Ergodic actions of compact quantum groups from solutions of the conjugate equations

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Dedicated by the first-named author to the memory of John E. Roberts, with admiration and devotion

Abstract We use a tensor C^*-category with conjugates and two quasitensor functors into the category of Hilbert spaces to define a *-algebra depending functorially on this data. If one of them is tensorial, we can complete in the maximal C^*-norm. A particular case of this construction allows us to begin with solutions of the conjugate equations and associate ergodic actions of quantum groups on the C^*-algebra in question. The quantum groups involved are $A_u(Q)$ and $B_u(Q)$.

1. Introduction

The theory of ergodic actions of compact quantum groups on unital C^*-algebras has recently attracted interest. In the group case, one of the first results was the theorem by Høegh-Krohn, Landstad, and Størmer [9] asserting that the multiplicity of an irreducible representation is always bounded by its dimension and that the unique G-invariant state is a trace.

Ergodic theory for group actions was later investigated by Wassermann [21]–[23], who classified all ergodic actions of $\mathrm{SU}(2)$ on von Neumann algebras, among other results. In particular, he proved the important result that $\mathrm{SU}(2)$ cannot act ergodically on the hyperfinite II_1 factor.

For compact quantum groups, ergodic theory on C^*-algebras was initiated by Boca. He generalized the Høegh-Krohn–Landstad–Størmer theorem, showing that the multiplicity of an irreducible is bounded instead by its quantum dimension. Woronowicz [25] noticed that the modular group of a compact quantum group is not always trivial; consequently, the invariant state cannot be a trace in general. Boca [4] described the modularity of this state for a general ergodic action.

Podleś [15] was interested in studying quantum spheres: he introduced subgroups and quotients for compact quantum groups and computed them for the

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quantum SU(2) and SO(3) groups. Some of the quantum spheres he found are not embeddable into the quotient spaces.

Later Wang [19] found many examples of ergodic quantum actions on C^*-algebras: for the quantum groups $A_u(Q)$ on type III_λ Powers factors, on the Cuntz algebras, on the injective factor of type III_1, and on the hyperfinite II_1 factor (this, by a Kac-type quantum group). He also found an example on a commutative C^*-algebra that is not a quotient.

Classifying the ergodic C^*-actions of the quantum group $\mathrm{S}_\mu\mathrm{U}(2)$ of Woronowicz is an open problem. Tomatsu [17] has classified all those which are embedded in the translation action of $\mathrm{S}_\mu\mathrm{U}(2)$.

Bichon, De Rijdt, and Vaes [3] have constructed examples of ergodic actions of $\mathrm{S}_\mu\mathrm{U}(2)$ not embeddable in the translation action, since the multiplicity of an irreducible is bigger than its integral dimension. The authors also introduced a new invariant, the quantum multiplicity $m(u)$ of an irreducible representation u in the action. This invariant reduces to the quantum dimension for the translation action. In general, one has the bounds: multiplicity$(u) \leq m(u) \leq$ q-dim(u). Even for quotient actions, the quantum multiplicity is not an integer in general. These examples were constructed by means of a generalization of the Tannaka–Kreĭn duality theorem for compact quantum groups (see [26]) to ergodic actions of full multiplicity.

The authors [11] in turn extended the duality theorem of [3] to general ergodic actions of compact quantum groups on unital C^*-algebras. To this aim, we introduced the notion of quasitensor functor between two tensor C^*-categories with conjugates, and we showed that quasitensor functors from the representation category of a compact quantum group G to the category of Hilbert spaces characterize the spectral functors of ergodic C^*-actions of G.

As an application, we constructed ergodic actions of $\mathrm{S}_\mu\mathrm{U}(d)$ starting from abstract tensor C^*-categories with a Hecke symmetry of parameter $q = \mu^2$. In particular, one gets ergodic actions of $\mathrm{S}_\mu\mathrm{U}(2)$ from a real or pseudoreal object y of a tensor C^*-category with intrinsic dimension $d(y) \geq 2$, with μ and $d(y)$ related by $d(y) = |\mu + \mu^{-1}|$ and μ positive if y is pseudoreal and negative otherwise. Our interest in braiding was motivated by low-dimensional quantum field theory, where braided tensor C^*-categories arise, albeit with a *unitary* braiding (see [8]).

The aim of this article is twofold. We first give an alternative notion of quasitensor functor, and we show the equivalence with that of [11], as well as the construction of the mentioned C^*-ergodic action of G. Furthermore, we apply this construction to obtain ergodic actions of compact quantum groups starting from solutions of the conjugate equation in a tensor C^*-category.

This article has a sequel, that is, [13], where we start precisely from the noncommutative space obtained here to construct Hilbert bimodule representations of compact quantum groups arising from tensor C^*-categories generated by an object of intrinsic dimension at least 2. In this sense, our ergodic actions should be regarded as virtual quantum subgroups.

Furthermore, as an application, we get ergodic actions of $\mathrm{S}_\mu\mathrm{U}(2)$ for negative or positive values of the deformation parameter uniquely determined by the intrinsic dimension. Ergodic actions of the quantum groups $B_u(Q)$ and $A_u(Q)$ of Wang appear as well.

The article is organized as follows. In Section 2 we review the main facts about compact quantum groups (see [27]), the main invariants of ergodic C^*-actions, and the duality theorem of [11].

In Section 3 we state our main results: the existence of ergodic C^*-actions of the compact quantum group $A_u(Q)$, in the notation of Wang, associated with an invertible positive matrix $Q \in M_n(\mathbb{C})$ with $\mathrm{Tr}(Q) = \mathrm{Tr}(Q^{-1})$ arising from normalized solutions $R, \overline{R}$ of the conjugate equations. Note that Q and R are related by $\mathrm{Trace}(Q) = R^* \circ R$.

For self-conjugate solutions of the conjugate equations we also get ergodic C^*-actions of the compact quantum group $B_u(Q)$, in the notation of Wang, associated with an invertible matrix $Q \in M_n(\mathbb{C})$ with $Q\overline{Q} = \pm 1$ and hence $\mathrm{Tr}\, Q^*Q = \mathrm{Tr}(Q^*Q)^{-1}$. Now Q and R are related by $\mathrm{Trace}(Q^*Q) = R^* \circ R$ and ± 1 distinguishes real from pseudoreal solutions.[1]

In Section 4, we generalize, at an algebraic level, the construction of the *-algebra carrying the ergodic action giving a formalism symmetric in two quasitensor functors and discuss its functorial properties. We then complete in the maximal C^*-norm when one of the two functors is tensorial. The general case will be considered elsewhere (see [14]). The action itself is defined in Section 5. Section 6 recalls some properties of the Temperley–Lieb categories associated with self-conjugate solutions of the conjugate equations, while Section 7 is devoted to related categories needed for treating general solutions of the conjugate equations. Section 8 treats the embeddings of these categories into the category of Hilbert spaces and the associated compact quantum groups and concludes with the proof of the main results.

2. Preliminaries

In this preliminary section we recall the main invariants associated with an ergodic action of a compact quantum group G on a unital C^*-algebra $\mathcal{C}$ and the duality theorem of [11].

2.1. Compact quantum groups

We follow Woronowicz [27] in defining a compact quantum group G to be a pair $G = (\mathcal{Q}, \Delta)$, where $\mathcal{Q}$ is a C^*-algebra with unit I and $\Delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$ is a unital coassociative *-homomorphism, the *coproduct*

$$\Delta \otimes \iota \circ \Delta = \iota \otimes \Delta \circ \Delta,$$

[1] Banica denotes $A_u(Q)$ by $A_u(F)$, where $Q = F^*F$, and $B_u(Q)$ by $A_o(F)$, where $Q = F^*$ (see [2, Définition 2]).

with $\iota : \mathcal{Q} \to \mathcal{Q}$ the identity map. To economize on brackets we will always evaluate tensor products before composition. The coproduct is required to satisfy the following nondegeneracy condition: the subspaces $I \otimes \mathcal{Q}\Delta(\mathcal{Q})$ and $\mathcal{Q} \otimes I\Delta(\mathcal{Q})$ are dense in $\mathcal{Q} \otimes \mathcal{Q}$.

A unitary *representation* of G on a finite-dimensional Hilbert space H_u is a linear map $u : H_u \to H_u \otimes \mathcal{Q}$ satisfying the group homomorphism property, nondegeneracy, and unitarity, expressed, respectively, by

$$u \otimes \iota \circ u = \iota \otimes \Delta \circ u,$$

$$u(H_u)(I \otimes \mathcal{Q}) = H_u \otimes \mathcal{Q},$$

$$\big(u(\psi), u(\psi')\big)_{\mathcal{Q}} = (\psi, \psi')I,$$

where on the left-hand side of the last relation we have the natural $\mathcal{Q}$-valued inner product of the right Hilbert module $H_u \otimes \mathcal{Q}$

$$(\psi \otimes q, \psi' \otimes q')_{\mathcal{Q}} := (\psi, \psi')q^* q', \quad \psi, \psi' \in H_u, q, q' \in \mathcal{Q}.$$

The *coefficients* of u are elements of $\mathcal{Q}$ defined by $u_{\psi,\psi'} := \psi^* \otimes I u(\psi')$, where $\psi^* : H_u \to \mathbb{C}$ is the annihilation operator $\psi^*\psi' := (\psi, \psi')$. Representation coefficients span a dense *-subalgebra of $\mathcal{Q}$. If u and v are two representations, we can form the *tensor product* representation $u \otimes v$ on the tensor product Hilbert space $H_u \otimes H_v$, defined by

$$(u \otimes v)_{\psi\otimes\phi,\psi'\otimes\phi'} := u_{\psi,\psi'} v_{\phi,\phi'}. \tag{2.1}$$

A *conjugate* of u is a unitary representation $\overline{u}$ with an antilinear invertible $j : H_u \to H_{\overline{u}}$ such that

$$\overline{u}_{\phi,j\psi} = (u_{j^*\phi,\psi})^*. \tag{2.2}$$

A conjugate $\overline{u}$ of u is defined up to unitary equivalence. Every representation has a conjugate representation (see [25]). The category $\mathrm{Rep}(G)$ with objects being unitary representations of G and arrows being the intertwining operators

$$(u, v) := \{A : H_u \to H_v : v \circ A = A \otimes I \circ u\}$$

is a tensor C^*-category with conjugates (and also subobjects and direct sums) in the sense of [10]. Furthermore, $\mathrm{Rep}(G)$ embeds naturally as a tensor *-subcategory of the category $\mathcal{H}$ of Hilbert spaces. Conversely, any tensor *-subcategory of $\mathcal{H}$ with conjugation, subobjects, and direct sums is the representation category of a compact quantum group (see [26]). The quantum groups of interest here are the Woronowicz deformations $\mathrm{S}_\mu\mathrm{U}(2)$ by a nonzero real parameter μ (see [26]) and the Van Daele–Wang orthogonal groups and unitary groups $B_u(Q)$ and $A_u(Q)$ associated with an invertible matrix $Q \in M_n(\mathbb{C})$ (see [18]).

2.2. Ergodic C^*-actions

Consider a unital C^*-algebra $\mathcal{C}$ and a compact quantum group G. An *action* of G on $\mathcal{C}$ is a unital *-homomorphism $\delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{Q}$ satisfying the group representation property

$$\iota \otimes \Delta \circ \delta = \delta \otimes \iota \circ \delta$$

and the nondegeneracy property requiring that $\delta(\mathcal{C})I \otimes \mathcal{Q}$ be dense in $\mathcal{C} \otimes \mathcal{Q}$. The *spectrum* of δ, $\mathrm{sp}(\delta)$, is defined to be the set of all unitary representations u of G for which there is a faithful linear map $T : H_u \to \mathcal{C}$ intertwining the representation u with the action δ:

$$\delta \circ T = T \otimes \iota \circ u.$$

In other words, if the u_{ij}'s are the coefficients of u in some orthonormal basis of H, we are requiring the existence of a spectral multiplet of linearly independent elements $c_1, \dots, c_d \in \mathcal{C}$, with d the dimension of u, transforming like u under the action $\delta(c_i) := \sum_j c_j \otimes u_{ji}$. The linear span of all the c_i's, denoted $\mathcal{C}_{\mathrm{sp}}$, as u varies in the spectrum, is a dense *-subalgebra of $\mathcal{C}$ (see [15]).

The action δ is called *ergodic* if the fixed-point algebra

$$\mathcal{C}^\delta = \{c \in \mathcal{C} : \delta(c) = c \otimes I\}$$

reduces to the complex numbers: $\mathcal{C}^\delta = \mathbb{C}I$. The simplest example of an ergodic action is the translation action of G on $\mathcal{C} = \mathcal{Q}$ with $\delta = \Delta$. Another simple class of examples is the adjoint actions on $B(H_u)$, where u is an irreducible unitary representation. The spectrum then consists of the subrepresentations of $u \otimes \overline{u}$.

If an action δ is ergodic, the spectral multiplets transforming like u form Hilbert spaces. In fact, for any representation u, consider the space

$$L_u := \{T : H_u \to \mathcal{C}, \delta \circ T = T \otimes \iota \circ u\}.$$

If $S, T \in L_u$, then $\langle S, T\rangle := \sum_i T(\psi_i)S(\psi_i)^*$, where (ψ_i) is an orthonormal basis of H_u, is an element of the fixed-point algebra $\mathcal{C}^\delta$ and, hence, a complex number. It is known that L_u is finite-dimensional and is therefore a Hilbert space with the above inner product. This Hilbert space is nonzero precisely when u contains a subrepresentation $v \in \mathrm{sp}(\delta)$. In particular, for an irreducible u, the conditions $u \in \mathrm{sp}(\delta)$ and $L_u \neq 0$ are equivalent. The dimension of L_u is called the *multiplicity* of u and denoted $\mathrm{mult}(u)$.

The complex conjugate vector space $\overline{L_u}$, endowed with the conjugate inner product

$$\langle \overline{S}, \overline{T}\rangle := \langle T, S\rangle = \sum_i S(\psi_i)T(\psi_i)^*,$$

is called the *spectral space* associated with u. If for example δ is the translation action on $\mathcal{Q}$, then any $\psi \in H_u$ defines an element of L_u by

$$T_\psi(\psi') := \psi^* \otimes I u(\psi').$$

Hence, the spectral space $\overline{L_u}$ can be identified with H_u through the unitary map

$$\psi \in H_u \to \overline{T_\psi} \in \overline{L_u}.$$

For a general ergodic action, we introduce certain maps whose coefficients generate the dense *-subalgebra $\mathcal{C}_{\mathrm{sp}}$, as the representations do in the case of the translation action.

For any $u \in \mathrm{Rep}(G)$, define the map

$$c_u : H_u \to \overline{L_u} \otimes \mathcal{C}$$

associated with the spectral space $\overline{L_u}$ by

$$c_u(\psi) := \sum_k \overline{T_k} \otimes T_k(\psi), \tag{2.3}$$

where T_k is any orthonormal basis of L_u. Clearly c_u is determined by its coefficients

$$c^u_{\overline{T},\psi} := \overline{T}^* \otimes I c_u(\psi) = T(\psi), \quad \psi \in H_u, T \in L_u. \tag{2.4}$$

The c_u's are called *multiplicity maps* in [11]. In the example of the translation action identifying $\overline{L_u}$ and H_u we have $c_u = u$. We can represent c_u as a rectangular matrix whose kth row is given by the multiplet $T_k = (T_k(\psi_1) \cdots T_k(\psi_d))$ transforming like u under δ.

It is known that the set of all coefficients $\{c^u_{\overline{T_k},\psi_j} = T_k(\psi_j), j, k\}$ of the multiplicity maps in orthonormal bases forms a linear basis for the dense *-subalgebra $\mathcal{C}_{\mathrm{sp}}$ (see [3], [11]) when u varies in a complete set of irreducible representations of $\mathrm{sp}(\delta)$, generalizing a well-known property of matrix coefficients of a compact quantum group (see [27]).

