 - 5P^3 - 12P^2 + 14P - 12) > 0 \end{aligned}$$

for $P > 2$. Hence, by the first case of (3.3) in Proposition 6, we have $\Phi(\mu) = R(a,b,c)/3P = (a+b+c)/3P = (1-\mu)P^2 - 1$.

Subcase when $\mu_2 < \mu < \mu_a$. First note that $\mu_1' - \mu_b = P(P+2)(2P-1)/6(P^2-1)(P^2-2) > 0$. We also have $\mu_b < 2/3 < \mu_0^-$. Thus, we observe that $\mu_b < \mu_2 = \min\{\mu_0^-, \mu_1'\}$, which implies that $b < 0$ in this case. Therefore, $|ab| - |c|(|b| + 4|a|) = -ab + c(-b+4a) = -9P^{-2}F(\mu)$, where

$$F(\mu) = 2(P^2-1)(P^2-2)^2\mu^2 - P^2(3P^4-12P^2+14)\mu + P^4(P^2-2).$$

The discriminant of $F(\mu)$ is $D = P^4V(P)$, where $V(P)$ is given in Theorem 2. One can see that the polynomial D in P has exactly two zeros P_1, P_2 in the interval $2 < P < +\infty$ with $P_1 \approx 2.05313 < P_2 \approx 2.82343$ and that $D \geq 0$ on $P > 2$ if and only if either $2 < P \leq P_1$ or $P_2 \leq P$. The axis of symmetry of $F(\mu)$ is $\mu = \mu_F := P^2(3P^4 - 12P^2 + 14)/4(P^2-1)(P^2-2)^2$. Since

$$\mu_F - 1 = \frac{P^2(-P^6 + 8P^4 - 18P^2 + 16)}{4(P^2-1)(P^2-2)^2} > 0 \quad (2 < P \leq 2.2),$$

we have $F(\mu) > F(1) = 2(P^2 - 4) > 0$ for $\mu < 1$ and $2 < P \leq P_1$. Since $F(\mu) > 0$ for all $\mu \in \mathbb{R}$ when $P_1 < P < P_2$, we conclude that $|ab| - |c|(|b| + 4|a|) = -9P^{-2}F(\mu) < 0$ for $\mu < \mu_a (< 1)$ and $2 < P < P_2$.

Solving the equation $F(\mu) = 0$, we write the solutions as

$$\mu_3^{\pm} = \frac{P^2(3P^4 - 12P^2 + 14) \pm P^2\sqrt{P^8 - 16P^6 + 84P^4 - 176P^2 + 132}}{4(P^2-1)(P^2-2)^2}$$

for $P \in [P_2, +\infty)$. Note that $F(\mu) > 0$ for $\mu \in (-\infty, \mu_3^-) \cup (\mu_3^+, +\infty)$ and that $F(\mu) \leq 0$ for $\mu \in [\mu_3^-, \mu_3^+]$. As above, we compute

$$\mu_a - \mu_F = \frac{P^2(P^6 - 9P^4 + 22P^2 - 18)}{4(P^2-1)^2(P^2-2)^2} > 0 \quad (2.5 < P)$$

and

$$F(\mu_a) = \frac{P^4(P^2-2)(P^2-3)}{(P^2-1)^3} > 0,$$

both of which imply that $\mu_3^+ < \mu_a$ for $P_2 \leq P$. On the other hand, for $2 < P$, we see that

$$\begin{aligned} F(\mu_1') &= -\frac{P^2(P-2)(6P^4 - P^3 - 38P^2 - 28P + 4)}{18(P^2-2)} \\ &= -\frac{P^2(P-2)U(P)}{18(P^2-2)} \leq 0 \end{aligned}$$

if and only if $P_* \leq P$, where P_* is the unique zero of $U(P)$ in $2 < P < +\infty$ as was introduced above. Hence, $\mu_3^- \leq \mu_1' = \mu_2 \leq \mu_3^+$ when $P_* \leq P$, and either $\mu_1' < \mu_3^-$ or $\mu_3^+ < \mu_1'$ when $P_2 \leq P < P_*$. In view of the fact that

$$\begin{aligned} (\mu_3^- - \mu_1')|_{P=P_2} &= \frac{P_2^2(3P_2^4 - 12P_2^2 + 14)}{4(P_2^2-1)(P_2^2-2)^2} - \frac{P_2(3P_2+2)}{6(P_2^2-2)} \\ &= \frac{P_2(3P_2^5 - 4P_2^4 - 18P_2^3 + 12P_2^2 + 30P_2 - 8)}{12(P_2^2-1)(P_2^2-2)^2} \\ &\approx 0.049 > 0, \end{aligned}$$

we can conclude, by continuity, that $\mu_2 = \mu_1' < \mu_3^-$ for $P_2 \leq P < P_*$. (In particular, we see that $\mu_0 = \mu_1' = \mu_3^-$ at $P = P_*$. Look around the point $(P_*, \mu_2(P_*))$ in Figure 1. We wonder if this is just an incidence.)

