On the doubly Feller property of resolvent

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Abstract In this article, we show the stability of the doubly Feller property of the resolvent for Markov processes, generalizing a work by Kai-Lai Chung on the stability of the doubly Feller property of a semigroup with multiplicative functionals. The stability of the doubly Feller property of the resolvent under time change is also presented.

1. Introduction

Let (E,d) be a locally compact separable metric space, $E_\partial := E \cup \{\partial\}$ its one-point compactification, $\mathcal{B}(E)$ its Borel σ-field on E, and $\mathcal{B}(E_\partial)$ its Borel σ-field on E_∂. It is well known that $\mathcal{B}(E_\partial) = \mathcal{B}(E) \cup \{B \cup \{\partial\} \mid B \in \mathcal{B}(E)\}$. Any function f defined on E is extended to E_∂ by setting $f(\partial) = 0$. Denote by $\mathcal{B}_b(E)$ (resp., by $C_b(E)$) the family of bounded Borel functions on E (resp., the family of bounded continuous functions on E), and denote by $C_0(E)$ (resp., by $C_\infty(E)$) the family of continuous functions on E with compact support (resp., the family of continuous functions on E vanishing at infinity).

We consider a Hunt process $\mathbf{X} = (\Omega, \mathcal{F}_t, \mathcal{F}_\infty, X_t, \zeta, \mathbf{P}_x)_{x \in E_\partial}$ defined on E_∂ and denote by $(P_t)_{t \geq 0}$ (resp., $(R_\alpha)_{\alpha > 0}$) its transition semigroup (resp., its resolvent kernel), that is, $P_t f(x) = \mathbf{E}_x[f(X_t)] = \int_\Omega f(X_t(\omega)) \mathbf{P}_x(d\omega)$ (resp., $R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt$) for $f \in \mathcal{B}_b(E_\partial)$. Here $\zeta := \inf\{t \geq 0 \mid X_t = \partial\}$ is the lifetime of $\mathbf{X}$ and ∂ is a cemetery point of $\mathbf{X}$, that is, $X_t = \partial$ for all $t \geq \zeta$ under $\mathbf{P}_x$ for $x \in E$. The transition semigroup $(P_t)_{t \geq 0}$ of $\mathbf{X}$ is said to have the *Feller property* if the following two conditions are satisfied.

(i) For each $t > 0$ and $f \in C_\infty(E)$, we have $P_t f \in C_\infty(E)$.
(ii) For each $f \in C_\infty(E)$ and $x \in E$, we have $\lim_{t \to 0} P_t f(x) = f(x)$.

The resolvent $(R_\alpha)_{\alpha > 0}$ of $\mathbf{X}$ is said to have the *Feller property* if the following two conditions are satisfied.

(i)$'$ For each $\alpha > 0$ and $f \in C_\infty(E)$, we have $R_\alpha f \in C_\infty(E)$.
(ii)$'$ For each $f \in C_\infty(E)$ and $x \in E$, we have $\lim_{\alpha \to \infty} \alpha R_\alpha f(x) = f(x)$.

It is known that (i) and (ii) together imply the following.

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(iii) For each $f \in C_\infty(E)$, we have $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$.

Since

$$R_\beta f = \int_0^\infty e^{-\beta t} P_t f\, dt$$

and

$$P_t f = \lim_{\beta\to\infty} e^{-t\beta} \sum_{n=0}^\infty \frac{(t\beta)^n}{n!} (\beta R_\beta)^n f, \quad f \in C_\infty(E),$$

hold in $(C_\infty(E), \|\cdot\|_\infty)$, the condition (i) is equivalent to (i)$'$. It is easy to see that (ii) implies (ii)$'$. Conversely, the conditions (i)$'$ and (ii)$'$ imply (ii). Indeed, it is known that these conditions together imply the following.

(iii)$'$ For each $f \in C_\infty(E)$, we have $\lim_{\beta\to\infty} \|\beta R_\beta f - f\|_\infty = 0$.

From (iii)$'$, we have

$$\begin{aligned}\big|P_t f(x) - f(x)\big| &\le \big|P_t f(x) - \beta R_\beta P_t f(x)\big| + \big|\beta R_\beta P_t f(x) - f(x)\big| \\ &\le 2\|\beta R_\beta f - f\|_\infty \\ &\quad + \beta\Big|(e^{\beta t} - 1)\int_t^\infty e^{-\beta s} P_s f(x)\, ds - \int_0^t e^{-\beta s} P_s f(x)\, ds\Big|.\end{aligned}$$

Hence, $\overline{\lim}_{t\to 0} |P_t f(x) - f(x)| \le 2\|\beta R_\beta f - f\|_\infty \to 0$ as $\beta \to \infty$. Consequently, the Feller property of $(P_t)_{t\ge 0}$ is equivalent to the Feller property of $(R_\alpha)_{\alpha>0}$. So we can say that $\mathbf{X}$ has the *Feller property* if (i) and (ii) or (i)$'$ and (ii)$'$ hold.

The semigroup $(P_t)_{t\ge 0}$ is said to have the *strong Feller property* if

(iv) for each $f \in \mathcal{B}_b(E)$ and $t > 0$, we have $P_t f \in C_b(E)$.

The resolvent $(R_\alpha)_{\alpha>0}$ is said to have the *strong Feller property* if

(iv)$'$ for each $f \in \mathcal{B}_b(E)$ and $\alpha > 0$, we have $R_\alpha f \in C_b(E)$.

When E is compact, ∂ is an isolated point of E_∂; hence, any function f on E_∂ with $f(\partial) = 0$, which is continuous on E, belongs to $C_\infty(E)$. In this case, the strong Feller property of the semigroup (resp., the resolvent) implies (i) (resp., (i)$'$).

REMARK 1.1

It is well known that the strong Feller property of $(P_t)_{t\ge 0}$ implies the strong Feller property of $(R_\alpha)_{\alpha>0}$, but the converse assertion is not true. Indeed, the following semigroups are not strong Feller, but their resolvents enjoy strong Feller.

(1) The shift semigroup $(P_t)_{t\ge 0}$ on $\mathbb{R}$ defined by $P_t f(x) := f(x + t\ell)$, $x \in \mathbb{R}$, $\ell \in \mathbb{R}\setminus\{0\}$, does not enjoy the strong Feller property, but the resolvent enjoys the strong Feller property in view of $R_\alpha f(x) = \frac{e^{\frac{\alpha x}{\ell}}}{\ell}\int_x^\infty e^{-\frac{\alpha y}{\ell}} f(y)\, dy$ (resp., $R_\alpha f(x) = \frac{e^{\frac{\alpha x}{\ell}}}{-\ell}\int_{-\infty}^x e^{-\frac{\alpha y}{\ell}} f(y)\, dy$) for $f \in \mathcal{B}_b(\mathbb{R})$ and $\ell > 0$ (resp., $\ell < 0$).

(2) The semigroup $(P_t)_{t\ge 0}$ of space-time Brownian motion (B_t,t) under $\mathbf{P}_{(x,\tau)}:=\mathbf{P}_x^{(1)}\otimes\mathbf{P}_\tau^{(2)}$ for $(x,\tau)\in\mathbb{R}^2$ defined by

$$P_tf(x,\tau):=\mathbf{E}_{(x,\tau)}\big[f(B_t,t)\big]=\mathbf{E}_x^{(1)}\otimes\mathbf{E}_\tau^{(2)}\big[f(B_t,t)\big],\quad f\in\mathcal{B}_b(\mathbb{R}^2),$$

does not enjoy the strong Feller property, but the resolvent enjoys the strong Feller property. Here $\mathbf{P}_x^{(1)}$ is the law for 1-dimensional Brownian motion starting from x and $\mathbf{P}_\tau^{(2)}$ is the law for uniform motion to the right starting from τ with speed 1, that is, $\mathbf{E}_\tau^{(2)}[g(t)]=g(\tau+t)$. More generally, the product semigroup of a strong Feller semigroup and the semigroup of uniform motion to the right is not a strong Feller semigroup, but its resolvent enjoys the strong Feller property. Indeed, for $x_n\to x$, $\tau_n\to\tau$, $f\in\mathcal{B}_b(\mathbb{R}^2)$, we see that

$$\begin{aligned}
&\big|R_\alpha f(x_n,\tau_n)-R_\alpha f(x,\tau)\big|\\
&\quad\le|e^{\alpha\tau_n}-e^{\alpha\tau}|\int_{\tau_n}^\infty e^{-\alpha s}\mathbf{E}_{(x_n,0)}\big[|f|(B_{s-\tau_n},s)\big]\,ds\\
&\qquad+e^{\alpha\tau}\bigg|\int_{\tau_n}^\infty e^{-\alpha s}\mathbf{E}_{(x_n,0)}\big[f(B_{s-\tau_n},s)\big]\,ds-\int_\tau^\infty e^{-\alpha s}\mathbf{E}_{(x,0)}\big[f(B_{s-\tau},s)\big]\,ds\bigg|\\
&\quad\le|e^{\alpha\tau_n}-e^{\alpha\tau}|\|f\|_\infty/\alpha\\
&\qquad+e^{\alpha\tau}\int_0^\infty e^{-\alpha s}\big|\mathbf{1}_{]\tau_n,\infty[}(s)P_{s-\tau_n}^{(1)}\big(f(\cdot,s)\big)(x_n)-\mathbf{1}_{]\tau,\infty[}(s)P_{s-\tau}^{(1)}\big(f(\cdot,s)\big)(x)\big|\,ds\\
&\quad\to 0\quad\text{as } n\to\infty.
\end{aligned}$$

