Quadratic numerical semigroups and the Koszul property

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Abstract Let H be a numerical semigroup. We give effective bounds for the multiplicity $e(H)$ when the associated graded ring $\operatorname{gr}_{\mathfrak{m}} K[H]$ is defined by quadrics. We classify Koszul complete intersection semigroups in terms of gluings. Furthermore, for several classes of numerical semigroups considered in the literature (arithmetic, compound, special almost-complete intersections, 3-semigroups, symmetric or pseudosymmetric 4-semigroups) we classify those which are Koszul.

Introduction

Let K be a field. A standard graded K-algebra R with graded maximal ideal $\mathfrak{m}$ is called *Koszul* if the R-module $K \cong R/\mathfrak{m}$ has an R-linear resolution. It is known that if I, the defining ideal of R, has a Gröbner basis of quadrics, then R is Koszul, and also that if R is Koszul, then I is generated by quadrics. Although it is in general difficult to certify that an algebra is Koszul, the properties of this class of rings make it an interesting endeavor. We refer to the survey articles [15] and [9] for more details.

Due to the promise of a rich theory, it is of interest to study the Koszul property for a larger class of rings. Inspired by an idea of Fröberg [14], the first author, Reiner, and Welker [19] consider the Koszul property for the associated graded ring of an affine semigroup ring with respect to the maximal multigraded ideal. For instance, it is proved that for a 2-dimensional normal affine semigroup ring its associated graded ring is Koszul (see [19, Proposition 5.3]).

In this article we focus on the case of 1-dimensional affine semigroup rings, that is, those coming from numerical semigroups. Recall that a numerical semigroup H is a subset of the nonnegative integers that is closed under addition and contains 0, and $\mathbb{N} \setminus H$ is finite or, equivalently, the greatest common divisor of all elements in H equals 1. We denote by $G(H)$ the unique minimal system of gener-

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Kyoto Journal of Mathematics, Vol. 57, No. 3 (2017), 585–612

DOI 10.1215/21562261-2017-0007,

Received October 30, 2015. Accepted April 25, 2016.

2010 Mathematics Subject Classification: Primary 13A30; Secondary 16S37, 16S36, 13C40, 13H10, 13P10.

Stamate's work supported by a grant of the Romanian Ministry of Education, National Research Council–Executive Agency for Higher Education, Research, Development, and Innovation Funding (CNCS–UEFISCDI), project number PN-II-RU-PD-2012-3–0656.

ators for H. The multiplicity and the embedding dimension of H are defined as $e(H) = \min G(H)$ and $\operatorname{emb\,dim}(H) = |G(H)|$, respectively. If $n = \operatorname{emb\,dim}(H)$, we say that H is an n-semigroup. We denote $K[H] = \bigoplus_{h\in H} Kt^h \subset K[t]$ as the semigroup ring associated to H. The tangent cone of $K[H]$ is the associated graded ring $\operatorname{gr}_{\mathfrak{m}} K[H] = \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ with respect to the maximal ideal $\mathfrak{m} = (t^h : h \in H, h \neq 0)K[H]$.

If $G(H) = \{a_1,\dots,a_n\}$, then the toric ideal I_H is defined as the kernel of the K-algebra map $\phi : K[x_1,\dots,x_n] \to K[H]$ letting $\phi(x_i) = t^{a_i}$ for $i = 1,\dots,n$. It is known that I_H is generated by the binomials $f = \prod_{i=1}^n x_i^{\alpha_i} - \prod_{i=1}^n x_i^{\beta_i}$, where $\alpha_i, \beta_i \geq 0$ for all $i = 1,\dots,n$ and $\sum_{i=1}^n \alpha_i a_i = \sum_{i=1}^n \beta_i a_i$. It is enough to use only such binomials where $\alpha_i\beta_i = 0$ for all $i = 1,\dots,n$.

For a nonzero polynomial f its *initial form* f^* is the homogeneous component of f of least degree and the initial degree of f is defined as $\deg f^*$. For an ideal I we let $I^* = (f^* : f \in I, f \neq 0)$. A *standard basis* for I is a set of polynomials in I whose initial forms generate I^*. It is known and easy to see that a standard basis is also a generating set for I.

With this notation, one can check that $\operatorname{gr}_{\mathfrak{m}} K[H] \cong K[x_1,\dots,x_n]/I_H^*$. From the algorithms that may be used to compute I_H^* (see [12] or [13]) one gets that I_H^* is generated by monomials and possibly homogeneous binomials.

We are interested in numerical semigroups H such that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Koszul. In general, even if I_H^* is quadratic, the tangent cone $R = \operatorname{gr}_{\mathfrak{m}} K[H]$ may not be Koszul. For instance, one can check with Singular [10] that, for $H = \langle 12,14,15,16,18,19\rangle$, I_H^* is generated in degree 2 and $\beta_{4,5}^R(K) = 1$; hence, the resolution of K over R is not linear.

We say that H is a *Koszul, quadratic*, or *G-quadratic semigroup* if $\operatorname{gr}_{\mathfrak{m}} K[H]$ is a Koszul ring, I_H^* is generated in degree 2, or (possibly after a suitable change of coordinates) it has a Gröbner basis of quadrics with respect to some term order, respectively. Note that, by [25], the quadratic property depends only on the generators of the semigroup and it does not depend on the field K. When discussing the Koszul property of H we work over a fixed field K, although we do not know of any semigroup H where the Koszul property depends on the field of coefficients.

For some of our arguments to work we need to assume that the field K is infinite.

The ideal I is called a *complete intersection ideal* (CI ideal for short) if it is minimally generated by $\operatorname{height} I$ elements. In case $\mu(I) = 1 + \operatorname{height} I$, one says that I is an *almost-CI ideal.* We say that a numerical semigroup H is an (almost) CI if I_H has that property. Note that, in general, if I_H is an (almost) CI ideal, that property may no longer hold for I_H^*. However, if I_H^* is generated in degree 2, we prove in Lemma 1.5 that I_H is almost-CI if and only if I_H^* is of the same kind.

Let $n = \operatorname{emb\,dim}(H)$. It is easy to see that $n \leq e(H)$. In Section 1 we show that if H is quadratic, then there is also an upper bound; namely, $e(H) \leq 2^{n-1}$. It is shown that if either one of these bounds is reached, then H is a Koszul semigroup. The upper bound is reached if and only if I_H^* is generated by a regular sequence of quadrics. These results are valid more generally for 1-dimensional Cohen–Macaulay local rings with infinite residue field, and our proofs are given in this generality.

Numerical experiments with Singular [10] make us believe that not all the values in the interval $[n, 2^{n-1}]$ are possible for the multiplicity of a quadratic semigroup H. If $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not a CI, under the extra assumption that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen–Macaulay, in Theorem 1.9 we prove that $e(H) \leq 2^{n-1} - 2^{n-3}$, and if equality holds, then H is G-quadratic and it is an almost-CI semigroup.

Interesting classes of semigroups arise from semigroups of smaller embedding dimension by the so-called *gluing* construction. If H_1, H_2 are numerical semigroups and c_1, c_2 are coprime integers such that $c_1 \in H_2 \setminus G(H_2)$ and $c_2 \in H_1 \setminus G(H_1)$, we say that $H = \langle c_1 H_1, c_2 H_2 \rangle$ is obtained by gluing H_1 and H_2. The most prominent result in this direction is Delorme's characterization of CI numerical semigroups (see [11]). Namely, any such semigroup is obtained by a sequence of gluings starting from $\mathbb{N}$ (see Theorem 2.13).

If $c_1 = 2$ and $H_2 = \mathbb{N}$, we say that $H = \langle 2H_1, c_2 \rangle$ is obtained from H_1 by a *quadratic gluing*. As a main result of Section 2 we complement Delorme's theorem by showing that any quadratic CI numerical semigroup is obtained by a sequence of quadratic gluings (see Theorem 2.14). This result is a consequence of Theorem 2.3 and Corollary 2.7, where we show that the semigroup $H = \langle 2L, \ell \rangle$ is quadratic, Koszul, or G-quadratic if and only if L has the respective property.

In Section 3 we apply the methods described so far to study the occurrence of quadratic or Koszul members in several important families of numerical semigroups for which the defining equations of the toric ring are better understood. In Propositions 3.2 and 3.4, respectively, we show that the multiplicity of a quadratic semigroup generated by an arithmetic sequence and a geometric sequence is either very small, compared with the embedding dimension, or as large as is allowed by Theorem 1.1.

This extremal property resembles another extremal behavior for these classes of semigroups. When H is generated by a geometric sequence, the Betti numbers in the resolution of the tangent cone $\operatorname{gr}_{\mathfrak{m}} K[H]$ are the smallest possible fixing the embedding dimension (because now $\operatorname{gr}_{\mathfrak{m}} K[H]$ is a CI; see Proposition 3.4). In previous joint work we conjectured that, for a given width of H, the largest Betti numbers for $\operatorname{gr}_{\mathfrak{m}} K[H]$ are obtained by some arithmetic sequences (see [20, Conjecture 2.1]).

In the rest of Section 3 we describe completely the quadratic 3-generated and the quadratic symmetric or pseudosymmetric 4-semigroups. We refer to Section 3.4 for the exact definitions. It is worth mentioning that these quadratic semigroups are also G-quadratic.

1. Bounds for the multiplicity

In this section we present some restrictions for the multiplicity of a quadratic numerical semigroup.

THEOREM 1.1

Let H be a quadratic numerical semigroup minimally generated by $n > 1$ elements, and let $K[H]$ be its semigroup ring. Then,

(a) $n \le e(H) \le 2^{n-1}$;
(b) $e(H) = n \Longleftrightarrow I_H^*$ *has a linear resolution;*
(c) $e(H) = 2^{n-1} \Longleftrightarrow I_H^*$ *is a CI ideal* $\Longleftrightarrow I_H$ *is a CI ideal.*

More generally, this theorem, formulated for semigroup rings, is true for any 1-dimensional local Cohen–Macaulay ring $(A, \mathfrak{m})$ with a presentation $A = B/I$, where $(B, \mathfrak{n})$ is a regular local ring with infinite residue field $S/\mathfrak{n}$ and where $I \subseteq \mathfrak{n}^2$. The next sequence of propositions shows this result in this generality.

Let $\widehat{K[H]}$ be the local ring obtained as the $\mathfrak{m}$-adic completion of $K[H]$. Theorem 1.1 follows from the fact that $\operatorname{gr}_{\mathfrak{m}} K[H] \cong \operatorname{gr}_{\mathfrak{m}} \widehat{K[H]}$ and $e(H)$ coincides with the multiplicity of $\widehat{K[H]}$.