Bichon, De Rijdt, and Vaes [3] introduce a new numerical invariant, the *quantum multiplicity* $m(u)$ of the representation u, in the following way. If $j : H_u \to H_{\overline{u}}$ defines a conjugate representation of u in the sense recalled in the previous section, then we can associate an invertible antilinear $J : L_u \to L_{\overline{u}}$ with J by setting $J(T)(\phi) := T(j^{-1}(\phi))^*$. Its inverse $J^{-1} : L_{\overline{u}} \to L_u$ is given by $J^{-1}(S)(\psi) = S(j(\psi))^*$. If u is irreducible, then $m(u)^2 := \mathrm{Trace}(JJ^*)\,\mathrm{Trace}((JJ^*)^{-1})$. One has

$$\mathrm{mult}(u) \leq m(u) \leq d(u),$$

an inequality which strengthens the inequality $\mathrm{mult}(u) \leq d(u)$ previously obtained by Boca [4] when generalizing the Hoegh-Krohn–Landstad–Stormer theorem [9] $\mathrm{mult}(u) \leq \dim(u)$ in the group case. If u is reducible, we define $m(u)$ as the infimum of all the above trace values, derived from all possible solutions of the conjugate equations for u. Then the inequality

$$\dim(L_u) \leq m(u) \leq d(u) \tag{2.5}$$

holds for all representations u. Notice that $m(u)$ takes the smallest possible value $\dim(L_u)$ precisely when for some j the associated J is a scalar multiple of an antiunitary. Examples of ergodic actions of $\mathrm{S}_\mu\mathrm{U}(2)$ where $\dim(u) < \mathrm{mult}(u) < m(u) = d(u)$ have been constructed in [3].

2.3. The spectral functor of an ergodic action and quasitensor functors

It has been shown in [11] that ergodic actions of compact quantum groups on unital C^*-algebras have a duality theory resembling the duality theory of Woronowicz for compact quantum groups: an ergodic G-action on $\mathcal{C}$ has a dual object allowing one to reconstruct the G-action on the maximal completion of $\mathcal{C}_{\mathrm{sp}}$. Furthermore, the dual objects of ergodic actions have been characterized.

The map $u \mapsto \overline{L_u}$ can be extended to a *functor*

$$\overline{L} : \mathrm{Rep}(G) \to \mathcal{H}$$

from the category of representations of G to the category $\mathcal{H}$ of Hilbert spaces. This functor is defined on arrows as follows.

If $A \in (u,v)$ and $T \in L_v$, then $T \circ A : H_u \to \mathcal{C}$ lies in L_u. Hence, if we identify $\overline{L_u}$ canonically with the dual vector space of L_u, then any arrow $A \in (u,v)$ in $\mathrm{Rep}(G)$ induces a linear map $\overline{L}_A \in (\overline{L_u}, \overline{L_v})$ via the natural pairing between L_u and $\overline{L_u}$

$$\overline{L}_A : \varphi \in \overline{L}_u \to \big(T \in L_v \to \varphi(T \circ A)\big) \in \overline{L_v}.$$

The spectral functor $\overline{L}$ and the multiplicity maps c_u are related as

$$\overline{L}_A \otimes I \circ c_u = c_v \circ A, \quad A \in (u,v),$$

for any $u, v \in \mathrm{Rep}(G)$. In terms of the matrix coefficients of c_u this reads

$$c^u_{\overline{L_{A^*}\, S},\psi} = c^v_{\overline{S},A\psi}, \quad A \in (u,v), \psi \in H_u, S \in L_v. \tag{2.6}$$

Taking the tensor C^*-category structure of $\mathrm{Rep}(G)$ and $\mathcal{H}$ into account, one can see that $\overline{L}$ is a *-functor but *not* a tensor *-functor, in general.

In fact, for $u, v \in \mathrm{Rep}(G)$, the tensor product Hilbert space $\overline{L_u} \otimes \overline{L_v}$ is in general just a subspace of $\overline{L_{u\otimes v}}$, in the sense that there is a natural isometric inclusion

$$\tilde{\overline{L}}_{u,v} : \overline{L_u} \otimes \overline{L_v} \to \overline{L_{u\otimes v}}$$

identifying a simple tensor $\overline{S} \otimes \overline{T}$ with the complex conjugate of the element of $L_{u\otimes v}$ defined by

$$\psi \otimes \phi \in H_u \otimes H_v \to S(\psi)T(\phi).$$

The main result of [11] characterizes the set of all ergodic action duals $(\overline{L}, \tilde{\overline{L}})$ algebraically among all *-functors

$$\tau : \mathrm{Rep}(G) \to \mathcal{H}$$

endowed with isometries $\tilde{\tau}_{u,v} : \tau_u \otimes \tau_v \to \tau_{u\otimes v}$. These are precisely the *quasitensor functors*, defined below.

We shall refer to [5] for the notion of an abstract (strict) tensor C^*-category $\mathcal{T}$. The tensor product between objects u and v will be denoted by $u \otimes v$ and between arrows S and T by $S \otimes T$. The tensor unit object will be denoted ι. We shall assume that ι is an irreducible object: $(\iota,\iota) = \mathbb{C}$, unless otherwise specified. The nth tensor power of an object u will be denoted u^n. When we refer to a tensor

C^*-category of Hilbert spaces, we mean that the objects are finite-dimensional Hilbert spaces and contain Hilbert spaces of any finite dimension. The spaces of arrows are all linear operators between the Hilbert spaces in question.

Let $\mathcal{T}$ and $\mathcal{R}$ be strict tensor C^*-categories. A *-functor $\tau : \mathcal{T} \to \mathcal{R}$ together with a collection of isometries $\tilde{\tau}_{u,v} \in (\tau_u \otimes \tau_v, \tau_{u\otimes v})$, for objects $u, v \in \mathcal{T}$, is called a *quasitensor* if

$$\tau_\iota = \iota, \tag{2.7}$$

$$\tilde{\tau}_{u,\iota} = \tilde{\tau}_{\iota,u} = 1_{\tau_u}, \tag{2.8}$$

$$\tilde{\tau}^*_{u,v\otimes w} \circ \tilde{\tau}_{u\otimes v,w} = 1_{\tau_u} \otimes \tilde{\tau}_{v,w} \circ \tilde{\tau}^*_{u,v} \otimes 1_{\tau_w} \tag{2.9}$$

and if

$$\tau(S \otimes T) \circ \tilde{\tau}_{u,v} = \tilde{\tau}_{u',v'} \circ \tau(S) \otimes \tau(T), \tag{2.10}$$

for any other pair of objects u', v' and arrows $S \in (u, u')$, $T \in (v, v')$. In particular, a *tensor* functor τ is quasitensor with $\tilde{\tau}_{u,v} := 1_{\tau_u \otimes \tau_v}$, as $\tau_{u\otimes v} = \tau_u \otimes \tau_v$ for all objects u, v, (2.7) and (2.8) hold by assumption, and (2.9) and (2.10) are trivially satisfied. More generally, if all the isometries $\tilde{\tau}_{u,v}$ are unitary, we recover the known notion of a *relaxed tensor* functor. This definition of a quasitensor functor differs from that given in [11], and the equivalence is established in the Appendix.

Given a quasitensor functor $(\mu, \tilde{\mu})$ into the category of Hilbert spaces, let τ denote the embedding functor of the category of finite-dimensional unitary representations of a compact quantum group G into the category of Hilbert spaces. Then, as shown in [11], there is a canonical ergodic action of G on a C^*-algebra ${}_\mu\mathcal{C}_\tau$. If μ is the spectral functor of an ergodic action of G on a C^*-algebra $\mathcal{B}$, then μ is isomorphic to the spectral functor of the action on ${}_\mu\mathcal{C}_\tau$ and the dense spectral subalgebras of $\mathcal{B}$ and ${}_\mu\mathcal{C}_\tau$ are canonically isomorphic. However, $\mathcal{B}$ and ${}_\mu\mathcal{C}_\tau$ need not be isomorphic. Note that ${}_\mu\mathcal{C}_\tau$ is the completion of its dense spectral subalgebra in the maximal C^*-norm, and this may not be the case for $\mathcal{B}$.

3. The main results

In this section we state our main results.

THEOREM 3.1

Let x be an object of a tensor C^*-category with irreducible tensor unit ι, and let $R \in (\iota, x^2)$ satisfy $R^* \otimes 1_x \circ 1_x \otimes R = \pm 1_x$ and $\|R\|^2 \geq 2$. For any integer $2 \leq n \leq \|R\|^2$, let $Q \in M_n(\mathbb{C})$ be any invertible matrix satisfying

$$Q\overline{Q} = \pm I, \quad \mathrm{Trace}(Q^*Q) = \mathrm{Trace}\big((Q^*Q)^{-1}\big) = \|R\|^2.$$

Then there is an ergodic action of the compact quantum group $B_u(Q)$ of Wang [20] on a unital C^*-algebra $\mathcal{C}$ with spectral spaces $\overline{L}_{u^r} = (\iota, x^r)$, $r \geq 0$, and $\overline{L}_{\sum_k \psi_k \otimes Q^* \psi_k} = R$, where the sum is taken over an orthonormal basis ψ_k of the Hilbert space of u. If $m(u) = \dim(\iota, x)$, u being the defining representation of

$B_u(Q)$, then

$$m(u^r) = \dim(\iota, x^r).$$

In particular, if we choose $n = 2$, we get an ergodic action of $\mathrm{S}_\mu\mathrm{U}(2)$ for a nonzero $-1 < \mu < 1$ determined by $|\mu + \mu^{-1}| = \|R\|^2$, where $\mu > 0$ if and only if x is pseudoreal.

In the examples derived from subfactors and treated in [12], we do have $m(u) = \dim(\iota, x)$.

THEOREM 3.2

Let x be an object of a tensor C^*-category with irreducible tensor unit ι, and let $R \in (\iota, \overline{x} \otimes x)$ and $\overline{R} \in (\iota, x \otimes \overline{x})$ satisfy $R^* \otimes 1_{\overline{x}} \circ 1_{\overline{x}} \otimes R = 1_{\overline{x}}$, $\overline{R}^* \otimes 1_x \circ 1_x \otimes R = 1_x$, and $\|R\|^2 = \|\overline{R}\|^2 \geq 2$. For any integer $2 \leq n \leq \|R\|^2$, let $Q \in M_n(\mathbb{C})$ be any positive invertible matrix satisfying

$$\mathrm{Trace}(Q) = \mathrm{Trace}(Q^{-1}) = \|R\|^2.$$

Then there is an ergodic action of the compact quantum group $A_u(Q)$ of Wang on a unital C^*-algebra $\mathcal{C}$ with spectral spaces $\overline{L}_{q(u,\overline{u})} = (\iota, q(x, \overline{x}))$, where q is a monomial in two variables and u is the defining representation of $A_u(Q)$. If $m(u) = \dim(\iota, x)$, then

$$m\big(q(u, \overline{u})\big) = \dim\big(\iota, q(x, \overline{x})\big),$$

for each q.

The proofs involve two main steps. The first is to embed the tensor *-subcategory generated by R or by R and $\overline{R}$ into the category of Hilbert spaces. The second step is to define the ergodic action by applying the duality theorem for ergodic actions of compact quantum groups on unital C^*-algebras proved in [11]. The construction of the C^*-algebra will be given in Section 4 in greater generality than in [11], and the G-action is explained in Section 5.

4. C^*-algebras from pairs of quasitensor functors

Let $\mathcal{A}$ be a tensor C^*-category with conjugates, and let $(\mu, \tilde{\mu}) : \mathcal{A} \to \mathcal{M}$ and $(\tau, \tilde{\tau}) : \mathcal{A} \to \mathcal{T}$ be quasitensor functors. We let ${}^\circ_\mu\mathcal{C}_\tau$ be the linear space $\sum_u (\mu_u, \iota) \otimes (\iota, \tau_u)$, the sum being taken over the objects of $\mathcal{A}$, quotiented by the linear subspace generated by elements of the form

$$M \circ \mu(A) \otimes T - M \otimes \tau(A) \circ T.$$

Thus, we may write ${}^\circ_\mu\mathcal{C}_\tau = \sum_u (\mu_u, \iota) \otimes_{\mathcal{A}} (\iota, \tau_u)$. We next define a product on ${}^\circ_\mu\mathcal{C}_\tau$ by setting, for $L \in (\mu_u, \iota)$, $M \in (\mu_v, \iota)$, $S \in (\iota, \tau_u)$, and $T \in (\iota, \tau_v)$,

$$(L \otimes S)(M \otimes T) := (L \otimes M) \circ \tilde{\mu}^*_{u,v} \otimes \tilde{\tau}_{u,v} \circ (S \otimes T).$$

It is easy to check that the product is well defined and associative.

When either $(\tau,\tilde{\tau})$ or $(\mu,\tilde{\mu})$ is minimal in the sense defined in the Appendix, ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ reduces to the complex numbers. The reason is that this algebra does not change if we complete $\mathcal{A}$ under direct sums and subobjects and extend $(\tau,\tilde{\tau})$. Every object of $\mathcal{A}$ is then a direct sum of irreducibles, and it becomes clear that we can restrict the sum over u in the definition of ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ to a representative set of irreducibles. But then $(\iota,\mu_u)=0$ unless $u=\iota$ so ${}^{\circ}_{\mu}\mathcal{C}_{\tau}=(\iota,\iota)$.

Tensor C^*-categories with conjugates have been studied in [10]. We recall the notion of conjugate object $\overline{u}$ of u. This object is defined, up to unitary equivalence, by the existence of two intertwiners $R\in(\iota,\overline{u}\otimes u)$ and $\overline{R}\in(\iota,u\otimes\overline{u})$ satisfying the conjugate equations

$$\overline{R}^*\otimes 1_u\circ 1_u\otimes R=1_u, \tag{4.1}$$

$$R^*\otimes 1_{\overline{u}}\circ 1_{\overline{u}}\otimes \overline{R}=1_{\overline{u}}. \tag{4.2}$$

The *intrinsic dimension* $d(u)$ of u is the infimum of all possible $\|R\|\|\overline{R}\|$.

If G is a compact quantum group, then $\mathrm{Rep}(G)$ is a tensor C^*-category with conjugates: for any representation u with conjugate representation $\overline{u}$ defined by the antilinear intertwiner $j:H_u\to H_{\overline{u}}$ as in (2.2), the elements $R:=\sum\psi_j\otimes j^{-1}\psi_j$ and $\overline{R}:=\sum_k\phi_k\otimes j\phi_k$ are intertwiners in $(\iota,\overline{u}\otimes u)$ and $(\iota,u\otimes\overline{u})$, respectively, and satisfy the conjugate equations. Hence, every representation has an associated intrinsic dimension $d(u)$, also called the *quantum dimension*, given by

$$d(u)^2=\inf\big(\|R\|\|\overline{R}\|\big)^2=\inf\mathrm{Trace}(j^*j)\,\mathrm{Trace}\big((j^*j)^{-1}\big). \tag{4.3}$$

Notice that $d(u)\geq\dim(u)$ with equality if and only if j is antiunitary. In terms of the quantum group, the condition $\dim(u)=d(u)$ for all u is equivalent to requiring the coinverse κ to be involutive.

