Singularities and Kodaira dimension of moduli scheme of stable sheaves on Enriques surfaces

Kimiko Yamada

Abstract We shall show that the Kodaira dimension of the moduli scheme of stable sheaves on any Enriques surface is zero when its expected dimension is 7 or more. A similar result holds also for bielliptic surfaces. We prove this by estimating the cup-product structure of Ext^1.

1. Introduction

Let X be a nonsingular projective minimal surface over $\mathbb{C}$, and let H be an ample line bundle on X. There is a coarse moduli scheme $M(H)$ (resp., $\bar{M}(H)$) of H-stable (resp., H-semistable) sheaves with fixed Chern classes $(r, c_1, c_2) \in \mathbb{Z}_{>0} \times \mathrm{NS}(X) \times \mathbb{Z}$, where we assume $r > 1$. Gieseker [1] and Maruyama [9] proved that $\bar{M}(H)$ is projective over $\mathbb{C}$, and $M(H)$ is its open subscheme. Suppose that X is a minimal surface with $\kappa(X) = 0$. In other words, X is an Enriques, hyperelliptic, K3 or Abelian surface. Then K_X is numerically equivalent to zero, and there is a covering map $\pi : \tilde{X} \to X$, where $\tilde{X}$ is a nonsingular projective surface with $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. Let d be the degree of π. The main result of this article is as follows.

THEOREM 1.1

Let X be a minimal surface with $\kappa(X) = 0$. Suppose that

$$2rc_2 - (r-1)c_1^2 - r^2\chi(\mathcal{O}_X) \geq 3d,$$

that is,

$$\exp\dim M(H) = \mathrm{ext}^1(E,E) - \mathrm{ext}^2(E,E)^\circ \geq 3d + 1 + p_g(X).$$

Then the moduli scheme $M(H)$ of H-stable sheaves with Chern classes (r, c_1, c_2) is locally complete intersection (l.c.i.), normal, 1-Gorenstein, and admits only canonical singularities.

Kyoto Journal of Mathematics, Vol. 53, No. 1 (2013), 145–153
DOI 10.1215/21562261-1966098,
Received October 21, 2011. Revised July 3, 2012. Accepted July 18, 2012.
2010 Mathematics Subject Classification: Primary 14J60; Secondary 14B05, 14D20, 14Exx.
Author's work supported by Japan Society for the Promotion of Science Grants-in-Aid for Young Scientists (B), no. 23740037.

We know that some positive multiple of the canonical class of $M(H)$ equals $\mathcal{O}_{M(H)}$ by the Grothendieck–Riemann–Roch theorem, so we have a corollary.

COROLLARY 1.2

If the hypothesis in the theorem above holds and, in addition, the greatest common divisor of r, $c_1 \cdot \mathcal{O}_X(1)$, and $c_1^2/2 - c_2$ equals 1, then $M(H)$ is known to be projective by [9, Theorem 6.11], and its Kodaira dimension is zero.

The new point of this article is that we consider stable sheaves on Enriques surfaces and observe how good singularities of its moduli schemes are. When X is K3, Abelian, or Fano, $M(H)$ is nonsingular, and its Kodaira dimension has been studied (see [9, Proposition 6.9.], [10], [13]). However, $M(H)$ has singularities in general; how good are they? Thus far, the works on formality are known on this problem. Goldman and Millson [2] discussed deformation of flat bundles; note that its first and second Chern classes are zero. The author does not know whether such a result is valid also for general stable (or Hermite–Einstein) bundles or not. Formality discusses the effect of the degree-two part of the defining ideal of the moduli space; we would rather estimate the degree-two part itself. As to moduli of stable sheaves on an Enriques surface, Kim dealt with it in [6] and obtained a finite birational map from $M(H)$ to a Lagrangian subvariety of the moduli space of sheaves on K3 surfaces. The singularities of $M(H)$ were not considered.