Here we use that the strong Feller property of the semigroup $(P_t^{(1)})_{t\ge0}$ of 1-dimensional Brownian motion implies the continuity of $]0,\infty[\times\mathbb{R}\ni(t,x)\mapsto P_t^{(1)}g(x)$ for any $g\in\mathcal{B}_b(\mathbb{R})$. On the other hand, $P_tf(x,\tau)=P_t^{(1)}f_1(x)f_2(\tau+s)$ for $f=f_1\otimes f_2$, $f_i\in\mathcal{B}_b(\mathbb{R})$ $(i=1,2)$, does not enjoy $P_tf\in C_b(\mathbb{R}^2)$ for $f_2\notin C_b(\mathbb{R})$.

Under (iv)$'$, we see that $R_1 1\in C_\infty(E)$ implies (i)$'$. Moreover, under (iv), $R_1 1\in C_\infty(E)$ implies (i) (see [1, Proposition 1]). The semigroup $(P_t)_{t\ge0}$ or $\mathbf{X}$ is said to have the *doubly Feller property* if it enjoys both the Feller property and the strong Feller property. The Hunt process $\mathbf{X}$ is said to have the *doubly resolvent Feller property* if its resolvent enjoys both the Feller property and the strong Feller property. $\mathbf{X}$ is said to be a *Feller process* (resp., *strong Feller process*, *doubly Feller process*) if it enjoys the Feller property (resp., strong Feller property, doubly Feller property). $\mathbf{X}$ is said to be a *resolvent strong Feller process* (resp., *doubly resolvent Feller process*) if it enjoys the resolvent strong Feller property (resp., doubly resolvent Feller property).

In [4], the stability of the doubly Feller property of a semigroup (of a part process) with multiplicative functionals was proved. In [2], another criterion for the stability was presented, and its stability was discussed under Feynman–Kac and Girsanov transformations. In this article, we show that the same conditions as in [4] also remain valid for the stability of the doubly Feller property of the resolvent of a part process (see Section 3), and with multiplicative functionals (see Sections 4, 5). The same conditions as in [2] also remain valid for the stability of the doubly Feller property of the resolvent of a part process with multiplicative functionals (see Section 6). Moreover, we prove that the stability of the (doubly) Feller property of the resolvent under time change holds in the framework of symmetric Markov processes (see Section 7).

We close this section with the following convention: for $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

2. Preliminary lemmas

For each $B \in \mathcal{B}(E)$, denote by σ_B the *hitting time to* B, $\sigma_B := \inf\{t > 0 \mid X_t \in B\}$, and denote by τ_B the *first exit time from* B, $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$. Note that $\tau_B = \sigma_{E \setminus B} \wedge \zeta$. It is known that, for each $t > 0$, $\{\sigma_B < t\} \in \mathcal{F}_t$, and the function $x \mapsto \mathbf{P}_x(\sigma_B < t)$ is universally measurable and is denoted by $\mathcal{B}^*(E)$. The following lemma is a counterpart of [4, Lemma 1] and its proof is similar to that of [4, Lemma 1].

LEMMA 2.1 (CF. [4, LEMMA 1])

If $\mathbf{X}$ has the resolvent strong Feller property, then for each $f \in \mathcal{B}_b^*(E)$ and $\alpha > 0$, we have $R_\alpha f \in C_b(E)$.

The following result is proved in [4, Lemma 2] (see [5] or see [3, p. 73, Exercise 2]). Though the treatment of the first exit time τ_B in [4] is slightly different from ours, its proof remains valid.

LEMMA 2.2 (SEE [4, LEMMA 2])

Let $\mathbf{X}$ be a Feller process. For each nonempty open subset B of E and its compact subset K of B, we have

$$\lim_{t \to 0} \sup_{x \in K} \mathbf{P}_x(\tau_B \le t) = 0; \qquad \textit{in particular}, \qquad \lim_{t \to 0} \sup_{x \in K} \mathbf{P}_x(\zeta \le t) = 0.$$

Proof

Since $\tau_B = \sigma_{E \setminus B} \wedge \zeta$ and $\lim_{t \to 0} \sup_{x \in K} \mathbf{P}_x(\sigma_{E \setminus B} \le t) = 0$ holds for any compact subset K of B by [5, (2.5) Lemma], it suffices to prove only the latter assertion. By the Feller property of $(P_t)_{t>0}$, for each $f \in C_\infty(E)$ and $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$\sup_{t \le t_0} \|P_t f - f\|_\infty < \varepsilon. \tag{2.1}$$

Choose $f \in C_0(E)$ with $0 \le f \le 1$ on E and $f = 1$ on K. Since $P_t f \le 1$, (2.1) yields

$$\inf_{t \le t_0} \inf_{x \in K} P_t f(x) > 1 - \varepsilon. \tag{2.2}$$

By (2.2), we have $1 - \varepsilon < \mathbf{E}_x[f(X_{t_0}) : t_0 < \zeta] \le \mathbf{P}_x(t_0 < \zeta)$; hence, $\sup_{x \in K} \mathbf{P}_x(\zeta \le t_0) < \varepsilon$. ∎

The following lemma is a counterpart of [4, Lemma 3].

LEMMA 2.3 (CF. [4, LEMMA 3])

Suppose that $\mathbf{X}$ *enjoys the strong Feller property of the resolvent. Then for each* $B\in\mathcal{B}(E_\partial)$ *and* $\alpha>0$, $x\mapsto\mathbf{E}_x[\int_0^{\sigma_B}e^{-\alpha t}\,dt]$ *is upper semicontinuous on* E.

Proof

It follows that

$$\begin{aligned}&\alpha\mathbf{E}_x\Big[\int_0^{\sigma_B}e^{-\alpha t}\,dt\Big]\\&\quad=\begin{cases}1-\alpha\uparrow\lim_{\beta\uparrow\infty}\beta R_{\beta+\alpha}\phi_B^\alpha(x) & \text{if } \partial\notin B,\\ 1+\alpha\downarrow\lim_{\beta\uparrow\infty}\beta R_{\beta+\alpha}(\phi_B^\alpha(\partial)-\phi_B^\alpha)(x)-\alpha\phi_B^\alpha(\partial) & \text{if } \partial\in B,\end{cases}\end{aligned}$$

where $\phi_B^\alpha(x):=\mathbf{E}_x[\int_{\sigma_B}^\infty e^{-\alpha t}\,dt]$ is a bounded universally measurable function on E_∂. Note that $\partial\notin B$ (resp., $\partial\in B$) implies $\phi_B^\alpha(\partial)=0$ (resp., $\phi_B^\alpha(\partial)=\mathbf{E}_\partial[\int_0^\infty e^{-\alpha t}\,dt]=\frac{1}{\alpha}$). If $\partial\in B$, then $\phi_B^\alpha(\partial)-\phi_B^\alpha$ is a bounded nonnegative universally measurable function on E vanishing at infinity. Then we can obtain the assertion by Lemma 2.1. □

The following theorem seems to be unknown.

THEOREM 2.1

Let $\mathbf{X}$ *be a Hunt process that has the doubly resolvent Feller property. Then for each nonempty open subset* B *of* E *and* $\alpha>0$, $x\mapsto\mathbf{E}_x[e^{-\alpha\tau_B}]$ *and* $x\mapsto\mathbf{E}_x[e^{-\alpha\sigma_{E\setminus B}}]$ *belong to* $C_b(B)$, *and* $x\mapsto\mathbf{E}_x[e^{-\alpha\zeta}]$ *belongs to* $C_b(E)$.