Let $\mathcal{A}$ be a tensor C^*-category, and pick for each object u of $\mathcal{A}$ a solution $R_u,\overline{R}_u$ of the conjugate equations. We agree to take $R_\iota=\overline{R}_\iota=1_\iota$. There is an associated conjugation on $\mathcal{A}$ defined, for $A\in(v,u)$, by

$$A^\bullet:=R_v^*\otimes 1_{\overline{u}}\circ 1_{\overline{v}}\otimes A^*\otimes 1_{\overline{u}}\circ 1_{\overline{v}}\otimes\overline{R}_u.$$

Also $A^\bullet\in(\overline{v},\overline{u})$ can be defined by the equation

$$1_v\otimes A^\bullet\circ\overline{R}_v=A^*\otimes 1_{\overline{u}}\circ\overline{R}_u.$$

If $B\in(w,v)$, then $(A\circ B)^\bullet=A^\bullet\circ B^\bullet$. If we use the product solutions of the conjugate equations for defining the conjugate of a product, then $(A\otimes B)^\bullet=B^\bullet\otimes A^\bullet$. In fact, if $A\in(u,u')$ and $B\in(v,v')$, then

$$(A\otimes B)^\bullet=R^*_{u\otimes v}\otimes 1_{\overline{v}'\otimes\overline{u}'}\circ 1_{\overline{v}\otimes\overline{u}}\otimes A^*\otimes B^*\otimes 1_{\overline{v}'\otimes\overline{u}'}\circ 1_{\overline{v}\otimes\overline{u}}\otimes\overline{R}_{u'\otimes v'}.$$

Substituting in the product form of the solutions we get

$$\begin{aligned}(A\otimes B)^\bullet&=(R_v^*\circ 1_{\overline{v}}\otimes R_u^*\otimes 1_v)\otimes 1_{\overline{v}'\otimes\overline{u}'}\circ 1_{\overline{v}\otimes\overline{u}}\otimes A^*\otimes B^*\otimes 1_{\overline{v}'\otimes\overline{u}'}\circ 1_{\overline{v}\otimes\overline{u}}\\&\quad\otimes(1_{u'}\otimes\overline{R}_{v'}\otimes 1_{\overline{u}'}\circ\overline{R}_{u'})\\&=(R_v^*\circ 1_{\overline{v}}\otimes B^*\circ 1_{\overline{v}}\otimes R_u^*\otimes 1_{v'})\otimes 1_{\overline{v}'\otimes\overline{u}'}\circ 1_{\overline{v}\otimes\overline{u}}\\&\quad\otimes(1_u\otimes\overline{R}_{v'}\otimes 1_{\overline{u}'}\circ A^*\otimes 1_{\overline{u}'}\circ\overline{R}_{u'})\end{aligned}$$

$$= (R_v^* \circ 1_{\overline{v}} \otimes B^*) \otimes 1_{\overline{v}' \otimes \overline{u}'} \circ 1_{\overline{v}} \otimes \overline{R}_{v'} \otimes 1_{\overline{u}'} \circ 1_{\overline{v}} \otimes R_u^* \otimes 1_{\overline{u}'} \circ 1_{\overline{v} \otimes \overline{u}}$$
$$\otimes (A^* \otimes 1_{\overline{u}'} \circ \overline{R}_{u'})$$
$$= B^\bullet \otimes 1_{\overline{u}'} \circ 1_{\overline{v}} \otimes A^\bullet = B^\bullet \otimes A^\bullet.$$

A computation shows that the inverse of $A \to A^\bullet$ is

$$A = \overline{R}_v^* \otimes 1_u \circ 1_v \otimes A^{\bullet *} \otimes 1_u \circ 1_v \otimes R_u.$$

Now

$$\hat{R}_u := \tilde{\mu}_{\overline{u},u}^* \circ \mu(R_u), \qquad \hat{\overline{R}}_u := \tilde{\mu}_{u,\overline{u}}^* \circ \mu(\overline{R}_u) \tag{4.4}$$

is a solution of the conjugate equations for μ_u since

$$\hat{R}_u^* \otimes 1_{\mu_{\overline{u}}} \circ 1_{\mu_{\overline{u}}} \otimes \hat{\overline{R}}_u = \mu(R_u^*) \otimes 1_{\mu_{\overline{u}}} \circ \tilde{\mu}_{\overline{u},u} \otimes 1_{\mu_{\overline{u}}} \circ 1_{\mu_{\overline{u}}} \otimes \tilde{\mu}_{u,\overline{u}}^* \circ 1_{\mu_{\overline{u}}} \otimes \mu(\overline{R}_u)$$
$$= \mu(R_u^*) \otimes 1_{\mu_{\overline{u}}} \circ \tilde{\mu}_{\overline{u}\otimes u,\overline{u}}^* \circ \tilde{\mu}_{\overline{u},u\otimes\overline{u}} \circ 1_{\mu_{\overline{u}}} \otimes \mu(\overline{R}_u)$$
$$= \mu(R_u^* \otimes 1_{\overline{u}}) \circ \mu(1_{\overline{u}} \otimes \overline{R}_u) = 1_{\mu_{\overline{u}}}$$

with the other relation following similarly. Therefore, there exists a conjugation defined on the full subcategory of $\mathcal{M}$ whose objects are the images of those of $\mathcal{A}$ under μ.

REMARK

If μ is not injective on objects, then this conjugation is not well defined. This plays no role in the following, since $\hat{R}_u$ is labeled by u rather than μ_u. In what follows, the tensor category $\mathcal{M}$ can, if desired, be replaced by a tensor C^*-category whose objects are those of $\mathcal{A}$ and where the arrows from u to v are arrows from μ_u to μ_v with the obvious algebraic operations. The *-functor μ then becomes an isomorphism on objects.

By changing the solution of the conjugate equations using an invertible X (see the Appendix), as we would expect, the corresponding change in $\hat{R}_u$ and $\hat{\overline{R}}_u$ is induced by $\mu(X)$. The solutions of the conjugate equations for τ_u, defined analogously, will be denoted by $\tilde{R}_u, \tilde{\overline{R}}_u$.

Given $A \in (v,u)$,

$$\mu(A^\bullet) = \mu(R_v^* \otimes 1_{\overline{u}} \circ 1_{\overline{v}} \otimes A^* \otimes 1_{\overline{u}} \circ 1_{\overline{v}} \otimes \overline{R}_u)$$
$$= \mu(R_v^*) \otimes 1_{\mu_{\overline{u}}} \circ \tilde{\mu}_{\overline{v}\otimes v,\overline{u}}^* \circ \tilde{\mu}_{\overline{v},v\otimes\overline{u}} \circ 1_{\mu_{\overline{v}}} \otimes \mu(A^* \otimes 1_{\overline{u}} \circ \overline{R}_u)$$
$$= \mu(R_v^*) \otimes 1_{\mu_{\overline{u}}} \circ \tilde{\mu}_{\overline{v},v} \otimes 1_{\mu_{\overline{v},v}} \circ \tilde{1}_{\mu_{\overline{v}}} \circ \tilde{\mu}_{v,\overline{u}}^* \circ 1_{\mu_{\overline{v}}} \otimes \mu(A^* \otimes 1_{\overline{u}} \circ \overline{R}_u)$$
$$= \hat{R}_v^* \otimes 1_{\mu_{\overline{u}}} \circ 1_{\mu_{\overline{v}}} \otimes \mu(A^*) \otimes 1_{\mu_{\overline{u}}} \circ 1_{\mu_{\overline{v}}} \otimes \tilde{\mu}_{u,\overline{u}}^* \circ 1_{\mu_{\overline{v}}} \otimes \overline{R}_u$$
$$= \hat{R}_v^* \otimes 1_{\mu_{\overline{u}}} \circ 1_{\mu_{\overline{v}}} \otimes \mu(A^*) \otimes 1_{\mu_{\overline{u}}} \circ 1_{\mu_{\overline{u}}} \otimes \hat{\overline{R}}_v.$$

Thus, $\mu(A^\bullet) = \mu(A)^\bullet$.

When $(\mu, \tilde{\mu})$ and $(\tau, \tilde{\tau})$ are quasitensor *-functors, we define an involution on ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ by setting

$$(M \otimes T)^* := M^\bullet \otimes T^\bullet.$$

This is well defined since, for example,

$$\begin{aligned}\big(M \circ \mu(A) \otimes T\big)^* &= \big(M \circ \mu(A)\big)^\bullet \otimes T^\bullet = M^\bullet \circ \mu(A^\bullet) \otimes T^\bullet \\ &= M^\bullet \otimes \tau(A^\bullet) \circ T^\bullet = \big(M \otimes \tau(A) \circ T\big)^*.\end{aligned}$$

If we change the solution of the conjugate equations by using an invertible X (see the Appendix), then $(M \otimes T)^*$ becomes $M^\bullet \circ \mu(X^*) \otimes \tau(X^{-1*}) \circ T^\bullet = (M \otimes T)^*$. In other words, the involution is independent of the choice of solutions of the conjugate equations in $\mathcal{A}$. Thus, to check that we really have an involution, it suffices to pick $R_{\overline{u}} = \overline{R}_u$ and $\overline{R}_{\overline{u}} = R_u$ when computing the second adjoint. In this case, the above computation of the inverse of $A \mapsto A^\bullet$ implies that we have an involution.

PROPOSITION 4.1

The product and involution defined above make ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ *into a* **-algebra.*

Proof

It suffices to show that

$$\begin{aligned}(N^\bullet \otimes M^\bullet \circ \tilde{\mu}^*_{\overline{v},\overline{u}}) \otimes (\tilde{\tau}_{\overline{v},\overline{u}} \circ T^\bullet \otimes S^\bullet) &= (N^\bullet \otimes T^\bullet) \otimes (M^\bullet \otimes S^\bullet) \\ &= (M \otimes N \circ \tilde{\mu}^*_{u,v})^\bullet \otimes (\tilde{\tau}_{u,v} \circ S \otimes T)^\bullet.\end{aligned}$$

As the involution is independent of the choice of solutions of the conjugate equations, we may suppose, in evaluating this expression, that $\hat{R}_{u\otimes v} = \tilde{\mu}^*_{\overline{v}\otimes\overline{u},u\otimes v} \circ \mu(1_{\overline{v}} \otimes R_u \otimes 1_v \circ R_v)$ with an analogous expression for $\tilde{R}_{u\otimes v}$. Now

$$\begin{aligned}&(M \otimes N \circ \tilde{\mu}^*_{u,v})^\bullet \\ &\quad= \hat{R}^*_{u\otimes v} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \otimes \tilde{\mu}_{u,v} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \otimes M^* \otimes N^* \\ &\quad= \mu(R^*_v) \circ \mu(1_{\overline{v}} \otimes R^*_u \otimes 1_v) \circ \tilde{\mu}_{\overline{v}\otimes\overline{u},u\otimes v} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \otimes \tilde{\mu}_{u,v} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \otimes M^* \otimes N^* \\ &\quad= \mu(R^*_v) \circ \mu(1_{\overline{v}} \otimes R^*_u \otimes 1_v) \circ \tilde{\mu}_{\overline{v}\otimes\overline{u}\otimes u,v} \circ \tilde{\mu}_{\overline{v}\otimes\overline{u},u} \otimes 1_{\mu_v} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \otimes M^* \otimes N^* \\ &\quad= \mu(R^*_v) \circ \tilde{\mu}_{\overline{v},v} \circ 1_{\mu_{\overline{v}}} \otimes \mu(R^*_u) \otimes 1_{\mu_v} \circ \tilde{\mu}^*_{\overline{v},\overline{u}\otimes u} \otimes 1_{\mu_v} \circ \tilde{\mu}_{\overline{v}\otimes\overline{u},u} \otimes 1_{\mu_v} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \\ &\qquad \otimes M^* \otimes N^* \\ &\quad= \hat{R}^*_v \circ 1_{\mu_{\overline{v}}} \otimes N^* \circ 1_{\mu_{\overline{v}}} \otimes \mu(R^*_u) \circ 1_{\mu_{\overline{v}}} \otimes \tilde{\mu}_{\overline{u},u} \circ \tilde{\mu}_{\overline{v},\overline{u}} \otimes 1_{\mu_u} \circ 1_{\mu_{\overline{v}\otimes\overline{u}}} \otimes M^* \\ &\quad= (\hat{R}^*_v \circ 1_{\mu_{\overline{v}}} \otimes N^*) \otimes (\hat{R}^*_u \circ 1_{\mu_{\overline{u}}} \otimes M^*) \circ \tilde{\mu}^*_{\overline{v},\overline{u}} = (N^\bullet \otimes M^\bullet \circ \tilde{\mu}^*_{\overline{v},\overline{u}}).\end{aligned}$$

This proves the result since the term involving S and T can be treated in the same way. $\square$

REMARK

Note that ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ depends only on the images of μ and τ. However, the images will not in general be tensor categories, and the existence of ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ depends on having two quasitensor functors.

In the following, c will denote the support of ι (see Appendix).

COROLLARY 4.2

We have that $\tilde{\tau}^{\bullet}_{u,v} \circ c_{\tau_{\overline{v}}} \otimes c_{\tau_{\overline{u}}} = \tilde{\tau}^{*\bullet *}_{u,v} \circ c_{\tau_{\overline{v}}} \otimes c_{\tau_{\overline{u}}} = \tilde{\tau}_{\overline{v},\overline{u}} \circ c_{\tau_{\overline{v}}} \otimes c_{\tau_{\overline{u}}}$.

Proof

Let $S_i \in (\iota, \tau_u)$, $T_j \in (\iota, \tau_v)$ be orthonormal bases. In the proof of Proposition 4.1 we have seen that

$$\tilde{\tau}^{\bullet}_{u,v} \circ T^{\bullet}_j \otimes S^{\bullet}_i = (\tilde{\tau}_{u,v} \circ S_i \otimes T_j)^{\bullet} = \tilde{\tau}_{\overline{v},\overline{u}} \circ T^{\bullet}_j \otimes S^{\bullet}_i.$$

Multiplying on the right by $T^{*\bullet}_j \otimes S^{*\bullet}_i$, summing over i and j, and using the fact that, by Lemma A.5, $c^{\bullet}_{\tau_u} = c_{\tau_{\overline{u}}}$, we get the one equality. The other equality follows similarly from Proposition 4.1 but by using the part involving the functor μ. $\square$

We now make use of the irreducibility of the tensor unit ι in $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{T}$ and define a linear functional h on ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ by setting, for $M \in (\mu_u, \iota)$ and $T \in (\iota, \tau_u)$,

$$h(M \otimes T) := \big(M \circ \mu(c_u)\big) \otimes T = \sum_i \big(M \circ \mu(V_{u,i})\big) \otimes \big(\tau(V^*_{u,i}) \circ T\big),$$

where $V_{u,i}$ is an orthonormal basis in (ι, u). The first expression shows that h is well defined, and the second shows that it takes values in $\mathbb{C}$.

The next task is to show that h is a faithful positive linear functional, and it would be natural to argue in terms of irreducibles. However, $\mathcal{A}$ does not necessarily have sufficient irreducibles, and there are two ways to proceed. Let $\mathcal{B}$ denote the completion of $\mathcal{A}$ under subobjects. Then $\mathcal{B}$ has sufficient irreducibles, and we have a canonical inclusion functor from $\mathcal{A}$ to $\mathcal{B}$. The quasitensor functors $(\mu, \tilde{\mu})$ and $(\tau, \tilde{\tau})$ from $\mathcal{A}$ can be extended to quasitensor functors $(\nu, \tilde{\nu})$ and $(\sigma, \tilde{\sigma})$ from $\mathcal{B}$. A variant of Proposition 4.4 below shows that ${}^{\circ}_{\mu}\mathcal{C}_{\tau}$ and ${}^{\circ}_{\nu}\mathcal{C}_{\sigma}$ are canonically isomorphic. After this we may suppose that $\mathcal{A}$ has sufficient irreducibles. To avoid giving the details involved, we give an alternative proof in which minimal projections in $\mathcal{A}$ are used in place of irreducibles, every unit being a sum of minimal projections.

As we have seen that the involution on our algebra is independent of the choice of conjugate, we suppose in the following computation that $u \mapsto R_u$ is a standard choice of solutions of the conjugate equations (see Appendix). The conjugation then commutes with the adjoint and maps projections into projections. If $E \in (u, u)$ is a minimal projection, then $E^{\bullet}$ is a minimal projection in $(\overline{u}, \overline{u})$. Setting $R_E := E^{\bullet} \otimes E \circ R_u$ and $\overline{R}_E := E \otimes E^{\bullet} \circ \overline{R}_u$, we have

$$E \otimes R^*_E \circ \overline{R}_E \otimes E = E, \qquad E^{\bullet} \otimes \overline{R}^*_E \circ R_E \otimes E^{\bullet} = E^{\bullet},$$

the form taken by the conjugate equations for minimal projections. If E is any projection in (u, u), we let $V_{E,i}$, $i = 1, 2, \dots, n_E$, be a maximal set of mutually orthogonal isometries in (ι, u) with $V_{E,i} \circ V_{E,i}^* \leq E$ and we set $c_E := \sum_i V_{E,i} \circ V_{E,i}^*$. Two minimal projections are equivalent if they are connected by a partial isometry, and we pick a set $\hat{E}$ of minimal projections, one from each equivalence class. Note that $c_{E^\bullet \otimes F} = 0$ if E and F are inequivalent minimal projections, whereas $c_{E^\bullet \otimes E} = \|R_E\|^{-2} R_E \circ R_E^*$.

