The approximate pseudorandom walk accompanied by the pseudostochastic process corresponding to a higher-order heat-type equation

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Abstract As is well known, a standard random walk is approximate to the stochastic process corresponding to the heat equation. Lachal constructed the approximate pseudorandom walk which is accompanied by the pseudostochastic process corresponding to an even-order heat-type equation. We have two purposes for this article. The first is to construct the approximate pseudorandom walk which is accompanied by the pseudostochastic process corresponding to an odd-order heat-type equation. The other is to propose a construction method for the approximate pseudorandom walk which is accompanied by the pseudostochastic process corresponding to an even-order heat-type equation. This method is different from that of Lachal.

1. Introduction

Many authors have studied pseudostochastic processes of the fourth-order heat-type equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4}. \tag{1.1}$$

Hochberg [3] defined a pseudostochastic process whose density is a fundamental solution of (1.1) and studied its stochastic integral and sojourn time. After Nishioka [10]–[12], Nakajima and Sato [7], Sato [13], and Lachal [4], [5] studied the distribution of the sojourn time and the joint distribution of a first hitting time and a first hitting place.

Nishioka [11] first gave the joint distribution of a first hitting time and a first hitting place of the stochastic pseudoprocess corresponding to (1.1). Sato [13] constructed the pseudorandom walk which approximates the pseudostochastic process corresponding to (1.1). He studied the joint distribution of a first hitting time and a first hitting place of it by the approximate pseudorandom walk.

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Moreover, Lachal [4] studied the joint distribution of a first hitting time and a first hitting place of a stochastic pseudoprocess whose density is a fundamental solution of

$$\frac{\partial u}{\partial t} = \pm \frac{\partial^{2l} u}{\partial x^{2l}}, \quad l = 1, 2, \ldots.$$

Lachal [5] constructed the pseudorandom walk which approximates the pseudostochastic process corresponding to an even-order heat-type equation.

But not many authors have studied pseudostochastic processes of odd-order heat-type equations compared to those who have studied even-order heat-type equations. Beghin, Hochberg, and Orsingher [1], Beghin, Orsingher, and Ragozina [2], and Nikitin and Orsingher [9] calculated the joint distribution of the maximum and position, the distribution of the sojourn time, and the other conditional distribution of the stochastic pseudoprocess corresponding to a third-order heat-type equation. Shimoyama [14] gave the joint distribution of a first hitting time and a first hitting place of the stochastic pseudoprocess corresponding to a third-order heat-type equation. Nakajima and Sato [8] gave the joint distribution of a first hitting time and a first hitting place of the stochastic pseudoprocess corresponding to a third-order heat-type equation through the approximate random walk.

There are two purposes for this article. The first purpose is to construct a pseudorandom walk which approximates the pseudostochastic process corresponding to an odd-order heat-type equation:

$$\frac{\partial u}{\partial t} = \pm \frac{\partial^{2l+1} u}{\partial x^{2l+1}}, \quad l = 1, 2, \ldots.$$

The second purpose is to construct a pseudorandom walk which approximates the pseudostochastic process corresponding to an even-order heat-type equation which is based on the concept of a subordinator.

The article is organized as follows. In Section 2, we construct a pseudorandom walk whose total variation may be different from one. This pseudorandom walk approximates the pseudostochastic process whose density is a fundamental solution of the odd-order heat-type equation (2.21). This construction method is more intuitive than that in the next section. In Section 3, we construct the same pseudorandom walk as in the previous section by another method. This construction method is more technical than the method of the previous section, but is more general. In Section 4, we give examples in the case of third- and fifth-order heat-type equations. In Section 5, we construct an even-order pseudorandom walk which approximates the pseudostochastic process with the concept of a subordinator.

2. A construction of a pseudorandom walk for an odd-order heat-type equation

In this section, we construct a pseudorandom walk which corresponds to the equation

$$\frac{\partial u}{\partial t} = A_{2l+1}\frac{\partial^{2l+1}u}{\partial x^{2l+1}}, \quad l = 1,2,\ldots, \tag{2.1}$$

where A_{2l+1} is a real nonzero constant. For $l \in \mathbf{N}$, we consider pseudoindependent random variables $\{\xi_j^+\}$ and $\{\xi_k^-\}$ as

$$P(\xi_j^+ = 1) = p_j, \qquad P(\xi_j^+ = 0) = 1 - p_j, \quad j = 1,2,\ldots,l+1,$$

and

$$P(\xi_k^- = 0) = r_k, \qquad P(\xi_k^- = -1) = 1 - r_k, \quad k = 1,2,\ldots,l,$$

where p_j and r_k may be complex numbers whose concrete values will be determined later. We set X^+ as

$$X^+ = \sum_{j=1}^{l+1}\xi_j^+ + \sum_{k=1}^{l}\xi_k^-,$$

and $X_1^+, X_2^+, X_3^+, \ldots$ are independent copies of X^+.

For $n \geq 0$, we define a pseudorandom walk Z_n^+ as

$$Z_n^+ \equiv x + \sum_{k=1}^{n} X_k^+.$$

We define the expectation of Z_n^+ as Sato [13] did. For any measurable function f, we set the expectation of f as an analogy of "the usual probability theory." That is,

$$E\big[f(Z_n^+)\big] = \sum_m f(m)P[Z_n^+ = m],$$

$$E_x\big[f(Z_n^+)\big] = E\big[f(Z_n^+) \mid Z_0^+ = x\big],$$

and

$$P_x[Z_n^+ = m] \equiv P[Z_n^+ = m \mid Z_0^+ = x].$$

We denote the characteristic function of X_1^+ by

$$M(\theta) \equiv E[e^{i\theta X_1^+}]. \tag{2.2}$$

Then the characteristic function $M_x(\theta) \equiv E_x[e^{i\theta Z_1^+}]$ for Z_1^+ starting at x is

$$\begin{aligned} M_x(\theta) &\equiv E_x[e^{i\theta Z_1^+}] \\ &= E[e^{i\theta(x+X_1^+)}] \\ &= e^{i\theta x}M(\theta). \end{aligned}$$

We examine $M(\theta)$:

$$\begin{aligned}
M(\theta) &\equiv E[e^{i\theta X_1^+}] \\
&= E[e^{i\theta(\sum_{j=1}^{l+1}\xi_j^+ + \sum_{k=1}^{l}\xi_k^-)}] \\
&= \prod_{j=1}^{l+1}(1-p_j+p_je^{i\theta})\prod_{k=1}^{l}\{r_k+(1-r_k)e^{-i\theta}\} \\
&= e^{-i\theta l}\prod_{j=1}^{l+1}\{1+p_j(e^{i\theta}-1)\}\prod_{k=1}^{l}\{1+r_k(e^{i\theta}-1)\} \\
&= e^{-i\theta l}\Big\{1+\sum_{j=1}^{2l+l}q_j(e^{i\theta}-1)+\sum_{j\neq k}q_jq_k(e^{i\theta}-1)^2 \\
&\qquad +\cdots+\prod_{j=1}^{2l+1}q_j(e^{i\theta}-1)^{2l+1}\Big\},
\end{aligned}$$

where we set $\{q_j\}$ as

$$\begin{aligned}
&q_1 = p_1, \qquad q_2 = p_2, \qquad \ldots, \qquad q_{l+1} = p_{l+1}, \\
&q_{l+2} = r_1, \qquad q_{l+2} = r_2, \qquad \ldots, \qquad q_{2l+1} = r_l.
\end{aligned}$$