Proof

It suffices to prove $\psi_B^\alpha:=\mathbf{E}_\cdot[e^{-\alpha\tau_B}]\in C_b(B)$. The proofs for the other cases are similar. First we prove that for each compact subset K of E

$$\lim_{\beta\to\infty}\sup_{x\in K}\psi_B^\beta(x)=0. \tag{2.3}$$

Indeed, by Lemma 2.2 and

$$\psi_B^\beta(x)=\mathbf{E}_x[e^{-\beta\tau_B}]=\mathbf{E}_x\Big[\int_{\tau_B}^\infty\beta e^{-\beta t}\,dt\Big]=\beta\int_0^\infty e^{-\beta t}\mathbf{P}_x(\tau_B\le t)\,dt,$$

we have (2.3). From (2.3) and

$$\begin{aligned}nR_{n+\alpha}\psi_B^\alpha(x)&=n\mathbf{E}_x\Big[\int_0^{\tau_B}e^{-nt}e^{-\alpha\tau_B}\,dt\Big]+n\mathbf{E}_x\Big[\int_{\tau_B}^\infty e^{-nt}e^{-\alpha(t+\tau_B\circ\theta_t)}\,dt\Big]\\&=\psi_B^\alpha(x)-\psi_B^{\alpha+n}(x)+n\mathbf{E}_x\Big[\int_{\tau_B}^\infty e^{-nt}e^{-\alpha(t+\tau_B\circ\theta_t)}\,dt\Big],\end{aligned}$$

we have

$$\sup_{x\in K}\left|\psi_B^\alpha(x) - nR_{n+\alpha}\psi_B^\alpha(x)\right| \le \sup_{x\in K}\psi_B^{\alpha+n}(x) + \sup_{x\in K} n\mathbf{E}_x\Big[\int_{\tau_B}^\infty e^{-nt}e^{-\alpha(t+\tau_B\circ\theta_t)}\,dt\Big]$$
$$\le \sup_{x\in K}\psi_B^{\alpha+n}(x) + \sup_{x\in K}\psi_B^n(x) \to 0 \quad \text{as } n\to\infty.$$

Therefore, we obtain the assertion by Lemma 2.1. □

3. Resolvent strong Feller property of the part of processes

Let B be an open subset of E with $B \neq E$, and let $B_\partial := B\cup\{\partial\}$. Define

$$X_t^B := \begin{cases} X_t & \text{if } t < \tau_B,\\ \partial & \text{if } t \ge \tau_B.\end{cases}$$

The process $\mathbf{X}^B = (\Omega, X_t^B, \mathbf{P}_x)$ is called "the process $\mathbf{X}$ killed outside B" or "the part of $\mathbf{X}$ on B." Its state space is B_∂, and its transition semigroup $(P_t^B)_{t\ge0}$ is given by

$$\text{(3.1)}\qquad \begin{cases} P_t^B(x,A) := \mathbf{P}_x(X_t\in A, t<\tau_B) & \text{if } x\in B \text{ and } A\in\mathcal{B}(E),\\ P_t^B(x,\{\partial\}) := 1 - P_t^B(x,E) & \text{if } x\in B, P_t^B(\partial,\{\partial\}) := 1.\end{cases}$$

If $\mathbf{X}$ is a Hunt process, it can be verified that $\mathbf{X}^B$ is also a Hunt process. We denote by $\mathcal{B}_b(B)$, $\mathcal{B}_b^*(B)$, $C_b(B)$ the indicated classes of functions restricted to B.

The following theorem is the main result of this section.

THEOREM 3.1 (CF. [4, THEOREM 1])

Suppose that the Hunt process $\mathbf{X}$ has the doubly resolvent Feller property. Let B be a nonempty proper open subset of E. Then $\mathbf{X}^B$ has the resolvent strong Feller property.

Proof

It suffices to prove that, for $f\in\mathcal{B}_b^+(B)$, $\phi_B^\alpha(x) := \alpha\mathbf{E}_x[\int_{\tau_B}^\infty e^{-\alpha t}f(X_t)\,dt]$ belongs to $C_b(B)$. The proofs for other cases are similar. From (2.3) and

$$\begin{aligned} nR_{n+\alpha}\phi_B^\alpha(x) &= n\alpha\mathbf{E}_x\Big[\int_0^{\tau_B} e^{-nt}\int_{\tau_B}^\infty e^{-\alpha u}f(X_u)\,du\,dt\Big]\\ &\quad + n\alpha\mathbf{E}_x\Big[\int_{\tau_B}^\infty e^{-nt}\int_{t+\tau_B\circ\theta_t}^\infty e^{-\alpha u}f(X_u)\,du\,dt\Big]\\ &= \phi_B^\alpha(x) - \alpha\mathbf{E}_x\Big[e^{-n\tau_B}\int_{\tau_B}^\infty e^{-\alpha u}f(X_u)\,du\Big]\\ &\quad + n\alpha\mathbf{E}_x\Big[\int_{\tau_B}^\infty e^{-nt}\int_{t+\tau_B\circ\theta_t}^\infty e^{-\alpha u}f(X_u)\,du\,dt\Big],\end{aligned}$$

we have for each compact set K of B

$$\sup_{x\in K}\left|\phi_B^\alpha(x) - nR_{n+\alpha}\phi_B^\alpha(x)\right| \le 2\|f\|_\infty\sup_{x\in K}\psi_B^n(x)\to 0 \quad\text{as } n\to\infty.$$

Therefore, we obtain the assertion by Theorem 2.1. □

4. Doubly resolvent Feller property of the part of processes

Let $\partial^* B$ be the boundary of B in E_∂. Note that $\partial^* B = \partial B \cup \{\partial\}$ holds; in particular, $\partial B = \partial^* B \cap E$, where $\partial B := \overline{B} \setminus B$ is the usual boundary of B in E. When f is defined only on B, we extend it to E_∂ by setting it to be zero outside B. We set $C_\infty(B) := \{f \in C(B) \mid \lim_{x\to z\in\partial^* B} f(x) = 0\}$. The open set B is said to be *regular* if, for each $z \in \partial^* B \cap E$, we have $\mathbf{P}_z(\sigma_{E\setminus B} = 0) = 1$.

The third main result of this article is the following.

THEOREM 4.1 (CF. [4, THEOREM 1])

Suppose that the Hunt process $\mathbf{X}$ has the doubly resolvent Feller property. Let B be a nonempty proper regular open subset of E. Then $\mathbf{X}^B$ also has the doubly resolvent Feller property. If further B is relatively compact, then for each $\alpha > 0$ and $f \in \mathcal{B}_b(B)$, we have $R_\alpha^B f \in C_\infty(B)$.

Proof

Define $R_\alpha^B f(x) := \mathbf{E}_x[\int_0^{\tau_B} e^{-\alpha t} f(X_t)\,dt]$ for $f \in \mathcal{B}_b(B)$. It suffices to prove $R_\alpha^B f \in C_\infty(B)$ for $f \in C_\infty(B)$ and $\alpha R_\alpha^B f(x) \to f(x)$ as $\alpha \to \infty$ for $f \in C_\infty(B)$ and $x \in B$. Since $\partial^* B$ may contain ∂, let us first consider this case. We then have, as $x \in B$, $x \to \partial$,

$$R_\alpha^B |f|(x) \le R_\alpha |f|(x) \to 0,$$

since $f \in C_\infty(B)$ can be regarded as a function in $C_\infty(E)$. On the other hand, if $x \to z \in \partial^* B \cap E$, then we have by Lemma 2.3 and the regularity assumption

$$\varlimsup_{x\to z} R_\alpha^B 1(x) = \varlimsup_{x\to z} \mathbf{E}_x\Big[\int_0^{\tau_B} e^{-\alpha t}\,dt\Big] \le \mathbf{E}_z\Big[\int_0^{\tau_B} e^{-\alpha t}\,dt\Big] = 0, \tag{4.1}$$

and consequently, as $x \in B$, $x \to z$, $|R_\alpha^B f(x)| \le \|f\|_\infty \mathbf{E}_x[\int_0^{\tau_B} e^{-\alpha t}\,dt] \to 0$. Thus, we have $R_\alpha^B f \in C_\infty(B)$ by Theorem 3.1. Since $\mathbf{P}_x(\tau_B > 0) = 1$ for $x \in B$, we have that for $x \in B$

$$\begin{aligned}\big|\alpha R_\alpha f(x) - \alpha R_\alpha^B f(x)\big| &\le \alpha \mathbf{E}_x\Big[\int_{\tau_B}^\infty e^{-\alpha t}\big|f(X_t)\big|\,dt\Big] \\ &\le \mathbf{E}_x[e^{-\alpha\tau_B}]\|f\|_\infty \to 0 \quad \text{as } \alpha \to \infty.\end{aligned}$$

The Feller property of $\mathbf{X}$ implies $\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = f(x)$; hence,

$$\lim_{\alpha\to\infty} \alpha R_\alpha^B f(x) = f(x).$$

Finally, $R_\alpha^B 1 \in C_\infty(B)$ under the relative compactness of B is already shown in (4.1). In this case, $\partial \notin \partial^* B = \partial B$. From $R_\alpha^B 1 \in C_\infty(B)$ and Theorem 3.1, we obtain $R_\alpha^B f \in C_\infty(B)$ for $f \in \mathcal{B}_b(B)$ under the relative compactness of B. □

Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L^2(E;\mathfrak{m})$, where $\mathfrak{m}$ is a positive Radon measure with full topological support. Let B be an open subset of E. The part space $(\mathcal{E}_B, \mathcal{F}_B)$, defined by $\mathcal{F}_B := \{u \in \mathcal{F} \mid u = 0 \text{ q.e. on } E \setminus B\}$ and $\mathcal{E}_B(u,v) := \mathcal{E}(u,v)$ for $u, v \in \mathcal{F}_B$, is a regular Dirichlet form on $L^2(B;\mathfrak{m})$.