NOTATION

All schemes are of finite type over $\mathbb{C}$, but it seems that the greater part of this article holds also for an algebraically closed field k with $(r, \mathrm{char}(k)) = 1$. For sheaves E and F on a projective scheme over $\mathbb{C}$, $\hom(E,F)$ and $\mathrm{ext}^i(E,F)$ mean $\dim \mathrm{Hom}(E,F)$ and $\dim \mathrm{Ext}^i(E,F)$, respectively. For homomorphisms $f : F \to G$ and $g : G' \to E$, f^* and f_* mean natural pullback and pushforward homomorphisms of Exts, respectively. For a line bundle L, $\mathrm{Ext}^i(E, E \otimes L)^\circ$ denotes the kernel of trace map $\mathrm{Ext}^i(E, E \otimes L) \to H^i(L)$.

2. Proof of Theorem 1.1

Let X be a nonsingular complex projective surface with arbitrary Kodaira dimension.

DEFINITION 2.1

For a nonzero torsion-free sheaf F, we denote $\chi(F(nH))/r(F)$ by $P(F(n))$. A coherent sheaf E on X is *stable* if E is torsion-free and for every proper coherent subsheaf F of E we have

$$P\big(F(n)\big) < P\big(E(n)\big) \quad (n \gg 0).$$

DEFINITION 2.2

A normal variety V has only *canonical singularities* if it satisfies the following two conditions:

(a) the Weil divisor rK_V is Cartier for some integer $r \geq 1$;

(b) if $f: W \to V$ is a resolution of V and $\{E_i\}$ the family of all exceptional prime divisors of f, then

$$K_W = f^*(K_V) + \sum a_i E_i \quad \text{with } a_i \geq 0.$$

FACT 2.3 ([4], COROLLARY 1.7, [12], [11])

Let $(X, x) \subset (\mathbb{C}^{n+1}, 0)$ be a hypersurface singularity defined by $t_0^{a_0} + t_1^{a_1} + \cdots + t_n^{a_n}$. This singularity is canonical if and only if $\sum_{i=0}^{n} \frac{1}{a_i} > 1$.

As to singularities of $M(H)$, we shall use the following fact in Kuranishi theory.

FACT 2.4 ([8])

Let E be a stable sheaf on an arbitrary nonsingular surface. Denote $\dim \operatorname{Ext}^1(E, E) = d + b$ and $\dim \operatorname{Ext}^2(E, E)^\circ = b$, and let $f_1, \ldots, f_b$ be a basis of $\operatorname{Hom}(E, E(K_X))^\circ$. Then the local ring R of $M(H)$ at E is isomorphic to $\mathbb{C}[[t_1, \ldots, t_{d+b}]]/(F_1, \ldots, F_b)$, where F_i is a power series starting with a degree-two term, which comes from

$$F_f : \operatorname{Ext}^1(E, E) \otimes \operatorname{Ext}^1(E, E) \longrightarrow \operatorname{Ext}^2(E, E) \longrightarrow \mathbb{C} \tag{1}$$

defined by $F_f(\alpha \otimes \beta) = \operatorname{tr}(f \circ \alpha \circ \beta + f \circ \beta \circ \alpha)$ and its dual map

$$F_f^\vee : \mathbb{C} \longrightarrow \operatorname{Ext}^1(E, E)^\vee \otimes \operatorname{Ext}^1(E, E)^\vee \longrightarrow \operatorname{Sym}^2\big(\operatorname{Ext}^1(E, E)^\vee\big), \tag{2}$$

where f equals f_i.

From now on, we shall suppose that K_X is numerically equivalent to zero and let E be a stable sheaf on X. If $M(H)$ is singular at E, then $\operatorname{ext}^2(E, E) = \hom(E, E(K_X)) = 1$ since the stability of E implies that $\hom(E, E(K_X)) \leq 1$ in this case, and so $\operatorname{Hom}(E, E(K_X))$ is spanned by one element, say, f.