That is,

$$M(\theta) = e^{-i\theta l}\{1+Q_1(e^{i\theta}-1)+Q_2(e^{i\theta}-1)^2+\cdots+Q_{2l+1}(e^{i\theta}-1)^{2l+1}\},$$

where

$$\text{(2.3)} \qquad Q_1 = \sum_{i=1}^{2l+l} q_i, \qquad Q_2 = \sum_{i\neq j} q_iq_j, \qquad \ldots, \qquad Q_{2l+1} = \prod_{i=1}^{2l+1} q_i.$$

If we give $Q_1,\ldots,Q_{2l+1}$, then we have $q_1,\ldots,q_{2l+1}$ as solutions of the equation

$$\lambda^{2l+1} - Q_1\lambda_{2l} + \cdots + (-1)^l Q_l\lambda^{l+1} + \cdots + Q_{2l}\lambda - Q_{2l+1} = 0.$$

Now, we shall determine $Q_1,\ldots,Q_{2l+1}$. For $\epsilon > 0$, we set

$$\text{(2.4)} \qquad Z_n^{+\epsilon} = x + \epsilon\sum_{k=1}^{n} X_k^+.$$

By (2.1), we have the property

$$\text{(2.5)} \qquad \begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon^{2l+1}}\big(E_x[e^{i\theta Z_1^{+\epsilon}}]-e^{i\theta x}\big) &= e^{i\theta x}\lim_{\epsilon\to 0}\frac{M(\epsilon\theta)-1}{\epsilon^{2l+1}} \\
&= A_{2l+1}\frac{d^{2l+1}}{dx^{2l+1}}e^{i\theta x} \\
&= A_{2l+1}(i\theta)^{2l+1}e^{i\theta x};
\end{aligned}$$

that is,

$$\text{(2.6)} \qquad \lim_{\epsilon\to 0}\frac{M(\epsilon\theta)-1}{(i\epsilon\theta)^{2l+1}} = A_{2l+1}.$$

For the sake of simplicity, we use the following notation:

$$F(x) \equiv e^{-lx}\{1 + Q_1(e^x - 1) + Q_2(e^x - 1)^2 + \cdots + Q_{2l+1}(e^x - 1)^{2l+1}\}. \tag{2.7}$$

That is, $F(i\theta) = M(\theta)$.

By (2.6), $F(x)$ must satisfy

$$F(x) = 1 + A_{2l+1}x^{2l+1} + O(x^{2l+2}). \tag{2.8}$$

By (2.7), setting $e^x - 1 = t$, we have

$$(t+1)^l F\bigl(\log(t+1)\bigr) = 1 + Q_1 t + Q_2 t^2 + \cdots + Q_{2l+1}t^{2l+1}. \tag{2.9}$$

By (2.8),

$$\begin{aligned}
&(t+1)^l F\bigl(\log(t+1)\bigr)\\
&\quad = (t+1)^l + A_{2l+1}(t+1)^l\bigl(\log(t+1)\bigr)^{2l+1} + O\Bigl(\bigl(\log(t+1)\bigr)^{2l+2}\Bigr) \qquad (2.10)\\
&\quad = (t+1)^l + A_{2l+1}t^{2l+1} + O(t^{2l+2}).
\end{aligned}$$

On equating (2.9) and (2.10), we get $\{Q_m\}$ as

$$Q_m = \begin{cases} \binom{l}{m}, & 1 \le m \le l,\\ 0, & l+1 \le m \le 2l,\\ A_{2l+1}, & m = 2l+1. \end{cases}$$

Therefore, $p_1, p_2, \ldots, p_{l+1}, r_1, r_2, \ldots, r_l$ are solutions of

$$\lambda^{l+1}(\lambda - 1)^l - A_{2l+1} = 0. \tag{2.11}$$

Moreover, we have

$$M(\theta) = 1 + A_{2l+1}(e^{i\theta} - 1)^{2l+1}e^{-i\theta l}. \tag{2.12}$$

LEMMA 2.1

Under the condition

$$0 < (-1)^l A_{2l+1} \le 2^{-(2l+1)}, \tag{2.13}$$

(2.11) has three real solutions and $(2l-2)$ *complex solutions.*

Proof

Let

$$f(\lambda) \equiv \lambda^{l+1}(\lambda - 1)^l.$$

Then we have

$$f'(\lambda) = \lambda^l(\lambda - 1)^{l-1}\{(2l+1)\lambda - (l+1)\},$$

which means that the condition for the existence of three real zeros is

$$0 < (-1)^l A_{2l+1} < (-1)^l f\Bigl(\frac{l+1}{2l+1}\Bigr).$$

Since $(-1)^l f(\frac{1}{2})$ is smaller than the right-hand side, the statement gives a sufficient condition. $\square$

Let v_1, v_2, v_3 be real solutions, and let $s_1, \bar{s}_1, \dots, s_{l-1}, \bar{s}_{l-1}$ be complex conjugate solutions of (2.11). Then we have

$$
\begin{aligned}
M(\theta) &= e^{-i\theta l} \prod_{j=1}^{2l+1} \{1 + q_j(e^{i\theta} - 1)\} \\
&= e^{-i\theta l} \prod_{j=1}^{3} (1 + v_j(e^{i\theta} - 1)) \prod_{k=1}^{l-1} \{1 + s_k(e^{i\theta} - 1)\}\{1 + \bar{s}_k(e^{i\theta} - 1)\} \\
&= \{v_3 + (1 - v_3)e^{-i\theta}\} \prod_{j=1}^{2} (1 - v_j + v_j e^{i\theta}) \\
&\quad \times \prod_{k=1}^{l-1} \{1 - s_k + s_k e^{i\theta}\}\{\bar{s}_k + (1 - \bar{s}_k)e^{-i\theta}\} \\
&= \{v_3 + (1 - v_3)e^{-i\theta}\} \prod_{j=1}^{2} (1 - v_j + v_j e^{i\theta}) \\
&\quad \times \prod_{k=1}^{l-1} \{2\Re s_k - 2|s_k|^2 + |s_k|^2 e^{i\theta} + (1 + |s_k|^2 - 2\Re s_k)e^{-i\theta}\}.
\end{aligned}
$$

Thus, we obtain the following.

PROPOSITION 2.2

Under the condition (2.13), we define the real pseudorandom variable Y_k $(k = 1, \dots, l-1)$ as

$$
\begin{aligned}
P(Y_k = 0) &= 2\Re s_k - 2|s_k|^2, \\
P(Y_k = 1) &= |s_k|^2, \qquad (2.14) \\
P(Y_k = -1) &= 1 + |s_k|^2 - 2\Re s_k
\end{aligned}
$$

and the three real pseudorandom variables X_j as

$$P(X_j = 0) = 1 - v_j, \qquad P(X_j = 1) = v_j \quad (j = 1, 2),$$

$$P(X_3 = 0) = v_3, \qquad P(X_3 = -1) = 1 - v_3,$$

where $\{v_j\}, \{s_k\}$ are solutions of (2.11) as above. Then $Z^+ = \sum_{j=1}^{3} X_j + \sum_{k=1}^{l-1} Y_k$ is a pseudorandom variable, and the sum of its independent copies constitutes our pseudorandom walk with a real-valued measure on the cylinder sets up to each finite time step.

