# A remark on Kovács's vanishing theorem

Osamu Fujino

**Abstract** We give an alternative proof of Kovács's vanishing theorem. Our proof is based on the standard arguments of the minimal model theory. We do not need the notion of Du Bois pairs. We reduce Kovács's vanishing theorem to the well-known relative Kawamata–Viehweg–Nadel vanishing theorem.

The following theorem is the main theorem of this paper, which we call Kovács's vanishing theorem.

THEOREM 1 (CF. [Kv, THEOREM 1.2])

*Let $(X, \Delta)$ be a log-canonical pair, and let $f : Y \to X$ be a proper birational morphism from a smooth variety $Y$ such that $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}\Delta$ is a simple normal crossing divisor on $Y$. In this situation, we can write*

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

*We put $E = \sum_{a_i=-1} E_i$. Then we have*

$$R^i f_* \mathcal{O}_Y(-E) = 0$$

*for every $i > 0$.*

In this short paper, we reduce Kovács's vanishing theorem to the well-known relative Kawamata–Viehweg–Nadel vanishing theorem by taking a dlt blowup. Our proof makes Kovács's vanishing theorem more accessible. From our viewpoint, Theorem 1 is a variant of the relative Kawamata–Viehweg–Nadel vanishing theorem.

Throughout this paper, we will work over an algebraically closed field $k$ of characteristic zero and freely use the standard notation of the minimal model theory.

REMARK 2

In [Kv], Kovács proved a rather general vanishing theorem for Du Bois pairs (see [Kv, Theorem 6.1]), and he used it to derive Theorem 1. For the details, see [Kv].

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Before we give a proof of Theorem 1, we make a brief remark.

REMARK 3

In [Kv, Theorem 1.2], $X$ is assumed to be $\mathbb{Q}$-factorial. Therefore, the statement of Theorem 1 is slightly better than the original one (cf. [Kv, Theorem 1.2]). However, we can check that Theorem 1 follows from [Kv, Theorem 1.2].

The following remark is important and seems to be well known to the experts (see, e.g., [Kv, Lemma 6.5.1]).

REMARK 4

The sheaf $R^i f_* \mathcal{O}_Y(-E)$ is independent of the choice of $f : Y \to X$ for every $i$. It can be checked easily by the standard arguments based on the weak factorization theorem (see [Kv, Lemma 6.5.1]). For related topics, see [F2, Lemma 4.2].

Let us start the proof of Theorem 1. It is essentially the same as the proof of [F3, Theorem 4.14] (see also [F2, Proposition 2.4]).

*Proof of Theorem 1*

By shrinking $X$, we may assume that $X$ is quasi-projective. We take a dlt blowup $g : Z \to X$ (see, e.g., [F1, Section 4]). This means that $g$ is a projective birational morphism, $K_Z + \Delta_Z = g^*(K_X + \Delta)$, and $(Z, \Delta_Z)$ is a $\mathbb{Q}$-factorial dlt pair. By using Szabó's resolution lemma, we take a resolution of singularities $h : Y \to Z$ with the following properties.

(1) $\mathrm{Exc}(h) \cup \mathrm{Supp}\, h_*^{-1}\Delta_Z$ is a simple normal crossing divisor on $Y$.
(2) $h$ is an isomorphism over the generic point of any lc center of $(Z, \Delta_Z)$.

We can write

$$K_Y + h_*^{-1}\Delta_Z = h^*(K_Z + \Delta_Z) + F.$$

We put $f = g \circ h : Y \to X$. In this situation, $E = \llcorner h_*^{-1}\Delta_Z \lrcorner$. Note that $\ulcorner F \urcorner$ is effective and $h$-exceptional by the construction. We also note that $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ is not necessarily a simple normal crossing divisor on $Y$ in the above construction. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(-E + \ulcorner F \urcorner) \to \mathcal{O}_Y(\ulcorner F \urcorner) \to \mathcal{O}_E(\ulcorner F|_E \urcorner) \to 0.$$