LEMMA 2.5

Let us choose coordinates t_i in Fact 2.4 so that $F_f^\vee = t_1^2 + \cdots + t_N^2$ with some $0 \leq N \leq d + b$. If the codimension of

$$\operatorname{Sing}\big(M(H)\big) := \big\{[E] \,\big|\, \operatorname{ext}^2(E, E)^\circ \neq 0\big\}$$

in $M(H)$ is more than 1 and $N \geq 3$, then the singularity of $M(H)$ at E is canonical.

Proof

The assumption implies that $F_f^\vee \neq 0$, and thus $M(H)$ is l.c.i. by Fact 2.4. Since $M(H)$ is smooth in codimension one, $M(H)$ is normal and 1-Gorenstein. In order to determine whether the singularity of the l.c.i. ring R at $x \in \operatorname{Spec}(R)$ is canonical or not, it is sufficient to look at its completion at x. This is because the singularities of R are canonical if and only if they are rational singularities (see

[5, Corollary 5.2.15]), and a variety V has only rational singularities if and only if V^{an} does so (see [7, Corollary 5.11]). Since the rationality of singularities is stable under small deformation (see [5, Section 8]), this lemma follows from Fact 2.3. $\square$

As to the covering $\pi : \tilde{X} \to X$ of X, we have a free action of a finite group G with $|G| = d$ on $\tilde{X}$ such that π equals the quotient map by G. It holds that $\pi_*(\mathcal{O}_{\tilde{X}}) = \bigoplus_{i=0}^{d-1} K_X^{-i}$.

LEMMA 2.6

If E is an H-stable sheaf on X, then $\pi^*(E)$ is π^*H-semistable.

Proof

Otherwise, π^*E has a nontrivial Harder–Narasimhan filtration (HNF) with respect to π^*H. Let F be the first part of HNF. By the uniqueness of HNF, F has a natural structure of a G-subsheaf of $\pi^*(E)$, and so F descends to a subsheaf F_0 of E. This F_0 breaks the stability of E, since K_X is numerically equivalent to 0. $\square$

LEMMA 2.7

The natural map $\mathrm{Ext}^1_X(E,E) \to \mathrm{Ext}^1_{\tilde{X}}(\pi^*(E), \pi^*(E))$ induces the isomorphism $\mathrm{Ext}^1(E,E) \simeq \mathrm{Ext}^1(\pi^*(E), \pi^*(E))^G$.

Proof

Let $E_\bullet \to E$ be a locally free resolution of E:

$$\begin{aligned}
&\mathrm{Ext}^1\bigl(\pi^*(E), \pi^*(E)\bigr)\\
&\quad = H^1\bigl(\mathbf{R}\Gamma_{\tilde{X}}(\pi^*(E_\bullet^\vee \otimes E_\bullet))\bigr)\\
&\quad = H^1\bigl(\mathbf{R}\Gamma_X \mathbf{R}\pi_*(\pi^*(E_\bullet^\vee \otimes E_\bullet))\bigr) = H^1\bigl(\mathbf{R}\Gamma_X(E_\bullet^\vee \otimes E_\bullet \otimes \pi_*\mathcal{O}_{\tilde{X}})\bigr)\\
&\quad = \bigoplus_{i=0}^{d-1} H^1\bigl(\mathbf{R}\Gamma_X(E_\bullet^\vee \otimes E_\bullet \otimes K_X^{-i})\bigr) = \bigoplus_{i=0}^{d-1} \mathrm{Ext}^1(E, E \otimes K_X^{-i}).
\end{aligned}$$

Since $\pi_*(\mathcal{O}_{\tilde{X}})^G = \mathcal{O}_X$, this lemma holds. $\square$