We set

$$p(n, m) \equiv P_0[Z_n^+ = m].$$

Then

$$\sum_m p(n,m)e^{im\theta} = E_0[e^{i\theta Z_n^+}]$$
$$= E[e^{i\theta \sum_{k=1}^n X_k^+}]$$
$$= E[e^{i\theta X_1^+}]\cdots E[e^{i\theta X_n^+}]$$
$$= \{M(\theta)\}^n.$$

Since $p(n,m)$ is a Fourier coefficient, we have

$$p(n,m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\theta)^n e^{-im\theta}\,d\theta \tag{2.15}$$

and

$$\lim_{m\to\infty} p(n,m) = 0.$$

PROPOSITION 2.3

For $m = -l, -l+1, \ldots, l+1$, we have

$$P_0(Z_1^+ = m) = \delta_{m,0} + (-1)^{l-m+1}\binom{2l+1}{l+m}A_{2l+1},$$

where $\delta_{m,0}$ is the Kronecker delta.

Proof

By (2.12), we have

$$\begin{aligned}
P_0(Z_1^+ = m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\theta)e^{-im\theta}\,d\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\{1 + A_{2l+1}(e^{i\theta}-1)^{2l+1}e^{-i\theta l}\}e^{-im\theta}\,d\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{1 + A_{2l+1}\sum_{j=0}^{2l+1}\binom{2l+1}{j}(-1)^{2l+1-j}e^{i(j-l)\theta}\right\}e^{-im\theta}\,d\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{e^{-im\theta} + A_{2l+1}\sum_{j=0}^{2l+1}\binom{2l+1}{j}(-1)^{2l+1-j}e^{i(j-l-m)\theta}\right\}d\theta.
\end{aligned}$$

We get the result. □

PROPOSITION 2.4

Under the condition (2.13) we have the property

$$\left|M(\theta)\right| \le 1 \tag{2.16}$$

and $|M(\theta)|$ is an even function of θ and decreasing in $[0,\pi]$. Let

$$K \equiv (-1)^l 2^l A_{2l+1}.$$

Then we have

$$\left|M(\theta)\right| \le 1 - \frac{K}{2}(1-\cos\theta)^{l+1}. \tag{2.17}$$

Proof
We simply write A as A_{2l+1}. Then we have

$$\begin{aligned} M(\theta) &= 1 + A(e^{i\theta}-1)^{2l+1}e^{-i\theta l} \\ &= 1 + A(e^{i\theta}-1)\left\{(e^{i\theta}-1)(1-e^{-i\theta})\right\}^l \\ &= 1 - (-1)^l 2^l A(1-e^{i\theta})(1-\cos\theta)^l \\ &= 1 - K(1-\cos\theta)^{l+1} + iK(1-\cos\theta)^l \sin\theta. \end{aligned} \tag{2.18}$$

Then

$$\begin{aligned} \left|M(\theta)\right|^2 &= \left\{1-K(1-\cos\theta)^{l+1}\right\}^2 + \left\{K(1-\cos\theta)^l \sin\theta\right\}^2 \\ &= 1 - 2K(1-\cos\theta)^{l+1} + K^2(1-\cos\theta)^{2l}\left\{(1-\cos\theta)^2 + \sin^2\theta\right\} \\ &= 1 - 2K(1-\cos\theta)^{l+1} + 2K^2(1-\cos\theta)^{2l+1}. \end{aligned} \tag{2.19}$$

Let

$$f(x) \equiv 1 - 2Kx^{l+1} + 2K^2x^{2l+1}.$$

It is clear that $f(x) = |M(\theta)|^2 \ge 0$ for $0 \le x \le 2$, and then we decide the range of K under $f(x) \le 1$ for $0 \le x \le 2$. Since

$$f(x) - 1 = 2x \cdot Kx^l(Kx^l - 1) \le 0,$$

we get

$$0 \le Kx^l \le 1,$$

which proves $K2^l \le 1$ and then

$$0 < (-1)^l A_{2l+1} \le 2^{-2l}. \tag{2.20}$$

The condition (2.13) is stronger than this inequality, and we have

$$K2^l \le \frac{1}{2}.$$

The solution x_0 of $f'(x) = 0$ satisfies

$$Kx_0^l = \frac{l+1}{2l+1} \ge \frac{1}{2}.$$

Thus, we get $x_0 \ge 2$, and $f(x)$ is decreasing in $[0,2]$.

We have

$$\begin{aligned} \left|M(\theta)\right|^2 &= 1 - 2K(1-\cos\theta)^{l+1}\left\{1 - K(1-\cos\theta)^l\right\} \\ &= 1 - 2C_\theta(1-\cos\theta)^{l+1} \\ &\le \left\{1 - C_\theta(1-\cos\theta)^{l+1}\right\}^2, \end{aligned}$$

where

$$C_\theta = K\{1 - K(1-\cos\theta)^l\}.$$

Since $K2^l \le 1/2$, we have $C_\theta \ge K/2$. □

In the following, we usually suppose that the condition (2.13) is satisfied. For example, we have

$$|p(n,m)| \le 1$$

under this condition.

PROPOSITION 2.5

We have

$$\lim_{r\to 1-0} \sum_{n=0}^{\infty} r^n P(Z_n^+ = 0) = \infty.$$

Proof

We set $K = (-2)^l A_{2l+1}$. Let $0 < r < 1$ and $\epsilon > 0$ be sufficiently small. By (2.15) and Proposition 2.4, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} r^n p(n,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-rM(\theta)}\,d\theta \\
&= \frac{1}{2\pi}\int_{|\theta|<\epsilon} \frac{1}{1-rM(\theta)}\,d\theta + \frac{1}{2\pi}\int_{\epsilon\le|\theta|\le\pi} \frac{1}{1-rM(\theta)}\,d\theta \\
&= I_1 + I_2 \quad \text{(say)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{r\to 1-0} I_1 &= \frac{1}{2\pi}\int_{|\theta|<\epsilon} \frac{1}{1-M(\theta)}\,d\theta \\
&= \frac{1}{2\pi}\frac{1}{2K}\int_{|\theta|<\epsilon} \frac{1-\cos\theta + i\sin\theta}{(1-\cos\theta)^{l+1}}\,d\theta \\
&= \frac{1}{2\pi}\frac{1}{2K}\int_{|\theta|<\epsilon} \frac{1}{(1-\cos\theta)^{l}}\,d\theta \\
&= \infty.
\end{aligned}$$

By (2.17), it is easy to see that I_2 is bounded as $r \to 1-0$. □

REMARK 2.6

By this proposition we may claim

$$E\big[\#\{n : n \ge 0, Z_n^+ = 0\}\big] = \infty,$$

where $\#A$ means the number of a set A. However, our random walk $\{Z_n^+\}$ generally takes a signed measure, and it may stick on 0. Then we cannot conclude that it is recursive as in usual probability theory.