Similarly, we have $|c|(|b| - 4|a|) - |ab| = -c(-b - 4a) + ab = 9P^{-1}G(\mu)$, where

$$G(\mu) = 2P^2(P^2-1)\mu^2 - (3P^4 - 4P^2 - 2)\mu + P^2(P^2-2).$$

Solving the equation $G(\mu)=0$, we write the solutions as

$$\mu_4^{\pm}=\frac{3P^4-4P^2-2\pm\sqrt{P^8-12P^4+16P^2+4}}{4P^2(P^2-1)},\quad 2<P.$$

Here, we note that $P^8-12P^4+16P^2+4=(P^4-6)^2+16P^2-32>132$ for $2<P$. We now compute $G(\mu_a)=P^2(P^-2)(P^2-3)/(P^2-1)^3>0$. Since the axis $\mu=\mu_G:=(3P^4-4P^2-2)/4P^2(P^2-1)$ of $G(\mu)$ satisfies $\mu_G<3/4<\mu_a$, we have $\mu_4^+<\mu_a$. On the other hand, since

$$\mu_4^- -\frac{1}{2}=\frac{-P^4-2P^2-2-\sqrt{P^8-12P^4+16P^2+4}}{4P^2(P^2-1)}<0,$$

we get $\mu_4^-<1/2<\mu_2$ for $2<P$. We now show that $\mu_0^-<\mu_4^+$ for $2<P$, from which the inequality $\mu_2<\mu_4^+$ will follow. Since $16P^8+8P^6-543P^4+1160P^2+16-(4P^4-8P^2-8)^2=3(P^2-4)(24P^4-85P^2+4)>0$, we have

$$72\mu_0^+>4+P^2+4P^4+\sqrt{(4P^4-8P^2-8)^2}=8P^4-7P^2-4>0$$

for $P>2$. Therefore,

$$\mu_0^-=\frac{P^2(P^2-2)}{9\mu_0^+}<\frac{8P^2(P^2-2)}{8P^4-7P^2-4}.$$

On the other hand, since $P^8-12P^4+16P^2+4=(P^4-6)^2+16(P^2-2)>(P^4-6)^2$, we obtain

$$\mu_4^+>\frac{3P^4-4P^2-2+(P^4-6)}{4P^2(P^2-1)}=\frac{(P^2+1)(P^2-2)}{P^2(P^2-1)}.$$

Because

$$\frac{(P^2+1)(P^2-2)}{P^2(P^2-1)}-\frac{8P^2(P^2-2)}{8P^4-7P^2-4}=\frac{(P^2-2)(9P^4-11P^2-4)}{P^2(P^2-1)(8P^4-7P^2-4)}>0$$

for $P>2$, the inequality $\mu_0^-<\mu_4^+$ follows as required.

We now summarize the above observations. Let $D=\{(P,\mu):2<P,\mu_2(P)<\mu<\mu_a(P)\}$. Here, we write μ_2 and so on as functions of P. We divide D into three parts D_1, D_2, D_3 according to whether the first, second, or third case occurs in (3.3), respectively. Then, $D_1=\{(P,\mu):P_2\le P<P_*,\mu_3^-(P)\le\mu\le\mu_3^+(P)\}\cup\{(P,\mu):P_*\le P,\mu_2(P)<\mu\le\mu_3^+(P)\}$ and $D_2=\{(P,\mu):\mu_4^+(P)\le\mu<\mu_a(P)\}$. Since D_1 and D_2 are disjoint, we necessarily have that $\mu_3^+<\mu_4^+$ for $P_2\le P$. Note here that $\Phi(\mu)=(a-b+c)/3P=P^2-3-\mu(P^2-2)^2/P^2$ for $(P,\mu)\in D_1$, that $\Phi(\mu)=(-a-b-c)/3P=(\mu-1)P^2+1$ for $(P,\mu)\in D_2$, and that

$$\begin{aligned}\Phi(\mu)&=\frac{(a-c)}{3P}\sqrt{1-\frac{b^2}{4ac}}\\&=(1-\mu)P(P^2-2)\sqrt{\frac{P^2-4\mu}{4\mu\{(1-\mu)(P^2-1)^2-1\}}}\end{aligned}$$

for $(P,\mu)\in D_3$.