Now, let $f : E \to E(K_X)$ be a nonzero traceless homomorphism. Since K_X is numerically equivalent to zero and E is stable, this f is an isomorphism, and so $\det(f) \neq 0$. Fix an isomorphism $\mathcal{O}_{\tilde{X}} \simeq \pi^*(K_X)$, and let $t \in \Gamma(\pi^*(K_X))$ be the image of 1 by this. When we denote the eigenpolynomial of π^*f by $P_{\pi^*f}(t)$, we can decompose it into eigenvalues

$$P_{\pi^*f}(t) = \prod_i (t - a_i)^{n_i},$$

where a_i are elements in $H^0(K_{\tilde{X}})$ which differ from each other, from the fact that $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. Let us fix a_1 and pick any $g \in G$. Then $\det(f) \neq 0$ implies $a_1 \neq 0$, and $g(\pi^*(f)) = \pi^*(f)$ implies $g(a_1) = a_i$ with some i.

LEMMA 2.8

If $g \neq e$, then $g(a_1) \neq a_1$.

Proof

Otherwise, one can indicate the orbit O_{a_1} as $\{a_1, \dots, a_m\}$ with some $m < |G| = d$. Then $a_1 \cdot \cdots \cdot a_m$ lies in $\Gamma(mK_{\tilde{X}})^G = \Gamma(mK_X)$, but we know that $\Gamma(mK_X) = 0$ for $m < d$. This contradicts the fact that $\det(f) \neq 0$. $\square$

LEMMA 2.9

We have $\prod_{g \in G}(\pi^* f - g(a_1)) = 0$ in $\mathrm{Hom}(\pi^* E, \pi^* E(dK_X))$.

Proof

This homomorphism f^G lies in $\mathrm{Hom}(\pi^* E, \pi^* E(dK_X))^G = \mathrm{Hom}(E, E(dK_X))$. If $f^G \neq 0$, then f^G should be injective since E is stable. However, f^G is not injective obviously. $\square$

For $g \in G$, denote $\mathrm{Ker}(\pi^* f - g(a_1)) \subset \pi^* E$ by F_g.

LEMMA 2.10

The natural map $\bigoplus_{g \in G} F_g \to \pi^* E$ is isomorphic.

Proof

Let us consider the map

$$\prod_{g \neq e} \big(\pi^* f - g(a_1)\big) : F_e \hookrightarrow \pi^* E \to F'_e := \mathrm{Im} \prod_{g \neq e} \big(\pi^* f - g(a_1)\big). \tag{3}$$

If $\alpha \in F_e$ belongs to the kernel of this map, then $0 = \prod_{g \neq e}(\pi^* f - g(a_1))(\alpha) = \prod_{g \neq e}(a_1 - g(a_1))(\alpha)$, and then Lemma 2.8 and $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$ deduce that $\alpha = 0$. Thus the map (3) is injective. Next, the map $f^{-(d-1)} : \pi^* E((d-1)K_X) \to \pi^* E$ induces the map $f^{-(d-1)} : F'_e \to F_e$ by Lemma 2.9. One can check that $f^{-(d-1)} \circ \prod_{g \neq e}(\pi^* f - g(a_1)) : F_e \to F_e$ is the multiple map by a nonzero constant and so is isomorphic by Lemma 2.8. Thereby, the map (3) is bijective, and hence we have $\pi^* E = F_e \oplus (\pi^* E / F_e)$. Repeating this, we can show this lemma. $\square$

Since $F_g \neq 0$, $P(F_g(n)) = P(\pi^* E(n))$ by Lemmas 2.6 and 2.10. The homomorphism $\pi^* f - g(a_1)$ induces exact sequences

$$0 \longrightarrow F_g \xrightarrow{i_g} \pi^* E \xrightarrow{p_g} G_g \longrightarrow 0 \tag{4}$$

and

$$0 \longrightarrow G_g \xrightarrow{j_g} \pi^* E(K_X) \xrightarrow{q_g} Q_g \longrightarrow 0. \tag{5}$$