Let $R = \operatorname{gr}_{\mathfrak{m}} A$. Then $R \cong S/I^*$, where $S = \operatorname{gr}_{\mathfrak{n}} B$ is a polynomial ring and I^* is the ideal of initial forms of I. We say that A is *quadratic* if I^* is generated by quadrics.

PROPOSITION 1.2

If A is quadratic, then

$$\operatorname{emb\,dim} A \le e(A) \le 2^{\operatorname{emb\,dim} A - 1}.$$

If $e(A) = \operatorname{emb\,dim}(A)$, *we say that A has* minimal multiplicity.

Proof

Since A is Cohen–Macaulay and its residue field $K = A/\mathfrak{m}$ is infinite, a classical result of Abhyankar [2] gives that $e(A) \ge \operatorname{emb\,dim}(A) - \dim A + 1 = \operatorname{emb\,dim}(A)$.

Let $n = \operatorname{emb\,dim}(R)$. Since K is infinite, there exists $x \in R_1$ such that $\ell(0 : x) < \infty$ (see [18, Lemma 4.3.1]). Denote $\bar{R} = R/(x)$. Then $H_R(t) = Q(t)/(1-t)$ with $Q(1) = e(R)$. From the exact sequence

$$0 \to (0 :_R x) \to R(-d) \xrightarrow{x} R \to \bar{R} \to 0$$

we obtain that

$$H_{\bar{R}}(t) = (1-t)H_R(t) + H_L(t) = Q(t) + H_L(t).$$

This yields

$$e(R) = Q(1) \le Q(1) + H_L(1) = H_{\bar{R}}(1) = e(\bar{R}). \tag{1}$$

Since R is quadratic, we get that $\bar{R} \cong \bar{S}/J$, where $\bar{S}$ is a polynomial ring in $n-1$ variables and where J is generated by quadrics. As K is infinite, J contains a regular sequence $q_1, \dots, q_{n-1}$ of quadrics. It follows that

$$e(R) \le e(\bar{R}) \le e\bigl(\bar{S}/(q_1, \dots, q_{n-1})\bigr) = 2^{n-1}. \tag{2}$$

□

PROPOSITION 1.3

The local ring A has minimal multiplicity if and only if I^* has a 2-linear S-resolution.

Proof

As in the proof of Proposition 1.2 we denote $\bar{R} = R/(x)$ for some $x \in R_1$ with $\ell(0:x) < \infty$, and we write $\bar{R} = \bar{S}/J$. We use the fact that an $\bar{\mathfrak{m}}$-primary ideal in $\bar{S}$ has a linear resolution if and only if it is a power of the graded maximal ideal $\bar{\mathfrak{m}}$ of $\bar{S}$ (see [6, Exercise 4.1.17(b)]).

We denote $n = \operatorname{emb\,dim}(A)$. Suppose that $e(A) = n$. Then R is Cohen–Macaulay by a result of J. Sally (see [28, Theorem 2]). This implies that x is regular on R; hence, $e(\bar{R}) = e(R)$. This is only possible if $J = \bar{\mathfrak{m}}^2$, which has a 2-linear resolution over $\bar{S}$ by the remark before. Since x is regular on R, it follows that I^* itself is quadratic and it has a linear resolution over S.

Conversely, assume that I^* has a 2-linear S-resolution. Therefore, I^* and J are generated by quadrics. If $e(R) > n$, by (1) we get $e(\bar{R}) > n$, which implies that $J \subsetneq \bar{\mathfrak{m}}^2$. Therefore, $\operatorname{reg} \bar{R} > 1$. By [12, Proposition 20.20],

$$\operatorname{reg} R = \max\{\operatorname{reg}(0:_R x), \operatorname{reg} \bar{R}\} > 1;$$

hence, I^* does not have a linear resolution over S, which is a contradiction. □

PROPOSITION 1.4

Assume that A is quadratic. The following statements are equivalent:

(a) $e(A) = 2^{\operatorname{emb\,dim} A - 1}$;
(b) *I^* is a CI ideal;*
(c) *I is a CI ideal.*

Proof

(a) ⇒ (b) Since $e(A) = 2^{\operatorname{emb\,dim} A - 1}$, by (2) it follows that J is generated by a regular sequence of quadrics and $e(R) = e(\bar{R})$. The latter implies that x is regular on R; therefore, I^* is generated by a regular sequence.

(b) ⇒ (a) This follows from the fact that I is generated by a regular sequence of $n - 1$ quadrics.

The equivalence of (b) and (c) is a consequence of Lemma 1.5. □

LEMMA 1.5

Let $(B, \mathfrak{n})$ be a regular local ring, and let $I \subseteq \mathfrak{n}^2$ be any ideal such that I^* is generated in degree 2.

(a) *Let $\mathcal{F} \subset I$ be a finite set. Then $\mathcal{F}$ is a (minimal) standard basis for I if and only if it is a (minimal) generating set for I.*

(b) *The ideal I is an (almost) CI ideal if and only if I^* is an (almost) CI ideal.*

Proof

(a) Let $\mathcal{F}=\{f_1,\dots,f_r\}$ be a minimal standard basis for I. As a general fact, $\mathcal{F}$ is also a generating set for I. Assume that $\mathcal{F}$ is not a minimal generating set. Without loss of generality we may write $f_1=\sum_{i=2}^r g_i f_i$ with $g_i\in B$, for $i=2,\dots,n$. Then,

$$f_1^*=\sum_{\substack{i=2\\ g_i f_i\notin \mathfrak{n}^3}}^{r} g_i^* f_i^*, \tag{3}$$

contradicting the fact that $\mathcal{F}$ is a minimal generating set for I^*.

Conversely, assume that $\mathcal{F}=\{f_1,\dots,f_r\}$ is a minimal generating system for I. Since I^* is generated in degree 2, it suffices to show that $f^*\in(f_1^*,\dots,f_r^*)$ for $f\in I$ with $\deg f^*=2$. We may write $f=\sum_{i=1}^r g_i f_i$ with $g_i\in B$, $i=1,\dots,r$. Then,

$$f^*=\sum_{\substack{i=1\\ g_i f_i\notin \mathfrak{n}^3}}^{r} g_i^* f_i^*,$$

because $\deg f^*=2$.

Part (b) follows from part (a) and the fact that $\operatorname{height} I=\operatorname{height} I^*$. □

There are further restrictions for the multiplicity of a quadratic semigroup H if we assume that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen–Macaulay. Before proving them, we list in the next lemma some useful arithmetic properties of the generators of a quadratic numerical semigroup.

LEMMA 1.6

Let H be a numerical semigroup minimally generated by $a_1<a_2<\cdots<a_n$ with $n>1$. If H is quadratic, then

- (a) *there exist $k,\ell\geq 2$ such that $a_1\mid a_k+a_\ell$;*
- (b) *$2a_i\in\langle a_1,\dots a_{i-1},a_{i+1},\dots a_n\rangle$, for all $2\leq i\leq n$.*

Proof

We may pick $\mathcal{B}=\{f_1,\dots,f_r\}$, a minimal standard basis of I_H consisting of binomials. Since H is quadratic, $\deg f_j^*=2$ for $j=1,\dots,r$.

For all $i=1,\dots,n$, let c_i be the smallest positive integer such that $c_i a_i$ is a sum of the other generators. Then for any i there exists $1\leq n_i\leq r$ such that $f_{n_i}=x_i^{c_i}-\cdots$.

Assume that $f_{n_1}=x_1^{c_1}-\prod_{j\neq 1}x_j^{r_j}\in\mathcal{B}$, which gives the relation

$$c_1a_1=\sum_{j\neq 1} r_j a_j \quad \text{with the } r_j\text{'s nonnegative integers.} \tag{4}$$

Since $a_1=e(H)$ we get $c_1>\sum_{j\neq 1} r_j$ and $f_{n_1}^*=\prod_{j\neq 1}x_j^{r_j}$. As $\deg f_{n_1}^*=2$, using (4) we conclude that there exist $k,\ell>1$ such that $c_1a_1=a_k+a_\ell$.

Let $i>1$. Then $g_i=x_1^{a_i}-x_i^{a_1}$ is in I_H and $g_i^*=x_i^{a_1}\in I_H^*$. Therefore, there exists a pure power of x_i, namely, x_i^2, among the terms of $f_1^*,\dots,f_r^*$. On the other hand, $x_i^{c_i}$ is the smallest pure power of x_i occurring in any binomial in I_H. We get that $1 < c_i \le 2$; hence, $c_i = 2$. This concludes the proof. □

EXAMPLE 1.7

A Singular [10] computation and also Propositions 3.6 and 3.2 show that the semigroups $H_1 = \langle 3,4,5\rangle$ and $H_2 = \langle 4,5,6\rangle$ are quadratic. With notation as in Lemma 1.6 we notice that $a_1 \mid a_2 + a_3$ and $a_1 \mid 2a_3$, respectively. Therefore, the indices k and ℓ in Lemma 1.6(a) may be distinct or the same.

REMARK 1.8

Lemma 1.6(a) appeared as [31, Proposition 5.11]. There, it was derived using the topological properties of the intervals in a quadratic semigroup, as described in [25].

THEOREM 1.9

Let H be a quadratic numerical semigroup minimally generated by $a_1 < \cdots < a_n$ such that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen–Macaulay. The following hold.

(a) *Either $n \le e(H) \le 2^{n-1} - 2^{n-3}$ or $e(H) = 2^{n-1}$.*
(b) *If $e(H) = 2^{n-1} - 2^{n-3}$, then I_H^* is an almost-CI ideal.*

In the situation of (b), *I_H^* has a quadratic Gröbner basis with respect to the degree reverse lexicographic order induced by $x_n > \cdots > x_1$.*

Proof

(a) Assume that $e(H) < 2^{n-1}$. Let $S = K[x_1,\dots,x_n]$. Since S/I_H^* is Cohen–Macaulay, we get that x_1 is regular on S/I_H^*. Going modulo x_1 we have

$$e(H) = e(S/I_H^*) = e\big(S/(x_1, I_H^*)\big) = e\big(K[x_2,\dots,x_n]/J\big),$$

where J denotes the image of the ideal I_H^* through the K-algebra map sending x_1 to 0 and keeping the other variables unchanged.

