A refinement of Foreman's four-vertex theorem and its dual version

Gudlaugur Thorbergsson and Masaaki Umehara

Abstract A strictly convex curve is a C^∞-regular simple closed curve whose Euclidean curvature function is positive. Fix a strictly convex curve Γ, and take two distinct tangent lines l_1 and l_2 of Γ. A few years ago, Brendan Foreman proved an interesting four-vertex theorem on semiosculating conics of Γ, which are tangent to l_1 and l_2, as a corollary of Ghys's theorem on diffeomorphisms of S^1. In this paper, we prove a refinement of Foreman's result. We then prove a projectively dual version of our refinement, which is a claim about semiosculating conics passing through two fixed points on Γ. We also show that the dual version of Foreman's four-vertex theorem is almost equivalent to the Ghys's theorem.

1. Introduction

The well-known four-vertex theorem asserts that *for a given convex curve Γ in the Euclidean plane $\mathbf{R}^2$, there exist at least four distinct points p_1, p_2, p_3, p_4 on Γ such that the osculating circles C_1, C_2, C_3, C_4 at these four points meet the curve Γ with multiplicity greater than or equal to* 4 (cf. [4]). The definition of *multiplicity* of intersection points (resp., order of contact) of two regular curves is given in [8]. The *order of contact* is by definition one less than the multiplicity of intersection points.

Moreover, Kneser [3] showed that one can take C_1, C_2 to be inscribed and C_3, C_4 to be circumscribed. (In [3], the assertion is proved for any simple closed regular curve.) Bose [1] improved this by showing that the number of the inscribed osculating circles s^+ (resp., the number of the circumscribed osculating circles s^-) satisfies the following so-called *Bose formula* for a given generic convex curve

$$s^+ - t^+ = 2 \quad (\text{resp., } s^- - t^- = 2), \tag{1.1}$$

where t^+ (resp., t^-) is the number of inscribed triple tangent circles (resp., the number of circumscribed triple tangent circles). (See Fact 2.1 and [10] for details and its history.) Moreover, the authors proved in [7] that the four points

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p_1, p_2, p_3, p_4 in Kneser's theorem can be chosen so that

$$p_1 \prec p_3 \prec p_2 \prec p_4 (\prec p_1), \tag{1.2}$$

where $\prec$ means the cyclic order on Γ. The result in (1.2) also holds for simple closed curves as in the case of (1.1).

A variant of the four-vertex theorem is known for diffeomorphisms of $S^1 (= \mathbf{P}^1)$: let f be a diffeomorphism of the real projective line $\mathbf{P}^1$. Then for each point p there exists a unique projective transformation $T_p \in \mathrm{PSL}(2, \mathbf{R})$ whose 2-jet at p coincides with that of f. Let us call T_p the *osculating map* of f at p. Then the following assertion holds.

FACT (GHYS'S THEOREM)
For each diffeomorphism f of $\mathbf{P}^1$, there exist four distinct points p_1, p_2, p_3, p_4 on $\mathbf{P}^1$, where T_{p_j} has the same 3-jet as f at p_j for each $j = 1, \ldots, 4$. In particular, the Schwarzian derivative of f vanishes at p_j.

Such a point p_j is called a *projective point* of f. In [7], the authors showed that one can take these four points p_j $(j = 1, \ldots, 4)$ on $\mathbf{P}^1$ so that each $f \circ T_{p_j}^{-1}$ has a connected fixed-point set, and they gave refinements of Ghys's theorem along the lines of (1.1) and (1.2). We call such a point p_j a *clean projective point.* The original four-vertex theorem for convex curves can be proved via Ghys's theorem (see [6]).

Since convexity is invariant under projective transformations, we may assume that a convex curve Γ lies in a certain affine plane $\mathbf{R}^2 (\subset \mathbf{P}^2)$. In the projective plane $\mathbf{P}^2$, the curve Γ bounds a closed contractible domain $\bar{D}_\Gamma$, which coincides with the interior domain bounded by Γ in the affine plane $\mathbf{R}^2$. Instead of osculating circles, one can consider osculating conics and "sextactic points." For any five distinct points on a strictly convex curve Γ, there exists a unique regular conic ω passing through the five points. Letting the five points all converge to p, the conic converges to a uniquely defined regular conic that is called the *osculating conic of* Γ *at* p. The osculating conic meets Γ with multiplicity at least five in p. If it meets with multiplicity at least six at p, then p is called a *sextactic point.* A strictly convex curve has at least six sextactic points (see [4]). This assertion was improved in [5], where it was shown that three of these sextactic points can be chosen so that the corresponding osculating conics are inscribed and the other three such that the corresponding osculating conics are circumscribed. A modern proof of this formula can be found in [8] and a refinement is given in [9, Theorem 1.2].

We now fix a strictly convex curve Γ and two distinct tangent lines l_1 and l_2 of Γ. Let $\Omega(l_1, l_2)$ be the set of regular conics which are tangent to both l_1 and l_2. For each point p on the curve Γ, there is a unique conic $\omega_p \in \Omega(l_1, l_2)$ which meets Γ at p with multiplicity three. (If p lies on l_1 or l_2, then the conic ω_p should meet Γ at p with multiplicity at least four.) The conic ω_p is called the

$\Omega(l_1, l_2)$-*osculating conic* of Γ at p. A point p on the curve Γ is called an $\Omega(l_1, l_2)$-*vertex* if ω_p hyperosculates at p; that is, ω_p meets Γ at p with multiplicity at least four (resp., at least five) if p does not lie on l_1 or l_2 (resp., if p lies on l_1 or l_2).