COROLLARY 4.1

Suppose that $\mathbf{X}$ *is an* $\mathfrak{m}$*-symmetric Hunt process associated to a regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $L^2(E;\mathfrak{m})$ *and that it has the doubly resolvent Feller property. Let* B *be a nonempty regular open proper subset of* E*, which is connected and relatively compact. Then, the embedding* $\mathcal{F}_B$ *in* $L^2(B;\mathfrak{m})$ *is compact; equivalently, the semigroup* $(T_t^B)_{t\geq 0}$ *associated to* $(\mathcal{E}_B,\mathcal{F}_B)$ *on* $L^2(B;\mathfrak{m})$ *is a compact operator on* $L^2(B;\mathfrak{m})$.

For the proof of Corollary 4.1, we need the following lemma.

LEMMA 4.1

Suppose that $\mathbf{X}$ *is an* $\mathfrak{m}$*-symmetric Hunt process associated to a regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $L^2(E;\mathfrak{m})$*, and suppose it has the resolvent strong Feller property. Here* $\mathfrak{m}$ *is a positive Radon measure with full topological support. Assume* E *is connected. Then* $\mathbf{X}$ *is* $\mathfrak{m}$*-irreducible, that is, any invariant set* B *satisfies* $\mathfrak{m}(B)=0$ *or* $\mathfrak{m}(E\setminus B)=0$.

Proof

Suppose that B is invariant in the sense that $R_\alpha \mathbf{1}_B u=\mathbf{1}_B R_\alpha u$ $\mathfrak{m}$-a.e. for $u\in L^2(E;\mathfrak{m})$ and $\alpha>0$. By taking a bounded strictly positive $u\in L^2(E;\mathfrak{m})$, $\mathbf{1}_B$ has a bounded continuous $\mathfrak{m}$-version f. Since $\mathfrak{m}$ has full topological support, we get $f^2=f$. Then the set $O:=\{x\in E\mid f(x)=1\}$ is an open and closed set which is an $\mathfrak{m}$-version of B. Owing to the connectedness of E, we see $O=\emptyset$ or $O=E$, implying $\mathfrak{m}(B)=0$ or $\mathfrak{m}(E\setminus B)=0$. □

Proof of Corollary 4.1

Let B be a regular open proper subset of E which is connected and relatively compact. By Theorem 4.1, the $\mathfrak{m}$-symmetric Hunt process $\mathbf{X}^B$ associated to $(\mathcal{E}_B,\mathcal{F}_B)$ on $L^2(B;\mathfrak{m})$ is a doubly resolvent Feller process satisfying $R_\alpha^B 1\in C_\infty(B)$. By Lemma 4.1, $\mathbf{X}^B$ is $\mathfrak{m}$-irreducible. Thus, $\mathbf{X}^B$ belongs to the class (T) in [12]. Here a symmetric Markov process $\mathbf{X}$ is said to be in class (T) if it is an $\mathfrak{m}$-irreducible resolvent strong Feller process and satisfies $R_\alpha 1\in C_\infty(E)$. Applying [12, Theorem 4.4 and Corollary 4.1], we obtain the conclusion. □

5. Doubly resolvent Feller property with multiplicative functionals, I

Let $(Z_t)_{t\geq 0}$ be a multiplicative functional associated with $\mathbf{X}$. Namely, for each $x\in E$, $\mathbf{P}_x$-a.s., $Z_0=1$, $0\leq Z_t<\infty$, $Z_t\in\mathfrak{F}_t$ for $t\geq 0$, and

$$Z_{t+s}=Z_s\cdot(Z_t\circ\theta_s),\quad \text{for all } t,s\geq 0. \tag{5.1}$$

We now impose a set of special conditions on $(Z_t)_{t\geq 0}$ as follows.

(a) For some $t>0$, $a_t:=\sup_{x\in E}\sup_{s\in[0,t]}\mathbf{E}_x[Z_s]<\infty$.

It follows from this, condition (5.1), and the Markov property that $(a_t)_{t\geq 0}$ is submultiplicative, that is, $a_{t+s}\leq a_t\cdot a_s$ for $t,s\in[0,\infty[$. Indeed, since $(a_t)_{t\geq 0}$ is increasing and $a_t \ge \mathbf{E}_x[Z_0] = 1$ for any $t \ge 0$,

$$a_{t+s} = \sup_{x\in E}\sup_{u\in[0,t+s]} \mathbf{E}_x[Z_u] = \sup_{x\in E}\sup_{u\in[0,s]} \mathbf{E}_x[Z_u] \vee \sup_{x\in E}\sup_{u\in[s,t+s]} \mathbf{E}_x[Z_u]$$
$$= a_s \vee \sup_{x\in E}\sup_{u\in[0,t]} \mathbf{E}_x\big[\mathbf{E}_{X_s}[Z_u]Z_s\big] \le a_s \vee (a_s \cdot a_t) \le a_s \cdot a_t.$$

Hence, by induction, (a) is in fact true for all (finite) $t > 0$. The submultiplicativity of $(a_t)_{t\ge 0}$ implies that $t \mapsto \log a_t^{1/t}$ is decreasing and $a_t \le a_{t_0}^{t/t_0}$ for any $t \ge t_0$ with any given $t_0 > 0$. Hence, for $\alpha > \alpha_0 := \inf_{s\in]0,\infty[} \log a_s^{1/s} \ge 0$ and taking $t_0 > 0$ with $\alpha > \log a_{t_0}^{1/t_0}$, we see that

$$\int_0^\infty e^{-\alpha t} a_t\, dt \le \int_0^{t_0} e^{-\alpha t} a_t\, dt + \int_{t_0}^\infty e^{-\alpha t} a_{t_0}^{t/t_0}\, dt < \infty.$$

(b) For each $t > 0$, there exists a number $p = p(t) > 1$ such that $\sup_{x\in E} \mathbf{E}_x[Z_t^p] < \infty$.

(c)^w For each compact subset K of E, we have

$$\lim_{t\to 0}\sup_{x\in K} \mathbf{E}_x\big[|Z_t - 1|\big] = 0.$$

(c)^s $\lim_{t\to 0}\sup_{x\in E} \mathbf{E}_x[|Z_t - 1|] = 0$.

REMARK 5.1

(1) The condition (c)^s is stronger than (c)^w.

(2) The condition (c)^s implies (a). Indeed, under (c)^s, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{t\in[0,\delta]}\sup_{x\in E} \mathbf{E}_x[|Z_t - 1|] \le \varepsilon$. Then we have $\sup_{t\in[0,\delta]}\sup_{x\in E} \mathbf{E}_x[|Z_t|] \le \varepsilon + 1 < \infty$, which shows (a).

Now we proceed to extend the results in the previous section to a process with multiplicative functionals. We begin by defining $(Q_t)_{t\ge 0}$ as follows: for $f \in \mathcal{B}_b(E)$,

$$Q_t f(x) := \mathbf{E}_x\big[Z_t f(X_t)\big].$$

By means of (5.1), we can verify that $(Q_t)_{t\ge 0}$ forms a semigroup, not necessarily sub-Markovian. But we have, for each $t > 0$,

$$\|Q_t f\|_\infty \le \big(\sup_{x\in E} \mathbf{E}_x[Z_t]\big)\|f\|_\infty \le a_t\|f\|_\infty$$

by (a), so that each Q_t maps $\mathcal{B}_b(E)$ into $\mathcal{B}_b(E)$. For $\alpha > \alpha_0 := \inf_{s\in]0,\infty[} \log a_s^{1/s}$ and $f \in \mathcal{B}_b(E)$, we set

$$S_\alpha f(x) := \mathbf{E}_x\Big[\int_0^\infty e^{-\alpha t} Z_t f(X_t)\, dt\Big]$$

and call $(S_\alpha)_{\alpha>\alpha_0}$ the *resolvent of the transformed process* from $\mathbf{X}$ by the multiplicative functional $(Z_t)_{t\ge 0}$.

THEOREM 5.1 (CF. [4, THEOREM 2])

Suppose that $\mathbf{X}$ has the resolvent strong Feller property and that conditions (a) and (c)^w hold. Then, for $f \in \mathcal{B}_b(E)$ and $\alpha > \alpha_0$, $S_\alpha f \in C_b(E)$.