By Lemma 1.6, for $j = 2,\dots,n$ there exist distinct polynomials $g_j = x_j^2 - m_j$ in J, where $m_j = 0$ or $m_j = x_i x_k$ with $i < j < k$. Therefore, with respect to the degree reverse lexicographic order induced by $x_1 < x_2 < \cdots$ we have that $\operatorname{in}_<(g_j) = x_j^2$, for $2 \le j \le n$.

By Theorem 1.1 we get that I_H^* is not a CI; hence, $\mu(I_H^*) = \mu(J) \ge n$. So in addition to $g_2,\dots,g_n$ there is at least one more generator f in J, $\deg f = 2$, and without loss of generality we may assume that f is either a monomial or a homogeneous binomial whose terms are not pure powers. We let $T = K[x_2,\dots,x_n]/(x_2^2,\dots,x_n^2)$ and let g be the residue class of $\operatorname{in}_<(f)$ in T. Hence,

$$\begin{aligned} e(H) &= \ell\big(K[x_2,\dots,x_n]/J\big) = \ell\big(K[x_2,\dots,x_n]/\operatorname{in}_<(J)\big) \\ &\le \ell\big(K[x_2,\dots,x_n]/\big(x_2^2,\dots,x_n^2,\operatorname{in}_<(f)\big)\big) \\ &= \ell\big(T/(g)\big) = \ell\big((0:_T g)\big). \end{aligned}$$

Let us denote $x_U = \prod_{k\in U} x_k$, for all $U \subset [2,n]$, where $x_\emptyset = 1$.

If $g=x_ix_j$, with $i\neq j$, a K-basis for $(0:_T g)$ is given by the monomials

$$\begin{aligned}&\{x_ix_U : U\subset[2,n]\setminus\{i,j\}\}\cup\{x_jx_V : V\subset[2,n]\setminus\{i,j\}\}\\ &\quad\cup\{x_ix_jx_W : W\subset[2,n]\setminus\{i,j\}\}.\end{aligned}$$

Hence, $e(H)\leq \dim_K(0:_T g)=3\cdot 2^{n-3}=2^{n-1}-2^{n-3}$.

(b) From the above arguments we note that the equality $e(H)=2^{n-1}-2^{n-3}$ holds if and only if $\mathrm{in}_<(J)=(x_2^2,\dots,x_n^2,\mathrm{in}_<(f))$; that is, $\{g_2,\dots,g_n,f\}$ is a (clearly reduced) Gröbner basis of J. Therefore, $\mu(J)=n$, which reads as I_H^* being an almost-CI ideal.

Clearly, $g_1,\dots,g_{n-1}$ and f may be lifted to S to quadratic polynomials $f_1,\dots,f_n$ in I_H^*, respectively, such that $\mathrm{in}_<(f_i)=\mathrm{in}_<(g_i)$ for $1\leq i\leq n-1$ and $\mathrm{in}_<(f_n)=\mathrm{in}_<(f)$. Let $\mathcal{F}=\{f_1,\dots,f_n\}$.

We claim that $\mathcal{F}$ is a Gröbner basis for I_H^*. Clearly, $(\mathrm{in}_<(f_1),\dots,\mathrm{in}_<(f_n))\subseteq \mathrm{in}_<(I_H^*)$. For the reverse inclusion it is enough to show that these two ideals have the same Hilbert series.

Indeed, since x_1 is regular on S/I_H^* and on $S/(\mathrm{in}_<(f_1),\dots,\mathrm{in}_<(f_n))$ we may write

$$\begin{aligned}H_{S/\mathrm{in}_<(I_H^*)}(t)&=H_{S/I_H^*}(t)=\frac{1}{1-t}H_{S/(x_1,I_H^*)}(t)\\ &=\frac{1}{1-t}H_{K[x_2,\dots,x_n]/J}(t),\\ H_{S/(\mathrm{in}_<(f_1),\dots,\mathrm{in}_<(f_n))}(t)&=\frac{1}{1-t}H_{K[x_2,\dots,x_n]/\mathrm{in}_<(J)}(t)\\ &=\frac{1}{1-t}H_{K[x_2,\dots,x_n]/J}(t).\end{aligned}$$

This ends the proof. □

In Example 2.15 we present a family of semigroups satisfying the hypotheses of Theorem 1.9(b).

REMARK 1.10

Note that the converse to the implication in Theorem 1.9(b) is not true. One may check with Singular [10] that $H=\langle 11,13,14,15,19\rangle$ is quadratic and almost-CI, but it is not a Koszul semigroup.

It is natural to ask the following.

QUESTION 1.11

Do the conclusions of Theorem 1.9 stay true for any quadratic numerical semigroup H, without assuming that $\mathrm{gr}_{\mathfrak{m}} K[H]$ is Cohen–Macaulay?

The answer is positive if $\operatorname{emb\,dim}(H)\leq 5$, as shown by the second author in [32, Proposition 1.5, Theorem 1.8]. Moreover, for $\operatorname{emb\,dim}(H)\leq 7$, it follows from

Rossi and Valla's work [27, Theorem 5.9] that if H is quadratic and not a CI, then $e(H) \leq 2^{n-1} - 2^{n-3}$.

PROPOSITION 1.12

Let H be a quadratic semigroup with $\operatorname{emb\,dim}(H) = n$. *Assume that $e(H)$ attains one of the extremal values, namely, $e(H) \in \{n, 2^{n-1}\}$, or that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen–Macaulay and $e(H) = 2^{n-1} - 2^{n-3}$. Then H is G-quadratic, and in particular, it is a Koszul semigroup.*

Proof

In the case in which $e(H) = n$, as noted in the proof of Proposition 1.3 we have that $R = \operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen–Macaulay, x_1 is regular on R, and $(x_1, I_H^*) = (x_1, (x_2, \ldots, x_n)^2)$, which is a monomial ideal. Therefore, $R/(x_1)$ is G-quadratic, and by using the work of Conca [8, Lemma 4.(2)], we conclude that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is G-quadratic, too.

In the case in which $e(H) = 2^{n-1}$, by Proposition 1.4 we have that I_H^* is a CI. On the other hand, by Lemma 1.6, for $j = 2, \ldots, n$ there exist distinct $n-1$ quadratic polynomials $f_j = x_j^2 - m_j$ in I_H^*, where $m_j = 0$ or $m_j = x_i x_k$ with $i < j < k$. Therefore, $I_H^* = (f_2, \ldots, f_n)$. With respect to the degree reverse lexicographic order induced by $x_1 > x_2 > \cdots$ we have that $\gcd(\operatorname{in}_<(f_i), \operatorname{in}_<(f_j)) = \gcd(x_i^2, x_j^2) = 1$ for all $2 \leq i < j \leq n$; hence, $\{f_2, \ldots, f_n\}$ is a quadratic Gröbner basis for I_H^*.

The case $e(H) = 2^{n-1} - 2^{n-3}$ was discussed in Theorem 1.9(b). $\square$

REMARK 1.13

It was proven in [19, Theorem 5.2] that if Λ is any affine semigroup such that $K[\Lambda]$ is Cohen–Macaulay and of minimal multiplicity, then $\operatorname{gr}_{\mathfrak{m}} K[\Lambda]$ is Koszul. With essentially the same argument as in the proof of Proposition 1.12 one obtains that $\operatorname{gr}_{\mathfrak{m}} K[\Lambda]$ is G-quadratic.

REMARK 1.14

The fact from Proposition 1.12 that "if $e(H) = n$, then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Koszul" is folklore, and it is also mentioned in the survey [9].

By applying [9, Section 6, Proposition 8] it follows that if $e(H) = n + 1$ and the Cohen–Macaulay type τ of $K[H]$ satisfies $\tau < n - 1$, then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Koszul. The proof can be easily continued to conclude that H is G-quadratic.

REMARK 1.15

Let H be a numerical semigroup with $e(H) = 2^{\operatorname{emb\,dim}(H)-1}$. It is easy to see that if I_H^* is CI, then I_H^* is generated in degree 2. However, if we assume that I_H is a CI, can we also derive that I_H^* is CI and, hence, quadratic?

2. Quadratic semigroups and gluings

The following construction on numerical semigroups was introduced by Watanabe [33] and extended later by Delorme [11] and Rosales [26], who seems to have coined the name *gluing*.

DEFINITION 2.1

Given the numerical semigroups H_1 and H_2 and the integers $c_1, c_2 > 1$, the semigroup $H = \langle c_1 H_1, c_2 H_2 \rangle$ is called a *gluing of* H_1 *and* H_2 if $c_1 \in H_2$, $c_2 \in H_1$, and $\gcd(c_1, c_2) = 1$.

We are interested in the situation when one of the glued semigroups is $\mathbb{N}$ itself.

DEFINITION 2.2

Given the numerical semigroup L and the integers $c > 1$ and ℓ such that $\ell \in L \setminus G(L)$ and $\gcd(c, \ell) = 1$, the numerical semigroup $H = \langle cL, \ell\mathbb{N} \rangle$ is called a *simple gluing* of L.

If moreover $c = 2$, we call $\langle 2L, \ell \rangle$ a *quadratic gluing.*

For the rest of the article, when we describe a semigroup as $\langle cL, \ell \rangle$ we assume that it is obtained from a simple gluing as in Definition 2.2. If in Definition 2.2 we allow $\ell \in G(L)$, then (exactly) one of the generators of H is superfluous and $\operatorname{emb\,dim}(H) = \operatorname{emb\,dim}(L)$ (cf. [11, Proposition 10.(ii)]). We want to avoid this case to have $\operatorname{emb\,dim}(H) = \operatorname{emb\,dim}(L) + 1$.

Consider the simple gluing $H = \langle cL, \ell \rangle$. Assume that $\operatorname{emb\,dim}(L) = n - 1$. We may write

$$\ell = \sum_{i=1}^{n-1} \lambda_i a_i \tag{5}$$

as a sum of the minimal generators $a_1, \ldots, a_{n-1}$ for L and such that $\sum_{i=1}^{n-1} \lambda_i \geq 2$ is maximal. This gives a so-called *gluing relation*

$$f = x_n^c - \prod_{i=1}^{n-1} x_i^{\lambda_i} \in I_H. \tag{6}$$

The largest value of $\sum_{i=1}^{n-1} \lambda_i$ such that (5) holds is called the *order* of ℓ in L, and it is denoted $\operatorname{ord}_L(\ell)$.