PROPOSITION 4.3

We have that h is a faithful positive linear functional on ${}_\mu^\circ\mathcal{C}_\tau$. The associated Gelfand–Naimark–Segal representation is bounded.

Proof

Given an object u of $\mathcal{A}$ there are partial isometries U_i with $U_i^* \circ U_i \in \hat{E}$ and $\sum_i U_i \circ U_i^* = 1_u$. If $M \in (\mu_u, \iota)$ and $T \in (\iota, \tau_u)$, then $M \otimes T = \sum_i M \circ \mu(U_i) \otimes \tau(U_i^*) \circ T)$. Thus, any element of ${}_\mu^\circ\mathcal{C}_\tau$ is a sum of elements of the form $M \otimes T$, where $M = M \circ \mu(E)$ and $\tau(E) \circ T$ for some $E \in \hat{E}$. Given $L \otimes S$ with $L = L \circ \mu(F)$ and $S = \tau(F) \circ S$, by the remarks above, $h((L \otimes S)^*(M \otimes T)) = 0$ if $E \neq F$, whereas if $E = F$, then

$$\begin{aligned} &h\big((L \otimes S)^*(M \otimes T)\big) \\ &\quad = \|R_E\|^{-2}\big(\hat{R}_E^* \circ \mu(E^\bullet) \otimes L^* \otimes M \circ \hat{R}_E\big)\big(\hat{R}_E^* \circ \mu(E^\bullet) \otimes S^* \otimes T \circ \hat{R}_E\big) \\ &\quad = \|R_E\|^{-2}\big(\phi_E(L^* \circ M)\big)\big(\phi_E^*(S^* \circ T)\big), \end{aligned}$$

with ϕ_E the scalar product on $\{X : X \circ \mu(E) = X = \mu(E) \circ X\}$ and $\{Y : Y \circ \tau(E) = Y = \tau(E) \circ Y\}$ induced by $\hat{R_E} := \mu(E^\bullet) \otimes \mu(E) \circ \hat{R}_u$ and $\tilde{R_E} := \tau(E^\bullet) \otimes \tau(E) \circ \tilde{R}_u$ as in the Appendix. To complete the proof it is enough to show that $h(X^*X) \geq 0$ for $X := \sum_{m,n} \lambda_{m,n} L_m \otimes S_n$ when L_m is an orthonormal basis in $(\mu_u, \iota) \circ \mu(E)$, S_n is an orthonormal basis in $\tau(E) \circ (\iota, \tau_u)$ with respect to the scalar products ϕ_E, and $\lambda_{m,n} \in \mathbb{C}$, and that $h(X^*X) = 0$ if and only if $X = 0$. But the above computation shows that

$$\begin{aligned} h(X^*X) &= \sum_{m,n,p,q} \|R_E\|^{-2} \overline{\lambda}_{m,n} \lambda_{p,q} \phi_E(L_m^* \circ L_p) \phi_E(S_m^* \circ S_q) \\ &= \sum_{m,n} \|R_E\|^{-2} |\lambda_{m,n}|^2, \end{aligned}$$

as required. The boundedness of the GNS representation of ${}_\mu^\circ\mathcal{C}_\tau$ associated to h can be proved with arguments generalizing those for the case of Tannaka–Kreĭn duality for compact quantum groups of [26]. See also the computations in the proof of [11, Lemma 8.3]. We refrain from giving details. □

The previous proposition implies, in particular, that ${}_\mu^\circ\mathcal{C}_\tau$ has a nontrivial C^*-norm. In general, the maximal C^*-norm may not be finite. However, this will be the case if either μ or τ is a (not necessarily strict) tensor functor, and this suffices for the purposes of [13]. To see this, we may extend the arguments of [11], where finiteness of the maximal C^*-norm is explicitly shown, to the setting of this article. However, the corresponding algebra $\mathcal{C}_{\mathcal{F}}$ of that paper was introduced in a slightly different way, by means of a complete set of irreducible objects. The following argument should make it clear that the two approaches are in fact equivalent.

We define a set of linear functionals on ${}_{\mu}^{\circ}\mathcal{C}_{\tau}$. Pick a maximal set $E_k \in (u_k, u_k)$, $k \in K$, of inequivalent minimal projections in $\mathcal{A}$. Then for each u, pick partial isometries W_i, $i \in I_u$, such that $W_i^* \circ W_i = E_{f_u(i)}$, where $f_u : I_u \to K$ and $\sum_{i \in I_u} W_i \circ W_i^* = 1_u$. Given $k \in K$, $M, M' \in (\iota, \mu_{u_k})$, and $N, N' \in (\iota, \mu_v)$, we set

$$\omega_{T,M}(N^* \otimes S) := \sum_{i \in I_v, f_v(i)=k} \big(N^* \circ \mu(W_i) \circ M\big)\big(T^* \circ \tau(W_i^*) \circ S\big).$$

This expression is independent of the choice of the partial isometries and is understood to be zero if $f_v^{-1}(k)$ is the empty set. We must check that $\omega_{T,M}$ is well defined. To this end, let $A \in (v, w)$, and let $P \in (\iota, \mu_w)$. Then

$$\begin{aligned}
&\omega_{T,M}\big(P^* \otimes \tau(A) \circ S\big) \\
&\quad = \sum_{\ell \in I_w, f_w(\ell)=k} \big(P^* \circ \mu(W_\ell) \circ M\big)\big(T^* \circ \tau(W_\ell^*) \circ \tau(A) \circ S\big) \\
&\quad = \sum_{i \in I_v, f_v=k, \ell \in K_v, f_w=k} \big(P^* \circ \mu(W_\ell) \circ M\big) \\
&\qquad \times \big(T^* \circ \tau(W_\ell^*) \circ \tau(A) \circ \tau(W_i) \circ \tau(W_i^*) \circ S\big) \\
&\quad = \sum_{i \in I_w; f_v(i)=k} \big(P^* \circ \mu(A) \circ \mu(W_i) \circ M\big)\big(T^* \circ \tau(W_i^*) \circ S\big) \\
&\quad = \omega_{T,M}\big(P^* \circ \mu(A) \otimes S\big),
\end{aligned}$$

as required. Note that $\omega_{1_\iota,1_\iota}$ is just the Haar state h. One would expect $\omega_{M,M}$ to be a positive linear functional, and imitating the proof in the case of h should shed light on the question.

Since, for $M, T \in (\iota, \mu_u)$ and W_i, $i \in I_u$, as above,

$$(M^* \otimes T) = \sum_{i \in I_u} \big(M^* \circ \mu(W_i)\big) \otimes \big(\tau(W_i^*) \circ T\big),$$

every element of ${}_{\mu}^{\circ}\mathcal{C}_{\tau}$ is a sum of elements of the form $(N^* \otimes S)$, where $\mu(E_k) \circ N = N$ and $\tau(E_k) \circ S = S$ for some $k \in K$. If $M = \mu(E_j) \circ M$ and $T = \tau(E_j) \circ T$, then $\omega_{T,M}(N^* \otimes S) = \delta_{jk}(T^* \circ S)(N^* \circ M)$. We now claim that the set of linear functionals $\omega_{T,M}$ with $M = \mu(E_k) \circ M$ and $T = \tau(E_k) \circ T$ separates sums of elements of the form $N^* \otimes S$ where $N = \mu(E_k) \circ N$ and $S = \tau(E_k) \circ S$. Any such sum X may be written in the form $\sum_{i,j} \lambda_{ij}(M_i^* \otimes T_j)$, where M_i and T_j are orthonormal bases of the range of $\mu(E_k)$ and $\tau(E_k)$. Additionally, $\omega_{M_n,M_p}(X) = \lambda_{np}$. Thus, the above set of linear functionals separates ${}_{\mu}^{\circ}\mathcal{C}_{\tau}$. We let ${}_{\mu}\mathcal{C}_{\tau}$ denote the C^*-algebra obtained by completing ${}_{\mu}^{\circ}\mathcal{C}_{\tau}$ in the maximal C^*-norm.

We now investigate the functorial properties of the above construction. Let $(\eta,\tilde\eta):\mathcal{A}_1\to\mathcal{A}_2$, $(\mu_1,\tilde\mu_1):\mathcal{A}_1\to\mathcal{M}$, $(\mu_2,\tilde\mu_2):\mathcal{A}_2\to\mathcal{M}$, $(\tau_1,\tilde\tau_1):\mathcal{A}_1\to\mathcal{T}$, and $(\tau_2,\tilde\tau_2):\mathcal{A}_2\to\mathcal{T}$ be quasitensor functors with $(\mu_1,\tilde\mu_1)=(\mu_2,\tilde\mu_2)\circ(\eta,\tilde\eta)$ and $(\tau_1,\tilde\tau_1)=(\tau_2,\tilde\tau_2)\circ(\eta,\tilde\eta)$. Then the above equalities imply that there is a well-defined natural unital multiplicative map η_* from ${}_{\mu_1}\mathcal{C}_{\tau_1}$ to ${}_{\mu_2}\mathcal{C}_{\tau_2}$. Since the adjoint is independent of the choice of solutions of the conjugate equations we may suppose that if R_u is chosen in $\mathcal{A}_1$, then $\eta(R_u)$ is chosen in $\mathcal{A}_2$, and a computation now shows that $\hat R_{\eta_u}=\hat R_u$, so that η_* is a unital morphism. Obviously, $\eta\mapsto\eta_*$ is a covariant functor.

PROPOSITION 4.4

If $(\eta,\tilde\eta):\mathcal{A}_1\to\mathcal{A}_2$ is full and each object of $\mathcal{A}_2$ is a direct sum of the images of projections under η, then η_* is an isomorphism.

Proof

As the C^*-algebras in question are obtained by completion in the maximal C^*-norm it will suffice to show that we have an isomorphism before completion. Given $M\in(\iota,\mu_{2,x})$ and $T\in(\iota,\tau_{2,x})$, pick projections E_i in $\mathcal{A}_1$ and partial isometries W_i in $\mathcal{A}_2$ such that $\sum_i W_i\circ W_i^*=1_x$ and $W_i^*\circ W_i=\eta(E_i)$. Then $\eta_*\sum_i(M^*\circ\mu_2(W_i)\otimes\tau_2(W_i^*)\circ T)=(M^*\otimes T)$ and η_* is surjective. If X is in the kernel of η_*, pick $T=\tau_1(E_k)\circ T$ and $M=\mu_1(E_k)\circ M$, where E_k is a minimal projection in $\mathcal{A}_1$. Then $\eta(E_k)$ is a minimal projection in $\mathcal{A}_2$, $T=\tau_2\eta(E_k)\circ T$, and $M=\mu_2\eta(E_k)\circ M$. Thus, $\omega_{T,M}(X)=\omega_{T,M}(\eta_*(X))=0$, so $X=0$. $\square$

As a second example of functorial properties we suppose that $\xi:(\sigma,\tilde\sigma)\to(\tau,\tilde\tau)$ is a unitary tensor natural transformation. Thus, $\xi_\iota=1_\iota$, $\xi_v\circ\sigma(A)=\tau(A)\circ\xi_u$ for $A\in(u,v)$, and $\xi_{u\otimes v}\circ\tilde\sigma_{u,v}=\tilde\tau_{u,v}\circ\xi_u\otimes\xi_v$. We now set

$$\xi_*(M\otimes S):=M\otimes\xi_u\circ S,\quad M\in(\mu_u,\iota),S\in(\iota,\sigma_u).$$

Obviously, if $A\in(u,v)$, then $\xi_*(M\otimes\sigma(A)\circ S)=\xi_*(M\circ\mu(A)\otimes S)$. Thus, ξ_* can be considered as a linear map from ${}^{\circ}_{\mu}\mathcal{C}_\sigma$ to ${}^{\circ}_{\mu}\mathcal{C}_\tau$:

$$\begin{aligned}
&\xi_*(M\otimes S)\xi_*(M'\otimes S')\\
&\quad=M\otimes M'\circ\tilde\mu^*_{u,u'}\otimes\tilde\tau_{u,u'}\circ(\xi_u\ \circ S)\otimes(\xi_{u'}\circ S')\\
&\quad=(M\otimes M'\circ\tilde\mu^*_{u,u'}\otimes\xi_{u\otimes u'}\circ\tilde\sigma_{u,u'}\circ S\otimes S')=\xi_*\big((M\otimes S)(M'\otimes S')\big).
\end{aligned}$$

Thus, ξ_* is multiplicative:

$$\xi_*\big((M\otimes S)^*\big)=\xi_*(M^\bullet\otimes S^\bullet)=(M^\bullet\otimes\xi_{\overline{u}}\circ S^\bullet),$$

whereas

$$\big(\xi_*(M\otimes S)\big)^*=\big(M^\bullet\otimes(\xi_u\circ S)^\bullet\big).$$

Thus, it suffices to show that $\xi_u^\bullet=\xi_{\overline{u}}$:

$$\xi_{\overline{u}}^*\circ\xi_u^\bullet=R^*_{\sigma_u}\otimes 1_{\sigma_{\overline{u}}}\circ 1_{\sigma_{\overline{u}}}\otimes\xi_u^*\otimes\xi_{\overline{u}}^*\circ 1_{\sigma_{\overline{u}}}\otimes\overline{R}_{\tau_u}.$$

Now

$$\xi_u^* \otimes \xi_{\overline{u}}^{\circ} \tilde{\tau}_{u,\overline{u}} * = \tilde{\sigma}_{u,\overline{u}}^* \circ \xi_{u \otimes \overline{u}}^*, \qquad \overline{R}_{\tau_u} = \tilde{\tau}_{u,\overline{u}}^* \circ \tau(\overline{R}_u),$$

and $\xi_{u\otimes\overline{u}}^* \circ \tau(\overline{R}_u) = \sigma(\overline{R}_u)$. This gives $\xi_u^* \otimes \xi_{\overline{u}}^* = 1_{\overline{u}}$, and hence, $\xi_u^\bullet = \xi_{\overline{u}}$, as required. Thus, ξ_* is an isomorphism and hence extends to an isomorphism from ${}_\mu\mathcal{C}_\sigma$ to ${}_\mu\mathcal{C}_\tau$.

As a third example of functorial properties, we consider quasitensor functors $(\mu, \tilde{\mu}) : \mathcal{A} \to \mathcal{M}$, $(\tau, \tilde{\tau}) : \mathcal{A} \to \mathcal{T}$, and $(\sigma, \tilde{\sigma}) : \mathcal{T} \to \mathcal{S}$ and define, for $M \in (\mu_u, \iota)$ and $S \in (\iota, \tau_u)$, $\sigma_*(M \otimes S) := (M \otimes \sigma(S))$. This obviously defines a linear map σ_* from ${}_\mu^\circ\mathcal{C}_\tau$ to ${}_\mu^\circ\mathcal{C}_{\sigma\tau}$. Then σ_* is multiplicative since

$$\widetilde{\sigma \circ \tau}_{u,v} \circ \sigma(S) \otimes \sigma(T) = \sigma(\tilde{\tau}_{u,v}) \circ \sigma(S \otimes T),$$

where $T \in (\iota, \tau_v)$. Furthermore, σ_* commutes with the adjoint since σ commutes with conjugation. Hence, σ_* extends to a morphism from ${}_\mu\mathcal{C}_\tau$ to ${}_\mu\mathcal{C}_{\sigma\tau}$.

PROPOSITION 4.5

We have that σ_* is faithful and if σ maps (ι, τ_u) onto $(\iota, \sigma\tau_u)$ for each object u of $\mathcal{A}$, then σ_* is an isomorphism.