Finally, setting $\mu_4=\mu_4^+$ for simplicity, we complete the proof of Theorem 2.

5. Variability region of $a_3 - a_2^2$

We first note that the class $\mathcal{C}o_p$ is not rotationally invariant for $0 < p < 1$ due to the presence of a pole at p. It is therefore more natural to consider the variability region of the Fekete–Szegő functional Λ_μ over $\mathcal{C}o_p$ rather than its modulus only. The present section will be devoted to the study of the variability region of $\Lambda_1(f) = a_3 - a_2^2$ because of its importance. Let

$$W_p = \{\Lambda_1(f) : f \in \mathcal{C}o_p\}$$

for $0 < p < 1$.

In the following, we fix $p \in (0,1)$ and put $P = p + 1/p > 2$. Let

$$f_\zeta(z) = \frac{z - T_p(p\zeta)z^2}{(1 - z/p)(1 + pz)} = \sum_{n=1}^{\infty} \frac{1 - p^{2n}\zeta}{p^{n-1}(1 - p^2\zeta)} z^n = \sum_{n=1}^{\infty} A_n(\zeta) z^n$$

for $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$. Here, T_p is defined in (2.1). One can check that f_ζ belongs to $\mathcal{C}o_p$ and corresponds to $\varphi(z) = T_p(\zeta T_p(z))$ through (1.3). As Avkhadiev and Wirths [5] pointed out, the function f_ζ with $|\zeta| = 1$ is extremal in important problems for the class $\mathcal{C}o_p$. Indeed, they proved that the closed disk $A_n(\overline{\mathbb{D}})$ is the variability region of the coefficient functional $a_n(f)$ for $f \in \mathcal{C}o_p$ (see also [12]). We now compute

$$\Lambda_1(f_\zeta) = A_3(\zeta) - A_2(\zeta)^2 = -\frac{(1 - p^2)^2\zeta}{(1 - p^2\zeta)^2} = -(P^2 - 4)K(p^2\zeta),$$

where $K(z) = z/(1 - z)^2$ is the Koebe function. One might expect that the image

$$\Omega_p = \{-(P^2 - 4)K(p^2 z) : |z| \le 1\}$$

would coincide with the variability region W_p. By accident, the form of $A_3 - A_2^2$ is the same as the second Hankel determinant $a_2a_4 - a_3^2$ of order 2 for f_ζ; namely,

$$A_2(\zeta)A_4(\zeta) - A_3(\zeta)^2 = -\frac{(1 - p^2)^2\zeta}{(1 - p^2\zeta)^2} = A_3(\zeta) - A_2(\zeta)^2.$$

The authors [12] investigated the set Ω_p in the context of the second Hankel determinant and found that $\Omega_p \subset \Omega_q$ for $0 < q < p < 1$ and that

$$\bigcup_{0<p<1} \Omega_p = \mathbb{D} \cup \{-1\} \qquad \text{and} \qquad \bigcap_{0<p<1} \Omega_p = \{-(1+z)^2/4 : |z| \le 1\}.$$

Note that $\{-(1+z)^2/4 : |z| \le 1\}$ is a closed Jordan domain bounded by a cardioid (see Figure 2). We [12] also observed that the variability region of $a_2a_4 - a_3^2$ for $\mathcal{C}o_p$ is properly larger than Ω_p. In the case of $a_3 - a_2^2$, rather surprisingly, the expected result partially holds, and a phase transition occurs.

THEOREM 7
Let $0 < p < 1$. The variability region W_p of $a_3 - a_2^2$ for $\mathcal{C}o_p$ satisfies $\Omega_p \subset W_p \subset \overline{\mathbb{D}}$. Moreover, $W_p = \Omega_p$ for $0 < p \le p_0$ and $W_p \ne \Omega_p$ for $p_0 < p < 1$, where

$$p_0 = \frac{1 + \sqrt{37} - \sqrt{2(1 + \sqrt{37})}}{6} \approx 0.553175.$$

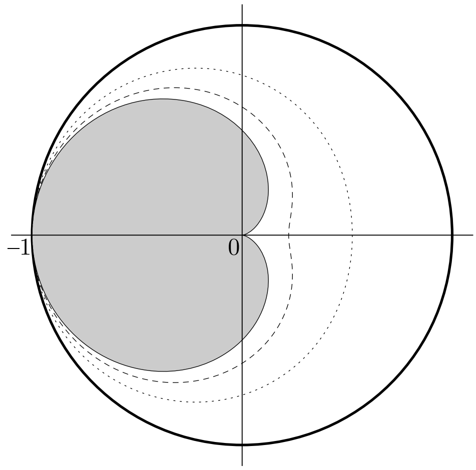

Figure 2. A couple of Ω_p's (the inside of dotted and dashed curves), the intersection cardioid, and the unit disk.