By convention, unless it is otherwise specified, when we work with the toric ideal of H we assume that ℓ corresponds to the last variable x_n and the rest of the generators of H are ordered as in L. If we denote $S = K[x_1, \ldots, x_n]$, it is easy to see (e.g., in the proof of [33, Lemma 1]) that

$$I_H = (I_L S, f). \tag{7}$$

The next result describes the transfer of quadraticity (and of the Koszul property) via gluings.

THEOREM 2.3

Consider the numerical semigroup $H = \langle cL, \ell\rangle$*, where* $\gcd(c, \ell) = 1$*,* $c > 1$*, and* $\ell \in L \setminus G(L)$*. Let* f *be a gluing relation as in (6).*

If $c \le \operatorname{ord}_L(\ell)$*, then the following hold:*

(a) $I_H^* = (I_L^* S, f^*)$;

(b) *H is quadratic* $\Longleftrightarrow c = 2$ *and L is quadratic;*

(c) *H is Koszul* $\Longleftrightarrow c = 2$ *and L is Koszul;*

(d) I_H^* *has a quadratic Gröbner basis* $\Longleftrightarrow c = 2$ *and* I_L^* *has a quadratic Gröbner basis.*

With notation as above, the condition $c \le \operatorname{ord}_L(\ell)$ implies, using (5), that $c \cdot e(L) \le \operatorname{ord}_L(\ell) \cdot e(L) \le \ell$. Since $\gcd(c, \ell) = 1$ and $c > 1$ we obtain

$$e\big(\langle cL, \ell\rangle\big) = c \cdot e(L) < \ell. \tag{8}$$

For the proof of Theorem 2.3 and later results, we need the following technical lemmas. The first one follows from [17, p. 185, Lemma, part (a)].

LEMMA 2.4

Let I be an ideal in the polynomial ring $S = K[x_1, \dots, x_n]$ *such that* $I \subset \mathfrak{m} = (x_1, \dots, x_n)$*. If* $f \in S$ *is such that* $\deg f^* > 0$ *and* f^* *is regular on* S/I^**, then*

$$(I, f)^* = (I^*, f^*).$$

As an immediate corollary of Lemma 2.4 we obtain the next statement.

LEMMA 2.5

Let $f_1, \dots, f_r$ *be a regular sequence in* $S = K[x_1, \dots, x_n]$ *such that* $f_1^*, \dots, f_r^*$ *is a regular sequence, too. Then*

$$(f_{i_1}, \dots, f_{i_s})^* = (f_{i_1}^*, \dots, f_{i_s}^*),$$

for $s = 1, \dots, r$ *and* $1 \le i_1 < \cdots < i_s \le r$.

LEMMA 2.6

Consider the ideal $I \subset K[x_1, \dots, x_{n-1}] \subset S = K[x_1, \dots, x_n]$ *and the polynomial* $f = x_n^c - m$ *with* $c > 0$ *and* $m \in K[x_1, \dots, x_{n-1}]$*. Then*

(a) *f is regular on S/IS, and*

(b) *if* $\deg m \ge c$*, then* f^* *is regular on* S/I^*S *and* $(IS, f)^* = (I^*S, f^*)$.

Proof

Let $<$ be the lexicographic term order on S induced by $x_n > x_{n-1} > \cdots > x_1$. Then $\operatorname{in}_<(f) = x_n^c$, and under the extra requirement from (b), we also have $\operatorname{in}_<(f^*) = x_n^c$. Since the variable x_n does not appear in the monomial generators for $\operatorname{in}_<(IS)$ and $\operatorname{in}_<(IS^*)$ we get that x_n^c is regular on $S/\operatorname{in}_<(I)$ and on

$S/\operatorname{in}_<(IS^*)$. By [12, Proposition 15.15] we obtain part (a) and the first half of (b).

The second part of (b) results from Lemma 2.4. □

We may now go back to the proof of Theorem 2.3.

Proof of Theorem 2.3
We apply Lemma 2.6 to I_L and the gluing relation (6) to conclude that (a) holds.

For (b), if $\mathcal{G}=\{f_1,\dots,f_r\}$ is any minimal standard basis for I_L, by (a) we get that $\mathcal{G}'=\mathcal{G}\cup\{f\}$ is a standard basis for I_H.

We claim that $\mathcal{G}'$ is minimal. Indeed, since the variable x_n appears in $\mathcal{G}'$ only in f^*, we cannot remove f from $\mathcal{G}'$. If we could remove some f_j, say, f_r, then $f_r^*=\sum_{i=1}^{r-1}h_if_i^*+hf^*$ for $h,h_i\in S$, $i=1,\dots,r$. By Lemma 2.6, f^* is regular on S/I^*S and we get $h\in I^*S$, which contradicts the minimality of $\mathcal{G}$.

Therefore, I_H^* is generated in degree 2 if and only if I_L^* is generated in degree 2 and $\deg f^*=2$. This gives (b).

For part (c) we remark that by (b) for any of the two implications that need to be checked we have $\deg f^*=c=2$. According to [3, Lemma 2], since f^* is regular on S/I^*S and of degree 2, the ring S/I_L^*S is Koszul if and only if

$$\frac{S/I_L^*S}{(f^*)S/I_L^*S}\cong S/(I_L^*S,f^*)=S/I_H^*\cong \operatorname{gr}_{\mathfrak{m}} K[H]$$

is Koszul. Clearly, S/I_L^*S is Koszul if and only if $K[x_1,\dots,x_{n-1}]/I_L^*$ (which is isomorphic to $\operatorname{gr}_{\mathfrak{m}} K[L]$) is Koszul; hence, (c) holds.

For part (d) we first assume that I_H^* has a quadratic Gröbner basis $\mathcal{G}$ with respect to some term order $<$. Without loss of generality we may assume that $\mathcal{G}$ is reduced, and because I_H^* is generated by binomials and/or monomials, it is well known that $\mathcal{G}$ consists of monomials and/or binomials of degree 2.

Note that the variable x_n may not appear with exponent different from 2 in any monomial term of any polynomial in $\mathcal{G}$. For an exponent 3 or larger, that is clear by the quadraticity of $\mathcal{G}$. Also, if we assume that there exists a relation $x_nx_k-\prod_{i=1}^{n-1}x_i^{\mu_i}\in I_H$ with $k<n$ and $\sum_{i=1}^{n-1}\mu_i\geq 2$, we get $\ell+ca_k=c\sum_{i=1}^{n-1}\mu_ia_i$, which is false since ℓ and c are coprime.

By (8), arguing as in the proof of Lemma 1.6 we obtain $x_n^{ca_1}\in I_H^*$, and also $x_n^{ca_1}\in \operatorname{in}_<(I_H^*)$. Therefore, there exists $g=x_n^2-m$ in $\mathcal{G}$ such that $\operatorname{in}_<(g)=x_n^2$ and m is a monomial or 0. If $g_1=x_n^2-m_1$ were another element in $\mathcal{G}$ containing x_n^2, we could reduce it further with g, but this contradicts the fact that $\mathcal{G}$ is a reduced Gröbner basis. Consequently, the variable x_n does not divide any monomial term of any element of $\mathcal{G}'=\mathcal{G}\setminus\{g\}$.

Let $J=(\mathcal{G}')$. Clearly $J\subset I_L^*S$. It is easy to see, by the Buchberger criterion, that $\mathcal{G}'$ is a reduced Gröbner basis for J. Hence,

$$\operatorname{in}_<(I_H^*)=\operatorname{in}_<(J)+(x_n^2).$$

Since f^* is regular on S/I_L^*S and x_n^2 is regular on $S/\operatorname{in}(J)$, using part (a) we obtain that

$$\operatorname{Hilb}_{S/I_H^*}(t) = (1-t^2)\operatorname{Hilb}_{S/I_L^*S}(t),$$
$$\operatorname{Hilb}_{S/\operatorname{in}_<(I_H^*)}(t) = (1-t^2)\operatorname{Hilb}_{S/\operatorname{in}_<(J)}(t).$$

According to Macaulay's theorem (see [13, Theorem 2.6]) we have $\operatorname{Hilb}_{S/I_H^*}(t) = \operatorname{Hilb}_{S/\operatorname{in}_<(I_H^*)}(t)$ and $\operatorname{Hilb}_{S/J}(t) = \operatorname{Hilb}_{S/\operatorname{in}_<(J)}(t)$. Hence, S/I_L^*S and S/J have the same Hilbert series, which together with $J \subseteq I_L^*S$ gives $J = I_L^*S$.

Consequently, $\mathcal{G}'$ is the reduced Gröbner basis for I_L^*S and also for I_L^*, and we are done. Indeed, if $q \in I_L^*S$, then $\operatorname{in}_<(q)$ is not divisible by x_n^2, and hence, it is divisible by $\operatorname{in}_<(g')$ for some $g' \in \mathcal{G}'$.

For the converse, assume that $c = 2$ and that I_L^* has a quadratic Gröbner basis $\mathcal{G}'$ with respect to some term order $<'$ on $K[x_1, \ldots, x_{n-1}]$. Let $<$ be the block order $(\text{lex}, <')$ where we first apply the lexicographic term order on the variable x_n and for ties we apply $<'$ on the rest of the monomial in the variables $x_1, \ldots, x_{n-1}$. Then $\operatorname{in}_<(f^*) = x_n^2$.

We claim that $\mathcal{G} = \mathcal{G}' \cup \{f^*\}$ is a Gröbner basis for I_H^*. Indeed, $\mathcal{G}'$ is already a Gröbner basis; hence, the only S-pairs to be checked involve f^* and $g \in \mathcal{G}'$. Since their leading terms are coprime, $S(f^*, g) \xrightarrow{\mathcal{G}} 0$. This finishes the proof. □

Since the hypothesis of Theorem 2.3 implies that $\operatorname{ord}_L(\ell) \geq 2$, we obtain the following corollary.

COROLLARY 2.7

Let L be any numerical semigroup, and let $\ell \in L \setminus G(L)$ be an odd integer. Then the semigroup $\langle 2L, \ell\rangle$ is quadratic (Koszul) if and only if L is quadratic (Koszul).

We will also need the following consequence.