Proof

Under the above condition, σ_* is obviously surjective. It therefore suffices to prove that σ_* is faithful on ${}_\mu^\circ\mathcal{C}_\tau$. To this end, let us denote the image of an element $X \in \bigoplus_u (\mu_u, \iota) \otimes (\iota, \tau_u)$ in $\bigoplus (\mu_u, \iota) \otimes_{\mathcal{A}} (\iota, \tau_u)$ by $\hat{X}$, and suppose that $\sigma_*(\hat{X}) = 0$. Then $\sigma_*(X)$, the image of X in $\bigoplus_u (\mu_u, \iota) \otimes (\iota, \sigma\tau_u)$, is of the form

$$\sum_i \big(M_i \otimes \big(\sigma\tau(A_i) \circ \sigma(S_i)\big) - \big(M_i \circ \mu(A_i)\big) \otimes \sigma(S_i)\big).$$

Set $Y := \bigoplus_i (M_i \otimes (\tau(A_i) \circ S_i) - (M_i \circ \mu(A_i)) \otimes S_i)$. Then $\sigma_*(X) = \sigma_*(Y)$ and $\hat{Y} = 0$. Hence, it suffices to show that $\hat{X} = 0$ when $\sigma_*(X) = 0$, but this follows since σ is faithful on the Hilbert spaces (ι, τ_u). □

We now recall the *-functor $q : \mathcal{T} \to \mathcal{H}$, discussed in the Appendix, taking an object x of $\mathcal{T}$ to the Hilbert space (ι, x) and $X \in (x, y)$ onto the map $T \mapsto X \circ T$. We have that q extends uniquely to a quasitensor functor $(q, \tilde{q})$, and $\tilde{q}$ is minimal. By the above result, q_* is an isomorphism, and in this sense, it suffices to consider quasitensor functors $(\mu, \tilde{\mu})$ and $(\tau, \tilde{\tau})$ taking values in the category of Hilbert spaces.

5. The genesis of ergodic actions

In this section we explain how to get actions of quantum groups on the C^*-algebras constructed from a pair of quasitensor functors in the last section. To this end, we suppose that we have an action η of a quantum group G on the category $\mathcal{T}$ leaving the objects of $\mathcal{T}$ invariant. Regarding the C^*-algebra $\mathcal{Q}$ of G as a C^*-category with a single object, η is a *-functor from $\mathcal{T}$ to $\mathcal{T} \otimes \mathcal{Q}$

with $\eta \otimes 1_{\mathcal{Q}} \circ \eta = 1_{\mathcal{T}} \otimes \Delta \circ \eta$, Δ being the coproduct. Since $\mathcal{T} \otimes \mathcal{Q}$ is not a tensor C^*-category, we cannot require η to be a tensor *-functor. The natural condition is to require instead that $\eta(S \otimes T) = \eta(S) \top \eta(T)$, where $\top$ indicates that we take a tensor product in the first component and a product in the second à la Woronowicz. This is the product used when defining the tensor product of representations of a quantum group. We further suppose that the arrows of $\mathcal{A}$ intertwine this action, that is, that $\eta(\tau(A) \circ T) = \tau(A) \otimes I \circ \eta(T)$ and that $\eta(\tilde{\tau}_{u,v}) = \tilde{\tau}_{u,v} \otimes I$ for each pair u, v of objects of $\mathcal{A}$.

PROPOSITION 5.1

There is a unique action α of G on ${}_\mu\mathcal{C}_\tau$ such that $\alpha(M \otimes T) := M \otimes \eta(T)$.

Proof

Obviously, α is well defined, and a simple computation shows that it is multiplicative. To show that it commutes with the adjoint, we must show that $\eta(S^\bullet) = \eta(S)^{\bullet \otimes *}$. Now

$$\eta(S^\bullet) = \eta(S^* \otimes 1_{\tau_{\overline{u}}} \circ \tilde{\overline{R}}_u) = \eta(S^*) \top 1_{\tau_{\overline{u}}} \otimes I \circ \tilde{\overline{R}}_u \otimes I = \eta(S)^{\bullet \otimes *}.$$

Finally, α extends to ${}_\mu\mathcal{C}_\tau$ by continuity and is trivially an action.

Let E_G denote the conditional expectation defined by averaging η over G, and let $E_{G,u}$ denote the projection obtained by restricting to (ι, τ_u). If $A \in (u, v)$, then $E_{G,v}\tau(A)E_{G,u} = \tau(A)E_{G,u}$. Thus, $E_{G,v}\tau(A) = \tau(A)E_{G,u}$. Thus, $E_{G,\cdot}$ is a natural transformation from τ to τ as is $\tau(c)$, where c denotes the central support of ι in $\mathcal{A}$ as before. If $E_{G,\cdot} = \tau(c)$, then the above action will be ergodic.

It remains to understand how to get appropriate actions of a quantum group G on $\mathcal{T}$ and when the induced action on ${}_\mu\mathcal{C}_\tau$ is ergodic. Now if we suppose that τ is a tensor *-functor into the category of Hilbert space, then, as $\mathcal{A}$ has conjugates, the duality theorem of Woronowicz [26] gives us an action η of a compact quantum group G_τ on the Hilbert spaces (ι, τ_u) and, hence, on the category $\mathcal{T}$. The C^*-algebra of G_τ is ${}_\tau\mathcal{C}_\tau$, showing how our construction generalizes that of Woronowicz. The action is defined by

$$\eta_u(T) = \sum_i T_i \otimes (T_i^* \otimes T), \quad T \in (\iota, \tau_u),$$

where the sum is taken over an orthonormal basis. The arrows of the form $\tau(A)$ intertwine this action, the conditional expectation defined by averaging over G_μ is $\tau(c)$, and $\eta(S \otimes T) = \eta(S) \top \eta(T)$. Thus, we have an induced action α of G_τ on ${}_\mu\mathcal{C}_\tau$:

$$\alpha(M \otimes T) = \sum_i (M \otimes T_i) \otimes (T_i^* \otimes T).$$

This action is ergodic, since the conditional expectation coincides with h, which is the unique invariant state. To put ourselves in the setting of [11], we replace μ by $q\mu$, which can then be identified with the spectral functor of the ergodic action. The spectral space $\overline{L_u}$ associated with the representation u can be identified, as a Hilbert space, with (ι, μ_u) through the map that takes $M \in (\iota, \mu_u)$ to the complex conjugate of the map of L_u:

$$T \in (\iota, \tau_u) \to M^* \otimes T \in {}_\mu\mathcal{C}_\tau. \qquad \square$$

REMARK

Under certain circumstances, the above generalizes to the case of a quasitensor functor τ, but since it is not needed here, we will give details in a separate article.

6. Self-conjugate solutions of the conjugate equations

To treat self-conjugate solutions of the conjugate equations we consider two tensor *-categories. The first, $\mathcal{T}_{rd}$, for real solutions, has objects that are powers y^n, $n \in \mathbb{N}_0$, of a generating object y and whose arrows are generated by a single arrow $S \in (\iota, y^2)$ satisfying $S^* \otimes 1_y \circ 1_y \otimes S = 1_y$ and $S^* \circ S = d$. The second, $\mathcal{T}_{pd}$, for pseudoreal solutions, has objects that are powers z^n, $n \in \mathbb{N}_0$, of a generating object z and whose arrows are generated by a single arrow S satisfying $S^* \otimes 1_z \circ 1_z \otimes S = -1_z$ and $S^* \circ S = d$. In both cases we suppose that $d \neq 0$.

As $\mathcal{T}_{rd}$ and $\mathcal{T}_{pd}$ are defined in terms of generators and relations they will satisfy the corresponding universal properties. But the analogous universal properties are satisfied by the Temperley–Lieb categories (see [28]), usually defined without reference to a *-operation. Hence, the categories $\mathcal{T}_{rd}$ and $\mathcal{T}_{pd}$ are Temperley–Lieb categories corresponding to parameters $\pm d$ with a *-operation defined by a solution S of self-conjugate solutions of the conjugate equations. As such, the following assertions are well known. The units and generating objects of these categories are irreducible, and the spaces of arrows are finite-dimensional. The categories are simple except at roots of unity, $d = 2\cos\frac{\pi}{\ell}$, $\ell = 3, 4, \ldots$, when they have a single nonzero proper ideal (see [7]). They are tensor C^*-categories when $d \geq 2$, and at roots of unity their quotients by the unique nonzero proper ideal are tensor C^*-categories having the universal property, but now for tensor C^*-categories.

We define a left inverse ψ for the generating object y of $\mathcal{T}_{rd}$ by

$$\psi_{m,n}(Y) := S^* \otimes 1_{y^{m-1}} \circ 1_y \otimes Y \circ 1_{y^{n-1}} \otimes S, \quad Y \in (y^n, y^m).$$

Iterating ψ we get a mapping $\mathrm{Tr} : (y^n, y^n) \to (\iota, \iota)$, the Markov trace. A right inverse for y is obtained by dualizing the above definition with respect to $\otimes$, and iterating again defines the Markov trace.

Left and right inverses for the generating object z of $\mathcal{T}_{pd}$ can be defined analogously, and their iterates again yield the Markov trace.

7. General solutions of the conjugate equations

We define $\mathcal{T}_d$ for $d \neq 0$ to be the tensor *-category whose objects are the words in x and $\overline{x}$ and whose arrows are generated by two arrows $R \in (\iota, \overline{x} \otimes x)$ and $\overline{R} \in (\iota, x \otimes \overline{x})$ subject to the relations $1_x \otimes R^* \circ \overline{R} \otimes 1_x = 1_x$, $1_{\overline{x}} \otimes \overline{R}^* \circ R \otimes 1_{\overline{x}} = 1_{\overline{x}}$, $R^* \circ R = d$, and $\overline{R}^* \circ \overline{R} = d$. We note that $\mathcal{T}_d$ has an involution $^\circ$ taking x to $\overline{x}$

and R to $\overline{R}$. We call an object *even* or *odd* according to whether it is a tensor product of an even or odd number of the objects x and $\overline{x}$. The space of arrows between an even and an odd object is zero.

As $\mathcal{T}_d$ has been defined in terms of generators and relations, it has a universal property: given any solutions $R', \overline{R}'$ of the conjugate equations in a tensor *-category $\mathcal{T}$ with $R'^* \circ R' = d$ and $\overline{R}'^* \circ \overline{R}' = d$, there is a unique tensor *-functor $\phi : \mathcal{T}_d \to \mathcal{T}$ such that $\phi(R) = R'$ and $\phi(\overline{R}) = \overline{R}'$. Yamagami [30] defines a tensor *-category in terms of oriented Kauffman diagrams and shows that it has the above universal property, so that this category is in fact isomorphic to $\mathcal{T}_d$. With his very different starting point, his proof of Theorem 7.1 below is quite different.

THEOREM 7.1

We have that ι and x are irreducible in $\mathcal{T}_d$.

Proof

If $X \in (\iota, \iota)$, then $X \to 1_x \otimes X$ is a faithful morphism from (ι, ι) to (x, x), since it suffices to show that $(\iota, x \otimes \overline{x})$ is 1-dimensional in $\mathcal{T}_d$. This will be the case if any intertwiner in $(\iota, x \otimes \overline{x})$ constructed as an algebraic expression in R, $\overline{R}$, 1_x, $1_{\overline{x}}$, and their adjoints reduces to a multiple of $\overline{R}$. A tensor product of the basic arrows 1_x, $1_{\overline{x}}$, R, $\overline{R}$, R^*, and $\overline{R}^*$ will be said to be a *term*. A term is *positive* if R^* and $\overline{R}^*$ are not involved and *negative* if R and $\overline{R}$ are not involved. By using the interchange law, any term can be written in the form $X_+ \circ X_-$, where X_+ is a positive term and X_- is a negative term. Now consider a composition of terms of the form $X_- \circ X_+$. We break these two terms into an equal number of pieces of minimal size such that the $\circ$-composition of the corresponding pieces is defined. We list the possible $\circ$-compositions of two pieces: $1_x \otimes R^* \circ \overline{R} \otimes 1_x = 1_x$; $R^* \otimes 1_{\overline{x}} \circ 1_{\overline{x}} \otimes \overline{R} = 1_{\overline{x}}$; $1_{\overline{x}} \otimes \overline{R}^* \circ R \otimes 1_{\overline{x}} = 1_{\overline{x}}$; $\overline{R}^* \otimes 1_x \circ 1_x \otimes R = 1_x$; $R^* \circ R = d$; $\overline{R}^* \circ \overline{R} = d$; $1_x \circ 1_x = 1_x$; $1_{\overline{x}} \circ 1_{\overline{x}} = 1_{\overline{x}}$; $1_{\overline{x} \otimes x} \circ R = R$; $1_{x \otimes \overline{x}} \circ \overline{R} = \overline{R}$; $R^* \circ 1_{\overline{x} \otimes x} = R^*$; $\overline{R}^* \circ 1_{x \otimes \overline{x}} = \overline{R}^*$; $1_{x \otimes \overline{x}} \otimes R^* \circ \overline{R} \otimes 1_{\overline{x} \otimes x} = R \otimes R^*$; $1_{\overline{x} \otimes x} \otimes R^* \circ R \otimes 1_{\overline{x} \otimes x} = R \otimes R^*$; $1_{x \otimes \overline{x}} \otimes \overline{R}^* \circ \overline{R} \otimes 1_{x \otimes \overline{x}} = \overline{R} \otimes \overline{R}^*$; $1_{\overline{x} \otimes x} \otimes \overline{R}^* \circ R \otimes 1_{x \otimes \overline{x}} = R \otimes R^*$. Thus, up to a scalar, $X_- \circ X_+ = Y_+ \circ Y_-$. Hence, up to a scalar any composition of terms can be written as a composition of positive and negative terms, where the negative terms appear on the right and the positive terms appear on the left. But such a composition is an arrow of $(\iota, x \otimes \overline{x})$ if and only if there are no negative terms and a single positive term $\overline{R}$. □

REMARK

Every composition of terms in $\mathcal{T}_d$ can also be written as a composition of positive and negative terms, the positive terms appearing on the right and the negative terms appearing on the left.

The universal property of $\mathcal{T}_d$ implies that there is a unique tensor *-functor $\phi : \mathcal{T}_d \to \mathcal{T}_{rd}$ such that $\phi_x = y$ and $\phi(R) = \phi(\overline{R}) = S$. As an aid to studying this functor we introduce the full subcategory $\mathcal{T}_d^a$ whose objects are of the form

$(x \otimes \overline{x})^n$, the even objects, or $(x \otimes \overline{x})^n \otimes x$, the odd objects, for $n \in \mathbb{N}_0$. This category is obviously not a tensor subcategory of $\mathcal{T}_d$, but it can be given the structure of a tensor *-category. Note first that there is no nonzero arrow between objects of different parity, so any nonzero arrow has a definite parity, and we define the tensor product as $X \otimes X'$ if X is even and $X \otimes X^{'\circ}$ if X is odd, where $X' \to X^{'\circ}$ acts as conjugation. As an object of $\mathcal{T}_d^a$, x is self-conjugate. Therefore, there is a unique tensor *-functor $\psi : \mathcal{T}_{rd}$ to $\mathcal{T}_d^a$ with $\psi_y = x$ and $\psi(S) = \overline{R}$. The restriction ϕ^a of ϕ to $\mathcal{T}_d^a$ is a tensor *-functor since $\phi(X) = \phi(X^\circ)$ and $\phi^a \psi = 1_{\mathcal{T}_{rd}}$. Thus, ϕ^a and ϕ are surjective on arrows. Obviously, ϕ is not full since $(x, \overline{x}) = 0$ and $(y, y) \neq 0$. Thus, in particular, we have proved the following result.

THEOREM 7.2

The canonical functor ϕ from $\mathcal{T}_d$ to $\mathcal{T}_{rd}$ is a tensor *-functor surjective on both objects and arrows but not full.

THEOREM 7.3

We have that $\psi : \mathcal{T}_{rd} \to \mathcal{T}_d^a$ is an isomorphism of tensor *-categories and $\phi : \mathcal{T}_d \to \mathcal{T}_{rd}$ is faithful.