A few years ago, Foreman [2] gave an elegant application of Ghys's theorem which we call in this paper *Foreman's theorem.* It says that *there exist four distinct* $\Omega(l_1, l_2)$-*vertices on* Γ. It is interesting because of the following reasons.

(i) The conics ω_{p_j} can neither be inscribed nor circumscribed.

(ii) Foreman showed that the convex curve Γ induces a diffeomorphism f_Γ on $\mathbf{P}^1$, and the four distinct projective points in Ghys's theorem correspond to the desired four points p_1, p_2, p_3, p_4 on Γ. However, even if p_j is a clean projective point, it is not clear whether or not the conic ω_{p_j} has such a global separation property.

We will consider the question of whether Foreman's theorem allows a refinement analogous to the two formulas (1.1) and (1.2). In fact, one can accomplish this as follows. We fix a base point b on Γ that neither lies on l_1 nor on l_2. The convex curve Γ is divided into two closed arcs by the two lines l_1 and l_2. We denote by Γ^+ (resp., Γ^-) the one of these two arcs that passes through b (resp., does not pass through b). We call Γ^+ the *future part* and Γ^- the *past part.* We then define two sets $\Omega^+(l_1, l_2)$ and $\Omega^-(l_1, l_2)$ as follows: a conic $\omega \in \Omega(l_1, l_2)$ belongs to $\Omega^+(l_1, l_2)$ (resp., $\Omega^-(l_1, l_2)$) if one can divide $\omega \in \Omega(l_1, l_2)$ into two closed arcs ω^+ and ω^- bounded by l_1 and l_2 such that

$$\Gamma^+ \subset \mathbf{P}^2 \setminus D_\omega, \quad \omega^- \subset \mathbf{P}^2 \setminus D_\Gamma, \qquad (\text{resp., } \Gamma^- \subset \mathbf{P}^2 \setminus D_\omega, \quad \omega^+ \subset \mathbf{P}^2 \setminus D_\Gamma).$$

The sets $\Omega^+(l_1, l_2)$ and $\Omega^-(l_1, l_2)$ do not depend on the order of two lines l_1, l_2, but depend on the base point b. If we put b on Γ^-, then $\Omega^-(l_1, l_2)$ changes into $\Omega^+(l_1, l_2)$. In Figure 1, the curve Γ and the conic ω are indicated by a simple closed curve and a circle, respectively. Namely, they are not indicated as the real conic and the convex curve, but instead they are "cartoons" which show more clearly how ω meets Γ. We frequently use this kind of figure for the sake of

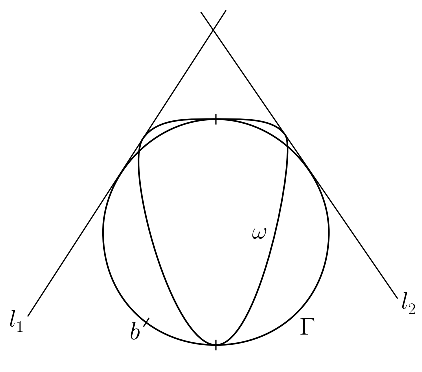

Figure 1. A conic in $\Omega^+(l_1, l_2)$

simplicity. Since the conics belonging to $\Omega^+(l_1,l_2)$ or $\Omega^-(l_1,l_2)$ are special, the union of these two subsets is only a proper subset of $\Omega(l_1,l_2)$. An $\Omega(l_1,l_2)$-vertex p is called a *clean* $\Omega^+(l_1,l_2)$-*vertex* (resp., a *clean* $\Omega^-(l_1,l_2)$-*vertex*) if ω_p belongs to $\Omega^+(l_1,l_2)$ (resp., $\Omega^-(l_1,l_2)$) and the intersection $\omega_p \cap \Gamma$ is a connected closed subset of Γ. The following result is our refinement of Foreman's theorem.

THEOREM 1.1

Let l_1 and l_2 be two distinct tangent lines of a strictly convex curve Γ. There exist four distinct points p_1,p_2,p_3,p_4 on Γ satisfying $p_1 \prec p_2 \prec p_3 \prec p_4$ and the following properties:

(1) *p_i is a clean $\Omega^+(l_1,l_2)$-vertex for $i=1,3$;*
(2) *p_j is a clean $\Omega^-(l_1,l_2)$-vertex for $j=2,4$.*

Moreover, an analogue of the formula (1.1) holds.

This theorem is proved in Section 1 by using the intrinsic circle systems introduced in [10] and [7].