THEOREM 2.7

We set $W_t^{+\epsilon} = \epsilon Z^+_{[t/\epsilon^{2l+1}]}$ *for any* $\epsilon > 0$*, where* $[x]$ *denotes its integer part. Then* $W_t^+ = \lim_{\epsilon\to 0} W_t^{+\epsilon}$ *is the pseudostochastic process whose transition probability density function is*

$$q^+(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\bigl\{i(-1)^l A_{2l+1}\mu^{2l+1}t - i\mu x\bigr\}\,d\mu.$$

This $q^+(t,x)$ *is the fundamental solution of the* $(2l+1)$*th-order diffusion equation*

$$\frac{\partial u}{\partial t} = A_{2l+1}\frac{\partial^{2l+1}u}{\partial x^{2l+1}}. \tag{2.21}$$

Proof

Let $x = m\epsilon$, $t = n\epsilon^{2l+1}$. Then

$$\begin{aligned}
p_\epsilon(t,x) &\equiv p(n,m)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\theta)^{t/\epsilon^{(2l+1)}} e^{-i\theta x/\epsilon}\,d\theta\\
&= \frac{1}{2\pi}\Bigl(\int_{\delta<|\theta|\le\pi} + \int_{|\theta|<\delta}\Bigr) M(\theta)^{t/\epsilon^{2l+1}} e^{-i\theta x/\epsilon}\,d\theta\\
&= J_1 + J_2 \quad \text{(say)},
\end{aligned}$$

where $\delta = \pi\epsilon^{\frac{l}{l+1}}$. Then we prove that

$$q^+(t,x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}p_\epsilon(t,x).$$

First, clearly,

$$q^+(0,x) = \delta(x),$$

where $\delta(x)$ is Dirac's delta function. Assume that $t > 0$ in the following. When $0 < \delta < |\theta| < \pi$, we note from (2.17) that

$$\bigl|M(\theta)\bigr| \le 1 - \frac{K}{2}(1-\cos\delta)^{l+1}.$$

We consider that

$$J_1 = \frac{1}{2\pi}\int_{\delta<|\theta|\le\pi} M(\theta)^{t/\epsilon^{2l+1}} e^{-i\theta x/\epsilon}\,d\theta.$$

Since $1 + x \le e^x$, we have

$$\begin{aligned}
\frac{1}{\epsilon}|J_1| &\le \frac{1}{\epsilon}\Bigl\{1 - \frac{K}{2}(1-\cos\delta)^{l+1}\Bigr\}^{t/\epsilon^{(2l+1)}}\\
&\le \frac{1}{\epsilon}\exp\Bigl\{-\frac{tK}{2}(1-\cos\delta)^{l+1}/\epsilon^{2l+1}\Bigr\}\\
&\sim \frac{1}{\epsilon}\exp\Bigl\{-\frac{tK}{2}\Bigl(\frac{\pi^2}{2}\Bigr)^{l+1}\epsilon^{-1}\Bigr\}\\
&\to 0 \quad \text{as } \epsilon\to 0.
\end{aligned}$$

Next, we consider that

$$J_2 = \frac{1}{2\pi}\int_{|\theta|<\delta} M(\theta)^{t/\epsilon^{2l+1}} e^{-i\theta x/\epsilon}\,d\theta$$
$$= \frac{\epsilon}{2\pi}\int_{-\pi\epsilon^{-1/(l+1)}}^{\pi\epsilon^{-1/(l+1)}} M(\epsilon\mu)^{t/\epsilon^{2l+1}} e^{-i\mu x}\,d\mu.$$

Let ϵ be sufficiently small. Let

$$\log(1+z) \equiv z + R(z).$$

Note that the inequality $|R(z)| \le |z|^2$ holds for sufficiently small z. Thus, we get

$$\log M(\theta) = -K(1-\cos\theta)^{l+1} + iK(1-\cos\theta)^l \sin\theta + R(\theta)$$

and

$$\big|R(\theta)\big| \le 2K^2(1-\cos\theta)^{2l+1}.$$

However, we see that

$$\frac{|R(\theta)|}{K(1-\cos\theta)^{l+1}} \le 2K(1-\cos\theta)^l.$$

Thus, we can neglect $R(\theta)$ in comparison to the first term which acts as a discount factor as follows. Noting that $K>0$, we have

$$\frac{2\pi}{\epsilon}J_2 = \int_{-\frac{\pi}{\epsilon^{1/(l+1)}}}^{\frac{\pi}{\epsilon^{1/(l+1)}}} M(\epsilon\mu)^{t/\epsilon^{2l+1}} e^{-i\mu x}\,d\mu$$
$$\simeq \int_{-\frac{\pi}{\epsilon^{1/(l+1)}}}^{\frac{\pi}{\epsilon^{1/(l+1)}}} \exp(-K2^{-l-1}\epsilon\mu^{2l+2}t + iK2^{-l}\mu^{2l+1}t - i\mu x)\,d\mu$$
$$\simeq \int_{-\infty}^{\infty} \exp(-K2^{-l-1}\epsilon\mu^{2l+2}t + iK2^{-l}\mu^{2l+1}t - i\mu x)\,d\mu$$
$$\to \int_{-\infty}^{\infty} \exp\{iK2^{-l}\mu^{2l+1}t - i\mu x\}\,d\mu \quad \text{as } \epsilon \to 0.$$

Thus, we get the conclusion. □

REMARK 2.8

It is natural to consider the case where $\{\xi_j^+\}$ for $j=1,2,\dots,l$ and $\{\xi_k^-\}$ for $k=1,2,\dots,l+1$. Clearly, this is a transform of x to $-x$.

We define

$$P(\xi_j^+ = 1) = 1-p_j, \qquad P(\xi_j^+ = 0) = p_j, \quad j=1,2,\dots,l,$$

and

$$P(\xi_k^- = 0) = 1-r_k, \qquad P(\xi_k^- = -1) = r_k, \quad k=1,2,\dots,l+1,$$

where p_j and r_k are solutions of the equation

$$\lambda^{l+1}(\lambda-1)^l = A_{2l+1},$$

where A_{2l+1} satisfies the condition (2.13). For $n \geq 0$, we define a pseudorandom walk Z_n^- as

$$X^- \equiv \sum_{j=1}^{l+1} \xi_j^+ + \sum_{k=1}^{l} \xi_k^- .$$

Then $X_1^-, X_2^-, \ldots,$ are independent copies of X^-, and

$$Z_n^- \equiv x + \sum_{k=1}^{n} X_k^- .$$

The characteristic function of $N(\theta)$ of Z_1^- starting at zero is

$$N(\theta) = 1 + A_{2l+1}(e^{-i\theta} - 1)^{2l+1} e^{i\theta l};$$

that is, $N(\theta)$ is the complex conjugate of $M(\theta)$. Moreover, the scaling limit of $\{Z_n^-\}$ is the pseudostochastic process whose transition probability density function is

$$q^-(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\bigl\{-i(-1)^l A_{2l+1}\mu^{2l+1} t - i\mu x\bigr\}\, d\mu.$$

This $q^-(t,x)$ is the fundamental solution of the $(2l+1)$th-order diffusion equation

$$\frac{\partial u}{\partial t} = -A_{2l+1} \frac{\partial^{2l+1} u}{\partial x^{2l+1}}.$$