Proof

If X is a term of $\mathcal{T}_d^a$, then $\phi^a(X)$ will be a term of $\mathcal{T}_{rd}$ and $\psi\phi^a(X) = X$. But every arrow of $\mathcal{T}_d^a$ is a linear combination of compositions of terms. Hence, ψ is an isomorphism. More generally, given a full subcategory $\mathcal{T}_d^s$ of $\mathcal{T}_d$ such that the restriction ϕ^s of ϕ to the objects of $\mathcal{T}_d^s$ is an isomorphism, the image of $\mathcal{T}_d^s$ under ϕ^s is a subcategory $\mathcal{T}_{rd}^s$ of $\mathcal{T}_{rd}$. Every term Y of $\mathcal{T}_{rd}^s$ is the image under ϕ^s of a unique term $\psi^s(Y)$ of $\mathcal{T}_d^s$. Note, however, that formally distinct terms of $\mathcal{T}_d$ can define the same arrow of $\mathcal{T}_d$, for example, $R \circ \overline{R}^* = \overline{R}^* \otimes R$. Nevertheless, ψ^s extends to a full functor from $\mathcal{T}_{rd}^s$ to $\mathcal{T}_d^s$. Then ϕ^s and ψ^s are isomorphisms since they are inverses of one another in restriction to terms. Using the linear isomorphism of $(x \otimes p, q)$ and $(p, \overline{x} \otimes q)$ where necessary, we conclude that ϕ is faithful. $\square$

We define left inverses ψ and $\overline{\psi}$ of x and $\overline{x}$ by

$$\psi_{p,q}(X) := R^* \otimes 1_p \circ 1_{\overline{x}} \otimes X \circ R \otimes 1_q, \quad X \in (x \otimes q, x \otimes p),$$

$$\overline{\psi}_{p,q}(X) := \overline{R}^* \otimes 1_p \circ 1_x \otimes X \circ \overline{R} \otimes 1_q, \quad X \in (\overline{x} \otimes q, \overline{x} \otimes p).$$

Iterating ψ and $\overline{\psi}$ appropriately we get a map $\mathrm{Tr} : (p, p) \to (\iota, \iota)$. Obviously, since the unit of $\mathcal{T}_d$ is irreducible by Theorem 7.1, $\mathrm{Tr}(X) = \mathrm{Tr}(\phi(X))$. Right inverses of x and $\overline{x}$ are defined by dualizing with respect to $\otimes$ and, when iterated appropriately, lead to the same map $\mathrm{Tr} : (p, p) \to (\iota, \iota)$. It follows that Tr is a trace in that $\mathrm{Tr}(X \circ X') = \mathrm{Tr}(X' \circ X)$ whenever the compositions are defined. Another way of looking at this trace is to note that $R, \overline{R}$ extend uniquely to a homomorphic choice $q \mapsto R_q$ of solutions of the conjugate equations in $\mathcal{T}_d$ (see the Appendix). The corresponding scalar product ψ_q on (q,q) is that associated with the trace.

Following [24], we call an arrow $X \in (p,q)$ of $\mathcal{T}_d$ *negligible* if $\mathrm{Tr}(X' \circ X) = 0$ for all $X' \in (q,p)$. Clearly, X is negligible if $\phi(X)$ is negligible in $\mathcal{T}_{rd}$ but the converse follows from Theorem 7.2. The set of negligible arrows is a tensor *-ideal, meaning that it is a *-subcategory and a tensor ideal of the category in the sense of [6, Section 1.1]. This is well known in the case of $\mathcal{T}_{rd}$ but holds in some generality.

PROPOSITION 7.4

Let $\mathcal{T}$ be a tensor *-category with conjugates and irreducible tensor unit, and let $u \mapsto R_u$ be a tracial and homomorphic choice of solutions of the conjugate equations with associated trace Tr. *Then the set $\mathcal{I}$ of negligible arrows in $\mathcal{T}$ is the maximal proper tensor *-ideal.*

Proof

If $X \in (u,v)$, then $\mathrm{Tr}(1_u) = R_u^* \circ 1_{\overline{u}\otimes u} \circ R_u = d_u$. So $\mathrm{Tr}(X' \circ X)^* = \mathrm{Tr}\, X^* \circ X'^* = \mathrm{Tr}\, X'^* \circ X^*$ so $\mathcal{I} = \mathcal{I}^*$. If $W \in (w,u)$ and $V \in (v,w)$, then $\mathrm{Tr}(V \circ X \circ W) = \mathrm{Tr}(W \circ V \circ X)$, so $X \in \mathcal{I}$ implies $X \circ W \in \mathcal{I}$. Now let $Z \in (t \otimes v, t \otimes u)$. Then, since $u \mapsto R_u$ is homomorphic,

$$\mathrm{Tr}(Z \circ 1_t \otimes X) = R_u^* \circ 1_{\overline{u}} \otimes (R_t^* \otimes 1_t \circ 1_{\overline{t}} \otimes Z \circ R_t \otimes 1_v \circ X) \circ R_u.$$

Thus, $X \in \mathcal{I}$ implies $1_t \otimes X \in \mathcal{I}$ and similarly for tensoring on the right. Then $\mathrm{Tr}(1_\iota) = 1$ so that $\mathcal{I}$ is a proper tensor *-ideal. Now if $\mathcal{J}$ is a proper tensor *-ideal and $X \in (u,v) \cap \mathcal{J}$, then $\mathrm{Tr}(X' \circ X) \in \mathcal{J}$ for all $X' \in (v,u)$. Thus, $X \in \mathcal{I}$, completing the proof. □

PROPOSITION 7.5

A tensor C^*-category with irreducible tensor unit and conjugates is simple.

Proof

A standard choice of solutions of the conjugate equations yields a trace independent of that choice (see Appendix), so the proof of Proposition 7.4 applies. Since the trace is faithful, every negligible arrow is zero. □

We now investigate the ideal structure of $\mathcal{T}_d$. As remarked above, an arrow X of $\mathcal{T}_d$ is negligible if and only if $\phi(X)$ is negligible. Thus, if $d \neq \cos\frac{\pi}{\ell}$ for $\ell = 3,4,\ldots$, then X is negligible if and only if $\phi(X) = 0$. Thus, by Theorem 7.3, $\mathcal{T}_d$ is simple for these values of d. If $d = \cos\frac{\pi}{\ell}$ for $\ell = 3,4,\ldots$, then again by Theorem 7.2, the ideal of negligible arrows in $\mathcal{T}_d$ is the unique nonzero proper tensor ideal. We have therefore proved the following result.

THEOREM 7.6

We have that $\mathcal{T}_d$ is simple for $d \neq 2\cos\frac{\pi}{\ell}$, while it has a single nonzero proper tensor ideal for $d = 2\cos\frac{\pi}{\ell}$.

As another consequence of Theorem 7.3, $\mathcal{T}_d$ will be a tensor C^*-category whenever $\mathcal{T}_{rd}$ is a tensor C^*-category, that is, whenever $d \geq 2$. If $d = 2\cos\frac{\pi}{\ell}$, $\ell = 3, 4, \ldots$, then its quotient by the unique nonzero proper ideal will be a tensor C^*-category having the universal property for normalized solutions of the conjugate equations with these values of d.

There is also a canonical functor $\phi : \mathcal{T}_d \to \mathcal{T}_{pd}$ with $\phi_x = \phi_{\overline{x}} = z$, $\phi(R) = S$, and $\phi(\overline{R}) = -S$. The results of this section have obvious analogues in this case.

8. Embedding the universal categories

There will be tensor *-functors from $\mathcal{T}_d$ to a tensor C^*-category $\mathcal{T}$ if $d \geq 2$ or if $d = 2\cos\frac{\pi}{\ell}$, for $\ell = 3, 4, \ldots$. For these values of d, given normalized solutions $R', \overline{R}'$ of the conjugate equations for x' in $\mathcal{T}$ with $R^{'*} \circ R' = d$, there is a unique tensor *-functor ϕ from $\mathcal{T}_d$ to $\mathcal{T}$ with $\phi_x = x'$, $\phi(R) = R'$, and $\phi(\overline{R}) = \overline{R}'$. When ϕ is injective on objects, the image of ϕ is a tensor C^*-category isomorphic to $\mathcal{T}_d$. Similarly, if $S' \in (\iota, y^{'2})$ is a real (or pseudoreal) solution of the conjugate equations in $\mathcal{T}$ with $S'^* \circ S' = d$, then there is a unique isomorphism ϕ from $\mathcal{T}_{rd}$ (or $\mathcal{T}_{pd}$) to the tensor C^*-subcategory generated by S taking y to y' and S to S'.

Obviously, a tensor C^*-category $\mathcal{T}$ with $(\iota, \iota) = \mathbb{C}$ can only be embeddable in the tensor C^*-category of Hilbert spaces if any normalized solution $R, \overline{R}$ of the conjugate equations in $\mathcal{T}$ with $R^* \circ R < 2$ is unitary. In particular, the quotient tensor C^*-categories $\mathcal{T}_d/\mathcal{I}$ for $d = 2\cos\frac{\pi}{\ell}$ cannot be embedded into the tensor C^*-category of Hilbert spaces.

We will now classify, for the possible values of d other than $d = 1$, the tensor *-functors from $\mathcal{T}_{rd}$ and $\mathcal{T}_{pd}$ to the category of Hilbert spaces up to a natural unitary tensor equivalence. They are determined by the parameters λ_i used by [3]. Let $\phi : \mathcal{T}_{rd} \to \mathcal{H}$ be a tensor *-functor. Then there are $0 < \lambda_i < 1$ with $\sum_1^k(\lambda_i^2 + \lambda_i^{-2}) + n - 2k = d$ and an orthonormal basis e_i of ϕ_σ such that

$$\phi(S) = \sum_1^k \lambda_i e_{i+k} \otimes e_i + \sum_1^k \lambda_i^{-1} e_i \otimes e_{i+k} + \sum_{2k+1}^n e_i \otimes e_i.$$

Similarly, if $\phi : \mathcal{T}_{pd} \to \mathcal{H}$ is a tensor *-functor, then there are $0 < \lambda_i \leq 1$ with $\sum_1^{\frac{n}{2}}(\lambda_i^2 + \lambda_i^{-2}) = d$ and an orthonormal basis e_i of ϕ_σ such that

$$\phi(S) = \sum_1^{\frac{n}{2}} \lambda_i e_{i+\frac{n}{2}} \otimes e_i - \sum_1^{\frac{n}{2}} \lambda_i^{-1} e_i \otimes e_{i+\frac{n}{2}}.$$

The following result can easily be proved (cf. [29]).

PROPOSITION 8.1

(a) *The parameters* λ_i *with* $0<\lambda_i<1$ *and* $\sum_1^k(\lambda_i^2+\lambda_i^{-2})+n-2k=d$ *classify the tensor* **-functors* $\phi:\mathcal{T}_{rd}\to\mathcal{H}$ *up to a natural unitary tensor equivalence.*

(b) *The parameters* λ_i *with* $0<\lambda_i\le 1$ *and* $\sum_1^{\frac{n}{2}}(\lambda_i^2+\lambda_i^{-2})=d$ *classify the tensor* **-functors* $\phi:\mathcal{T}_{pd}\to\mathcal{H}$ *up to a natural tensor unitary equivalence.*

In case (a), the spectrum of $j_y^*\circ j_y$ is $\{\lambda_i^2,\lambda_i^{-2}:1\le i\le k\}\cup\{1:2k+1\le i\le n\}$, where j_y is the antilinear invertible operator on H_y defined by $\phi(S):=\sum_i e_i\otimes j_ye_i$. In case (b) it is $\{\lambda_i^2,\lambda_i^{-2}:1\le i\le \frac{n}{2}\}$.

We next classify the tensor *-functors $\phi:\mathcal{T}_d\to\mathcal{H}$, $d\ne 1$, up to a natural tensor unitary equivalence. Any such functor ϕ determines an invertible antilinear operator j_x on H_x via $\phi(\overline{R}_x):=\sum_i e_i\otimes j_xe_i$. The following result can again be easily proved.

PROPOSITION 8.2

The natural tensor unitary equivalence classes of embeddings of $\mathcal{T}_d$ *into Hilbert spaces are classified by a monotone set of parameters* $0<\lambda_i$ *with* $\sum_{i=1}^n\lambda_i^2=\sum_{i=1}^n\lambda_i^{-2}=d$, *where* $\{\lambda_i^2\}$ *is just the eigenvalue list of* $j_x^*\circ j_x$.

A rather less natural description of these equivalence classes in terms of an invertible linear operator can be found in [30].

REMARK

As we have canonical tensor *-functors from $\mathcal{T}_d$ to $\mathcal{T}_{rd}$ and $\mathcal{T}_{pd}$ taking R and $\overline{R}$ onto S and $-S$, respectively, an embedding of $\mathcal{T}_{rd}$ or $\mathcal{T}_{pd}$ induces an embedding of $\mathcal{T}_d$, equivalent embeddings inducing equivalent embeddings. Thus, for each set of parameters in Proposition 8.1 there is a corresponding set of parameters in Proposition 8.2. The eigenvalue list of $j_y^*j_y$ in Proposition 8.1(a) and $j_z^*j_z$ in Proposition 8.1(b) has been indicated above. Recall that n is even in Proposition 8.1(b). We see that no two inequivalent embeddings of $\mathcal{T}_{rd}$ or $\mathcal{T}_{pd}$ can induce equivalent embeddings of $\mathcal{T}_d$, but for each embedding of $\mathcal{T}_{pd}$ there is an embedding of $\mathcal{T}_{rd}$ inducing an equivalent embedding of $\mathcal{T}_d$.

If $\mathcal{A}$ is a tensor C^*-category with conjugates and irreducible tensor unit, then by the duality theorem of Woronowicz [26] every embedding of $\mathcal{A}$ into the category of Hilbert spaces determines a compact quantum group whose category of finite-dimensional representations is equivalent to the completion of $\mathcal{A}$ under subobjects and direct sums. It can be easily shown that embeddings differing by a tensor unitary equivalence yield isomorphic quantum groups. In fact, this results from the discussion following Proposition 4.4.

We now describe the compact quantum groups arising from the embeddings of $\mathcal{T}_d$, $\mathcal{T}_{rd}$, and $\mathcal{T}_{pd}$. Obviously, all of these quantum groups have a representation theory generated by a single fundamental representation u. In the Hilbert space of the embedding, the arrows $R\in(\iota,\overline{u}\otimes u)$ and $\overline{R}\in(\iota,u\otimes\overline{u})$ are then intertwining operators between the associated representations of the compact quantum group. The conjugation $j_u : H_u \to H_{\overline{u}}$ is an invertible antilinear intertwiner j_u from u to $\overline{u}$. Thus, $j_u \otimes^* \circ u = \overline{u} \circ j_u$ or, in terms of matrix elements, $j_{mn} u^*_{np} = \overline{u}_{mr} j_{rp}$. This relation might be used to define the compact quantum groups involved. In fact, the quantum groups involved have been defined, less intrinsically, in terms of a linear operator Q in the notation of Wang [20] and F in that of Banica [1], [2].

We first treat the self-conjugate case, that is, embeddings of $\mathcal{T}_{rd}$ and $\mathcal{T}_{pd}$ so that $\overline{u} = u$ and, hence, $j_u^2 = \pm 1$. We let c be an antiunitary involution on the Hilbert space of u and set $Q := c j_u^*$. Then $QcQc = \pm 1$ and $Q^* \otimes 1 \circ c \otimes^* \circ u = u \circ Q^* \circ c$. Working in the basis where $c_{ij} = \delta_{ik}$, we get $u_{np} Q_{nm} = Q_{pn} u^*_{mn}$ or $u^t \circ Q = Q \otimes 1 \circ u^*$. These are the defining relations for the compact quantum group $B_u(Q)$ in the notation of Wang. Note that $Q^* Q = j_u j_u^*$. Thus, the isomorphism class of $B_u(Q)$ depends only on the eigenvalue list of Q^*Q, improving Wang's result. Banica uses the adjoint operator $F := Q^* = j_u c$ and denotes the quantum group by $A_o(F)$.

Turning to the embeddings of $\mathcal{T}_d$, Banica and Wang make use not of the conjugate representation $\overline{u}$ but of the equivalent nonunitary representation $\tilde{u}$, $\tilde{u}_{mn} := u^*_{mn}$, which depends on a choice of orthonormal basis in H_u. The antiunitary operator c leaving this basis fixed intertwines $\tilde{u}$ and u. Thus, $F := j_u c$ is a linear intertwiner from $\tilde{u}$ and $\overline{u}$, and by setting $Q := F^* F$, a computation shows that $u_{np} Q_{nr} = Q_{ps} u^*_{rs}$ or $u^t \circ Q = Q \otimes 1 \circ \tilde{u}$. This is the relation used by Wang to define the compact quantum group $A_u(Q)$ or $A_u(F)$ in the notation of Banica. Note that the eigenvalue list of Q coincides with that of $j_u^* j_u$ and is hence characteristic of the natural tensor unitary equivalence class of the embedding. As Wang showed, the quantum groups $A_u(Q)$ and $A_u(Q^{-1})$ are isomorphic, and this reflects the involution on $\mathcal{T}_d$ exchanging R and $\overline{R}$. The relations between the groups $A_u(Q)$ and the embeddings of $\mathcal{T}_d$ have already been established by Yamagami [30].