COROLLARY 2.8

Let L be a quadratic numerical semigroup, and let $\ell \in L \setminus G(L)$ be an odd integer. Denote $H = \langle 2L, \ell\rangle$. The following hold:

- (a) *H is a CI semigroup $\Longleftrightarrow$ L is a CI semigroup;*
- (b) *if I_L^* is an almost-CI, then I_H^* is an almost-CI, too.*

Proof

By Corollary 2.7, H is quadratic. Since $e(H) = 2e(L)$, part (a) follows from Theorem 1.1(c). For part (b), denote $\operatorname{emb\,dim}(L) = n-1$. Hence, $\mu(I_L^*) = n-1$, and by Theorem 2.3(a) we obtain $\mu(I_H^*) \leq n$. If $\mu(I_H^*) = n-1$, by part (a) we get that L is CI, which is false. Therefore, $\mu(I_H^*) = n$. □

REMARK 2.9

In general, if $\operatorname{gr}_{\mathfrak{m}} K[L]$ is a CI, then it it not always true that for $H = \langle cL, \ell\rangle$

its tangent cone $\mathrm{gr}_{\mathfrak{m}} K[H]$ is CI, too. If we let $H = \langle 4\langle 2,5\rangle, 7\rangle = \langle 7,8,20\rangle$, mapping the variables to the generators taken in increasing order, we have $I_H^* = (x_3^2, x_2x_3, x_1^4x_3, x_2^7)$, which is not even a Cohen–Macaulay ideal.

REMARK 2.10

If $c > \mathrm{ord}(\ell)$, we have less control over the output of the gluing, even if $\deg f^* = \mathrm{ord}_L(\ell) = 2$. Let $L = \langle 4,6,7,9\rangle$. Clearly, $\mathrm{ord}_L(8) = \mathrm{ord}_L(10) = 2$. It is easy to compute (e.g., using Singular [10] or CoCoa [1])

$$I_L^* = (x_2^2, x_2x_3 - x_1x_4, x_3^2, x_2x_4, x_3x_4, x_4^2)$$

and check that the listed generators are a quadratic Gröbner basis with respect to the degree reverse lexicographic order on $x_1 > x_2 > x_3 > x_4$.

We may also check that, for the gluing $H_1 = \langle 3L, 8\rangle = \langle 12,18,21,27,8\rangle$, the ideal

$$I_{H_1}^* = (x_1^2, x_2^2, x_2x_3 - x_1x_4, x_3^2, x_2x_4, x_3x_4, x_4^2)$$

has a quadratic Gröbner basis with respect to the same term order as above. However, for the gluing $H_2 = \langle 3L, 10\rangle = \langle 12,18,21,27,10\rangle$ the ideal

$$I_{H_2}^* = (x_1x_2, x_2^2, x_2x_3 - x_1x_4, x_3^2, x_2x_4, x_3x_4, x_4^2, x_1^3x_3 - x_4x_5^3, x_1^4 - x_2x_5^3)$$

is not generated in degree 2.

REMARK 2.11

Arbitrary gluings of quadratic numerical semigroups may not be quadratic anymore. If we consider the Koszul semigroups $H_1 = \langle 2,3\rangle$, $H_2 = \langle 2,5\rangle$, and the gluing $H = \langle 7H_1, 5H_2\rangle = \langle 14,21,10,15\rangle$, we have that $e(H) > 8$, and according to our Theorem 1.1, H is not quadratic. Note that by Delorme's Theorem 2.13, H is a CI semigroup.

EXAMPLE 2.12

Watanabe [33, Lemma 3] shows that for any odd integer $a > 0$ the semigroup

$$W_n(a) = \langle 2^n, 2^n + a, 2^n + 2a, 2^n + 4a, \ldots, 2^n + 2^i a, \ldots, 2^n + 2^{n-1}a\rangle$$

is a CI of $\mathrm{emb\,dim}(W_n(a)) = n + 1$. We prove that it is a Koszul semigroup.

It is easy to see by induction on n that $W_n(a)$ may be obtained by simple gluings by the rule $W_n(a) = \langle 2W_{n-1}(a), 2^n + a\rangle$ for any $n > 1$, starting from $W_1(a) = \langle 2, 2 + a\rangle$. Clearly, $W_1(a)$ is Koszul; hence, by induction using Theorem 2.3(c) we get that $W_n(a)$ is a Koszul semigroup for any n.

Our next result shows that the quadratic (and Koszul) semigroups H for which $K[H]$ is a CI are obtained from $\mathbb{N}$ by a sequence of quadratic gluings. We first recall Delorme's structure theorem for CI semigroup rings.

THEOREM 2.13 (DELORME [11, PROPOSITION 9])

Let H be a numerical semigroup. Then $K[H]$ is a CI if and only if either $H = \mathbb{N}$

or there exist numerical semigroups H_1, H_2 *and coprime integers* c_1, c_2 *such that* $H = \langle c_1 H_1, c_2 H_2\rangle$, $c_1 \in H_2 \setminus G(H_2)$, $c_2 \in H_1 \setminus G(H_1)$, *and* $K[H_1]$, $K[H_2]$ *are CIs.*

THEOREM 2.14

Let H *be a numerical semigroup such that* $K[H]$ *is a CI. The following are equivalent:*

(a) H *is obtained uniquely from* $\mathbb{N}$ *by a series of quadratic gluings* $H_0 = \mathbb{N}$, $H_1 = \langle 2H_0, \ell_1\rangle, \ldots, H_{n-1} = \langle 2H_{n-2}, \ell_{n-1}\rangle = H$, *where* ℓ_i *is an odd integer in* $H_{i-1} \setminus G(H_{i-1})$ *for* $i = 1, \ldots, n-1$;

(b) H *is Koszul;*

(c) H *is quadratic.*

Proof

For (a) $\Rightarrow$ (b) we start with H_0, which is Koszul, and we repeatedly use Theorem 2.3 to derive that $H_1, \ldots, H_{n-1} = H$ are Koszul, as well. The implication (b) $\Rightarrow$ (c) is clear.

For (c) $\Rightarrow$ (a) we assume that H is quadratic. We prove the existence of a chain of quadratic gluings by induction on $n = \operatorname{emb dim}(H)$. For $n = 1$ we have $H = \mathbb{N}$, and there is nothing to prove. If $n = 2$, then $H = \langle a, b\rangle$ with $a < b$ coprime. This gives $I_H = (x_1^b - x_2^a)$ and $I_H^* = (x_2^a)$. Since H is quadratic we get $a = 2$ and b is odd. Hence, $H = \langle 2\mathbb{N}, b\rangle$ is a simple gluing as desired.

Assume that all CI quadratic semigroups of embedding dimension smaller than n may be obtained as in (a). Let H be a quadratic CI semigroup with $\operatorname{emb dim}(H) = n$. By Delorme's Theorem 2.13, $H = \langle c_1 U, c_2 V\rangle$, where U and V are CI with $\operatorname{emb dim}(U) = r$ and $\operatorname{emb dim}(V) = n - r$.

If $r = 1$, then by Theorem 2.3(b) $c_2 = 2$ and V must be quadratic; that is, $H = \langle c_1, 2V\rangle$. By the induction hypothesis, we may obtain V from $\mathbb{N}$ via quadratic gluings, and we are done. The case $n - r = 1$ is treated similarly.

Assume that $r > 1$ and $n - r > 1$. If $I_U = (f_1, \ldots, f_{r-1})$ and $I_V = (f_r, \ldots, f_{n-2})$, then from the proof of Delorme's Theorem 2.13, $I_H = I_U + I_V + (f_{n-1})$, where the gluing relation f_{n-1} is obtained in a similar way as in (6). Since H is quadratic, by Lemma 1.5 we get that $f_1, \ldots, f_{n-1}$ is a minimal standard basis of I_H, and since $\operatorname{height} I_H = \operatorname{height} I_H^* = n - 1$ we get that $f_1, \ldots, f_{n-1}$ and $f_1^*, \ldots, f_{n-1}^*$ are regular sequences. By Lemma 2.5 we obtain that $f_1, \ldots, f_{r-1}$ and $f_r, \ldots, f_{n-2}$ form a standard basis for I_U and I_V, respectively. This gives that U and V are quadratic CI semigroups.

By Theorem 1.1, $e(U) = 2^{r-1}$ and $e(V) = 2^{n-r-1}$. Without loss of generality we may assume that $e(H) = c_1 e(U)$. Then $c_1 = 2^{n-r}$ and $c_2 \in U$ is odd. By the induction hypothesis we may obtain V from a quadratic gluing: $V = \langle 2W, \ell\rangle$, where W is quadratic and CI, and $\ell \in W \setminus G(W)$ is odd. Hence, $e(W) = 2^{n-r-2}$. By Delorme's Theorem 2.13 the numerical semigroup $Z = \langle 2^{n-r-1}U, c_2 W\rangle$ is CI, obtained by gluing the CI semigroups U and W. Note that we may write

$$H = \langle c_1 U, c_2 V\rangle = \big\langle 2^{n-r}U, c_2\langle 2W, \ell\rangle\big\rangle = \langle 2^{n-r}U, 2c_2 W, c_2 \cdot \ell\rangle = \langle 2Z, c_2 \cdot \ell\rangle.$$

Since $c_2 \cdot \ell$ is odd and $c_2 \cdot \ell \in Z \setminus G(Z)$ (because $\ell \in W \setminus G(W)$) we may apply Corollary 2.7 to obtain that Z is a quadratic numerical semigroup. Since Z is CI and $\operatorname{emb\,dim}(Z) = n-2$, by the induction hypothesis it may be obtained from $\mathbb{N}$ by quadratic gluings, so the same is true for H.

The uniqueness of the decomposition follows from the fact that there is exactly one odd minimal generator for H. Hence, it must be chosen as ℓ_{n-1}. ☐

As an application of the gluing construction we present an infinite family of quadratic almost-CI semigroups satisfying the upper bound in Theorem 1.9(b).

EXAMPLE 2.15

Let $H_4 = \langle 6,7,8,9\rangle$. We may read the defining equations of $\operatorname{gr}_{\mathfrak{m}} K[H_4]$ from the proof of Proposition 3.1, and we have that H_4 is Koszul and $\mu(I^*_{H_4}) = 4$. Hence, $I^*_{H_4}$ is an almost-CI ideal. We recursively construct the semigroups

$$H_{n+4} = \langle 2H_{n+3}, 3^{n+2}\rangle, \quad \text{for all } n > 0.$$

It is easy to check by induction and by using Theorem 2.3 that for all $n > 0$:

(a) $3^{n+2} \in H_{n+3}$; hence, H_{n+4} is obtained by a simple gluing from H_{n+3} and $\operatorname{emb\,dim}(H_{n+4}) = n+4$;

(b) $H_{n+4} = \langle 2^n \cdot 6, 2^n \cdot 7, 2^n \cdot 8, 2^n \cdot 9, 2^{n-1} \cdot 3^3, 2^{n-2} \cdot 3^4, \ldots, 2 \cdot 3^{n+1}, 3^{n+2}\rangle$; hence, $e(H_{n+4}) = 2^{n+3} - 2^{n+1}$;

(c) $I_{H_{n+4}} = (x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_2x_3 - x_1x_4, x_1^3 - x_4^2) + (x_i^3 - x_{i+1}^2 : 4 \le i \le n+3)$;

(d) $I^*_{H_{n+4}} = (x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_2x_3 - x_1x_4) + (x_i^2 : 4 \le i \le n+4)$; hence, it is an almost-CI ideal.