Proof
Fix a compact set K, and take $x \in K$. Since

$$\begin{aligned}
&\big|S_\alpha f(x) - \beta R_{\beta+\alpha} S_\alpha f(x)\big| \\
&\quad \le \big|S_\alpha f(x) - \beta S_{\beta+\alpha} S_\alpha f(x)\big| + \big|\beta S_{\beta+\alpha} S_\alpha f(x) - \beta R_{\beta+\alpha} S_\alpha f(x)\big| \\
&\quad \le \Big|\beta \int_0^\infty e^{-\beta s}\big(S_\alpha f(x) - \mathbf{E}_x\big[e^{-\alpha s} Z_s S_\alpha f(X_s)\big]\big)\, ds\Big| \\
&\qquad + \beta \mathbf{E}_x\Big[\int_0^\infty e^{-(\beta+\alpha)s} |Z_s - 1| S_\alpha |f|(X_s)\, ds\Big] \\
&\quad \le \beta \int_0^\infty e^{-\beta s}\Big|S_\alpha f(x) - \mathbf{E}_x\Big[\int_s^\infty e^{-\alpha t} Z_t f(X_t)\, dt\Big]\Big|\, ds \\
&\qquad + \beta \int_0^\infty e^{-(\beta+\alpha)s} \mathbf{E}_x\big[|Z_s - 1|\big]\, ds \cdot \|S_\alpha f\|_\infty \\
&\quad \le \beta \int_0^\infty e^{-\beta s} \mathbf{E}_x\Big[\int_0^s e^{-\alpha t} Z_t |f|(X_t)\, dt\Big]\, ds \\
&\qquad + \beta \int_0^\infty e^{-(\beta+\alpha)s} \sup_{x\in K} \mathbf{E}_x\big[|Z_s - 1|\big]\, ds \Big(\int_0^\infty e^{-\alpha t} a_t\, dt\Big) \|f\|_\infty \\
&\quad \le \beta \int_0^\infty e^{-\beta s}\Big(\int_0^s e^{-\alpha t} a_t\, dt\Big)\, ds \cdot \|f\|_\infty \\
&\qquad + \beta \int_0^\infty e^{-\beta s} \sup_{x\in K} \mathbf{E}_x\big[|Z_s - 1|\big]\, ds \Big(\int_0^\infty e^{-\alpha t} a_t\, dt\Big) \|f\|_\infty,
\end{aligned}$$

we have that under (c)^w and $\int_0^\infty e^{-\alpha t} a_t\, dt < \infty$ for $\alpha > \alpha_0$

$$\sup_{x\in K}\big|S_\alpha f(x) - \beta R_{\beta+\alpha} S_\alpha f(x)\big| \to 0 \quad \text{as } \beta \to \infty.$$

This implies the assertion. □

COROLLARY 5.1 (CF. [2, COROLLARY 1.2])

Suppose that $\mathbf{X}$ *has the resolvent strong Feller property and that condition (c)*^{s} *holds. Then, for* $f \in \mathcal{B}_b(E)$ *and* $\alpha > \alpha_0$, $S_\alpha f \in C_b(E)$.

THEOREM 5.2 (CF. [4, THEOREM 2])

Suppose that $\mathbf{X}$ *has the doubly resolvent Feller property, and suppose conditions (a), (b), and (c)*^{w} *hold. Then, for* $f \in C_\infty(E)$ *and* $\alpha > \alpha_0$, $S_\alpha f \in C_\infty(E)$ *and* $\lim_{\alpha\to\infty} \alpha S_\alpha f(x) = f(x)$ *for* $f \in C_\infty(E)$ *and* $x \in E$.

Proof
Under the Feller property of $\mathbf{X}$, we can show that $\lim_{x\to\partial} Q_t f(x) = 0$, for $f \in C_\infty(E)$ and $t > 0$, holds under (b) and that $\lim_{t\to 0} Q_t f(x) = f(x)$, for $f \in C_\infty(E)$, holds under (c)^w. Since $Q_t|f|(x) \le a_t \|f\|_\infty$ for $x \in E_\partial$ and $f \in C_\infty(E)$ and since $e^{-\alpha t} a_t$ is integrable on $[0,\infty[$ for $\alpha > \alpha_0$, we can apply the dominated convergence

theorem so that

$$\lim_{x\to\partial} S_\alpha|f|(x) = \lim_{x\to\partial}\int_0^\infty e^{-\alpha t}Q_t|f|(x)\,dt = \int_0^\infty e^{-\alpha t}\lim_{x\to\partial} Q_t|f|(x)\,dt = 0.$$

Then we conclude that $S_\alpha f \in C_\infty(E)$ for $\alpha > \alpha_0$ by Theorem 5.1. Next we show the latter assertion. For $\varepsilon > 0$, there is $\delta > 0$ such that $|Q_t f(x) - f(x)| < \varepsilon$ for any $t \in [0,\delta]$. Then, for $\alpha > \log a_\delta^{1/\delta}$ we have

$$\begin{aligned}
\bigl|\alpha S_\alpha f(x) - f(x)\bigr| &= \alpha\int_0^\delta e^{-\alpha t}\bigl|Q_t f(x) - f(x)\bigr|\,dt \\
&\qquad + \alpha\int_\delta^\infty e^{-\alpha t}\bigl|Q_t f(x) - f(x)\bigr|\,dt \\
&\le \varepsilon + \alpha\int_\delta^\infty e^{-\alpha t}(a_t + 1)\,dt\cdot\|f\|_\infty \\
&\le \varepsilon + \alpha\int_\delta^\infty e^{-\alpha t}(a_\delta^{t/\delta} + 1)\,dt\cdot\|f\|_\infty \\
&= \varepsilon + \alpha\Bigl(\frac{e^{-(\alpha-\log a_\delta^{1/\delta})\delta}}{\alpha - \log a_\delta^{1/\delta}} + \frac{e^{-\alpha\delta}}{\alpha}\Bigr)\|f\|_\infty.
\end{aligned}$$

Thus, we obtain $\overline{\lim}_{\alpha\to\infty}|\alpha S_\alpha f(x) - f(x)| \le \varepsilon$. $\square$

COROLLARY 5.2

Suppose that $\mathbf{X}$ has the doubly resolvent Feller property, and suppose conditions (b) and (c)^s hold. Then, for $f \in C_\infty(E)$ and $\alpha > \alpha_0$, $S_\alpha f \in C_\infty(E)$ and $\lim_{\alpha\to\infty}\alpha S_\alpha f(x) = f(x)$ for $f \in C_\infty(E)$ and $x \in E$.

Combining Theorems 4.1 and 5.2, we obtain the following result.

THEOREM 5.3 (CF. [4, THEOREM 3])

Let $\mathbf{X}$ be a doubly resolvent Feller process, let B be as in Theorem 4.1, and let $(Z_t)_{t\ge 0}$ be as in Theorem 5.2. Define for $x \in E$, $\alpha > \alpha_0$, and $f \in \mathcal{B}_b(B)$

$$S_\alpha^B f(x) := \mathbf{E}_x\Bigl[\int_0^{\tau_B} e^{-\alpha t}Z_t f(X_t)\,dt\Bigr].$$

Then we have $S_\alpha^B f \in C_b(B)$ for $f \in \mathcal{B}_b(B)$, and we have $S_\alpha^B f \in C_\infty(B)$ and $\lim_{\alpha\to\infty} S_\alpha^B f(x) = f(x)$ for $f \in C_\infty(B)$ and $x \in B$. Moreover, if B is relatively compact, then $S_\alpha^B f \in C_\infty(B)$ for $\alpha > \alpha_0$, $f \in \mathcal{B}_b(B)$.

6. Doubly resolvent Feller property with multiplicative functionals, II

Let $(Z_t)_{t\ge 0}$ be as in the previous section. In this section, we will give another criterion on the doubly resolvent Feller property. Throughout this section, we fix a nonempty open set B. The following conditions are dependent on B.

(a)$_B$ For some $t > 0$, $a_t^B := \sup_{x\in B}\sup_{s\in[0,t]}\mathbf{E}_x[Z_s : s < \tau_B] < \infty$.

(a)$^*_B$ There exists $p>1$ such that

$$a_t^B(p) := \sup_{x\in B}\sup_{s\in[0,t]} \mathbf{E}_x[Z_s^p : s<\tau_B] < \infty \quad \text{for some } t>0.$$

As shown in the previous section, we can deduce that $t\mapsto a_t^B$ is submultiplicative under (a)$_B$. This implies that $t\mapsto \log(a_t^B)^{1/t}$ is decreasing, $a_t^B \le (a_{t_0}^B)^{t/t_0}$ for all $t\ge t_0$ with given $t_0>0$, and for any $\alpha>\alpha_0^B := \inf_{s\in]0,\infty[}\log(a_s^B)^{1/s}\ge 0$, $\int_0^\infty e^{-\alpha t}a_t^B\,dt<\infty$. Under (a)$^*_B$, we have a similar statement including $\alpha > \alpha_0^B(p) := \inf_{s\in]0,\infty[}\log(a_s^B(p))^{1/s}\ge 0$, $\int_0^\infty e^{-\alpha t}a_t^B(p)\,dt<\infty$.

(b)^w_B For each $t>0$ and any compact subset K of B, there exists a number $p=p(K,t)>1$ such that $\sup_{x\in K}\mathbf{E}_x[Z_t^p]<\infty$.