PROPOSITION 2.11

One can find $h \in G$ with $h \neq e$ as follows: the dimension of the image of the *natural map,*

$$\text{(6)} \qquad \operatorname{Ext}^1\bigl(Q_e(-K_X), F_h\bigr) \xrightarrow{i_{h*}} \operatorname{Ext}^1\bigl(Q_e(-K_X), \pi^*E\bigr) \xrightarrow{q_e^*} \operatorname{Ext}^1(\pi^*E, \pi^*E),$$

which comes from the exact sequences (4) and (5), is $-\frac{1}{d}\chi(E,E)$ *or more.*

We shall prove this proposition later; we presume it proved for now. From now on, suppose that the hypothesis in Theorem 1.1 holds, which implies that $-\chi(E,E)/d - d^2 + 1 \geq 3$. Hence there is a nonzero element $\alpha \in \operatorname{Ext}^1(\pi^*E, \pi^*E)$ lying in the image of the map (6). For any $h \in G$, we have the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}^1\bigl(Q_e(-K_X), F_h\bigr) \arrow[r, "i_{h*}"'] \arrow[d, "q_e^*"] & \operatorname{Ext}^1\bigl(Q_e(-K_X), \pi^*E\bigr) \arrow[d, "q_e^*"'] & \\
\operatorname{Ext}^1(\pi^*E, F_h) \arrow[r, "i_{h*}"'] & \operatorname{Ext}^1(\pi^*E, \pi^*E) \arrow[r, "p_{h*}"'] \arrow[d, "j_e^*"] & \operatorname{Ext}^1(\pi^*E, G_h) \arrow[d, "j_{h*}"] \\
& \operatorname{Ext}^1\bigl(G_e(-K_X), \pi^*E\bigr) \arrow[d, "p_e^*"] & \operatorname{Ext}^1\bigl(\pi^*E, \pi^*E(K_X)\bigr) \\
& \operatorname{Ext}^1\bigl(\pi^*E(-K_X), \pi^*E\bigr) &
\end{tikzcd}$$

(7)

and it holds that $p_e^* \circ j_e^* = (\pi^*f - a_1)^*$ and $j_{h*} \circ p_{h*} = (\pi^*f - h(a_1))_*$. Thus $(\pi^*f - a_1)^*(\alpha) = 0$ and $(\pi^*f - h(a_1))_*(\alpha) = 0$. From the definition of pullback and pushforward, it implies that

$$\text{(8)} \qquad \alpha \circ \pi^*f = (\pi^*f)^*(\alpha) = a_1\alpha \qquad \text{and} \qquad \pi^*f \circ \alpha = (\pi^*f)_*(\alpha) = h(a_1)\alpha$$

in $\operatorname{Ext}^1(\pi^*E, \pi^*E(K_X))$. Here we denote $\sum_{g\in G} g(\alpha) \in \operatorname{Ext}^1(\pi^*(E), \pi^*(E))$ by α^G. If $\alpha^G = 0$, then $-\alpha \in \operatorname{Ker}(\pi^*f - a_1)$ and $\sum_{g\neq e} g(\alpha)$ should belong to $\operatorname{Ker}\bigl(\prod_{g\neq e}(\pi^*f - g(a_1))\bigr)$, but $\operatorname{Ker}(\pi^*f - a_1) \cap \operatorname{Ker}\bigl(\prod_{g\neq e}(\pi^*f - g(a_1))\bigr) = 0$ by Lemma 2.8. Thus $\alpha^G \neq 0$. From Lemma 2.7, α^G descends to a nonzero element $\bar{\alpha} \in \operatorname{Ext}^1(E,E)$. Since f is isomorphic, by the Serre duality we have some $\beta \in \operatorname{Ext}^1(E,E)$ such that

$$\text{(9)} \qquad \operatorname{tr}(f \circ \bar{\alpha} \circ \beta) \neq 0.$$

LEMMA 2.12

If β *satisfies (9), then* $\operatorname{tr}(f \circ \bar{\alpha} \circ \beta + f \circ \beta \circ \bar{\alpha}) \neq 0$.