Since $-E + F \sim_{\mathbb{R},h} K_Y + \{h_*^{-1}\Delta_Z\}$ and $F \sim_{\mathbb{R},h} K_Y + h_*^{-1}\Delta_Z$, we have

$$-E + \ulcorner F \urcorner \sim_{\mathbb{R},h} K_Y + \{h_*^{-1}\Delta_Z\} + \{-F\}$$

and

$$\ulcorner F \urcorner \sim_{\mathbb{R},h} K_Y + h_*^{-1}\Delta_Z + \{-F\}.$$

By the relative Kawamata–Viehweg vanishing theorem and the vanishing theorem of Reid–Fukuda type (see, e.g., [F3, Lemma 4.10]), we have

$$R^i h_* \mathcal{O}_Y(-E + \ulcorner F \urcorner) = R^i h_* \mathcal{O}_Y(\ulcorner F \urcorner) = 0$$

for every $i>0$. Therefore, we have a short exact sequence

$$0\to h_*\mathcal{O}_Y(-E+\ulcorner F\urcorner)\to\mathcal{O}_Z\to h_*\mathcal{O}_E(\ulcorner F|_E\urcorner)\to 0$$

and $R^ih_*\mathcal{O}_E(\ulcorner F|_E\urcorner)=0$ for every $i>0$. Note that $\ulcorner F\urcorner$ is effective and $h$-exceptional. Thus we obtain

$$\mathcal{O}_{\llcorner\Delta_Z\lrcorner}\simeq h_*\mathcal{O}_E\simeq h_*\mathcal{O}_E(\ulcorner F|_E\urcorner).$$

By the above vanishing result, we obtain $Rh_*\mathcal{O}_E(\ulcorner F|_E\urcorner)\simeq\mathcal{O}_{\llcorner\Delta_Z\lrcorner}$ in the derived category of coherent sheaves on $\llcorner\Delta_Z\lrcorner$. Therefore, the composition

$$\mathcal{O}_{\llcorner\Delta_Z\lrcorner}\xrightarrow{\alpha}Rh_*\mathcal{O}_E\xrightarrow{\beta}Rh_*\mathcal{O}_E(\ulcorner F|_E\urcorner)\simeq\mathcal{O}_{\llcorner\Delta_Z\lrcorner}$$

is a quasi-isomorphism. Apply $R\mathcal{H}om_{\llcorner\Delta_Z\lrcorner}(\_\_,\omega^\bullet_{\llcorner\Delta_Z\lrcorner})$ to

$$\mathcal{O}_{\llcorner\Delta_Z\lrcorner}\xrightarrow{\alpha}Rh_*\mathcal{O}_E\xrightarrow{\beta}\mathcal{O}_{\llcorner\Delta_Z\lrcorner},$$

where $\omega^\bullet_{\llcorner\Delta_Z\lrcorner}$ is the dualizing complex of $\llcorner\Delta_Z\lrcorner$. Then we obtain that

$$\omega^\bullet_{\llcorner\Delta_Z\lrcorner}\xrightarrow{a}Rh_*\omega^\bullet_E\xrightarrow{b}\omega^\bullet_{\llcorner\Delta_Z\lrcorner}$$

and that $b\circ a$ is a quasi-isomorphism by the Grothendieck duality, where $\omega^\bullet_E\simeq\omega_E[\dim E]$ is the dualizing complex of $E$. Hence, we have

$$h^i(\omega^\bullet_{\llcorner\Delta_Z\lrcorner})\subseteq R^ih_*\omega^\bullet_E\simeq R^{i+d}h_*\omega_E,$$

where $d=\dim E=\dim\llcorner\Delta_Z\lrcorner=\dim X-1$. By the vanishing theorem (see, e.g., [F3, Lemma 2.33] and [F4, Lemma 3.2]), $R^ih_*\omega_E=0$ for every $i>0$. Therefore, $h^i(\omega^\bullet_{\llcorner\Delta_Z\lrcorner})=0$ for every $i>-d$. Thus, $\llcorner\Delta_Z\lrcorner$ is Cohen–Macaulay. This implies $\omega^\bullet_{\llcorner\Delta_Z\lrcorner}\simeq\omega_{\llcorner\Delta_Z\lrcorner}[d]$. Since $E$ is a simple normal crossing divisor on $Y$ and $\omega_E$ is an invertible sheaf on $E$, every associated prime of $\omega_E$ is the generic point of some irreducible component of $E$. By $h$, every irreducible component of $E$ is mapped birationally onto an irreducible component of $\llcorner\Delta_Z\lrcorner$. Therefore, $h_*\omega_E$ is a pure sheaf on $\llcorner\Delta_Z\lrcorner$. Since the composition