3. Another construction of a pseudorandom walk for an odd-order heat-type equation

We set the random variable $\{\xi\}$ whose distribution is

$$P(\xi = m) = \alpha_m, \quad m = -l-1, -l, \ldots, l, l+1,$$

and take $\{\xi_j\}$ as independent copies of ξ. For $n \geq 0$, we define a pseudorandom walk Z_n as

$$Z_n = x + \sum_{j=1}^{n} \xi_j .$$

The characteristic function $M(\theta)$ of Z_1 starting at zero is

$$M(\theta) = E[e^{i\theta Z_1}] = \sum_{k=-l-1}^{l+1} \alpha_k e^{i\theta k}.$$

For our purpose, we need the property

$$M(\theta) = 1 + O(\theta^{2l+1}).$$

We deduce α_k satisfying this property. Then

$$\begin{aligned}\widetilde{M}(\theta) &= \sum_{k=-l-1}^{l+1} (\alpha_k - \delta_{k,0}) e^{i\theta k} \\ &= M(\theta) - 1 = O(\theta^{2l+1}).\end{aligned}$$

Thus,

$$\begin{aligned} e^{i(l+1)\theta}\widetilde{M}(\theta) &= \sum_{k=-l-1}^{l+1} (\alpha_k - \delta_{k,0}) e^{i\theta(k+l+1)} \\ &= \sum_{j=0}^{2(l+1)} (\alpha_{j-l-1} - \delta_{j-l-1,0}) e^{i\theta j} \\ &= \sum_{j=0}^{2(l+1)} \beta_j e^{i\theta j} \quad \text{(say)}. \end{aligned}$$

Therefore, we shall find $\{\beta_j\}$ which satisfy

$$\sum_{j=0}^{2(l+1)} \beta_j j^r = 0, \quad 0 \le r \le 2l.$$

Define $[j]_r$ as $[j]_0 = 1$, and define

$$[j]_r = j(j-1)\cdots(j-r+1) \quad (r \ge 1).$$

The Stirling numbers of the second kind can be defined as the coefficients of the expansion

$$j^n = \sum_{r=0}^{n} \left\{ {n \atop r} \right\} [j]_r. \tag{3.1}$$

Then the next lemma is well known.

LEMMA 3.1

We have that

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j j^r = (-1)^n \left\{ {r \atop n} \right\} n!. \tag{3.2}$$

In particular, the left-hand side of this equation takes value zero for $0 \le r \le n-1$.

The next proposition is easily obtained by this lemma and derived from the classical Newton's difference formula. However, since this is a basic formula for the construction of our pseudorandom walk, we state the result and give a proof.

PROPOSITION 3.2

Let n be any positive integer. Let $0 \le r \le n-1$, and let $a_j \ne 0$ for some j. We set $\binom{n}{k} = 0$ for $n < k$ or $k < 0$ as a matter of convenience.

(i) *We suppose that*

$$\sum_{j=0}^{n} a_j j^r = 0.$$

Then we have

$$a_j = a_n \binom{n}{j} (-1)^{n-j} = a_0 \binom{n}{j} (-1)^j.$$

(ii) *We suppose that*

$$\sum_{j=0}^{n+1} a_j j^r = 0.$$

Then we have

$$a_j = a_0 \binom{n}{j} (-1)^j + a_{n+1} \binom{n}{j-1} (-1)^{n+1-j}.$$

Moreover, we have

$$\sum_{j=0}^{n+1} a_j j^{n+1} = n! \big((-1)^n a_0 + a_{n+1} \big)$$

and

$$\sum_{j=0}^{n+1} a_j x^j = (-1)^n a_0 (x-1)^n + a_{n+1} x (x-1)^n$$

for case (i) as $a_{n+1} = 0$.

Proof

(i) We set $M(x) = \sum_{j=0}^{n} a_j (1+x)^j$. By the assumption,

$$\left(\frac{d}{dx} \right)^r M(x)|_{x=0} = \sum_{j=0}^{n} a_j [j]_r = 0$$

for $r \leq n-1$. Thus, we have

$$M(x) = \sum_{j=0}^{n} a_j (1+x)^j = a_n x^n.$$

Letting $w = 1 + x$, we get

$$\begin{aligned} M(x) &= \sum_{j=0}^{n} a_j w^j = a_n (w-1)^n \\ &= a_n \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} w^j. \end{aligned}$$

(ii) By the assumption, we also have for $0 \leq r \leq n-1$

$$\sum_{j=0}^{n+1} a_j (j-1)^r = 0.$$

Let

$$b_j = a_0 \binom{n}{j} (-1)^j$$

and $c_j = a_j - b_j$. Then by the above lemma,

$$\sum_{j=0}^{n+1} b_j(j-1)^r = \sum_{j=0}^{n} b_j(j-1)^r = 0 \quad \text{for } r \le n-1.$$

Thus,

$$\begin{aligned} 0 &= \sum_{j=0}^{n+1} c_j(j-1)^r \\ &= \sum_{j=1}^{n+1} c_j(j-1)^r = \sum_{j=0}^{n} c_{j+1} j^r. \end{aligned}$$

Therefore, according to (i),

$$c_{j+1} = c_{n+1}\binom{n}{j}(-1)^{n-j},$$

which proves the statement with $c_{n+1} = a_{n+1}$. The proof of the rest is easy. □

Set $n = 2l+1$. Owing to this proposition, we get

$$\beta_j = \beta_0\binom{n}{j}(-1)^j + \beta_{n+1}\binom{n}{j-1}(-1)^{n+1-j},$$

which agrees with Proposition 2.3 with $\beta_0 = 0$. This shows that there only appears the combination of $(\frac{d}{dx})^n$ and $-(\frac{d}{dx})^n$.

On the other hand, when $n = 2l+2$ and we suppose that

$$M(x) = O(x^n),$$

then we only see the difference formula for $(\frac{d}{dx})^n$ as Proposition 3.2(i). Though the study of the even-order case is easy and simple, we omit it here. We will give it in the last section by another method.

We continue the odd-order case $n = 2l+1$. We set $x = e^{i\theta}$. Then we have

$$\begin{aligned} x^{l+1}\widetilde{M}(x) &= \sum_{j=0}^{n+1} \beta_j x^j \\ &= (-1)^n\beta_0(x-1)^n + \beta_{n+1}x(x-1)^n \\ &= (-\beta_0 + \beta_{n+1}x)(x-1)^n. \end{aligned}$$

Thus, under the condition

$$\beta_0 \ne \beta_{n+1}$$

we get the desired property

$$\widetilde{M}(\theta) = (-\beta_0 + \beta_{n+1})(i\theta)^n + O(\theta^{n+1}).$$

In addition, we consider the condition that $|M(\theta)| \le 1$.