Thus, given a normalized solution of the conjugate equations $R, \overline{R}$ in a tensor C^*-category $\mathcal{M}$ we have a canonical tensor *-functor $\mu : \mathcal{T}_d \to \mathcal{M}$, and picking an embedding τ into the category of Hilbert spaces, we get an ergodic action of G_τ on ${}_\mu\mathcal{C}_\tau$. By choosing τ suitably, $G_\tau \simeq A_u(Q)$ for any $Q > 0$ with $\mathrm{Tr}(Q) = R^* \circ R$.

Similarly, given a real solution of the conjugate equations R in a tensor C^*-category $\mathcal{M}$, we have a canonical tensor *-functor $\mu : \mathcal{T}_{rd} \to \mathcal{M}$, and picking an embedding τ of $\mathcal{T}_{rd}$ into the category of Hilbert spaces, we get an ergodic action of G_τ on ${}_\mu\mathcal{C}_\tau$. By choosing τ suitably, $G_\tau \simeq B_u(Q)$ for any Q with $\mathrm{Tr}(Q^*Q) = \mathrm{Tr}(Q^*Q)^{-1} = R^* \circ R$ and $QcQc = I$. Since $R = \sum_k \psi_k \otimes j_u \psi_k$, where the sum is taken over an orthonormal basis of τ_u invariant under c, we have $R = \sum_k \psi_k \otimes Q^* \psi_k$.

Given a pseudoreal solution of the conjugate equations R in a tensor C^*-category $\mathcal{M}$, we have a canonical tensor *-functor $\tau : \mathcal{T}_{pd} \to \mathcal{M}$, and picking an embedding τ of $\mathcal{T}_{pd}$ into the category of Hilbert spaces, we get an ergodic action of G_τ on ${}_\mu\mathcal{C}_\tau$. By choosing τ suitably, $G_\tau \simeq B_u(Q)$ for any Q with $\mathrm{Tr}(Q^*Q) = \mathrm{Tr}(Q^*Q)^{-1} = R^* \circ R$ and $QcQc = -I$. The comment on ergodic actions of $\mathrm{S}_\mu\mathrm{U}(2)$

follows, since $\mathrm{S}_\mu\mathrm{U}(2)$ is isomorphic to $B_u(Q)$ with $QcQc = 1$ when $\mu > 0$ and the eigenvalue list of Q^*Q is $|\mu| \le |\mu^{-1}|$, and $\mathrm{S}_\mu\mathrm{U}(2)$ is isomorphic to $B_u(Q)$ with $QcQc = -1$ when $\mu < 0$ and the eigenvalue list of Q^*Q is $\mu < \mu^{-1}$. In this case, $R = -\sum_k \psi_k \otimes Q^*\psi_k$.

If we pick $v \mapsto R_v$ to be standard, then the condition $m(v) = \dim(\iota, \mu_v)$ is equivalent to saying that $\hat{R}_v$ is standard. The results on q-multiplicity now follow from Corollary A.10. This completes the proof of Theorems 3.1 and 3.2.

9. Outlook

The work reported on in this article is in the process of being extended in several directions. In Section 6, we introduced the tensor *-categories with conjugates $\mathcal{T}_{rd}$ and $\mathcal{T}_{pd}$ whose objects were tensor powers of a single object and described their embeddings into the tensor category of Hilbert spaces and the associated compact quantum groups. An interesting problem is to describe tensor C^*-categories without conjugates whose objects are again tensor powers of a single irreducible object but where the completion under subobjects has conjugates, their embeddings into Hilbert spaces, and the associated compact quantum groups. The compact quantum groups $\mathrm{S}_\mu\mathrm{U}(n)$, $n \ge 3$, are the prime examples that can be obtained in this way. We have not found a systematic way of producing further examples nor of classifying the underlying tensor C^*-categories. However, we have found compact quantum groups depending on two integers $n > 2$, the smallest integer $n > 0$ such that $\iota \le x^n$, where x is the generating object and d is the intrinsic dimension of x, and also the dimension of the Hilbert space of the corresponding representation of the compact quantum group.

Although not discussed in this article, our way of constructing ergodic actions leads to two ergodic actions on the C^*-algebra ${}_\mu\mathcal{C}_\tau$. Thus, if G_μ and G_τ denote the quantum groups with $\mathcal{C}(G_\mu) = {}_\mu\mathcal{C}_\mu$ and $\mathcal{C}(G_\tau) = {}_\tau\mathcal{C}_\tau$, then G_μ acts on the left and G_τ on the right on ${}_\mu\mathcal{C}_\tau$. The simplest well-known example of this phenomenon is when $\mu = \tau$, yielding the left and right actions of a quantum group on itself.

Finally, the C^*-algebras ${}_\mu\mathcal{C}_\tau$ may be used in a different way; we may define a suitable left action of the algebra on itself together with the obvious right action, making it into a ${}_\mu\mathcal{C}_\tau$-bimodule. Further bimodules can be constructed to reflect more fully the structure of the target tensor C^*-category $\mathcal{M}$ of μ. When μ is surjective on objects, $\mathcal{M}$ can be embedded in a tensor C^*-category of ${}_\mu\mathcal{C}_\tau$-bimodules, and this in turn leads to further insight on when $\mathcal{M}$ can be embedded into the tensor C^*-category of Hilbert spaces.

Appendix

In this section we show some properties of quasitensor functors and conjugation used in the article and begin by establishing the equivalence of the definition of quasitensor functor used here with that in [11].

Composing (2.9) on the left by $1_{\tau_u} \otimes \tilde{\tau}^*_{v,w}$ and on the right by $\tilde{\tau}_{u,v} \otimes 1_{\tau_w}$ gives

$$1_{\tau_u} \otimes \tilde{\tau}^*_{v,w} \circ \tilde{\tau}^*_{u,v\otimes w} \circ \tilde{\tau}_{u\otimes v,w} \circ \tilde{\tau}_{u,v} \otimes 1_{\tau_w} = 1_{\tau_u} \otimes 1_{\tau_v} \otimes 1_{\tau_w}.$$

Since we are dealing with isometries, this implies

$$\tilde{\tau}_{u\otimes v,w}\circ\tilde{\tau}_{u,v}\otimes 1_{\tau_w}=\tilde{\tau}_{u,v\otimes w}\circ 1_{\tau_u}\otimes\tilde{\tau}_{v,w}=:\tilde{\tau}_{u,v,w}, \tag{A.1}$$

the associativity condition. If we let $E_{u,v}\in(\tau_{u\otimes v},\tau_{u\otimes v})$ be the range projection of $\tilde{\tau}_{u,v}$ and $E_{u,v,w}\in(\tau_{u\otimes v\otimes w},\tau_{u\otimes v\otimes w})$ be the range projection of $\tilde{\tau}_{u,v,w}$, then by (2.9) and (A.1)

$$\begin{aligned} E_{u,v\otimes w}\circ E_{u\otimes v,w}&=\tilde{\tau}_{u,v\otimes w}\circ 1_{\tau_u}\otimes\tilde{\tau}_{v,w}\circ\tilde{\tau}^*_{u,v}\otimes 1_{\tau_w}\circ\tilde{\tau}^*_{u\otimes v,w}\\ &=\tilde{\tau}_{u\otimes v,w}\circ\tilde{\tau}_{u,v}\otimes 1_{\tau_w}\circ\tilde{\tau}^*_{u,v}\otimes 1_{\tau_w}\circ\tilde{\tau}^*_{u\otimes v,w}=E_{u,v,w}. \end{aligned} \tag{A.2}$$

Note that (A.1) and (A.2) replaced (2.9) in the definition of quasitensor functor in [11]. On the other hand, composing (A.2) on the left with $\tilde{\tau}^*_{u\otimes v,w}$ and on the right with $\tau_{u,v\otimes w}$ and using (A.1), we get (2.9). Thus, the two definitions are equivalent.

REMARK

We automatically have $\tilde{\tau}_{\iota,u}=\tilde{\tau}_{u,\iota}=1_{\tau_u}$ if the initial tensor C^*-category has conjugates or if every object is a direct sum of irreducibles.

Let us, informally, think of $\tau_u\otimes\tau_v$ as a subspace of $\tau_{u\otimes v}$. Equations (A.1) combined with (A.2) require the projection onto $\tau_{u\otimes v}\otimes\tau_w$ to take the subspace $\tau_u\otimes\tau_{v\otimes w}$ onto $\tau_u\otimes\tau_v\otimes\tau_w$. This property should be thought of as a variant of Popa's [16] commuting square condition for a square of inclusion of finite von Neumann algebras. In fact in that situation we have inclusions $N\subset M$, $Q\subset P$ such that $Q\subset N$ and $P\subset M$. Recall that this square is called a *commuting square* if $E^M_N(P)\subset Q$ (or, equivalently, if one of the following hold: $E^M_P(N)\subset Q$, $E^M_N E^M_P=E^M_P E^M_N=E^M_Q$).

PROPOSITION A.1

Let $(\sigma,\tilde{\sigma})$ and $(\tau,\tilde{\tau})$ be quasitensor functors, and suppose that $\rho:=\tau\sigma$ is defined. Set $\tilde{\rho}_{u,v}:=\tau(\tilde{\sigma}_{u,v})\circ\tilde{\tau}_{\sigma_u,\sigma_v}$. Then $(\rho,\tilde{\rho})$ is a quasitensor functor.

Proof

The proof just involves routine computations. It is given here for completeness. Obviously, $\tilde{\rho}_{\iota,u}$ and $\tilde{\rho}_{u,\iota}$ are units for any object u. We have

$$\begin{aligned} \tilde{\rho}^*_{u,v\otimes w}\circ\tilde{\rho}_{u\otimes v,w}&=\tilde{\tau}^*_{\sigma_u,\sigma_{v\otimes w}}\circ\tau(\tilde{\sigma}^*_{u,v\otimes w})\circ\tau(\tilde{\sigma}_{u\otimes v,w})\circ\tilde{\tau}_{\sigma_{u\otimes v},\sigma_w}\\ &=\tilde{\tau}^*_{\sigma_u,\sigma_{v\otimes w}}\circ\tau(1_{\sigma_u}\otimes\tilde{\sigma}_{v,w})\circ\tau(\tilde{\sigma}^*_{u,v}\otimes 1_{\sigma_w})\circ\tilde{\tau}_{\sigma_{u\otimes v},\sigma_w}\\ &=1_{\rho_u}\otimes\tau(\tilde{\sigma}_{v,w})\circ\tilde{\tau}^*_{\sigma_u,\sigma_v\otimes\sigma_w}\circ\tilde{\tau}_{\sigma_u\otimes\sigma_v,\sigma_w}\circ\tau(\tilde{\sigma}^*_{u,v})\otimes 1_{\rho_w}\\ &=1_{\rho_u}\otimes\tau(\tilde{\sigma}_{v,w})\circ 1_{\rho_u}\otimes\tilde{\tau}_{\sigma_v,\sigma_w}\circ\tilde{\tau}^*_{\sigma_u,\sigma_v}\otimes 1_{\rho_w}\circ\tau(\tilde{\sigma}^*_{u,v})\otimes 1_{\rho_w}\\ &=1_{\rho_u}\otimes\tilde{\rho}_{v,w}\circ\tilde{\rho}^*_{u,v}\otimes 1_{\rho_w}, \end{aligned}$$

completing the proof. □

We comment here on one particularly simple class of quasitensor functors. Let u be an object of a tensor C^*-category with irreducible tensor unit $(\iota,\iota)=\mathbb{C}$ For an object u pick an orthonormal basis A_i of the Hilbert space (ι,u), and set $c_u := \sum_i A_i \circ A_i^*$. Then c is the support of ι and is in the center of $\mathcal{A}$, meaning that if $T \in (u,v)$, then $T \circ c_u = c_v \circ T$. Note that

$$c_u \otimes c_v = c_{u\otimes v} \circ 1_u \otimes c_v = c_{u\otimes v} \circ c_u \otimes 1_v.$$

In fact,

$$c_{u\otimes v} \circ c_u \otimes 1_v = \sum_i c_{u\otimes v} \circ A_i \otimes 1_v \circ A_i^* \otimes 1_v = \sum_i A_i \otimes c_v \circ A_i^* \otimes 1_v = c_u \otimes c_v.$$

PROPOSITION A.2

Let $\tau : \mathcal{A} \to \mathcal{T}$ be a *-functor between tensor C^*-categories, where $\mathcal{T}$ has irreducible tensor unit and every object of $\mathcal{T}$ is a tensor product of objects in the image of τ. Let u,v be objects of $\mathcal{A}$, let A_i and B_j be orthonormal bases of the Hilbert spaces (ι,u) and (ι,v), respectively, and set

$$\tilde{\tau}_{u,v} = \sum_{i,j} \tau(A_i \otimes B_j) \circ \tau(A_i^*) \otimes \tau(B_j^*).$$

Then $\tilde{\tau}$ satisfies conditions (2.7)–(2.10) (but it may just be a partial isometry). We have that $(\tau,\tilde{\tau})$ is a quasitensor functor if and only if $\tau(c_u) = 1_{\tau_u}$ for all objects u of $\mathcal{A}$, $\mathcal{T}$ is a full tensor subcategory of a category of Hilbert spaces, and $\tilde{\tau}$ is the unique natural transformation making τ into a quasitensor functor.

Proof

It is easily checked that $\tilde{\tau}$ is a natural transformation and satisfies the associativity condition. Its initial projection is $\tau(c_u)\otimes\tau(c_v)$, and its final projection is $E_{u,v} = \tau(c_u \otimes c_v)$. Hence,

$$E_{u\otimes v,w} \circ E_{u,v\otimes w} = \tau(c_{u\otimes v} \otimes c_w \circ c_u \otimes c_{v\otimes w}) = \tau(c_u \otimes c_v \otimes c_w) = E_{u,v,w}.$$

Thus, $(\tau,\tilde{\tau})$ will be a quasitensor functor if and only if $\tau(c_u) = 1_{\tau_u}$ for all u. In particular, the support of the tensor unit of $\mathcal{T}$ is the unit, and every object of $\mathcal{T}$ is a direct sum of copies of the unit, so $\mathcal{T}$ is a full tensor subcategory of a category of Hilbert spaces. If $(\tau,\tilde{\tau})$ is to be a quasitensor functor, then $\tilde{\tau}_{u,v} \circ \tau(A)\otimes\tau(B) = \tau(A\otimes B)$ for all $A \in (\iota,u)$ and $B \in (\iota,v)$. □

The condition $\tau(c_u) = 1_{\tau_u}$ for all u is very strong. It also implies that τ_u is a zero object whenever $c_u = 0$, that is, whenever $(\iota,u) = 0$. Note that a general quasitensor functor has $E_{u,v} \geq \tau(c_u \otimes c_v)$ with equality characterizing the above special case. For this reason, we then say that $\tilde{\tau}$ is *minimal*. This case can be alternatively characterized by saying that the kernel of τ is precisely the set of arrows of $\mathcal{A}$ which are zero when composed with c. For if $T \in (u,v)$ and $T \circ c_u = 0$, then $\tau(T\circ c_u) = \tau(T) = 0$. Conversely if $\tau(T) = 0$, then $\tau(B_j^* \circ T \circ A_i) = 0$. Hence, $c_v \circ T \circ c_u = T \circ c_u = 0$. Thus, essentially what the functor τ does is to map u onto the Hilbert space (ι,u) and T onto the map $A \mapsto T \circ A$.