More generally than Question 1.11 we may ask the following.

QUESTION 2.16

For a given $n > 1$ what is the possible multiplicity of any quadratic (or Koszul) semigroup H with $\operatorname{emb\,dim}(H) = n$?

The results presented so far and in the next section show that there are examples of Koszul semigroups whenever $n \le e(H) \le 2n-2$, $e(H) = 2^{n-1} - 2^{n-3}$, or $e(H) = 2^{n-1}$. We remark that the gluing construction described in Corollary 2.7 allows us to construct new quadratic (or Koszul) semigroups with double multiplicity and of embedding dimension increased by 1.

3. Quadraticity in some families of semigroups

Let H be a quadratic numerical semigroup of embedding dimension n. By the results described so far we have that

$$n \le e(H) \le 2^{n-1}. \tag{9}$$

These bounds are tight. Indeed, if $e(H) = n$, then H is quadratic by Proposition 1.3 and even G-quadratic by Proposition 1.12. The upper bound is reached, for example, in Example 2.12.

In this section we study the quadratic property in some families of numerical semigroups that have been considered in the literature.

3.1. Koszul arithmetic and geometric sequences

A sequence $a_1 < a_2 < \cdots < a_n$ of nonnegative integers is called an *arithmetic sequence* or a *geometric sequence* if there exists a d such that $a_{i+1} = d + a_i$ or $a_{i+1} = da_i$, respectively, for $i = 1, \dots, n-1$.

We show that the multiplicity of quadratic semigroups generated by an arithmetic sequence is in the lower part of the interval in (9), while for geometric sequences the multiplicity is maximal.

The next statement about the tangent cone of a numerical semigroup generated by an arithmetic sequence is of interest by itself. We could not locate this result in the literature, so for the convenience of the reader we give a proof, including the references on which our proof is based.

PROPOSITION 3.1

If the numerical semigroup H is generated by an arithmetic sequence $a_1 < \cdots < a_n$, then I_H^* is minimally generated by its reduced Gröbner basis with respect to the degree reverse lexicographic order induced by $x_1 > x_2 > \cdots > x_n$.

Proof

Let $S = K[x_1, \dots, x_n]$. Patil [24] proved that, under our hypothesis on H, the generators of the toric ideal I_H depend on the unique positive integers a and b with $1 \le b \le n-1$ such that $a_1 = a(n-1) + b$. Namely,

$$I_H = (x_i x_{j+1} - x_{i+1} x_j : 1 \le i < j \le n-1) + (x_n^a x_{b+i} - x_1^{a+d} x_i : 1 \le i \le n-b).$$

It is shown in the proof of [20, Proposition 2.5] that these generators of I_H are also a standard basis of I_H^* (alternatively see [29, Corollary 2.4(iii)]). Hence,

$$I_H^* = (x_i x_{j+1} - x_{i+1} x_j : 1 \le i < j \le n-1) + (x_n^a x_{b+i} : 1 \le i \le n-b).$$

Denote $f_{ij} = x_i x_{j+1} - x_{i+1} x_j$ for $1 \le i < j \le n-1$ and $g_i = x_n^a x_{b+i}$ for $1 \le i \le n-b$.

We verify Buchberger's criterion (see [13, Theorem 2.14]) for

$$\mathcal{G} = \{f_{ij} : 1 \le i < j \le n-1\} \cup \{g_i : 1 \le i \le n-b\}.$$

We first note that the ideal J generated by the f_{ij}'s is the ideal of 2-minors of the matrix of indeterminates

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ x_2 & x_3 & \cdots & x_n \end{pmatrix},$$

and it is the defining ideal of the rational normal scroll. We may also view J as the binomial edge ideal attached to the complete graph K_n. Since K_n is a closed graph, by [7, Theorem 1.1] we obtain that the f_{ij}'s form a Gröbner basis for J with respect to our term order. We refer to [7] for the unexplained terminology.

Since the S-pair of two monomials is zero, all that is left to show is that $S(f_{ij}, g_k) \xrightarrow{\mathcal{G}} 0$. For $1 \leq i < j \leq n-1$ the leading monomial $\mathrm{in}_<(f_{ij}) = x_{i+1}x_j$ is coprime to x_n. If $\gcd(\mathrm{in}_<(f_{ij}), g_k) = 1$, then $S(f_{ij}, g_k) \xrightarrow{\mathcal{G}} 0$ (see [13, Proposition 2.15]). Otherwise, if $\gcd(x_{i+1}x_j, g_k) \neq 1$ or, equivalently, $x_{b+k} \mid x_{i+1}x_j$, then we get that $b+k \in \{i+1, j\}$ and $b+k < j+1$. Therefore, $S(f_{ij}, g_k) = x_n^a x_i x_{j+1} = x_i(x_n^a x_{j+1})$ is a multiple of an element in $\mathcal{G}$ and $S(f_{ij}, g_k) \xrightarrow{\mathcal{G}} 0$ in this case, as well. ☐

The next statement describes the Koszul arithmetic sequences.

PROPOSITION 3.2

Fix $n \geq 3$. *Let* a_1 *and* d *be positive integers such that* $n \leq a_1$ *and* $\gcd(a_1, d) = 1$. *If we let*

$$H = \langle a_1, a_1 + d, \ldots, a_1 + (n-1)d \rangle,$$

then the following are equivalent:

(a) I_H^* *has a quadratic Gröbner basis;*
(b) $\mathrm{gr}_{\mathfrak{m}} K[H]$ *is Koszul;*
(c) I_H^* *is generated by quadrics;*
(d) $n \leq a_1 \leq 2n-2$.

Proof

With notation as in the proof of Proposition 3.1 we observe that I_H^* is generated by quadrics if and only if $a = 1$; that is, $n \leq a_1 \leq 2n-2$. Hence, (c) ⇒ (d). By Proposition 3.1 in these cases I_H^* has a quadratic Gröbner basis. Hence, (d) ⇒ (a). The rest of the implications are known to hold in general. ☐

The class of compound semigroups was recently introduced by Kiers, O'Neill, and Ponomarenko [21].

DEFINITION 3.3 ([21])

Consider the integers $2 \leq a_i < b_i$ such that $\gcd(a_i, b_i \cdots b_n) = 1$ for $i = 1, \ldots, n$. Let $q_i = b_1 \cdots b_{i-1} a_i \cdots a_n$, for $i = 1, \ldots, n+1$. The sequence $q_1, \ldots, q_{n+1}$ is called a *compound sequence*, and the semigroup $H = \langle q_1, \ldots, q_n \rangle$ is called a *compound semigroup*.

With notation as above, if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$, then $q_1, \ldots, q_{n+1}$ is a geometric sequence. In what follows we show that for any compound semigroup H we have that I_H^* is CI and this allows us to identify the quadratic (equivalently, Koszul) compound semigroups.

PROPOSITION 3.4

Let $2 < a_i < b_i$ *be positive integers such that* $\gcd(a_i, b_i \cdots b_n) = 1$ *for* $i = 1, \dots, n$. *Let*

$$H = \langle a_1 a_2 \cdots a_n, b_1 a_2 \cdots a_n, b_1 b_2 a_3 \cdots a_n, \dots, b_2 \cdots b_n \rangle.$$

The following hold:

(a) *the ideal* I_H^* *is a CI;*
(b) I_H^* *is quadratic* $\Longleftrightarrow$ H *is Koszul* $\Longleftrightarrow$ $a_i = 2$ *for* $i = 1, \dots, n$.

Proof

For part (a) we observe that H is obtained from another compound semigroup by a simple gluing:

$$H = \big\langle a_1 a_2 \cdots a_n, b_1 \langle a_2 \cdots a_n, b_2 a_3 \cdots a_n, \dots, b_2 \cdots b_n \rangle \big\rangle,$$

which gives a first (gluing) relation $f_1 = x_1^{b_1} - x_2^{a_1}$ in I_H. We continue decomposing into compound semigroups of smaller embedding dimension, and in the end we get

$$I_H = (x_1^{b_1} - x_2^{a_1}, x_2^{b_2} - x_3^{a_2}, \dots, x_n^{b_n} - x_{n+1}^{a_n}). \tag{10}$$

By Delorme's Theorem 2.13 we get that I_H is a CI.

If for any $f \in S = K[x_1, \dots, x_{n+1}]$ we denote by $\bar{f}$ the polynomial $f(0, x_2, \dots, x_{n+1})$ in $K[x_2, \dots, x_{n+1}]$, then one has $\bar{I}_H = (x_2^{a_1}, x_3^{a_2}, \dots, x_{n+1}^{a_{n+1}})$. These generators form a standard basis for the homogeneous ideal $\bar{I}_H$, and each of these generators can be lifted to an element in I_H with the same initial degree. Therefore, by the criterion in [17, Theorem 1] (see also [20, Lemma 1.2]) we conclude that the generators in (10) are a standard basis. Hence, $I_H^* = (x_2^{a_1}, x_3^{a_2}, \dots, x_n^{a_{n+1}})$ and I_H^* is a CI.

From this it follows that I_H^* is quadratic if and only if $a_1 = \cdots = a_n = 2$. The remaining equivalence from (b) is given by Theorem 2.14. ☐

The next result shows a family of CIs that are never quadratic.