We have

$$M(x) = 1 + x^{-(l+1)}(x-1)^n(-\beta_0 + \beta_{n+1}x).$$

We set the variable $v = \frac{1}{1-x}$. Since $x = -(1-v)v^{-1}$ and $x - 1 = -v^{-1}$,

$$\begin{aligned} M(v) &= 1 - (-1)^{l+1}(1-v)^{-(l+1)}v^{l+1}(-v)^{-(2l+1)}\bigl\{\beta_0 + \beta_{n+1}(1-v)v^{-1}\bigr\} \\ &= 1 - (1-v)^{-(l+1)}v^{-(l+1)}\bigl\{(-1)^l\beta_0 v + (-1)^l\beta_{n+1}(1-v)\bigr\} \\ &= 1 - (1-v)^{-(l+1)}v^{-(l+1)}\bigl\{av + b(1-v)\bigr\} \quad \text{(say)}. \end{aligned}$$

On the other hand, by $x = e^{i\theta}$ we have

$$\begin{aligned} v(1-v) &= -\frac{e^{i\theta}}{(e^{i\theta}-1)^2} \\ &= \frac{1}{4\sin^2\frac{\theta}{2}} \\ &= S^{-1} \quad \text{(say)}. \end{aligned}$$

Thus,

$$M(v) = 1 - S^{l+1}\bigl\{av + b(1-v)\bigr\}.$$

Since $\bar{v} = 1 - v$ and $\overline{1-v} = v$, we have

$$\bar{M}(v) = 1 - S^{l+1}\bigl\{a(1-v) + bv\bigr\}.$$

Then

$$\bigl|M(v)\bigr|^2 = 1 - S^{l+1}(a+b) + S^{2(l+1)}|L|^2,$$

where

$$|L|^2 = \bigl|av + b(1-v)\bigr|^2 = (a-b)^2S^{-1} + ab.$$

Thus,

$$\begin{aligned} f(S) &= \bigl|M(v)\bigr|^2 - 1 \\ &= S^{l+1}\bigl(-(a+b) + S^l\bigl\{(a+b)^2 + ab(S-4)\bigr\}\bigr). \end{aligned}$$

We need that $|M(\theta)| \le 1$ and $|M(\theta)|$ is a decreasing function of θ. Then we see that $f(S) \le 0$ and $f(S)$ is decreasing in $S \in [0,4]$. From $S = 4$, we get $0 < a + b \le 2^{-2l}$ at least.

However, we state the stronger condition

$$0 < a + b \le 2^{-(2l+1)}. \tag{3.3}$$

Under this condition, considering the condition such that $f(S)$ is decreasing, we get

$$ab \ge -\frac{l(2l+1)}{4(3l+2)}(a+b)^2,$$

but we omit the details of the calculation. Now, we have the following theorem.

THEOREM 3.3
We suppose that $\alpha_{-l-1} \neq \alpha_{l+1}$,

$$0 < (-1)^l \alpha_{-l-1} + (-1)^l \alpha_{l+1} \leq 2^{-(2l+1)},$$

and

$$\alpha_{-l-1}\alpha_{l+1} \geq -\frac{l(2l+1)}{4(3l+2)}(\alpha_{-l-1} + \alpha_{l+1})^2.$$

Then we have the property $|M(\theta)| \leq 1$. *Moreover,* $|M(\theta)|$ *is an even function of* θ *and a decreasing function in* $[0, \pi]$. *The one-step variable* ξ *of our random walk has the following distribution:*

$$P(\xi = m) = \delta_{m,0} + \alpha_{-l-1}\binom{2l+1}{m+l+1}(-1)^{m+l+1} + \alpha_{l+1}\binom{2l+1}{m+l}(-1)^{m+l+1},$$

for $m = -l-1, -l, \dots, l+1$.

Moreover, we have

$$\begin{aligned} M(\theta) &= 1 + e^{-i\theta(l+1)}(e^{i\theta} - 1)^{2l+1}(-\alpha_{-l-1} + \alpha_{l+1}e^{i\theta}) \\ &= 1 - (-2)^l(\alpha_{l+1} + \alpha_{-l-1})(1 - \cos\theta)^{l+1} \\ &\quad + i(-2)^l(\alpha_{l+1} - \alpha_{-l-1})(1 - \cos\theta)^l \sin\theta. \end{aligned}$$

Hence, we get the next theorem through a proof similar to that of Theorem 2.7.

THEOREM 3.4
Let α_{l+1} *and* α_{-l-1} *satisfy Theorem 3.3. We set* $X_t^\epsilon = \epsilon Z_{[t/\epsilon^{2l+1}]}$ *for any* $\epsilon > 0$. *Then* $X_t = \lim_{\epsilon \to 0} X_t^\epsilon$ *is the pseudostochastic process whose transition probability density function is*

$$q(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\bigl\{i(-1)^l(\alpha_{l+1} - \alpha_{-l-1})\mu^{2l+1}t - i\mu x\bigr\}\, d\mu.$$

This $q(t,x)$ *is the fundamental solution of the* $(2l+1)$*th-order diffusion equation*

$$\frac{\partial u}{\partial t} = (\alpha_{l+1} - \alpha_{-l-1})\frac{\partial^{2l+1} u}{\partial x^{2l+1}}. \tag{3.4}$$

REMARK 3.5
Clearly, taking $\alpha_{l+1} = A_{2l+1}$ and $\alpha_{-l-1} = 0$ we have $Z_n = Z_n^+$. Similarly, we have $Z_n = Z_n^-$ by taking $\alpha_{l+1} = 0$ and $\alpha_{-l-1} = A_{2l+1}$.

4. Some examples

In this section, we give a pseudorandom walk $\{Z_n\}$ for the cases $l = 1$ and $l = 2$.

EXAMPLE 4.1 ($2L+1=3$)

In this case, we take $l=1$. By Theorem 3.3,

$$\begin{cases} P(Z_{n+1}-Z_n=2)=\alpha_2, \\ P(Z_{n+1}-Z_n=1)=-3\alpha_2-\alpha_{-2}, \\ P(Z_{n+1}-Z_n=0)=1+3\alpha_2+3\alpha_{-2}, \\ P(Z_{n+1}-Z_n=-1)=-\alpha_2-3\alpha_{-2}, \\ P(Z_{n+1}-Z_n=-2)=\alpha_{-2}, \end{cases}$$

where $\alpha_{-2}\neq\alpha_2$, $-2^{-3}\leq\alpha_{-2}+\alpha_2<0$, and $\alpha_{-2}\alpha_2\geq-\frac{3}{20}(\alpha_{-2}+\alpha_2)^2$.

By Theorem 3.4 its corresponding diffusion equation is

$$\frac{\partial u}{\partial t}=(\alpha_2-\alpha_{-2})\frac{\partial^3 u}{\partial x^3}.$$

Its fundamental solution is

$$q(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-i(\alpha_2-\alpha_{-2})\mu^3 t-i\mu x\}\,d\mu.$$

In particular, setting $\alpha_{-2}=0$ and $\alpha_2=A_3$, we get the pseudorandom walk which appeared in [8]. This pseudorandom walk can be decomposed as

$$Z_n-Z_{n-1}=X_n^+,\qquad X_n^+=\xi_1^+ +\xi_2^+ +\xi_1^-.$$

We set $\{\xi_1^+,\xi_2^+\}$ as

$$P(\xi_1^+=1)=p_1,\qquad P(\xi_1^+=0)=1-p_1,$$
$$P(\xi_2^+=1)=p_2,\qquad P(\xi_2^+=0)=1-p_2,$$

and we set ξ_1^- as

$$P(\xi_1^-=0)=r_1,\qquad P(\xi_1^-=-1)=1-r_1,$$

where p_1, p_2, and r_1 are solutions of

$$\lambda^2(\lambda-1)=A_3\quad\text{for }-1/2^3\leq A_3<0$$

by (2.13).