The real projective plane $\mathbf{P}^2$ has its dual projective plane $\mathbf{P}^2_*$. Under this duality, lines in $\mathbf{P}^2$ correspond to points in $\mathbf{P}^2_*$. A strictly convex curve Γ in $\mathbf{P}^2$ has a (unique) dual curve Γ^* in $\mathbf{P}^2_*$ which is also strictly convex. Then two distinct tangent lines l_1 and l_2 to Γ correspond to two distinct points o_1 and o_2 on Γ^*. The dual version of Foreman's four-vertex theorem is then as follows. Take two distinct points o_1 and o_2 on Γ. Let $\Omega(o_1,o_2)$ be the set of regular conics passing through both o_1 and o_2. For each point p on Γ, we define the $\Omega(o_1,o_2)$-*osculating conic* at p as in the case of $\Omega(\ell_1,\ell_2)$. (If p coincides with o_1 or o_2, then the conic ω_p should meet Γ at p with multiplicity at least four.) A point p on the curve Γ is called an $\Omega(o_1,o_2)$-*vertex* if the $\Omega(o_1,o_2)$-osculating conic hyperosculates at p.

We now fix a base point b such that $b \neq o_i$ $(i=1,2)$. The convex curve Γ is divided into two closed arcs by o_1,o_2. We denote by Γ^+ (resp., Γ^-) the one of these two arcs that passes through b (resp., does not pass through b). We call Γ^+ the *future part* and Γ^- the *past part*. We define two subsets $\Omega^+(o_1,o_2)$ and $\Omega^-(o_1,o_2)$ as follows. A conic $\omega \in \Omega(o_1,o_2)$ belongs to $\Omega^+(o_1,o_2)$ (resp., $\Omega^-(o_1,o_2)$) if one can divide $\omega \in \Omega(o_1,o_2)$ into two closed arcs ω^+ and ω^- bounded by o_1 and o_2 such that (see Figure 2)

$$\Gamma^+ \subset \mathbf{P}^2 \setminus D_\omega, \quad \omega^- \subset \mathbf{P}^2 \setminus D_\Gamma, \qquad (\text{resp., } \Gamma^- \subset \mathbf{P}^2 \setminus D_\omega, \quad \omega^+ \subset \mathbf{P}^2 \setminus D_\Gamma).$$

By definition, $\Omega^\pm(o_1,o_2)$ does not depend on the order of o_1,o_2. However, if we put the base point b on Γ^-, then $\Omega^+(o_1,o_2)$ changes into $\Omega^-(o_1,o_2)$.

An $\Omega(o_1,o_2)$-vertex p is called a *clean* $\Omega^+(o_1,o_2)$-*vertex* (resp., a *clean* $\Omega^-(o_1,o_2)$-*vertex*) if the $\Omega(o_1,o_2)$-osculating conic ω_p belongs to $\Omega^+(o_1,o_2)$ (resp., $\Omega^-(o_1,o_2)$) and the intersection $\omega_p \cap \Gamma$ is a connected closed subset of Γ. The following is the dual version of Theorem 1.1.

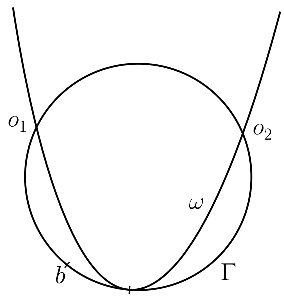

Figure 2. A conic in $\Omega^+(o_1, o_2)$

THEOREM 1.2

There exist four distinct points p_1, p_2, p_3, p_4 *on* Γ *satisfying* $p_1 \prec p_2 \prec p_3 \prec p_4$ *and the following properties:*

(1) p_i *is a clean* $\Omega^+(o_1, o_2)$*-vertex for* $i = 1, 3$;
(2) p_j *is a clean* $\Omega^-(o_1, o_2)$*-vertex for* $j = 2, 4$.

Moreover, an analogue of (1.1) holds. In particular, there exist four distinct $\Omega(o_1, o_2)$*-vertices on* Γ.

In Section 3, we show that the set of $\Omega(o_1, o_2)$-vertices on Γ^* corresponds under the duality exactly to the set of $\Omega(l_1, l_2)$-vertices on Γ, which proves the last assertion of the theorem. However, to prove all assertions of Theorem 1.2, we cannot use duality, since clean $\Omega(l_1, l_2)$-vertices on Γ might not correspond to clean $\Omega(o_1, o_2)$-vertices on Γ^* in general. We give two distinct proofs: one uses a method similar to the one in the proof of Theorem 1.1, and the other is an application of the refinement of Ghys’s theorem given in [7], whereas Theorem 1.1 does not follow directly from it.

In Section 4, we also consider the case that l is a tangent line and o is a point on a strictly convex curve Γ. We get similar results on $\Omega(l, o)$-vertices as in the case of $\Omega(l_1, l_2)$ and $\Omega(o_1, o_2)$.

2. Proof of Theorem 1.1

Before giving a proof of Theorem 1.1, we recall fundamental properties of intrinsic circle systems: we denote by $\prec$ the cyclic order of S^1. A family of nonempty closed subsets $F = \{F_p\}_{p \in S^1}$ is called an *intrinsic circle system* on S^1 if it satisfies the following three conditions:

(I1) $p \in F_p$ for each $p \in S^1$. If $q \in F_p$, then $F_p = F_q$;
(I2) If $p' \in F_p$, $q' \in F_q$ and $p \prec q \prec p' \prec q'$, then $F_p = F_q$ holds;
(I3) Let $\{p_n\}$ and $\{q_n\}$ be two sequences in S^1 converging to p and q in S^1, respectively. Suppose that $q_n \in F_{p_n}$ for each $n = 1, 2, 3, \ldots$. Then $q \in F_p$ holds.