PROPOSITION 3.5

Let $a_1, \dots, a_n$ *be pairwise coprime positive integers,* $n > 2$. *Let* $P = \prod_{i=1}^n a_i$. *The numerical semigroup* $H = \langle P/a_1, \dots, P/a_n \rangle$ *is a CI semigroup that is never quadratic.*

Proof

Without loss of generality assume that $a_1 > \cdots > a_n$. We prove the CI property by induction on n, and we identify a decomposition satisfying Delorme's Theorem 2.13. Letting $Q = P/a_1$, we may write

$$H = \big\langle P/a_1, a_1 \langle Q/a_2, \dots, Q/a_n \rangle \big\rangle.$$

Hence, H is obtained via a simple gluing from the semigroup $L = \langle Q/a_2, \dots, Q/a_n\rangle$, which is CI by the induction hypothesis. This gluing gives $I_H = (x_1^{a_1} - x_2^{a_2}, I_L)$, and after iterating this ungluing several times we obtain

$$I_H = (x_1^{a_1} - x_2^{a_2}, x_2^{a_2} - x_3^{a_3}, \dots, x_{n-1}^{a_{n-1}} - x_n^{a_n}).$$

We argue as in the proof of Proposition 3.4(b). Modulo x_1, $\bar{I}_H = (x_2^{a_2}, x_3^{a_3}, \dots, x_n^{a_n})$, whose (monomial) generators are a standard basis, and they may be naturally lifted to polynomials in I_H with the same initial degree. By the same criterion in [20, Lemma 1.2], I_H is generated by a standard basis and $I_H^* = (x_2^{a_2}, x_3^{a_3}, \dots, x_n^{a_n})$. Since $n > 2$ and the a_i's are coprime, it is clear that I_H^* is not generated in degree 2.

Second (partial) proof of the nonquadraticity. Given that H is CI, if it were also quadratic, by Theorem 1.1 we would have $P/a_1 = 2^{n-1}$. Since the a_i's are coprime, $P/a_1 = 2^{n-1}$ has at least $n-1$ distinct prime divisors, which is false for $n > 2$. ☐

3.2. Koszul 3-semigroups

We next describe the Koszul numerical semigroups of embedding dimension 3.

Let H be a quadratic numerical semigroup minimally generated by $a_1 < a_2 < a_3$. By Theorem 1.1, $e(H) \in \{3, 4\}$. For any of these two values, by Proposition 1.12 we get that H is Koszul.

Assume that $e(H) = 4$. By Theorem 1.1(c), H is a quadratic CI. Hence, by Theorem 2.14 it is obtained from $\mathbb{N}$ via quadratic gluings: $H = \langle 2\langle 2, c\rangle, \ell\rangle$, where c and ℓ are odd integers, $c > 1$, $\ell \in \langle 2, c\rangle \setminus \{2, c\}$, that is, $\ell = 2\alpha + c\beta$, $\beta = 2\gamma + 1$ is odd, and α and γ are not simultaneously equal to 0. Equivalently,

$$H = \langle 4, 2c, \ell\rangle = \langle 4, 2c, 2\alpha + c\beta\rangle = \big\langle 4, 2c, 2(\alpha + c\gamma) + c\big\rangle = \langle 4, 2c, 2a + c\rangle,$$

where a, c are positive integers with $c > 1$ odd. Here we denoted $a = \alpha + c\gamma$. Clearly, $a > 0$; otherwise, $H = \langle 4, c\rangle$ and $\operatorname{emb\,dim}(H) < 3$, a contradiction.

We group these findings into the next result.

PROPOSITION 3.6

Let H be a numerical semigroup with $\operatorname{emb\,dim}(H) = 3$. *The following are equivalent:*

(a) *H is a Koszul semigroup;*

(b) *H is a quadratic semigroup;*

(c) *$e(H) = 3$ or $e(H) = 4$ and $H = \langle 4, 2c, 2a + c\rangle$, where a, c are positive integers with $c > 1$ odd.*

T. Shibuta [30] has communicated to the second author that he could prove Proposition 3.6 by using [16] and [17].

3.3. Special almost-complete intersections

Let H be a numerical semigroup minimally generated by $a_1 < \cdots < a_n$, where $n > 2$. For $i = 1, \ldots, n$, let c_i be the smallest positive integer such that $c_i a_i$ is a sum of the other generators. This produces a binomial $f_i = x_i^{c_i} - m_i$ in I_H, where m_i is not divisible by x_i.

It is clear that $x_i^{c_i}$ is the least pure power of x_i that occurs as a term of any polynomial in I_H. If we are able to choose m_i that is not a pure power for any i, then the f_i's are distinct. If moreover they generate I_H, we say that H is a *special almost-CI semigroup*. By [16], any 3-generated numerical semigroup that is not a CI is a special almost-CI.

We note that, by Lemma 1.5, if H is quadratic and a special almost-CI numerical semigroup, then I_H^* is an almost-CI ideal.

PROPOSITION 3.7

Assume H is a quadratic and special almost-CI semigroup. Then I_H^* has a quadratic Gröbner basis with respect to the degree reverse lexicographic order if and only if $e(H) = 2^{n-1} - 2^{n-3}$. In particular, $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Koszul if $e(H) = 2^{n-1} - 2^{n-3}$.

Proof

With notation as above, if H is quadratic, by Lemma 1.6 we get that $c_1 > 2$ and $c_i = 2$ for all $i > 1$. By Lemma 1.5 we obtain that $I_H^* = (f_1^*, \ldots, f_n^*)$ and that it is a minimal generating set. Clearly, we have $\mathrm{in}_<(f_i^*) = x_i^2$ for $1 < i \le n$ and $\mathrm{in}_<(f_1^*) = x_j x_k$ for some $2 \le j < k < n$. We note that x_1 does not occur in any of the $\mathrm{in}_<(f_i^*)$'s. Hence,

$$\begin{aligned} e(H) = e(S/I_H^*) = e\big(S/\mathrm{in}_<(I_H^*)\big) &\le \ell\big(K[x_2, \ldots, x_n]/\big(\mathrm{in}_<(f_1^*), \ldots, \mathrm{in}_<(f_n^*)\big)\big) \\ &= \ell\big(K[x_2, \ldots, x_n]/(x_2^2, \ldots, x_n^2, x_j x_k)\big) \\ &= 2^{n-1} - 2^{n-3}, \end{aligned}$$

after a computation similar to the one in the proof of Theorem 1.5(b). We conclude that $e(H) = 2^{n-1} - 2^{n-3}$ if and only if $f_1^*, \ldots, f_n^*$ form a Gröbner basis for I_H^*. □

EXAMPLE 3.8

A quadratic special almost-CI of embedding dimension n need not be Koszul if $e(H) < 2^{n-1} - 2^{n-3}$. Indeed, let $H = \langle 11, 13, 14, 15, 19 \rangle$. Then

$$I_H = (x_1^3 - x_3 x_5, x_2^2 - x_1 x_4, x_3^2 - x_2 x_4, x_4^2 - x_1 x_5, x_5^2 - x_1 x_2 x_3).$$

As noticed in Remark 1.10, H is quadratic, but it is not a Koszul semigroup.

As an extension of Corollary 2.8 we have the following.

COROLLARY 3.9

Let L be a special almost-CI numerical semigroup that is quadratic, and let $\ell \in$

$L \setminus G(L)$ be an odd integer that is not a multiple of $e(L)$. Then $H = \langle 2L, \ell\rangle$ is a special almost-CI semigroup, too.

Proof

By Corollary 2.8 together with Lemma 1.5 we get that H is quadratic and that I_H is an almost-CI ideal. Let $n = \operatorname{emb\,dim}(H)$. If we denote by f the gluing relation, then using the convention that x_n corresponds to the new generator ℓ, we have that $I_H = (I_L, f)$, and the gluing relation f from (6) is of the form $x_n^2 - m$, where m is a monomial in the variables $x_1, \dots, x_{n-1}$.

We claim that we may choose f such that m is not a pure power. Indeed, if $m = x_1^c$ with $c > 1$, then ℓ is a multiple of $e(L)$, which contradicts our assumption. Assume that $m = x_i^c$ with $1 < i \le n-1$ and $c > 1$. By our assumption on L and Lemma 1.6, there exists an equation $f_i = x_i^2 - u$, where u is a monomial which is not a pure power. Then we can replace f by $x_n^2 - x_i^{c-2}u$. Since L is special almost-CI, we conclude that the same is true about H. □

REMARK 3.10

As a consequence of Corollary 3.9, starting from any quadratic special almost-CI semigroup, by gluing we can construct semigroups with these properties of any larger embedding dimension.

3.4. Symmetric and pseudosymmetric Koszul 4-semigroups

The *pseudo-Frobenius numbers* of the numerical semigroup H are the elements of the finite set

$$PF(H) = \{n \in \mathbb{Z} \setminus H : n + h \in H, \text{ for all } h \in H \setminus \{0\}\}.$$

The *Frobenius number* of H, usually defined as $g(H) = \max \mathbb{N} \setminus H$, also satisfies $g(H) = \max PF(H)$.

The semigroup H is called *symmetric* if for any integer n exactly one of n or $g(H) - n$ is in H. Algebraically, by a celebrated theorem of Kunz [23, Theorem, p. 749], H is symmetric if and only if $K[H]$ is a Gorenstein ring. One can check that H is symmetric if and only if $PF(H) = \{g(H)\}$. The semigroup H is called *pseudosymmetric* if $PF(H) = \{g(H)/2, g(H)\}$.

In the remainder of Section 3.4 we describe the 4-generated symmetric or pseudosymmetric numerical semigroups that are also Koszul.

3.4.1. The symmetric case

Let H be a symmetric numerical semigroup such that $\operatorname{emb\,dim}(H) = 4$. If H is CI and Koszul, then by Theorem 2.14 we have that H is obtained from $\mathbb{N}$ by a sequence of quadratic simple gluings. Using also Proposition 3.6 we have that $H = \langle 2\langle 4, 2c, 2a + c\rangle, \ell\rangle = \langle 8, 4c, 4a + 2c, \ell\rangle$, where a, c, ℓ are positive integers, $c, \ell > 1$ are odd, and $\ell \in \langle 4, 2c, 2a + c\rangle \setminus \{2a + c\}$.

If H is not CI, we employ the following characterization found by Bresinsky [5], as given by Barucci, Fröberg, and Şahin [4, Theorem 3].