EXAMPLE 4.2 ($2L+1=5$)

In this case, we take $l=2$. By Theorem 3.3,

$$\begin{cases} P(Z_{n+1}-Z_n=3)=\alpha_3, \\ P(Z_{n+1}-Z_n=2)=-5\alpha_3-\alpha_{-3}, \\ P(Z_{n+1}-Z_n=1)=10\alpha_3+5\alpha_{-3}, \\ P(Z_{n+1}-Z_n=0)=1-10\alpha_3-10\alpha_{-3}, \\ P(Z_{n+1}-Z_n=-1)=5\alpha_3+10\alpha_{-3}, \\ P(Z_{n+1}-Z_n=-2)=-\alpha_3-5\alpha_{-3}, \\ P(Z_{n+1}-Z_n=-3)=\alpha_{-3}, \end{cases}$$

where $\alpha_{-3}\neq\alpha_3$, $0<\alpha_{-3}+\alpha_3\leq 2^{-5}$, and $\alpha_{-3}\alpha_3\geq-\frac{5}{16}(\alpha_{-3}+\alpha_3)^2$.

By Theorem 3.4 its corresponding diffusion equation is

$$\frac{\partial u}{\partial t} = (\alpha_3 - \alpha_{-3}) \frac{\partial^5 u}{\partial x^5}.$$

Its fundamental solution is

$$q(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i(\alpha_3 - \alpha_{-3})\mu^5 t - i\mu x\}\, d\mu.$$

In particular, by setting $\alpha_{-3} = 0$ and $\alpha_3 = A_5$, this pseudorandom walk can be decomposed as

$$Z_{n+1} - Z_n = X_n^+, \qquad X_n^+ = \sum_{i=1}^{3} \xi_i^+ + \sum_{j=1}^{2} \xi_j^-.$$

We set $\{\xi_1^+, \xi_2^+, \xi_3^+\}$ as

$$\begin{aligned} P(\xi_1^+ = 1) &= v_1, & P(\xi_1^+ = 0) &= 1 - v_1, \\ P(\xi_2^+ = 1) &= v_2, & P(\xi_2^+ = 0) &= 1 - v_2, \\ P(\xi_3^+ = 1) &= p, & P(\xi_3^+ = 0) &= 1 - p, \end{aligned}$$

and we set $\{\xi_1^-, \xi_2^-\}$ as

$$\begin{aligned} P(\xi_1^- = 0) &= v_3, & P(\xi_1^- = -1) &= 1 - v_3, \\ P(\xi_2^- = 0) &= \bar{p}, & P(\xi_2^- = -1) &= 1 - \bar{p}, \end{aligned}$$

where v_1, v_2, v_3 are real solutions and $p, \bar{p}$ are complex solutions of

$$\lambda^3(\lambda - 1)^2 = A_5 \quad \text{for } 0 < A_5 \le 1/2^5.$$

5. A construction of a pseudorandom walk for an even-order heat-type equation

We construct a pseudorandom walk $\{Y_n\}$ which is approximate to the $2l$th-order pseudostochastic process $\{X_t\}$. This idea is basically due to Motoo [6].

We consider two pseudoindependent, identically distributed pseudorandom variables $\{\eta_j\}$ and $\{\tau_i\}$. The η_j's distribution is

$$P(\eta_j = 0) = 1 - 2p, \qquad P(\eta_j = \pm 1) = p,$$

and the τ_i's distribution is

$$P(\tau_i = 0) = r_0, \qquad P(\tau_i = 1) = r_1, \qquad \ldots, \qquad P(\tau_i = l) = r_l,$$

where p and r_k are real numbers and $\sum_{k=0}^{l} r_k = 1$.

Let $\eta_0 = x$ and $\tau_0 = 0$. We set $S_n = \sum_{j=0}^{n} \eta_j$ and $T_n = \sum_{i=0}^{n} \tau_i$. Define

$$Y_n = S_{T_n}.$$

The characteristic function of S_1 must be less than or equal to 1. That is,

$$\left|E_0[e^{i\theta S_1}]\right| = \left|1 - 2p(1 - \cos\theta)\right| \le 1.$$

Then p is satisfied with

$$0 < p \le 1/2.$$

The characteristic function of Y_1 is

$$\begin{aligned} E_x[e^{i\theta Y_1}] &= \sum_{j=0}^{l} E_x[e^{i\theta S_j} \mid T_1 = j] P(T_1 = j) \\ &= e^{i\theta x} \sum_{j=0}^{l} \big\{1 - 2p(1-\cos\theta)\big\}^j r_j. \end{aligned}$$

For $n \ge 1$, $\epsilon > 0$, we set $Y_n^\epsilon = \epsilon Y_n$ and $Y_0^\epsilon = x$. We shall take p and $\{r_j\}$ satisfying

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{1}{\epsilon^{2l}} \big(E_x[e^{i\theta Y_1^\epsilon}] - e^{i\theta x}\big) &= A_{2l} \frac{d^{2l}}{dx^{2l}} e^{i\theta x} \\ &= (-1)^l A_{2l} \theta^{2l} e^{i\theta x}. \end{aligned} \tag{5.1}$$

We consider $E[e^{i\theta Y_1^\epsilon}]$:

$$\begin{aligned} E[e^{i\theta Y_1^\epsilon}] &= \sum_{j=0}^{l} \big\{1 - 2p(1-\cos\theta\epsilon)\big\}^j r_j \\ &= \sum_{j=0}^{l} \sum_{k=0}^{j} \binom{j}{k} (-2p)^k (1-\cos\theta\epsilon)^k r_j \\ &= \sum_{k=0}^{l} (-2p)^k (1-\cos\theta\epsilon)^k \sum_{j=k}^{l} \binom{j}{k} r_j. \end{aligned}$$

Since $1 - \cos\theta\epsilon = \frac{1}{2}(\theta\epsilon)^2 + O(\epsilon^4)$ for sufficiently small ϵ, we have

$$E[e^{i\theta Y_1^\epsilon}] = \sum_{k=0}^{l} (-p)^k \big((\theta\epsilon)^{2k} + O(\epsilon^{4k})\big) \sum_{j=k}^{l} \binom{j}{k} r_j.$$

Therefore, we take p and $\{r_k\}$ that are satisfied with the equations

$$p^l r_l = A_{2l}, \tag{5.2}$$

$$\sum_{j=k}^{l} \binom{j}{k} (r_j - \delta_{j,0}) = 0, \quad \text{for } 0 \le k \le l-1. \tag{5.3}$$

Then we easily obtain

$$r_j = \delta_{j,0} + (-1)^{l+j} \binom{l}{j} p^{-l} A_{2l}, \quad 0 \le j \le l. \tag{5.4}$$

Then, we also get

$$M(\theta) = 1 + (-2)^l A_{2l} (1-\cos\theta)^l. \tag{5.5}$$

Clearly, we have the following result.