(b)^s_B For each $t>0$, there exists a number $p=p(t)>1$ such that $\sup_{x\in B}\mathbf{E}_x[Z_t^p]<\infty$.

(c)^w_B For any relatively compact open subset D of B, we have

$$\lim_{t\to 0}\sup_{x\in D}\mathbf{E}_x\big[|Z_t-1| : t<\tau_D\big]=0.$$

(c)^s_B $\lim_{t\to 0}\sup_{x\in B}\mathbf{E}_x[|Z_t-1| : t<\tau_B]=0$.

REMARK 6.1

(1) The condition (b)^s_B (resp., (c)^s_B) is stronger than (b)^w_B (resp., (c)^w_B).

(2) The condition (c)^s_B implies the condition (a)$_B$. Indeed, under (c)^s_B, for any given $\varepsilon>0$, there exists $\delta>0$ such that $\sup_{t\in[0,\delta]}\sup_{x\in B}\mathbf{E}_x[|Z_t-1| : t<\tau_B]\le\varepsilon$. Then we have $\sup_{t\in[0,\delta]}\sup_{x\in B}\mathbf{E}_x[|Z_t| : t<\tau_B]\le\varepsilon+1<\infty$, which shows (a)$_B$.

THEOREM 6.1 (CF. [2, THEOREM 1.4])

Let $\mathbf{X}$ *be a doubly resolvent Feller process, and let* B *be an open subset of* E. *Under* (a)$_B$, (b)^w_B, *and* (c)^w_B, *for any* $\alpha>\alpha_0^B := \inf_{s\in]0,\infty[}\log(a_s^B)^{1/s}\ge 0$, $S_\alpha^B f\in C_b(B)$ *for* $f\in\mathcal{B}_b(B)$.

REMARK 6.2

In [2, Theorem 1.4], condition (a)$_B$ is missing from the strong Feller property of $(Q_t^B)_{t\ge 0}$, and it should be added.

Proof of Theorem 6.1

Let D be a relatively compact open subset of B. The resolvent strong Feller property of the part process $\mathbf{X}^D$ holds under the doubly resolvent Feller property of $\mathbf{X}$ by Theorem 3.1. By applying Corollary 5.1 and (c)^w_B to $\mathbf{X}^D$, the resolvent $(S_\alpha^D)_{\alpha>\alpha_0^D}$ defined by $S_\alpha^D f(x) := \mathbf{E}_x[\int_0^{\tau_D} e^{-\alpha t}Z_t f(X_t)\,dt]$, $f\in\mathcal{B}_b(D)$, has the strong Feller property. Take $g\in\mathcal{B}_b(B)$. Let $\{D_n\}$ be an increasing sequence of relatively compact open sets converging to B. Then the quasileft continuity of $\mathbf{X}$ yields $\mathbf{P}_x(\lim_{n\to\infty}\tau_{D_n}=\tau_B)=1$ for any $x\in B$. Take a compact subset K of B.

Then there exists an $n_0 \in \mathbb{N}$ so that $K \subset D_n$ for all $n \geq n_0$ and

$$\sup_{x\in K}\left|S_\alpha^B g(x) - S_\alpha^{D_n} g(x)\right| \leq \|g\|_\infty \int_0^\infty e^{-\alpha t} \sup_{x\in K} \mathbf{E}_x[Z_t : \tau_{D_n} \leq t < \tau_B]\, dt.$$

Since $\sup_{x\in K} \mathbf{E}_x[Z_t : \tau_{D_n} \leq t < \tau_B] \leq a_t^B$ and $\int_0^\infty e^{-\alpha t} a_t^B\, dt < \infty$ for $\alpha > \alpha_0^B$, it suffices to show that, for each $t > 0$, $\lim_{n\to\infty} \sup_{x\in K} \mathbf{E}_x[Z_t : \tau_{D_n} \leq t < \tau_B] = 0$. By (b)$_B^w$ and

$$\sup_{x\in K} \mathbf{E}_x[Z_t : \tau_{D_n} \leq t < \tau_B] \leq \sup_{x\in K} \mathbf{E}_x[Z_t^p]^{\frac{1}{p}} \cdot \sup_{x\in K} \mathbf{P}_x(\tau_{D_n} \leq t < \tau_B)^{\frac{p-1}{p}},$$

we can obtain $S_\alpha^B g \in C_b(B)$ for $\alpha > \alpha_0^B$, because $\lim_{n\to\infty} \sup_{x\in K} \mathbf{P}_x(\tau_{D_n} \leq t < \tau_B) = 0$ is proved in [2, Theorem 1.4]. □

A positive additive functional (PAF) B_t in the classical sense is said to be in the *Kato class* of $\mathbf{X}$ if $\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x[B_t] = 0$. A PAF B_t in the classical sense is said to be in the *local Kato class* if, for any relatively compact open subset G of E, $(\mathbf{1}_G * B)_t := \int_0^t \mathbf{1}_G(X_s)\, dB_s$ is of Kato class (see [2, Section 2] for the PAF of (local) Kato class).

COROLLARY 6.1 (CF. [2, THEOREM 2.5])

Let $\mathbf{X}$ be a doubly resolvent Feller process. Let B_t be a PAF of local Kato class. Then the subprocess killed by e^{-B_t} enjoys the doubly Feller property of the resolvent.

Proof

First we prove the strong Feller property of the resolvent of the subprocess. By definition, $\mathbf{1}_G * B$ is of Kato class for any relatively compact open subset G of E. Set $Z_t := e^{-B_t}$. Then Z_t satisfies the conditions (a)$_G$ with $\alpha_0^G = 0$, (b)$_G^s$, and (c)$_G^s$ for such G. Hence, $S_\alpha^G \varphi \in C_b(G)$ for $\varphi \in \mathcal{B}_b(E)$ and $\alpha > 0$ by Theorem 6.1. Letting $G \uparrow E$, we see the lower semicontinuity of $x \mapsto S_\alpha\varphi(x)$ for nonnegative $\varphi \in \mathcal{B}_b(E)$. On the other hand, we know the continuity of $x \mapsto \mathbf{E}_x[\int_0^\infty e^{-(\mathbf{1}_G * B)_t}\varphi(X_t)\, dt]$ for nonnegative $\varphi \in \mathcal{B}_b(E)$ from Corollary 5.1, because $(\mathbf{1}_G * B)_t$ is of Kato class. Letting $G \uparrow E$, we see the upper semicontinuity of $S_\alpha\varphi$ for nonnegative $\varphi \in \mathcal{B}_b(E)$. Therefore, we obtain the continuity of $S_\alpha\varphi$ for nonnegative $\varphi \in \mathcal{B}_b(E)$ and, hence, for general $\varphi \in \mathcal{B}_b(E)$. The Feller property of the subprocess easily follows from $0 \leq S_\alpha\varphi \leq R_\alpha\varphi$ for nonnegative φ. □

THEOREM 6.2 (CF. [2, THEOREM 1.4])

Let $\mathbf{X}$ be a doubly resolvent Feller process, and let B be an open subset of E. Suppose that B is regular. Under (a)$_B$, (b)$_B^s$, and (c)$_B^s$, for any $\alpha > \alpha_0^B := \inf_{s\in]0,\infty[} \log(a_s^B)^{1/s} \geq 0$, we have $S_\alpha^B f \in C_\infty(B)$ and $\lim_{\alpha\to\infty} \alpha S_\alpha^B f(x) = f(x)$ for $f \in C_\infty(B)$ and $x \in B$. Suppose further that B is relatively compact, and assume (a)$_B^*$ or that there exists an open set C with $\overline{B} \subset C$ such that (c)$_C^s$ holds. Then $S_\alpha^B f \in C_\infty(B)$ for $f \in \mathcal{B}_b(B)$.