Quasitensor functors $(\tau, \tilde{\tau})$ with $\tilde{\tau}$ minimal are of no direct interest in this article as ${}_\mu\mathcal{C}_\tau$ reduces to the complex numbers. Indirectly, however, the minimal quasitensor functor $(q, \overline{q})$, defined below, plays a role in composition. Let $(\tau, \tilde{\tau})$ be a quasitensor functor $\mathcal{A} \to \mathcal{T}$, and let $q : \mathcal{T} \to \mathcal{H}$ denote the *-functor taking an object x of $\mathcal{T}$ to the Hilbert space (ι, x) and the arrow $T \in (x, y)$ to the map $A \mapsto T \circ A$. There is then a unique quasitensor functor $(q, \tilde{q})$ and $\tilde{q}$ is minimal. The composition $(q, \tilde{q}) \circ (\tau, \tilde{\tau})$ is then a quasitensor functor from $\mathcal{A}$ to $\mathcal{H}$ without the natural transformation being unitary, in general. When τ is actually a tensor *-functor, this class of examples was considered in [11] and includes as a special case the invariant vectors functor.

Note the following corollaries of the above discussion.

COROLLARY A.3

Let $\tau : \mathcal{A} \to \mathcal{T}$ be a *-functor between tensor C^*-categories where $\mathcal{A}$ is a category of Hilbert spaces. Then τ may be made into a quasitensor functor in a unique way and is then a relaxed tensor functor.

COROLLARY A.4

The tensor product on a C^*-category of Hilbert spaces is uniquely defined up to a natural unitary transformation. Any two tensor C^*-categories of Hilbert spaces are equivalent.

We next discuss properties of the conjugation on arrows $A \to A^\bullet$ defined in Section 4. This conjugation does not necessarily commute with the adjoint, so that $A^{*\bullet*}$ is, in general, an alternative conjugation. If we choose standard solutions of the conjugate equations, however, then $A^{*\bullet} = A^{\bullet *}$. In the next lemma, we prove two results that will be used later.

LEMMA A.5

Let A_i be an orthonormal basis of the Hilbert space (ι, u). Then $\sum_i A_i \otimes A_i^\bullet = c_u \otimes c_{\overline{u}} \circ \overline{R}_u$ and $\sum_i A_i^{*\bullet*} \otimes A_i = c_{\overline{u}} \otimes c_u \circ R_u$. Furthermore, $c_u^\bullet = c_{\overline{u}}$.

Proof

Let $A, B \in (\iota, u)$. Then $A \otimes B^\bullet = A \otimes 1_{\overline{u}} \circ B^\bullet = (A \circ B^*) \otimes 1_{\overline{u}} \circ \overline{R}_u$. Thus, $\sum_i A_i \otimes A_i^\bullet = c_u \otimes 1_{\overline{u}} \circ c_{u \otimes v} \circ \overline{R}_u = c_u \otimes c_{\overline{u}} \circ \overline{R}_u$. The second result can be proved similarly. Now $c_u^\bullet = \sum_i A_i^\bullet \circ A_i^{*\bullet}$. Thus, $c_{\overline{u}} c_u^\bullet = c_u^\bullet$ and similarly $c_u^\bullet c_{\overline{u}} = c_{\overline{u}}$. But both c and $c^\bullet$ lie in the center of $\mathcal{A}$, and the result follows. ☐

A choice of solutions of the conjugate equations determines a scalar product on each (u, u), and we write $\phi_u(A^* \circ B) := R_u^* \circ 1_{\overline{u}} \otimes (A^* \circ B) \circ R_u$.

If $X \in (\overline{u}, \tilde{u})$ is invertible, then $X \otimes 1_u \circ R_u$ and $1_u \otimes X^{*-1} \circ \overline{R}_u$ is another solution of the conjugate equations for u. By changing R_u using X and R_v using Y, the conjugation becomes $X^{-1*} \circ A^\bullet \circ Y^*$, while the scalar product on (u, u) is given by $\phi_u(A^* \circ B \circ (X^* \circ X)^{\bullet *})$.

Let $R_u, \overline{R}_u$ and $R_v, \overline{R}_v$ be solutions of the conjugate equations for u and v. Then $\overline{R}_u, R_u$ solves the conjugate equations for $\overline{u}$, and $1_{\overline{v}} \otimes R_u \otimes 1_v \circ R_v, 1_u \otimes \overline{R}_v \otimes 1_{\overline{u}} \circ \overline{R}_u$ solves the conjugate equations for $u \otimes v$. If these solutions always coincide with $R_{\overline{u}}, \overline{R}_{\overline{u}}$ and $R_{u\otimes v}, \overline{R}_{u\otimes v}$, respectively, then $u \mapsto R_u$ will be said to be a homomorphic choice of solutions of the conjugate equations.

PROPOSITION A.6

Let $u \mapsto R_u$ be a homomorphic choice of solutions of the conjugate equations. Then the associated conjugation $^\bullet$ is involutive.

Proof

Let $A \in (u, v)$. Then u, v are the conjugates of $\overline{u}, \overline{v}$ and

$$A^{\bullet\bullet} = R^*_{\overline{u}} \otimes 1_v \circ 1_u \otimes (1_{\overline{u}} \otimes \overline{R}^*_v \circ 1_{\overline{u}} \otimes A \otimes 1_{\overline{v}} \circ R_u \otimes 1_{\overline{v}}) \otimes 1_v \circ 1_u \otimes \overline{R}_{\overline{v}}.$$

But $R_{\overline{u}} = \overline{R}_u$ and $\overline{R}_{\overline{v}} = R_v$, so

$$\begin{aligned} A^{\bullet\bullet} &= \overline{R}^*_u \otimes 1_v \circ 1_u \otimes (1_{\overline{u}} \otimes \overline{R}^*_v \otimes 1_v \circ 1_{\overline{u}\otimes v} \otimes R_v \circ 1_{\overline{u}} \otimes A \circ R_u) \\ &= \overline{R}^*_u \otimes 1_v \circ 1_{u\otimes u} \otimes A \circ 1_u \otimes R_u = A. \end{aligned}$$

The following simple construction shows that, up to an equivalence of tensor C^*-categories, we may find such a homomorphic choice. Given a tensor C^*-category $\mathcal{T}$ with a choice $u \mapsto R_u$ of solutions of the conjugate equations, let $\mathcal{T}^\otimes$ be the tensor C^*-category whose objects are words in the objects u of $\mathcal{T}$ and their formal adjoints $\overline{u}$. The tensor product of objects is defined by juxtaposition: $(u_1, u_2, \dots, u_m) \otimes (v_1, v_2, \dots, v_n) := (u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m)$. The arrows are defined by setting

$$\big((u_1, u_2, \dots, u_m), (v_1, v_2, \dots, v_n)\big) := (u_1 \otimes u_2 \otimes \cdots \otimes u_m, v_1 \otimes v_2 \otimes \cdots \otimes v_n)$$

and are given the obvious algebraic operations. There is a tensor *-functor $\eta : \mathcal{T}^\otimes \to \mathcal{T}$ with $\eta_{(u_1,u_2,\dots,u_m)} = u_1 \otimes u_2 \otimes \cdots \otimes u_m$ acting as the identity on arrows. Here a formal conjugate $\overline{u}$ is mapped onto the conjugate of u in $\mathcal{T}$ determined by R_u. Obviously η is an equivalence of tensor C^*-categories. We now choose solutions of the conjugate equations for sequences of length 1 setting $R_{(u)} := R_u$ and $R_{(\overline{u})} := \overline{R}_u$ and extend in the unique way to get a homomorphic choice. □

LEMMA A.7

Let $u \mapsto R_u$ be a choice of solutions of the conjugate equations for u such that

$$\psi_u(A) := \overline{R}_u{}^* \circ 1_u \otimes A \circ \overline{R}_u, \quad A \in (\overline{u}, \overline{u}),$$

is tracial: $\psi_u(A^* \circ B) = \psi_v(B \circ A^*)$, $A, B \in (\overline{u}, \overline{v})$, and write for clarity $J_u S := S^\bullet$, $S \in (\iota, u)$. Then

$$\mathrm{Tr}(J^*_u \circ J_u) = \mathrm{Tr}(J^{-1*}_u \circ J^{-1}_u) = \dim(\iota, u).$$

Proof

If S_i is an orthonormal basis of (ι, u), then by Lemma A.5

$$\mathrm{Tr}(J_u^* J_u) = \sum_{i,j} (S_i \otimes S_i^\bullet)^* \circ (S_j \otimes S_j^\bullet) = \overline{R}_u^* \circ c_u \otimes c_{\overline{u}} \circ \overline{R}_u = \overline{R}_u^* \circ 1_u \otimes (c_{\overline{u}} \circ c_u^\bullet) \circ \overline{R}_u.$$

But by Lemma A.5, $c_u^\bullet = c_{\overline{u}}$ so we get $\mathrm{Tr}(J_u^* J_u) = \psi_u(c_{\overline{u}})$. Picking an orthonormal basis $\overline{S}_i$ for $(\iota, \overline{u})$ gives $c_{\overline{u}} = \sum_i \overline{S}_i \circ \overline{S}_i^*$. But ψ is tracial, so $\mathrm{Tr}(J_u^* J_u) = \sum_i \overline{S}_i^* \circ \overline{S}_i = \dim(\iota, \overline{u}) = \dim(\iota, u)$, as required. □

It has been shown in [10] that if ψ is tracial, then the corresponding conjugation commutes with the adjoint. As a consequence, if $S, T \in (\iota, u)$, then

$$(T, S) = (S, T)^* = (S, T)^\bullet = (S^* \circ T)^\bullet = S^{\bullet *} \circ T^\bullet = (S^\bullet, T^\bullet).$$

In other words, $^\bullet : (\iota, u) \to (\iota, \overline{u})$ is antiunitary.

Standard solutions $R, \overline{R}$ of the conjugate equations for u have special properties. They are unique up to a unitary, and products of standard solutions are again standard. Furthermore, ϕ_u is independent of the choice of standard solution, so we may replace $\overline{R}_u$ by $R_{\overline{u}}$ and get $\psi_u = \phi_{\overline{u}}$, and it is known that ϕ is tracial, that is, if $A, B \in (u, v)$, then $\phi_u(A^* \circ B) = \phi_v(B \circ A^*)$. Furthermore, $\phi_u(1_u) = R^* \circ R = \overline{R}^* \circ \overline{R} = d(u)$, the intrinsic dimension of u (see [10]).

LEMMA A.8

Let $u \mapsto R_u$ be a choice of standard solutions of the conjugate equations.

(a) *Let $W \in (v, u)$ be an isometry. Then $R_v = W^{\bullet *} \otimes W^* \circ R_u$ and $\overline{R}_v = W^* \otimes W^{\bullet *} \circ \overline{R}_u$.*

(b) *Let $W_i \in (u_i, u)$ be isometries with $\sum_i W_i \circ W_i^* = 1_u$. Then $R_u = \sum_i W_i^\bullet \otimes W_i \circ R_{u_i}$ and $\overline{R}_u = \sum_i W_i \otimes W_i^\bullet \circ \overline{R}_{u_i}$.*

(c) *There is a unitary $V \in (u, \overline{\overline{u}})$ such that $R_{\overline{u}} = V \otimes 1_{\overline{u}} \circ \overline{R}_u$ and $\overline{R}_{\overline{u}} = 1_{\overline{u}} \otimes V \circ R_u$.*

Proof

(a) This is proved as

$$\overline{R}_v = W^* \otimes 1_{\overline{v}} \circ 1_u \otimes W^{\bullet *} \circ \overline{R}_u = W^* \circ W \otimes 1_{\overline{v}} \circ \overline{R}_v = R_v.$$

The second equation can be proved similarly.

(b) We compute

$$\sum_i W_i^\bullet \otimes W_i \circ R_{u_i} = \sum_i 1_{\overline{u}} \otimes (W_i \circ W_i^*) \circ R_u = R_u.$$

The second equation can be proved similarly.

(c) Note that $\overline{R}_u, R_u$ is a standard solution of the conjugate equations for $\overline{u}$ and therefore differs from $R_{\overline{u}}, \overline{R}_{\overline{u}}$ by a unitary V as claimed. □

It must be remembered though that, even if $u \to R_u$ is standard, $\hat{R}_u$ and $\tilde{R}_u$ defined in Section 4 will not, in general, be standard, nor will the conjugation commute with the adjoint. Nevertheless, the following result holds.

LEMMA A.9

Let $u \mapsto R_u$ be a standard choice of solutions of the conjugate equations. Then the set of objects u such that $\hat{R}_u, \hat{\overline{R}}_u$ is a standard solution of the conjugate equations for μ_u is closed under tensor products, subobjects, direct sums, and conjugates.

Proof

Suppose that $\hat{R}_u, \hat{\overline{R}}_u$ is standard and that $W \in (v, u)$ is an isometry. Then by Lemma A.8, $\hat{R}_v = \mu(W^{\bullet *}) \otimes \mu(W^*) \circ \hat{R}_u$. Now

$$\hat{R}_v^* \circ \hat{R}_v = \hat{R}_u^* \circ \mu(E^\bullet) \otimes \mu(E) \circ \hat{R}_u = \hat{R}_u^* \circ 1_{\mu_{\overline{u}}} \otimes \mu(E) \circ \hat{\overline{R}}_u = \phi_{\mu_u}(E),$$

where ϕ_u is the standard left inverse of μ_u. By the tracial property of the standard left inverse, $\phi_{\mu_u}(E) = \phi_{\mu_u}(W^* \circ W) = \phi_{\mu_v}(1_{\mu_v}) = d(\mu_v)$. Similarly $\hat{\overline{R}}_v^* \circ \hat{\overline{R}}_v = d(\mu_v)$ and $\hat{R}_v, \hat{\overline{R}}_v$ are standard. Now suppose that $\hat{R}_{u_i}, \hat{\overline{R}}_{u_i}$ are standard and that $W_i \in (u_i, u)$ are isometries with $\sum_i W_i \circ W_i^* = 1_u$. Then by Lemma A.7, $\hat{R}_u = \sum_i W_i^\bullet \otimes W_i \circ \hat{R}_{u_i}$. Hence, $\hat{R}_u^* \circ \hat{R}_u = \sum_i \hat{R}_{u_i}^* \circ \hat{R}_{u_i} = \sum_i d(\mu_{u_i}) = d(\mu_u)$. Similarly, $\hat{\overline{R}}_u^* \circ \hat{\overline{R}}_u = d(\mu_u)$ so that $\hat{R}_u, \hat{\overline{R}}_u$ is standard. Again, if $\hat{R}_u$ is standard, by Lemma A.7, there is a unitary $V \in (u, \overline{\overline{u}})$ such that $\hat{R}_{\overline{u}} = \mu(V) \otimes 1_{\mu_{\overline{u}}} \circ \hat{\overline{R}}_u$. Thus, $\hat{R}_{\overline{u}}^* \circ \hat{R}_{\overline{u}} = \hat{\overline{R}}_u^* \circ \hat{\overline{R}}_u = d(\mu_u) = d(\mu_{\overline{u}})$ and similarly $\hat{\overline{R}}_{\overline{u}}^* \circ \hat{\overline{R}}_{\overline{u}} = d(\mu_{\overline{u}})$. Thus, $\hat{R}_{\overline{u}}$ is standard. The question of whether $\hat{R}_u$ is standard is obviously independent of the choice of standard solution R_u. If $R_{u\otimes v}$ is chosen to be of product form, then the same is true of $\hat{R}_{u\otimes v}$. Thus, $\hat{R}_u$ and $\hat{R}_v$ being standard implies that $\hat{R}_{u\otimes v}$ is standard. □

COROLLARY A.10

If $u \mapsto R_u$ is standard and μ_v is an irreducible generator of $\mathcal{M}$, then $u \mapsto \hat{R}_u$ is standard. If $d(u) = d(\mu_u)$, then $\hat{R}_u, \hat{\overline{R}}_u$ is standard.

Proof

The first statement follows since, μ_v being irreducible, $\hat{R}_v, \hat{\overline{R}}_v$ is automatically standard. Now $d(\mu_u) \le \hat{R}_u^* \circ \hat{R}_u = \mu(R_u^*) \circ E_{\overline{u},u} \circ \mu(R_u) \le R_u^* \circ R_u = d(u) = d(\mu_u)$, and the result follows. □

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