PROPOSITION 5.1

Suppose that A_{2l} satisfies

$$0 < (-1)^{l+1} A_{2l} \le 2^{-2l}. \tag{5.6}$$

Then we have the property $0 \le M(\theta) \le 1$ and $M(\theta)$ is decreasing in $[0,\pi]$.

Moreover, we can show the following theorem.

THEOREM 5.2

For $-l \le m \le l$,

$$P(Y_1 = m) = \delta_{m,0} + A_{2l} \sum_{k=0}^{l} \binom{l}{k} \binom{k}{\frac{k+m}{2}} (-2)^{l-k},$$

where $\binom{k}{\frac{k+m}{2}} = 0$ if $\frac{k+m}{2}$ is not an integer.

Proof

We have that

$$\begin{aligned} P(Y_1 = m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta} E[e^{i\theta Y_1}]\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta} \sum_{j=0}^{l} \big\{1 - 2p + p(e^{i\theta} + e^{-i\theta})\big\}^j r_j \, d\theta \\ &= \sum_{j=0}^{l} \sum_{k=0}^{j} \binom{j}{k} (1-2p)^{j-k} p^k r_j \\ &\quad \times \sum_{n=0}^{k} \binom{k}{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(2n-k-m)\theta}\, d\theta. \end{aligned}$$

Then we have

$$\begin{aligned} P(Y_1 = m) &= \sum_{j=0}^{l} \sum_{k=0}^{j} \binom{j}{k} \binom{k}{\frac{k+m}{2}} (1-2p)^{j-k} p^k r_j \\ &= \sum_{k=0}^{l} \binom{k}{\frac{k+m}{2}} p^k \sum_{j=k}^{l} \binom{j}{k} (1-2p)^{j-k} r_j. \end{aligned}$$

The inner sum is

$$\begin{aligned} &\sum_{j=k}^{l} \binom{j}{k} (1-2p)^{j-k} r_j \\ &\quad = \sum_{j=k}^{l} \binom{j}{k} (1-2p)^{j-k} \left(\delta_{j,0} + (-1)^{l+j} \binom{l}{j} p^{-l} A_{2l} \right) \end{aligned}$$

$$
\begin{aligned}
&= \delta_{k,0} + p^{-l} A_{2l} \binom{l}{k} \sum_{j=k}^{l} \binom{l-k}{j-k} (-1)^{l+j} (1-2p)^{j-k} \\
&= \delta_{k,0} + p^{-l} A_{2l} \binom{l}{k} (-2p)^{l-k} \\
&= \delta_{k,0} + p^{-k} A_{2l} (-2)^{l-k} \binom{l}{k}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
P(Y_1 = m) &= \sum_{k=0}^{l} \binom{k}{\frac{k+m}{2}} p^k \left(\delta_{k,0} + p^{-k} A_{2l} (-2)^{l-k} \binom{l}{k} \right) \\
&= \delta_{m,0} + A_{2l} \sum_{k=0}^{l} \binom{k}{\frac{k+m}{2}} \binom{l}{k} (-2)^{l-k}.
\end{aligned}
$$

□

THEOREM 5.3

Suppose that the condition (5.6) holds. We set $X_t^\epsilon = \epsilon Y_{[t/\epsilon^{2l}]}$ for any $\epsilon > 0$. Then $X_t = \lim_{\epsilon \to 0} X_t^\epsilon$ is a pseudostochastic process whose transition probability is

$$
q(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{(-1)^l A_{2l} \mu^{2l} t - i\mu x\} \, d\mu.
$$

This $q(t,x)$ is the fundamental solution of the $2l$th-order diffusion equation

$$
\frac{\partial u}{\partial t} = A_{2l} \frac{\partial^{2l} u}{\partial x^{2l}}. \tag{5.7}
$$

Proof

The proof is almost the same as the proof of Theorem 2.7, but is simpler than that. We use the same notation. Let $x = k\epsilon$, $t = n\epsilon^{2l}$. Then

$$
\begin{aligned}
p_\epsilon(t,x) &\equiv p(n,k) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} M(\theta)^{t/\epsilon^{2l}} e^{-ix\theta/\epsilon} \, d\theta \\
&= \frac{1}{2\pi} \Big(\int_{\delta < |\theta| \le \pi} + \int_{|\theta| < \delta} \Big) M(\theta)^{t/\epsilon^{2l+1}} e^{-i\theta x/\epsilon} \, d\theta \\
&= I_1 + I_2 \quad \text{(say)},
\end{aligned}
$$

where $\delta \equiv \pi \epsilon^{3/4}$. Then we shall prove that

$$
q(t,x) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} p_\epsilon(t,x)
$$

is the fundamental solution of (5.7).

First, clearly,

$$
q^+(0,x) = \delta(x),
$$

where $\delta(x)$ is Dirac's delta function. Let $t > 0$. We set $K \equiv (-1)^{l+1} 2^l A_{2l}$. We consider that

$$I_1 = \frac{1}{2\pi} \int_{\delta < |\theta| \le \pi} M(\epsilon\mu)^{t/\epsilon^{2l}} e^{-ix\theta/\epsilon} \, d\theta.$$

Since

$$M(\theta) = 1 - K(1 - \cos\theta)^l \le \exp\{-K(1 - \cos\theta)^l\},$$

we have

$$\begin{aligned} \frac{1}{\epsilon} I_1 &\le \frac{1}{\epsilon} \exp\{-tK(1 - \cos\delta)^l \epsilon^{-2l}\} \\ &\simeq \frac{1}{\epsilon} \exp\Big\{-tK\Big(\frac{\pi^2}{2}\Big)^l \epsilon^{-l/2}\Big\} \\ &\to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

Let ϵ be sufficiently small. We notice that

$$\begin{aligned} I_2 &= \frac{1}{2\pi} \int_{|\theta| \le \delta} M(\theta)^{t/\epsilon^{2l}} e^{-ix\theta/\epsilon} \, d\theta \\ &= \frac{1}{2\pi} \int_{|\mu| \le \pi\epsilon^{-1/4}} M(\epsilon\mu)^{t/\epsilon^{2l}} e^{-ix\mu} \, d\mu\epsilon. \end{aligned}$$

Then we have

$$M(\theta) = \exp\{-K(1 - \cos\theta)^l + R(\theta)\}$$

and

$$\big|R(\theta)\big| \le K^2 (1 - \cos\theta)^{2l}.$$

Therefore, we get

$$\begin{aligned} \big|R(\delta)\big| \epsilon^{-2l} &\le K^2 (1 - \cos\delta)^{2l} \epsilon^{-2l} \\ &\simeq K^2 \Big(\frac{\pi^2}{2}\Big)^{2l} \epsilon^l. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \frac{2\pi}{\epsilon} I_2 &= \int_{-\frac{\pi}{\epsilon^{1/4}}}^{\frac{\pi}{\epsilon^{1/4}}} \exp\Big\{-K\Big(\frac{1 - \cos\epsilon\mu}{\epsilon^2}\Big)^l t + O(\epsilon^l) - i\mu x\Big\} \, d\mu \\ &\to \int_{-\infty}^{\infty} \exp\{(-1)^l A_{2l} \mu^{2l} t - i\mu x\} \, d\mu \quad \text{as } \epsilon \to 0, \end{aligned}$$

which is absolutely convergent by Lebesgue's dominated convergence theorem. □

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