Proof
By Theorem 6.1, we already know that $S_\alpha^B f \in C_b(B)$ for $f \in C_\infty(B)$ and $\alpha > \alpha_0^B$. Recall the boundary $\partial^* B$ of B in E_∂. Since $\mathbf{X}^B$ is also a Feller process on B by Theorem 4.1, we then have that by (b)$_B^s$

$$Q_t^B|f|(x) \le \sup_{y\in B} \mathbf{E}_y[Z_t^p : t<\tau_B]^{\frac{1}{p}} \left(P_t^B|f|^{\frac{p}{p-1}}(x)\right)^{\frac{p-1}{p}} \to 0 \quad \text{as } B \ni x \to z \in \partial^* B.$$

Moreover, $|Q_t^B f(x)| \le \|f\|_\infty a_t^B$ and $\int_0^\infty e^{-\alpha t} a_t^B\, dt < \infty$ imply $S_\alpha^B f(x) = \int_0^\infty e^{-\alpha t} Q_t^B f(x)\, dt \to 0$ as $B \ni x \to z \in \partial^* B$. Next, we prove $\lim_{\alpha\to\infty} \alpha S_\alpha^B f(x) = f(x)$ for $f \in C_\infty(B)$ and $x \in B$. Owing to (c)$_B^s$, for any $\varepsilon > 0$, there is $\delta > 0$ such that $\sup_{t\in[0,\delta]} \sup_{x\in B} \mathbf{E}_x[|Z_t - 1| : t < \tau_B] \le \varepsilon$. Then

$$\begin{aligned}\left|\alpha S_\alpha^B f(x) - \alpha R_\alpha^B f(x)\right| &\le \|f\|_\infty \alpha \int_0^\infty e^{-\alpha t} \mathbf{E}_x\left[|Z_t - 1| : t < \tau_B\right] dt \\ &\le \varepsilon \|f\|_\infty + \|f\|_\infty \alpha \int_\delta^\infty e^{-\alpha t}\bigl(\sup_{x\in B} \mathbf{E}_x[Z_t : t<\tau_B] + 1\bigr)\, dt \\ &\le \|f\|_\infty \varepsilon + \|f\|_\infty \alpha \int_\delta^\infty e^{-\alpha t}\bigl((a_\delta^B)^{t/\delta} + 1\bigr)\, dt.\end{aligned}$$

In the same way as in the proof of Theorem 5.2, we have $\lim_{\alpha\to\infty} |\alpha S_\alpha^B f(x) - \alpha R_\alpha^B f(x)| \le \varepsilon \|f\|_\infty$. Therefore, we obtain the conclusion by Theorem 4.1. Finally we suppose the relative compactness of B. If (c)$_C^s$ holds for an open set $C \supset \overline{B}$, then $S_\alpha^C \phi \in C_b(C)$ for any $\phi \in \mathcal{B}_b(C)$ and $\alpha > \alpha_0^C$ by Corollary 5.1. Then $S_\alpha^B 1(x) = S_\alpha^C 1(x) - \uparrow \lim_{\beta\to\infty} \beta S_{\alpha+\beta}^C \phi_{B,C}^Z(x)$ yields the upper semicontinuity of $C \ni x \mapsto S_\alpha^B 1(x)$, where $\phi_{B,C}^Z(x) := \mathbf{E}_x[\int_{\tau_B}^{\tau_C} e^{-\alpha t} Z_t\, dt]$; in particular,

$$\varlimsup_{x\to z\in\partial B} S_\alpha^B 1(x) \le S_\alpha^B 1(z) = 0 \quad \text{for } \alpha > \alpha_0^C.$$

If (a)$_B^*$ holds, then for $\alpha > \alpha_0^B(p)$ with some $p > 1$,

$$\varlimsup_{x\to z\in\partial B} S_\alpha^B 1(x) \le \left(\int_0^\infty e^{-\alpha t} a_t^B(p)\, dt\right)^{\frac{1}{p}} \left(R_\alpha^B 1(z)\right)^{\frac{p-1}{p}} = 0, \tag{6.1}$$

where we use the upper semicontinuity of $x \mapsto R_\alpha^B 1(x)$ by Lemma 2.3. In view of the resolvent equation for $(S_\alpha^B)_{\alpha > \alpha_0^B}$, we have $\lim_{x\to z\in\partial B} S_\alpha^B 1(x) = 0$ for $\alpha > \alpha_0^B$. Consequently, $|S_\alpha^B f(x)| \le \|f\|_\infty S_\alpha^B 1(x) \to 0$ as $B \ni x \to z \in \partial B$ for $\alpha > \alpha_0^B$. $\square$

7. Doubly resolvent Feller property of time-changed process

In this section, we assume that $\mathbf{X}$ is an $\mathfrak{m}$-symmetric Markov process on E whose Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;\mathfrak{m})$ is regular. We assume that $\mathbf{X}$ satisfies the absolute continuity condition $(\mathbf{AC})$: $P_t(x,dy) \ll \mathfrak{m}(dy)$ for each $x \in E$ and $t > 0$. Under $(\mathbf{AC})$, we always have the α-order resolvent kernel $R_\alpha(x,y)$ (0-order resolvent or Green kernel $R(x,y)$ provided $\mathbf{X}$ is transient).

Let $S_1(\mathbf{X})$ be the family of positive smooth measures in the strict sense; that is, any $\nu \in S_1(\mathbf{X})$ is a Revuz measure of a positive continuous additive functional (PCAF) B_t in the strict sense (see [7, Theorem 5.1.7]):

$$\int_E f(x)\nu(dx) = \uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathbf{E}_{\mathfrak{m}}\Big[\int_0^t f(X_s)\,dB_s\Big].$$

A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *Kato class* of $\mathbf{X}$ if $\lim_{\alpha\to\infty}\sup_{x\in E} R_\alpha\nu(x) = 0$ and $\nu \in S_1(\mathbf{X})$ is said to be in the *local Kato class* of $\mathbf{X}$ if $\mathbf{1}_G\nu$ is of Kato class for any relatively compact open set G. Denote by $S_K^1(\mathbf{X})$ (resp., $S_{LK}^1(\mathbf{X})$) the family of measures of Kato class (resp., local Kato class).

For $\nu \in S_1(\mathbf{X})$ and its associated PCAF B in the strict sense, let $(\check{\mathbf{X}},\nu)$ be the time-changed process defined by $(\check{\mathbf{X}},\nu) := (\Omega, X_{\tau_t}, \mathbf{P}_x)_{x\in\tilde{Y}}$, where $\tau_t := \inf\{s>0 \mid B_s > t\}$ is the right continuous inverse of the PCAF B_t and $\tilde{Y}$ is the fine support of ν defined by $\tilde{Y} := \{x \in E \mid \mathbf{P}_x(R=0)=1\}$ with $R := \inf\{t>0 \mid B_t > 0\}$. It is known that $(\check{\mathbf{X}},\nu)$ is a ν-symmetric right process on $\tilde{Y}$, and it can be realized as a Hunt process on $Y := \mathrm{supp}[\nu]$ (consequently on $\tilde{Y}$) (see [7], [10]).

The following is proved in [9, Lemma 4.1] provided $\mathbf{X}$ is transient, but the proof can be done without transience.

LEMMA 7.1 (CF. [9, LEMMA 4.1])

Suppose that $\mathbf{X}$ enjoys the strong Feller property of the resolvent. Let ν be a nonnegative smooth measure such that $\nu \in S_K^1(\mathbf{X})$. Then the time-changed process $(\check{\mathbf{X}},\nu)$ enjoys the strong Feller property of the resolvent. More strongly,

$$\check{R}_\beta\varphi(x) := \mathbf{E}_x\Big[\int_0^\infty e^{-\beta B_t}\varphi(X_t)\,dB_t\Big]$$

satisfies $\check{R}_\beta\varphi \in C_b(E)$ for $\varphi \in \mathcal{B}_b(E)$ and $\beta > 0$.

Our main result in this section is the following.

THEOREM 7.1

Suppose that $\mathbf{X}$ enjoys the doubly Feller property of the resolvent. Assume $\nu \in S_{LK}^1(\mathbf{X})$. Then the time-changed process $(\check{\mathbf{X}},\nu)$ enjoys the doubly Feller property of the resolvent. More strongly, we have $\check{R}_\beta\varphi \in C_b(E)$ (resp., $\check{R}_\beta\varphi \in C_\infty(E)$) for $\varphi \in \mathcal{B}_b(E)$ (resp., $\varphi \in \mathcal{B}_b(E)$ having compact support) and $\beta > 0$.

Proof

Assume $\nu \in S_{LK}^1(\mathbf{X})$ and the doubly resolvent Feller property of $\mathbf{X}$. First we prove that $\check{R}_\beta\varphi \in C_b(E)$ for $\varphi \in \mathcal{B}_b(E)$. Let $\mathbf{X}^{\beta\nu}$ be the subprocess of $\mathbf{X}$ killed by $e^{-\beta B_t}$. Then $\mathbf{X}^{\beta\nu}$ also enjoys the doubly Feller property of the resolvent by Corollary 6.1. Denote by $R_\alpha^{\beta\nu}$ the α-order resolvent of $\mathbf{X}^{\beta\nu}$. We then see that

$$\beta R_n^{\beta\nu}\nu(x) = \beta\mathbf{E}_x\Big[\int_0^\zeta e^{-nt-\beta B_t}\,dB_t\Big] = \mathbf{E}_x\Big[1 - e^{-n\zeta-\beta B_\zeta} - \int_0^{n\zeta} e^{-\beta B_{t/n}}e^{-t}\,dt\Big]$$

uniformly converges to zero on each compact set K as $n\to\infty$, because $x \mapsto n\mathbf{E}_x[\int_0^\zeta e^{-\beta B_t}e^{-nt}\,dt]$ (resp., $x \mapsto \mathbf{E}_x[e^{-n\zeta}]$) is continuous by the doubly Feller property of the resolvent for $\mathbf{X}^{\beta\nu}$ (resp., by Theorem 2.1) and increasingly convergent to 1 (resp., decreasingly convergent to zero) pointwise as $n\to\infty$. Set