We call F_p the *intrinsic* F*-circle* at p. Fix an intrinsic circle system F. We indicate by $p \sim q$ that $F_p = F_q$ holds. It is clear that $\sim$ is an equivalence relation on S^1. We denote by $[p]$ the equivalence class of p, and we denote by $r([p])$ the number of connected components of F_p which does not depend on the choice of p. A point $p \in S^1$ is called an *F-vertex* if $r([p]) = 1$. Set

$$S := \{[p] \in F/\sim; r([p]) = 1\}, \qquad T := \{[p] \in F/\sim; r([p]) \geq 3\},$$

and

$$s := \sum_{[p]\in S} r(p), \qquad t := \sum_{[p]\in T} (r(p) - 2). \tag{2.1}$$

Then the following assertion is proved in [10, Theorem 2.7].

FACT 2.1 (THE ABSTRACT BOSE FORMULA)
If s is finite, then so is t. Moreover the identity $s - t = 2$ holds.

If we let C_p denote a maximal inscribed circle tangent to a given convex curve Γ at a point p, then the family of subsets of $S^1 := \Gamma$ defined by $F_p := C_p \cap \Gamma$ for $p \in S^1$ is a typical example of an intrinsic circle system. In this case, F-vertices are clean; that is, the osculating circles are inscribed at F-vertices.

A pair of intrinsic circle systems (F^+, F^-) on S^1 is called *compatible* if an F^+-vertex (resp., F^--vertex) is never an F^--vertex (resp., F^+-vertex), and F^+-vertices (resp., F^--vertices) cannot accumulate to an F^--vertex (resp., an F^+-vertex). The following assertion holds (see [7]).

FACT 2.2 (THE THEOREM OF FOUR SIGN CHANGES OF CLEAN VERTICES)
Let (F^+, F^-) be a compatible pair of intrinsic circle systems on S^1. Then there exist four points p_1, p_2, p_3, p_4 on S^1 satisfying (1.2) so that p_1, p_3 are F^+-vertices and p_2, p_4 are F^--vertices.

In [7], the refinements of the four-vertex theorem for plane curves and Ghys's theorem are both proved by constructing compatible pairs of intrinsic circle systems.

Let l_1 and l_2 be two distinct tangent lines on Γ. To prove Theorem 1.1, it is sufficient to construct a compatible pair of intrinsic circle systems (F^+, F^-) after identifying S^1 with Γ. For this purpose, we will prove the following.

PROPOSITION 2.3
For each p on Γ, there exists a unique conic ω_p^1 in $\Omega^+(l_1, l_2)$ (resp., ω_p^2 in $\Omega^-(l_1, l_2)$) such that

(1) *ω_p^1 (resp., ω_p^2) is the $\Omega(l_1, l_2)$-osculating conic at p, or*
(2) *$\{p\}$ is a proper subset of F_p^+ (resp., F_p^-), where*

$$F_p^+ := \big((\omega_p^1)^+ \cap \Gamma^+\big) \cup \big((\omega_p^1)^- \cap \Gamma^-\big),$$
$$F_p^- := \big((\omega_p^2)^+ \cap \Gamma^+\big) \cup \big((\omega_p^2)^- \cap \Gamma^-\big).$$

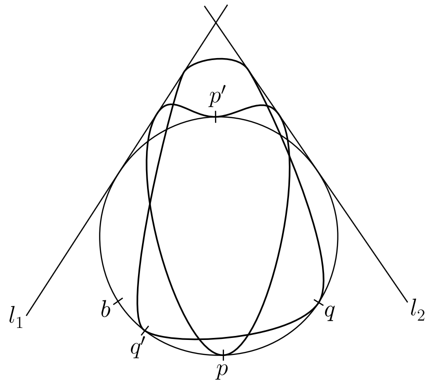

Figure 3. A typical arrangement

Once the proposition is proved, (F^+, F^-) gives a compatible pair of intrinsic circle systems on Γ. In fact, if F_p^+ is connected, then ω_p^1 coincides with the $\Omega(l_1,l_2)$-osculating conic at p. Hence, F^+-vertices (resp., F^--vertices) correspond to clean $\Omega^+(l_1,l_2)$-vertices (resp., clean $\Omega^-(l_1,l_2)$-vertices). By the uniqueness of ω_p^1, the three axioms (I1–I3) and the compatibility condition are all proved using the fact that two conics having more than four common points must coincide.