Proof
Letting $\sigma = T_{p^2}(\zeta)$, we have the representation

$$A_3(\zeta) - A_2(\zeta)^2 = -\frac{p^2}{(1+p^2)^2}\big[1-(p^2+p^{-2})\sigma+\sigma^2\big] = -P^{-2}h(\sigma),$$

where

$$h(\sigma) = 1 - t\sigma + \sigma^2, \quad t = P^2 - 2 > 2.$$

Hence, $\Omega_p = -P^{-2}h(\overline{\mathbb{D}})$. One can easily check that h is univalent on $\overline{\mathbb{D}}$. Let Δ_r be the image of the closed disk $|z| \le r$ under the mapping h for $0 \le r \le 1$. For $\zeta, \omega \in \partial\mathbb{D}$, the sharp inequality

$$\big|h(\zeta) - h(r\omega)\big| = |\zeta - r\omega||\zeta + r\omega - t| \ge (1-r)(t-1-r) = h(r) - h(1)$$

holds. Hence, the Euclidean distance δ_r between $\partial\Delta_r$ and $\partial\Delta_1$ is given as $(1-r)(t-1-r) = (1-r)(P^2-3-r)$ for $0 \le r \le 1$. Note that if $|w - h(\sigma)| \le \delta_r$ for some $\sigma \in \mathbb{C}$ with $|\sigma| = r$, then $w \in \Delta_1$.

Letting $\mu = 1$ in (4.1), we obtain the following representation of $\Lambda_1(f)$ for $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{C}o_p$:

$$\begin{aligned} a_3 - a_2^2 &= -P^{-2} + (2P^{-2}-1)\sigma_0 - P^{-2}\sigma_0^2 - \frac{(1-|\sigma_0|^2)\sigma_1}{3P} \\ &= -P^{-2}\big[h(-\sigma_0) + \big(1-|\sigma_0|^2\big)\sigma_1 P/3\big], \end{aligned}$$

for some $\sigma_0, \sigma_1 \in \overline{\mathbb{D}}$. Put $r = |\sigma_0|$. Then $h(-\sigma_0) \in \partial\Delta_r$. If $(1-r^2)P/3 \le \delta_r$, then we have $a_3 - a_2^2 \in -P^{-2}\Delta_1 = \Omega_p$. Since

$$\delta_r - (1-r^2)P/3 = \frac{1-r}{3}\big[3P^2 - (1+r)P - 9 - 3r\big] \ge \frac{1-r}{3}[3P^2 - 2P - 12],$$

we have $(1-r^2)P/3 \le \delta_r$ for $P \ge P_0 := (1+\sqrt{37})/3 \approx 2.36092$, which is the larger zero of the polynomial $3P^2 - 2P - 12$. Note that p_0 is determined by $P_0 = p_0 + 1/p_0$. Thus, we have shown that $W_p \subset \Omega_p$ for $0 < p \le p_0$.

We next assume that $2 < P < P_0$. Since $3P^2 - 2P - 12 < 0$, we can find an $r \in (0,1)$ such that $h(r) - h(1) - (1-r^2)P/3 = (1-r)[3P^2 - (1+r)P - 9 - 3r]/3 < 0$. We choose $\sigma_0 = -r$ and $\sigma_1 = 1$. Then there is a function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in $\mathcal{C}o_p$ satisfying (4.1) with $\mu = 1$:

$$a_3 - a_2^2 = -P^{-2}\big[h(r) + (1-r^2)P/3\big].$$

Therefore, we get

$$a_3 - a_2^2 = -P^{-2}\big[h(1) + \big\{h(r) - h(1) + (1-r^2)P/3\big\}\big] > -P^{-2}h(1) = 1 - 4P^{-2},$$

which implies that $a_3 - a_2^2 \in W_p \setminus \Omega_p$ because $\Omega_p \cap \mathbb{R} = [-1, 1-4P^{-2}]$. □

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