$$\omega_{\llcorner\Delta_Z\lrcorner}\to h_*\omega_E\to\omega_{\llcorner\Delta_Z\lrcorner}$$

is an isomorphism, which is induced by $a$ and $b$ above, we obtain $h_*\omega_E\simeq\omega_{\llcorner\Delta_Z\lrcorner}$. It is because $h_*\omega_E$ is generically isomorphic to $\omega_{\llcorner\Delta_Z\lrcorner}$. By the Grothendieck duality,

$$\begin{aligned}Rh_*\mathcal{O}_E&\simeq R\mathcal{H}om_{\llcorner\Delta_Z\lrcorner}(Rh_*\omega^\bullet_E,\omega^\bullet_{\llcorner\Delta_Z\lrcorner})\\&\simeq R\mathcal{H}om_{\llcorner\Delta_Z\lrcorner}(\omega^\bullet_{\llcorner\Delta_Z\lrcorner},\omega^\bullet_{\llcorner\Delta_Z\lrcorner})\simeq\mathcal{O}_{\llcorner\Delta_Z\lrcorner}\end{aligned}$$

in the derived category of coherent sheaves on $\llcorner\Delta_Z\lrcorner$. In particular, $R^ih_*\mathcal{O}_E=0$ for every $i>0$. Since $Z$ has only rational singularities, we have $R^ih_*\mathcal{O}_Y=0$ for every $i>0$ and $h_*\mathcal{O}_Y\simeq\mathcal{O}_Z$. Thus, we can easily check that $R^ih_*\mathcal{O}_Y(-E)=0$ for every $i>0$ by using the exact sequence

$$0\to\mathcal{O}_Y(-E)\to\mathcal{O}_Y\to\mathcal{O}_E\to 0.$$

Note that $h_*\mathcal{O}_E\simeq\mathcal{O}_{\llcorner\Delta_Z\lrcorner}$. We can also check that $h_*\mathcal{O}_Y(-E)=\mathcal{J}(Z,\Delta_Z)$, where $\mathcal{J}(Z,\Delta_Z)$ is the multiplier ideal sheaf associated to the pair $(Z,\Delta_Z)$. Note that

$\mathcal{J}(Z, \Delta_Z) = \mathcal{O}_Z(-\llcorner \Delta_Z \lrcorner)$ in our situation. Therefore,

$$R^i f_* \mathcal{O}_Y(-E) \simeq R^i g_* \mathcal{J}(Z, \Delta_Z)$$

for every $i$ by Leray's spectral sequence. By the relative Kawamata–Viehweg–Nadel vanishing theorem, $R^i g_* \mathcal{J}(Z, \Delta_Z) = 0$ for every $i > 0$. Thus we obtain $R^i f_* \mathcal{O}_Y(-E) = 0$ for every $i > 0$. Note that $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}\Delta$ is not necessarily a simple normal crossing divisor on $Y$ in the above construction. Let $\mathcal{I}_{\operatorname{Exc}(f)}$ be the defining ideal sheaf of $\operatorname{Exc}(f)$ on $Y$. Apply the principalization of $\mathcal{I}_{\operatorname{Exc}(f)}$. Then we obtain a sequence of blowups whose centers have simple normal crossings with $\operatorname{Exc}(h) \cup \operatorname{Supp} h_*^{-1}\Delta_Z$ (see, e.g., [Ko, Theorem 3.35]). In this process, $R^i f_* \mathcal{O}_Y(-E)$ does not change for every $i$ as in Remark 4 (see also [F2, Discussion 4.6]). Therefore, we may assume that $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}\Delta$ is a simple normal crossing divisor on $Y$. Remark 4 completes the proof of Theorem 1. □

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan; fujino@math.kyoto-u.ac.jp