We have indicated a typical impossible case of two conics in $\Omega^+(l_1,l_2)$ in Figure 3, where we find six intersections between the two conics. Since the roles of Γ^+ and Γ^- interchange if we move the base point b between Γ^+ and Γ^-, we may assume that p lies on Γ^+. Since the uniqueness of ω_p^1 and ω_p^2 follows from the fact that there is a unique conic passing through five given points, it is sufficient to show the existence of ω_p^1 and ω_p^2 as in Proposition 2.3. (To prove the uniqueness when $p \in l_1$ or $p \in l_2$, we also need the fact that ω_p must meet Γ at p with multiplicity at least three.) Fix a point p on Γ arbitrarily. We first consider the case that p neither lies on l_1 nor on l_2. We denote by m the tangent line of Γ at p. Let A (resp., B) be the point where l_1 (resp., l_2) meets Γ. By a suitable projective transformation of $\mathbf{P}^2$, we may assume that l_1 is the line $\{y=1\}$, l_2 is the x-axis, and m is the y-axis in the affine plane $(\mathbf{R}^2; x, y)$ contained in $\mathbf{P}^2$. Then A, B can be placed on the right-hand side of the y-axis (see Figure 4). Take a conic $\omega_0 \in \Omega^+(l_1,l_2)$ so that ω_0 is tangent to the y-axis at p from the left. Let $c(t) = (x(t), y(t))$ be a parameterization of the conic ω_0 such that $c(0)=p$ and $c(t+1) = c(t)$ for $t \in \mathbf{R}$. Define a family of conics by $c_\lambda(t) := (\lambda x(t), y(t))$, where $\lambda \in [0,\infty)$. Denote by C_λ the image of the curve c_λ. By definition, C_1 coincides with ω_0.

LEMMA 2.4

The family $\{C_\lambda\}_{\lambda\in(0,\infty)}$ satisfies the following properties.

(a) *There exist positive constants ε and δ such that $c_\lambda([-\varepsilon,\varepsilon])$ lies in $\bar{D}_\Gamma$ if $\lambda > \delta$ and $c_\lambda([-\varepsilon,\varepsilon])$ lies in $\mathbf{P}^2 \setminus D_\Gamma$ if $\lambda < 1/\delta$.*

(b) *If λ is sufficiently large, C_λ belongs to $\Omega^+(l_1,l_2)$.*

(c) *If λ is sufficiently small, C_λ belongs to $\Omega^-(l_1,l_2)$.*

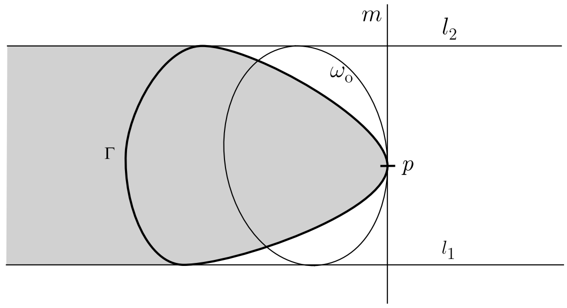

Figure 4. The figure after the projective transformation

Proof
In Figure 4, the convex domain $A_+(\Gamma)$ is marked in gray. The Euclidean curvature of C_λ at p is equal to $\lambda\kappa_0$, where κ_0 is the curvature of ω_0 at p. In particular, there exist positive constants ε and δ such that $c_\delta([-\varepsilon,\varepsilon])$ lies in $\bar{D}_\Gamma$ and $c_{1/\delta}([-\varepsilon,\varepsilon])$ lies in $\mathbf{P}^2 \setminus D_\Gamma$. If $\lambda > \delta$ (resp., $\lambda < 1/\delta$), then $c_\lambda([-\varepsilon,\varepsilon])$ lies on the left-hand side of c_δ (resp., right-hand side of $c_{1/\delta}$). Hence assertion (a) follows.

Next, we prove (b) (resp., (c)). It is obvious that C^-_λ (resp., $C^-_{1/\lambda}$) does not meet Γ^- for sufficiently large λ. So if (b) (resp., (c)) fails, then for each positive integer n, there exists a positive number λ_n and point $q_n(\neq p)$ on $C^+_{\lambda_n}$ (resp., C^-_{1/λ_n}) such that $q_n \in \Gamma^+$ and $\{\lambda_n\}$ diverges to ∞. Since Γ^+ is compact, we may assume that the sequence $\{q_n\}$ converges to a point $q_\infty \in \Gamma^+$. Since the x-component of $c_{\lambda_n}(t)$ $(t \notin \mathbf{Z})$ (resp., $c_{1/\lambda_n}(t)$) diverges to ∞ (resp., converges to 0) when $n \to \infty$, we get $q_\infty = p$. This is a contradiction, since q_n must lie on the left-hand side (resp., right-hand side) of Γ by (a). □

We now come to the proof of Proposition 2.3. By (b) and (c), we can set

$$\lambda_1 := \inf\bigl\{\lambda \in (0,\infty); C_\lambda \in \Omega^+(l_1,l_2)\bigr\},$$
$$\lambda_2 := \sup\bigl\{\lambda \in (0,\infty); C_\lambda \in \Omega^-(l_1,l_2)\bigr\}.$$