Proof

From the definition of $\bar{\alpha}$, we have

$$\begin{aligned}\operatorname{tr}\bigl(\pi^*(f\circ\bar\alpha\circ\beta)\bigr) &= \operatorname{tr}(\pi^* f\circ\pi^*\bar\alpha\circ\pi^*\beta)\\ &= \sum_{g\in G}\operatorname{tr}\bigl(\pi^* f\circ g(\alpha)\circ\pi^*\beta\bigr) = \sum_{g}\operatorname{tr}\bigl(gh(a_1)\cdot g(\alpha)\circ\pi^*(\beta)\bigr),\end{aligned}$$

where one gets the last equation by the action of g on the second at (8). From a basic property of trace (see [3, p. 257]), it holds that

$$\begin{aligned}-\operatorname{tr}\bigl(\pi^*(f\circ\beta\circ\bar\alpha)\bigr) &= \operatorname{tr}\bigl(\pi^*(\bar\alpha\circ f\circ\beta)\bigr)\\ &= \sum_{g}\operatorname{tr}\bigl(g(\alpha)\circ\pi^* f\circ\pi^*\beta\bigr) = \sum_{g}\operatorname{tr}\bigl(g(a_1)\cdot g(\alpha)\circ\pi^*(\beta)\bigr),\end{aligned}$$

where one gets the last equation by the action of g on the first in (8). Since $a_1\in\Gamma(\pi^*K_X)=\Gamma(\mathcal{O}_{\tilde X})$ is nowhere vanishing, we can set $\lambda = h(a_1)/a_1\in\Gamma(\mathcal{O}_{\tilde X})\simeq\mathbb{C}\simeq\Gamma(\mathcal{O}_{\tilde X})^G$, and thus $gh(a_1)/g(a_1)=g(h(a_1)/a_1)=h(a_1)/a_1=\lambda$ for any $g\in G$. Thereby $\operatorname{tr}(\pi^*(f\circ\bar\alpha\circ\beta))=-\lambda\operatorname{tr}(\pi^*(f\circ\beta\circ\bar\alpha))$, and then

$$\pi^*\operatorname{tr}(f\circ\bar\alpha\circ\beta+f\circ\beta\circ\bar\alpha)=\Bigl(1-\frac{1}{\lambda}\Bigr)\pi^*\operatorname{tr}(f\circ\bar\alpha\circ\beta)\neq 0,$$

since $\lambda\neq 0,1$ by Lemma 2.8. □

Summing up, we have found $\bar\alpha$ and β in $\operatorname{Ext}^1(E,E)$ such that $F_f(\bar\alpha,\beta)\neq 0$ for F_f at (1) if $-\frac{1}{d}\chi(E,E)>0$ by Proposition 2.11, and consequently we can find $t_1\in\operatorname{Ext}^1(E,E)$ such that $F_f(t_1,t_1)=1$. Then, replacing the argument above from $\operatorname{Ext}^1(E,E)$ to $\operatorname{Ker}F_f(t_1,\cdot)$, we can find t_2 such that $F_f(t_1,t_2)=0$ and $F_f(t_2,t_2)=1$, when $-\frac{1}{d}\chi(E,E)\geq 2$.

LEMMA 2.13

(a) *If E belongs to* $\operatorname{Sing}(M(H))$ *from Lemma 2.5, then $E\simeq\pi_*(F_g)$ for all $g\in G$.*

(b) *For any $g\in G$, F_g is π^*H-stable.*

(c) *When $-\frac{1}{d}\chi(E,E)\geq 3$, the codimension of* $\operatorname{Sing}(M(H))$ *in $M(H)$ is greater than* 1.