$g_n^{\beta\nu} := n(I - nR_{n+1}^{\beta\nu})R_1^{\beta\nu}(\varphi\nu)$. Then $R_1^{\beta\nu}(\varphi\nu) - R_1^{\beta\nu} g_n^{\beta\nu} = R_{n+1}^{\beta\nu}(\varphi\nu)$ uniformly converges to zero on each compact set K, which implies $R_1^{\beta\nu}(\varphi\nu) \in C_b(E)$. Owing to the generalized resolvent equation (see [11, Lemma 4.1.1]), we conclude that $\check{R}_\beta\varphi = R^{\beta\nu}(\varphi\nu) = R_1^{\beta\nu}(\varphi\nu) + R_1^{\beta\nu}(R^{\beta\nu}(\varphi\nu)) \in C_b(E)$ for general $\varphi \in \mathcal{B}_b(E)$. Next we prove the Feller property of $\check{\mathbf{X}}$. For $\phi \in C_b(E)$ and $x \in E$, we see for $x \in E$ that

$$\left|\beta\check{R}_\beta\phi(x) - \phi(x)\right| \le \mathbf{E}_x\Big[\mathbf{E}_{X_{\sigma_{\tilde{Y}}}}\Big[\int_0^\infty e^{-t}\left|\phi(X_{\tau_{t/\beta}}) - \phi(X_0)\right| dt\Big]\Big] \to 0 \quad \text{as } \beta \to \infty.$$

So it suffices to prove $\check{R}_\beta\varphi \in C_\infty(E)$ for $\varphi \in \mathcal{B}_b(E)$ having compact support for the Feller property of $(\check{\mathbf{X}}, \nu)$, because $C_0(E)$ is uniformly dense in $C_\infty(E)$. Fix $\varphi \in \mathcal{B}_b(E)$ having compact support $K := \operatorname{supp}[\varphi]$. We may assume that ν is non-trivial. Let $\tilde{Y}$ be the fine support of B_t, and set $C := \{x \in E \mid \mathbf{P}_x(\sigma_{\tilde{Y}} < \infty) > 0\}$. Then C is a Borel measurable $(T_t)_{t\ge 0}$-invariant set such that C is finely open and finely closed (see [7, (4.6.16)]). As seen in the proof of Lemma 4.1, $\mathbf{1}_C$ has a bounded continuous $\mathfrak{m}$-version f by virtue of the strong Feller property of the resolvent. Since both f and $\mathbf{1}_C$ are finely continuous and $\mathfrak{m}$ has full fine support, $\mathbf{1}_C$ coincides with f, which means that C is open and closed. Moreover, [7, (4.6.9), (4.6.10)] hold for $x \in E$, and the same equations hold with C being replaced by $E \setminus C$. Therefore, C is $\mathbf{X}$-invariant in the sense of [7]. Note that the restriction $\mathbf{X}^C$ to C of $\mathbf{X}$ is a part process on C. Hence, $\mathbf{X}^C$ is a doubly resolvent Feller process in view of Theorem 4.1. It is proved in [7, Lemma 6.2.6(ii) with Theorem 6.1.1] that the subprocess $\mathbf{X}^{\beta(\nu+\mathbf{1}_{C^c}\mathfrak{m})}$ killed by $e^{-\beta(B_t+\int_0^t \mathbf{1}_{C^c}(X_s)\,ds)}$ is transient. Replacing the state space E with C, we may always assume that the subprocess $\mathbf{X}^{\beta\nu}$ killed by $e^{-\beta B_t}$ is transient. Then there exists a strictly positive bounded function $g \in L^1(E;\mathfrak{m})$ such that $R^{\beta\nu}g \in \mathcal{B}_b(E)$ (see [8]), where $R^{\beta\nu}$ is the 0-order resolvent of $\mathbf{X}^{\beta\nu}$. Let $A_t := \int_0^t g(X_s)\,ds$ be a PCAF associated to the measure $g\mathfrak{m}$. Let $\mathbf{X}^{\beta\nu+\alpha g\mathfrak{m}}$ be the subprocess killed by $e^{-\alpha A_t - \beta B_t}$. By Corollary 6.1, $\mathbf{X}^{\beta\nu+\alpha g\mathfrak{m}}$ is a doubly resolvent Feller process. Let

$$\|R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}\|_{L(C_\infty(E))} := \sup_{f \in C_\infty(E), \|f\|_\infty = 1} \sup_{x \in E}\left|R_\alpha^{\beta\nu+\alpha g\mathfrak{m}} f(x)\right|$$

be the operator norm of $R_\alpha^{\beta\nu+\alpha g\mathfrak{m}} : C_\infty(E) \to C_\infty(E)$. If $\|\alpha R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}\|_{L(C_\infty(E))} = 1$, then there exist $f \in C_\infty(E)$ with $\|f\|_\infty = 1$ and $x_n \in E$ such that $1 - \frac{1}{n} \le \alpha R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}|f|(x_n) \le 1$. Taking a subsequence if necessary, we may assume $x_n \to x \in E_\partial$. So $\alpha R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}|f|(x) = 1$, because $R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}|f| \in C_\infty(E)$; consequently, $x \ne \partial$. Then $\alpha\int_0^\infty e^{-\alpha t}\mathbf{E}_x[1 - e^{-\alpha A_t - \beta B_t}|f|(X_t)]\,dt = 0$, which yields a contradiction from $\|f\|_\infty = 1$. Thus, we have $\|\alpha R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}\|_{L(C_\infty(E))} < 1$. This yields

$$R^{\beta\nu+\alpha g\mathfrak{m}}\mathbf{1}_K = \sum_{n=0}^\infty (\alpha R_\alpha^{\beta\nu+\alpha g\mathfrak{m}})^n R_\alpha^{\beta\nu+\alpha g\mathfrak{m}}\mathbf{1}_K \in C_\infty(E). \tag{7.1}$$

Since $\nu \in S^1_{LK}(\mathbf{X})$, we see that $\mathbf{1}_K\nu \in S^1_{K_\infty}(\mathbf{X}^{\beta\nu+\alpha g\mathfrak{m}})$. Here $S^1_{K_\infty}(\mathbf{X}^{\beta\nu+\alpha g\mathfrak{m}})$ is the class of Green-tight measures of Kato class with respect to $\mathbf{X}^{\beta\nu+\alpha g\mathfrak{m}}$ (see [6] or [9, Definition 4.1(2)] for the definition of Green-tight measures of Kato class). Then one can apply [6, Lemma 2.3(5)] with (7.1) to $\mathbf{X}^{\beta\nu+\alpha g\mathfrak{m}}$ so that for any $\phi \in \mathcal{B}_b(E)$

$$(7.2) \qquad R^{\beta\nu+\alpha g\mathfrak{m}}\phi\mathbf{1}_K\nu = \mathbf{E}_{\cdot}\left[\int_0^\infty e^{-\alpha A_t-\beta B_t}\phi(X_t)\mathbf{1}_K(X_t)\,dB_t\right] \in C_\infty(E).$$

Though [6, Lemma 2.3(5)] is proved under the doubly Feller property of $\mathbf{X}$, its proof remains valid for the doubly resolvent Feller property of $\mathbf{X}$. For $\phi \in \mathcal{B}_b(E)$, we define $V_{A,B}^{\alpha,\beta}\phi(x) := \mathbf{E}_x[\int_0^\infty e^{-\alpha A_t-\beta B_t}\phi(X_t)\,dB_t]$. Owing to the generalized resolvent equation (see [11, Lemma 4.1.1]) $V_{A,B}^{\gamma,\beta}\phi - V_{A,B}^{\alpha,\beta}\phi = (\alpha-\gamma)V_{B,A}^{\beta,\alpha}V_{A,B}^{\gamma,\beta}\phi$ $(\alpha,\beta,\gamma \geq 0)$, we have

$$\check{R}_\beta\varphi = V_{A,B}^{\alpha,\beta}\varphi + \alpha V_{B,A}^{\beta,\alpha}V_{A,B}^{0,\beta}\varphi = R^{\beta\nu+\alpha g\mathfrak{m}}\varphi\mathbf{1}_K\nu + \alpha R^{\beta\nu+\alpha g\mathfrak{m}}(gV_{A,B}^{0,\beta}\varphi).$$

From this,

$$\check{R}_\beta|\varphi|(x) \leq \mathbf{E}_x\left[\int_0^\infty e^{-\alpha A_t-\beta B_t}\mathbf{1}_K(X_t)\,dB_t\right]\cdot\|\varphi\|_\infty + \frac{\alpha}{\beta}\|\varphi\|_\infty\|R^{\beta\nu}g\|_\infty.$$

Letting $x \to \partial$ with (7.2) and $\alpha \to 0$, we obtain $\lim_{x\to\partial}\check{R}_\beta\varphi(x) = 0$. This proves $\check{R}_\beta\varphi \in C_\infty(E)$. □

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