Then by definition, $C_{\lambda_1} \in \Omega^+(l_1,l_2)$ (resp., $C_{\lambda_2} \in \Omega^-(l_1,l_2)$). If the osculating $\Omega(l_1,l_2)$-conic ω_p of Γ at p coincides with C_{λ_1} (resp., C_{λ_2}), then we are in case (1) of Proposition 2.3. So we may assume $C_{\lambda_1} \neq \omega_p$ (resp., $C_{\lambda_2} \neq \omega_p$). If $\{p\}$ is a proper subset of

$$\Lambda_1 := (C^+_{\lambda_1} \cap \Gamma^+) \cup (C^-_{\lambda_1} \cap \Gamma^-)$$
$$\bigl(\text{resp., } \Lambda_2 := (C^+_{\lambda_2} \cap \Gamma^+) \cup (C^-_{\lambda_2} \cap \Gamma^-)\bigr),$$

then we are in case (2) after setting $\omega_p^1 = C_{\lambda_1}$ (resp., $\omega_p^2 = C_{\lambda_2}$). Thus we may assume that $\Lambda_1 = \{p\}$ (resp., $\Lambda_2 = \{p\}$). Since C_{λ_1} (resp., C_{λ_2}) is not an osculating conic, the curvature of C_{λ_1} (resp., C_{λ_2}) at p must be greater than (resp., less than) that of osculating $\Omega(l_1,l_2)$-conic at p and then C_{λ_1} (resp., C_{λ_2}) lies on the left-hand side (resp., right-hand side) of Γ around p. Then for a sufficiently small $\varepsilon > 0$, $C_{\lambda_1-\varepsilon}$ (resp., $C_{\lambda_2+\varepsilon}$) must also belong to $\Omega^+(l_1,l_2)$ (resp., $\Omega^-(l_1,l_2)$), which contradicts the definition of λ_1, and proves Proposition 2.3 whenever p is not on $l_1 \cup l_2$.

Next, we consider the case that p lies on l_1 or l_2. After a suitable replacement of b, we may assume without loss of generality that p lies on l_1. Take a sequence $\{p_n\}_{n=1}^{\infty}$ on $\Gamma \setminus \{p\}$ converging to p. Then there exists a conic $\omega^1_{p_n} \in \Omega^+(l_1,l_2)$ and $\omega^2_{p_n} \in \Omega^+(l_1,l_2)$ satisfying one of the two properties of Proposition 2.3. If there exists a subsequence $\{\omega^1_{p_{i_n}}\}_{n=1}^{\infty}$ such that $\omega^1_{p_{i_n}}$ is the $\Omega(l_1,l_2)$-osculating conic, then $\{\omega^1_{p_{i_n}}\}_{n=1}^{\infty}$ converges to the $\Omega(l_1,l_2)$-osculating conic at p, and the limit conic must satisfy (1). So we may assume that each $\omega^1_{p_n}$ is not equal to the $\Omega(l_1,l_2)$-osculating conic at p_n. Then there exists a point q_n on Γ so that $q_n \in F^+_{p_n}$. Since Γ is compact, we may assume that $\{q_n\}$ converges to a point $q \in \Gamma$. If $q \neq p$, then the limit of $\{\omega^1_{p_n}\}_{n=1}^{\infty}$ is a regular conic satisfying (2) since $q \in F^+_p$. If $q = p$, then the limit of $\{\omega^1_{p_n}\}_{n=1}^{\infty}$ must converge to the $\Omega(l_1,l_2)$-osculating conic at p and satisfy (1). Finally, we can apply the same argument for $\omega^2_{p_n} \in \Omega^-(l_1,l_2)$ and can prove the limiting conic satisfies case (1) and case (2) of Proposition 2.3 which completes the proof.

3. The dual version

Let $\gamma(t)$ be a closed strictly convex curve in $\mathbf{P}^2$. Then it can be lifted to a spherical curve $\tilde{\gamma} : S^1 \to S^2$ whose image is contained in an open hemisphere. If we denote by

$$\tilde{\pi} : S^2 \to \mathbf{P}^2$$

the canonical covering projection, then $\pi \circ \tilde{\gamma} = \gamma$ holds. We denote by $\tilde{n}(t) \in T_{\tilde{\gamma}(t)}S^2$ the unit normal vector of the curve $\tilde{\gamma}$ pointing into the interior domain. This provides us with a map

$$\tilde{n} : S^1 \to S^2 (\subset \mathbf{R}^3),$$

such that $\tilde{n}(t)$ is orthogonal to $\tilde{\gamma}(t)$ and $\dot{\tilde{\gamma}}(t) (:= d\tilde{\gamma}/dt)$ in $\mathbf{R}^3$. We set

$$\gamma_* = \tilde{\pi} \circ \tilde{n} : S^1 \to \mathbf{P}^2.$$

Then $\gamma_*(t)$ is also a closed strictly convex curve. Using the canonical inner product on $\mathbf{R}^3$, one can identify the dual vector space of $\mathbf{R}^3$ with $\mathbf{R}^3$. Then the correspondence $\gamma \mapsto n$ realizes the duality of convex curves between $\mathbf{P}^2$ and $\mathbf{P}^2_*$. In fact, the tangent lines of (resp., the points on) γ correspond to the points on (the tangent lines of) $\gamma_*(t)$. Without loss of generality, the convex curve γ lies in the affine plane $\mathbf{R}^2 = \{[x,y,1] \in \mathbf{P}^2 ; x,y \in \mathbf{R}\}$. Then we can write $\gamma(t) := (x(t), y(t))$. Without loss of generality, we may also assume that the origin is contained in the interior domain. Then we can choose a parameter t so that

$$x(t)\dot{y}(t) - y(t)\dot{x}(t) > 0$$

for each t. Then the dual of $\gamma(t)$ in $\mathbf{P}^2_*$ is given by

$$\gamma_* = \frac{1}{x\dot{y} - y\dot{x}}(-\dot{y}, \dot{x}).$$

In particular, if $l := \gamma$ is a line such that $\dot{\gamma}$ is a constant unit vector $e := (\cos\theta, \sin\theta)$, then

$$\frac{1}{d}(-\sin\theta, \cos\theta) \quad \text{where } d := \det(\gamma, e) \tag{3.1}$$

is called the *dual point* of the line l, where d is the signed distance of the line l from the origin.