Proof

Since $\bigoplus_{g\in G}F_g\simeq\pi^*E$, we have an injective map $E\to\pi_*(F_e)$. Applying the Grothendieck–Riemann–Roch theorem and projection formula to $\pi_*(F_e)$, we have $d\operatorname{ch}(E)\equiv\sum_g\operatorname{ch}(\pi_*(F_g)))=d\operatorname{ch}(\pi_*(F_e))$ in $\operatorname{CH}(X)_{\mathbb{Q}}$, since $F_g=g^*(F_e)$ for all g. Thus the injective map $E\to\pi_*(F_e)$ is isomorphic. Clearly $\pi_*(F_g)=\pi_*(F_e)$, so we get (a). Next, F_e satisfies $P(F_e(n\pi^*H))=P(\pi^*E(nH))$ by Lemma 2.6. If F_e is not π^*H-stable, then there is a proper subsheaf $F'\subset F_e$ such that $P(F'(n\pi^*H))=P(\pi^*E(nH))$. Then $\bigoplus_{g\in G}g^*(F')\subset\pi^*(E)$ descends to a proper subsheaf of E, which becomes an H-destabilizer of E. This is a contradiction and gives (b). Next, as to (c), we can assume $d\geq 2$, for if $d=1$, then K_X is trivial, and then $M(H)$ is nonsingular by [10]. From (a), the moduli number of $\operatorname{Sing}(M(H))$ is not greater than $\operatorname{ext}^1_{\tilde X}(F_e,F_e)=-\chi(F_e,F_e)+2$, which is not greater than $-\chi(E,E)/d+2$ from equations (10) and (12) below. Thereby when

$-\frac{1}{d}\chi(E,E) \geq 3$ we can check that

$$\begin{aligned}\dim M(H) - \dim \operatorname{Sing}\bigl(M(H)\bigr) &\geq \operatorname{ext}^1(E,E) - \operatorname{ext}^2(E,E) - \operatorname{ext}^1(F_e,F_e)\\ &\geq -\chi(E,E) + 1 - 2 + \frac{1}{d}\chi(E,E)\\ &= \frac{1-d}{d}\chi(E,E) - 1 \geq 3(d-1) - 1 \geq 2,\end{aligned}$$

and this leads to (c). □

In such a way, we can describe $F_f^\vee$ at (2) as $F^\vee = t_1^2 + \cdots + t_N^2$ with $N \geq 3$ when $-\frac{1}{d}\chi(E,E) \geq 3$. Therefore Theorem 1.1 follows from Lemma 2.5, Lemma 2.13, and Proposition 2.11.

Proof of Proposition 2.11
From Lemma 2.10,

$$|G|\chi(E,E) = \chi(\pi^*E, \pi^*E) = \sum_{g\in G}\sum_{g'\in G}\chi(F_{g'}, F_g) = |G|\sum_{g\in G}\chi(F_e, F_g), \tag{10}$$

and so $\chi(E,E) = \sum_g \chi(F_e, F_g)$. From the Riemann–Roch theorem, we have

$$\begin{aligned}\chi(F_e,F_g) &= \bigl(r(F_g)c_1(F_e)^2 + r(F_e)c_1(F_g)^2\bigr)/2\\ &\quad - r(F_e)c_2(F_g) - r(F_g)c_2(F_e) - c_1(F_e)c_1(F_g) + r(F_e)r(F_g)\chi(\mathcal{O}_{\tilde{X}}).\end{aligned} \tag{11}$$

Since $F_g = g^*(F_e)$, it holds that $r(F_e) = r(F_g)$, $c_1(F_e)^2 = c_1(F_g)^2$, and $c_2(F_e) = c_2(F_g)$. By the Hodge index theorem $(c_1(F_e) - c_1(F_g))^2 = c_1(F_e)^2 + c_1(F_g)^2 - 2c_1(F_e)c_1(F_g) = 2(c_1(F_e)^2 - c_1(F_e)c_1(F_g)) \leq 0$, and hence (11) says that

$$\chi(F_e,F_e) - \chi(F_e,F_g) = -c_1(F_e)^2 + c_1(F_e)c_1(F_g) \geq 0. \tag{12}$$