THEOREM 3.11 (BRESINSKY [5, THEOREMS 5, 3])

The numerical semigroup $H = \langle a_1, a_2, a_3, a_4\rangle$ *is* 4*-generated symmetric, not a CI, if and only if there are integers* c_i*,* $1 \le i \le 4$*, and* α_{ij}*,* $ij \in \{21, 31, 32, 42, 13, 43, 14, 24\}$*, such that for* $0 < \alpha_{ij} < c_i$*, for all i,j,*

$$c_1 = \alpha_{21} + \alpha_{31}, \qquad c_2 = \alpha_{32} + \alpha_{42}, \qquad c_3 = \alpha_{13} + \alpha_{43}, \qquad c_4 = \alpha_{14} + \alpha_{24},$$

$$a_1 = c_2c_3\alpha_{14} + \alpha_{32}\alpha_{13}\alpha_{24}, \qquad a_2 = c_3c_4\alpha_{21} + \alpha_{31}\alpha_{43}\alpha_{24},$$

$$a_3 = c_1c_4\alpha_{32} + \alpha_{14}\alpha_{42}\alpha_{31}, \qquad a_4 = c_1c_2\alpha_{43} + \alpha_{42}\alpha_{21}\alpha_{13}.$$

Then $K[H] \cong S/(f_1, f_2, f_3, f_4, f_5)$*, where*

$$f_1 = x_1^{c_1} - x_3^{\alpha_{13}}x_4^{\alpha_{14}}, \qquad f_2 = x_2^{c_2} - x_1^{\alpha_{21}}x_4^{\alpha_{24}}, \qquad f_3 = x_3^{c_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}},$$

$$f_4 = x_4^{c_4} - x_2^{\alpha_{42}}x_3^{\alpha_{43}}, \qquad f_5 = x_3^{\alpha_{43}}x_1^{\alpha_{21}} - x_2^{\alpha_{32}}x_4^{\alpha_{14}}.$$

For quadratic, symmetric, and not CI semigroups we obtain the following classification result.

THEOREM 3.12

Let H *be a* 4*-generated semigroup that is symmetric and not a CI. The following are equivalent:*

(a) *H is Koszul;*
(b) *H is quadratic;*
(c) $e(H) = 5$*;*
(d) $H = \langle 5, 4a + b, 2a + 3b, 3a + 2b\rangle$ *for some positive integers* a, b *such that* $a - b$ *is not divisible by* 5.

Moreover, the integers a *and* b *in* (d) *are uniquely determined by* H.

Proof

The implication (a) ⇒ (b) is clear. For (b) ⇒ (c) assume that $H = \langle a_1, a_2, a_3, a_4\rangle$ is quadratic. Using Lemma 1.6 and Bresinsky's Theorem 3.11, without loss of generality we may assume that $c_1 > 2$ (i.e., $e(H) = a_1$) and $c_2 = c_3 = c_4 = 2$. The conditions $0 < \alpha_{ij} < c_i$ give $\alpha_{ij} = 1$ for $ij \in \{32, 42, 13, 43, 14, 24\}$; hence, $a_1 = c_2c_3\alpha_{14} + \alpha_{32}\alpha_{13}\alpha_{24} = 5$.

For (c) ⇒ (d), taking into account the restrictions in Theorem 3.11 we note that the equation

$$5 = c_2c_3\alpha_{14} + \alpha_{32}\alpha_{13}\alpha_{24}$$

holds only if $\alpha_{ij} = 1$ for $ij \in \{14, 32, 13, 24\}$ and if $c_2 = c_3 = 2$. The latter set of equalities yields $\alpha_{42} = \alpha_{43} = 1$. For brevity we denote $a = \alpha_{21}$ and $b = \alpha_{31}$. We plug these values into Bresinsky's theorem, and we get $a_2 = 4a + b$, $a_3 = 2a + 3b$, and $a_4 = 3a + 2b$.

We show that this parameterization is one-to-one. Let $a, b, a', b' > 0$ and $5 \nmid a-b$, $5 \nmid a'-b'$ such that

$$\langle 5, 4a+b, 2a+3b, 3a+2b\rangle = \langle 5, 4a'+b, 2a'+3b', 3a'+2b'\rangle.$$

Note that $4a+b, 3a+2b, 2a+3b$ and $4a'+b', 3a'+2b', 2a'+3b'$ are arithmetic sequences with common difference $b-a$, and $b'-a'$, respectively.

If $(b-a)(b'-a') < 0$, then $b-a = a'-b'$ and $5a+(b-a) = 4a+b = 2a'+3b' = 5b'+2(a'-b')$. Hence, $5 \mid b-a$, which is false.

If $(b-a)(b'-a') > 0$, then $b-a = b'-a'$ and $5a+(b-a) = 4a+b = 4a'+b' = 5a'+(b'-a')$. Hence, $(a,b) = (a',b')$, and we are done.

For (d) ⇒ (a) we first note by using Bresinsky's theorem that H is indeed symmetric and all the c_i's and the α_{ij}'s can be read from the proof of the implication (c) ⇒ (d).

Consequently, $I_H = (f_1, f_2, f_3, f_4, f_5)$, where

$$f_1 = x_1^{a+b} - x_3x_4, \qquad f_2 = x_2^2 - x_1^a x_4, \qquad f_3 = x_3^2 - x_1^b x_2,$$
$$f_4 = x_4^2 - x_2x_3, \qquad f_5 = x_3x_1^a - x_2x_4.$$

Modulo x_1 we get

$$\bar{I}_H = (x_3x_4, x_2^2, x_3^2, x_4^2 - x_2x_3, x_2x_4), \tag{11}$$

a monomial ideal whose generators (at the same time, a standard basis) may be lifted to the f_i's in I_H and keep the same initial degree. We apply the criterion in [20, Lemma 1.2] to conclude that x_1 is regular on S/I_H^* and $f_1, \ldots, f_5$ form a standard basis for I_H. Hence,

$$I_H^* = \begin{cases} (x_3x_4, x_2^2, x_3^2, x_4^2 - x_2x_3, x_2x_4) & \text{if } a \neq 1 \text{ and } b \neq 1, \\ (x_3x_4, x_2^2 - x_1x_4, x_3^2, x_4^2 - x_2x_3, x_3x_1 - x_2x_4) & \text{if } a = 1 \text{ and } b \neq 1, \\ (x_3x_4, x_2^2, x_3^2 - x_1x_2, x_4^2 - x_2x_3, x_2x_4) & \text{if } a \neq 1 \text{ and } b = 1. \end{cases} \tag{12}$$

It is easy to check that in each of these situations I_H^* has a quadratic Gröbner basis with respect to the degree reverse lexicographic order induced by $x_4 > x_3 > x_2 > x_1$. In particular, S/I_H^* is a Koszul ring. □

We verified with Singular [10] that in any of the three cases from (12) the ring S/I_H^* is Gorenstein. Together with Theorem 1.1(c) we obtain the following.

COROLLARY 3.13

Let H be a 4-generated symmetric and quadratic numerical semigroup. Then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Gorenstein.

3.4.2. The pseudosymmetric case

Four-generated pseudosymmetric semigroups were characterized by Komeda [22], where these were studied under the name *almost-symmetric*. In the formulation from [4], the following holds.

THEOREM 3.14 (KOMEDA [22, THEOREMS 6.4, 6.5])

The semigroup $H=\langle a_1,a_2,a_3,a_4\rangle$ *is pseudosymmetric if and only if there are positive integers* $c_i>1$, $1\le i\le 4$, *and* $0<\alpha_{21}<c_1-1$ *such that*

$$\begin{aligned}&a_1=c_2c_3(c_4-1)+1, \qquad a_2=\alpha_{21}c_3c_4+(c_1-\alpha_{21}-1)(c_3-1)+c_3,\\ &a_3=c_1c_4+(c_1-\alpha_{21}-1)(c_2-1)(c_4-1)-c_4+1,\\ &a_4=c_1c_2(c_3-1)+\alpha_{21}(c_2-1)+c_2,\end{aligned}$$

and $\gcd(a_1,a_2,a_3,a_4)=1$. *Then* $K[H]\cong S/(f_1,f_2,f_3,f_4,f_5)$, *where*

$$\begin{aligned}&f_1=x_1^{c_1}-x_3x_4^{c_4-1}, \qquad f_2=x_2^{c_2}-x_1^{\alpha_{21}}x_4, \qquad f_3=x_3^{c_3}-x_1^{c_1-\alpha_{21}-1}x_2,\\ &f_4=x_4^{c_4}-x_1x_2^{c_2-1}x_3^{c_3-1}, \qquad f_5=x_3^{c_3-1}x_1^{\alpha_{21}+1}-x_2x_4^{c_4-1}.\end{aligned}$$

The quadratic pseudosymmetric 4-semigroups are described by the following result.

PROPOSITION 3.15

Let $H=\langle a_1,a_2,a_3,a_4\rangle$ *be a pseudosymmetric numerical semigroup. The following are equivalent:*

(a) H *is Koszul;*
(b) H *is quadratic;*
(c) $H=\langle 5,3a+b+1,3b-a-2,a+2b+2\rangle$ *for some integers* $0<a<b-1$ *such that* $3a+b+1$ *is not a multiple of* 5.

Proof

For (b) ⇒ (c), by Lemma 1.6 and the restriction $0<\alpha_{21}<c_1-1$ in Komeda's Theorem 3.14, we get that $e(H)=a_1$ and $c_2=c_3=c_4=2$. This gives $a_1=5$, $a_2=3\alpha_{21}+c_1+1$, $a_3=3c_1-\alpha_{21}-2$, and $a_4=2c_1+\alpha_{21}+2$. Letting $a=\alpha_{21}$ and $b=c_1$ we obtain the desired description.

Note that $3a_2\equiv a_3 \bmod 5$ and $2a_2\equiv a_4 \bmod 5$. Therefore, $\gcd(a_1,a_2,a_3,a_4)=1$ precisely when $3a+b+1$ is not a multiple of 5.

For (c) ⇒ (a), by Komeda's theorem we have

$$I_H=(x_1^b-x_3x_4,x_2^2-x_1^ax_4,x_3^2-x_1^{b-a-1}x_2,x_4^2-x_1x_2x_3,x_3x_1^{a+1}-x_2x_4).$$

Modulo x_1 it becomes $\bar{I}_H=(x_3x_4,x_2^2,x_3^2,x_4^2,x_2x_4)$. These monomials can be lifted to polynomials in I_H with the same initial degree. Hence, by using the criterion in [20, Lemma 1.2], I_H is generated by a standard basis, and x_1 is a regular element (of degree 1) on S/I_H^*. From here we notice that I_H^* is generated in degree 2. Since $(S/I_H^*)/x_1(S/I_H^*)\cong K[x_2,x_3,x_4]/(x_3x_4,x_2^2,x_3^2,x_4^2,x_2x_4)$, which is Koszul, by [3, Lemma 2], S/I_H^* is Koszul as well. □

Acknowledgments. The use of CoCoA [1] and Singular [10] was vital for the development of this article. We wish to thank their respective teams of developers.

We thank M. E. Rossi and G. Valla for pointing us toward their work in [27], which gives evidence to support our Question 1.11, and we also thank M. E. Rossi for suggesting Remark 1.14.

Important progress on this research took place during the authors' visits to the coauthor's home institution. We are grateful for their hospitality.

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