PROPOSITION 3.1

Let γ_i $(i = 1, 2)$ be strictly convex curves in $\mathbf{R}^2$ that meet at a point p with multiplicity $m \geq 2$. Let p^* be the dual point corresponding to the common tangent line at p. Then their dual curves meet at p^* with the same multiplicity m.

Proof

Since the two curves meet at p with multiplicity m, the $(m-1)$-th jet of $\gamma_1(t)$ at $t = 0$ coincides with that of $\gamma_2(t)$. The crucial point is that the curvature function of the dual curve γ_i^* $(i = 1, 2)$ of γ_i is given by the formula

$$\kappa_i^* = -\frac{\det(\gamma_i, \dot{\gamma}_i)^3}{|\gamma_i|^3 \det(\dot{\gamma}_i, \ddot{\gamma}_i)},$$

which involves only the 2-jet of the curve γ_i. (The same phenomenon occurs in the proof of the Foreman theorem; see the formula for $\kappa(t)$ in [2].) Now the assertion follows from the lemma in the appendix. $\square$

Let (Γ^*, o_1, o_2) be the dual of (Γ, l_1, l_2). By Proposition 3.1, the $\Omega(l_1, l_2)$-vertices of Γ correspond one-to-one to the $\Omega(o_1, o_2)$-vertices of Γ^*. This is analogous to the correspondence between the $\Omega(l_1, l_2)$-vertices of Γ and the projective points of the associated diffeomorphism on S^1 found by Foreman. We get the following corollary.

COROLLARY 3.2

Let Γ be a strictly convex curve, and let o_1, o_2 be two distinct points on Γ. Then there exist at least four distinct $\Omega(o_1, o_2)$-vertices on Γ; namely, the last assertion of Theorem 1.2 holds.

We fix a strictly convex curve Γ and distinct points o_1, o_2 on Γ. To prove the whole statement of Theorem 1.2, we need to construct a compatible pair of intrinsic circle systems. For each $\omega \in \Omega^+(o_1, o_2)$ (resp., $\omega \in \Omega^-(o_1, o_2)$), we denote by $G^+(\omega)$ (resp., $G^-(\omega)$) the subset of Γ^+ (resp., Γ^-), where ω^+ (resp., ω^-) meets Γ^+ (resp., Γ^-) with multiplicity more than one.

PROPOSITION 3.3

For each p on Γ, there exists a unique conic ω_p^1 (resp., ω_p^2) in $\Omega^+(o_1, o_2)$ (resp., $\Omega^-(o_1, o_2)$) such that

(1) *ω_p^1 (resp., ω_p^2) is the $\Omega(o_1, o_2)$-osculating conic, or*

(2) $\{p\}$ *is a proper subset of* F_p^+ *(resp.,* F_p^-*), where*

$$F_p^+ := G^+(\omega_p^1) \cup G^-(\omega_p^1), \qquad F_p^- := G^+(\omega_p^2) \cup G^-(\omega_p^2).$$

Proof
The uniqueness of ω_p^1 and ω_p^2 is proved using the fact that two conics having more than four points in common must coincide. (To prove the uniqueness when $p = o_1$ or $p = o_2$, we also need the fact that ω_p must meet Γ at p with multiplicity at least three.) So it is sufficient to prove the existence of ω_p^1 and ω_p^2.

We first consider the case $p \neq o_1, o_2$. Let m be the tangent line of Γ at p. By a suitable projective transformation, we may assume that Γ lies in the affine plane $(\mathbf{R}^2; x, y)$ such that $p = (0,-1)$, $o_1 = (-1,0)$, $o_2 = (1,0)$, and m coincides with the line $y = -1$ such that Γ is tangent to $y = -1$ at $(0,-1)$. Then the family of conics passing through o_1, o_2 and tangent to the line $y = -1$ at $(0,-1)$ is given by

$$C_\lambda : x^2 + \lambda y^2 + (\lambda - 1)y - 1 = 0.$$

If $\lambda < -1$, then ω^+ of this conic does not pass through p, since C_λ is a hyperbola which is tangent to the line $y = -1$ from below. Hence C_λ for $\lambda < -1$ cannot be a candidate for ω_p^1 or ω_p^2. So we consider a family of conics $\{C_\lambda\}_{\lambda > -1}$ in $\Omega(o_1, o_2)$. If λ tends to -1, then C_λ collapses to the union of two lines $(x-y-1)(x+y+1) = 0$, and if λ goes to ∞, then C_λ collapses to a pair of parallel lines $y = 0, 1$. Using this property, one can easily show that there exists a constant δ such that $C_\lambda \in \Omega^+(o_1, o_2)$ if $(-1 <)\lambda < -1 + 1/\delta$ and $C_\lambda \in \Omega^-(o_1, o_2)$ if $\lambda > \delta$. So we can set

$$\lambda_1 := \sup\bigl\{\lambda \in (-1, \infty); C_\lambda \in \Omega^+(o_1, o_2)\bigr\},$$
$$\lambda_2 := \inf\bigl\{\lambda \in (-1, \infty); C_\lambda \in \Omega^-(l_1, l_2)\bigr\}.$$