Then some $h \neq e$ satisfies $\chi(F_e,F_h) \leq \chi(E,E)/d$; otherwise, all g satisfy $\chi(F_e,F_g) > \chi(E,E)/d$, which contradicts the fact that $\chi(E,E) = \sum_g \chi(F_e,F_g)$. For such $h \neq e$, $\operatorname{ext}^1(F_e,F_h) \geq -\frac{1}{d}\chi(E,E)$. As to the homomorphism (6), we remark that $Q_e = F_e(K_X)$ and that exact sequences (4) and (5) split since $\pi^*(K_X)$ is trivial. Therefore the following holds:

$$\begin{aligned}&\dim \operatorname{Im}\bigl(\operatorname{Ext}^1(Q_e(-K_X), F_h) \to \operatorname{Ext}^1(\pi^*E, \pi^*E)\bigr)\\ &\quad \geq \operatorname{ext}^1\bigl(Q_e(-K_X), F_h\bigr) = \operatorname{ext}^1(F_e,F_h) \geq -\frac{1}{d}\chi(E,E).\end{aligned}$$

□

Acknowledgments. The author would like to express sincere gratitude to Professors M. Inaba, D. Matsushita, S. Mukai, and Y. Namikawa for their invaluable suggestions and comments. Deep appreciation also goes to Professors O. Fujino and S. Mori for explaining the importance of singularities to the author.

References

[1] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. (2) **106** (1977), 45–60. MR 0466475. DOI 10.2307/1971157.

[2] W. Goldman and J. Millson, "Deformations of flat bundles over Kähler manifolds" in *Geometry and Topology (Athens, Ga., 1985)*, Lecture Notes in Pure and Appl. Math. **105**, Dekker, New York, 1987, 129–145. MR 0873290.

[3] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Aspects Math. **E31**, Vieweg, Braunschweig, 1997. MR 1450870.

[4] S. Ishii, *The canonical modifications by weighted blow-ups*, J. Algebraic Geom. **5** (1996), 783–799. MR 1486988.

[5] ———, *Introduction to Singularities* (in Japanese), Springer, Tokyo, 1997.

[6] H. Kim, *Moduli spaces of stable vector bundles on Enriques surfaces*, Nagoya Math. J. **150** (1998), 85–94. MR 1633143.

[7] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, with the collaboration of C. H. Clemens and A. Corti, translated from the 1998 Japanese original, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, Cambridge, 1998. MR 1658959. DOI 10.1017/CBO9780511662560.

[8] A. Laudal, "Matric Massey products and formal moduli, I" in *Algebra, Algebraic Topology and Their Interactions (Stockholm, 1983)*, Lecture Notes in Math. **1183**, Springer, Berlin, 1986, 218–240. MR 0846451. DOI 10.1007/BFb0075462.

[9] M. Maruyama, *Moduli of stable sheaves, II*, J. Math. Kyoto Univ. **18** (1978), 557–614. MR 0509499.

[10] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. **77** (1984), 101–116. MR 0751133. DOI 10.1007/BF01389137.

[11] M. Reid, "Young person's guide to canonical singularities" in *Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math. **46**, Amer. Math. Soc., Providence, 1987, 345–414. MR 0927963.

[12] K. Watanabe and T. Higuchi, *On a certain class of purely elliptic singularities in dimensions >2*, Sci. Rep. Yokohama Nat. Univ. Sect. I **30** (1983), 31–35. MR 0737860.

[13] K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), 817–884. MR 1872531. DOI 10.1007/s002080100255.

Department of Applied Mathematics, Faculty of Science, Okayama University of Science, Japan; yamada@xmath.ous.ac.jp