Then it can be proved that

$$\omega_p^1 := C_{\lambda_1}, \qquad \omega_p^2 := C_{\lambda_2}$$

satisfy the properties in Proposition 3.3. Finally, the case $p = o_1$ or $p = o_2$ can be proved by taking a limit $p_n \to p$ as in the proof of Proposition 2.3. □

Proof of Theorem 1.2
We let F^+ and F^- be the families of closed subsets of Γ as in part (2) of Proposition 3.3. It can be directly checked that (F^+, F^-) gives a compatible pair of intrinsic circle systems on Γ by using the fact that two conics having more than four points in common must coincide. Now the assertion follows from Fact 2.1 and Fact 2.2. □

We now suggest a different proof of Theorem 1.2. We can identify the pencils of lines through o_1 and o_2 with $\mathbf{P}^1(o_i)$ $(i = 1, 2)$. For each point $p \in \Gamma$, there exists a unique line $\varphi_i(p) \in \mathbf{P}^1(o_i)$ passing through o_i and p (see Fig. 5). (If $p = o_i$, then $\varphi_i(p)$ should be the tangent line of Γ at o_i.) Then, we get a diffeomorphism f_Γ :

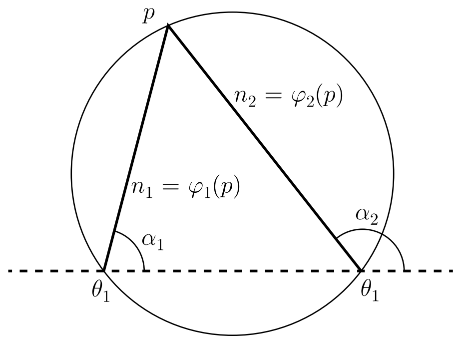

Figure 5. The correspondence f_Γ

$\mathbf{P}^1(o_1) \to \mathbf{P}^1(o_2)$ given by $f_\Gamma = \varphi_2 \circ (\varphi_1)^{-1}$. The following gives rise to another proof of Theorem 1.2.

THEOREM 3.4

The map f_Γ can be identified with the diffeomorphism given in [2] for the dual convex curve Γ^*. In particular, f_Γ is a projective transformation if and only if Γ is a conic. Moreover, a conic $\omega \in \Omega(o_1, o_2)$ meets Γ at p with multiplicity $m(\geq 1)$ if and only if the osculating map T_{f_Γ} has the same $(m-1)$-jet as T_ω at $\varphi_1(p)$. Furthermore, the intrinsic circle system associated to f_Γ given in [7] coincides exactly with that induced by Proposition 3.3.

Proof

Without loss of generality, we may assume that Γ lies in $\mathbf{R}^2$ with the origin inside of Γ, and $o_i = (x_i, -1)$ $(i = 1, 2)$. Let m be the line through o_1 and o_2. Under duality, o_1 and o_2 correspond to two tangent lines l_1 and l_2 of Γ^*, respectively. Then the dual point M on Γ^* of the line m is the intersection point of l_1 and l_2. Let n_1 (resp., n_2) be a line whose angle with m is α_i. Then n_i $(i = 1, 2)$ is an element of $P^1(o_i)$ which can be expressed by the homogeneous coordinates $[\cos\alpha_i, \sin\alpha_i]$. By (3.1) the dual point N_i of n_i lies on the line l_i whose signed Euclidean distance from M is proportional to $1/(x_i + \cot\alpha_i)$ which is just a projective action of the inhomogeneous coordinate $\cot\alpha_i$ of n_i in $P^1(o_i)$.

Suppose now that n_1 meets n_2 at a point p on Γ; namely, $n_i = \varphi_i(p)$ holds for $i = 1, 2$. Then the dual of p is just the tangent line of Γ^* passing through $N_1 \in l_1$ and $N_2 \in l_2$. Thus, f_Γ is just equal to the diffeomorphism given by Foreman [2] for the dual convex curve Γ^*.

Since Γ is a conic if and only if Γ^* is, the second assertion follows. By replacing Γ by $\omega \in \Omega(o_1, o_2)$, we get a diffeomorphism f_ω. If ω coincides with the $\Omega(o_1, o_2)$-osculating conic at p, f_ω is equal to the osculating map of f_Γ at $\varphi_1(p)$. The last assertion can easily be proved using the two facts that the intersections between ω and Γ correspond to the fixed points of $f_\Gamma \circ f_\omega^{-1}$ and that minimal (resp., maximal) projective points of f_Γ correspond to the points where $\Omega(o_1,o_2)$-osculating conic lies locally on the left-hand side (resp. right-hand side) of Γ. $\square$

4. The case of $\Omega(l,o)$

Let l and o be a tangent line and a point on a strictly convex curve Γ such that o does not lie in l. Let $\Omega(l,o)$ be the set of regular conics passing through o and having contact with l. For each point p on Γ, we can define the $\Omega(l,o)$-*osculating conic* at p as in the case of $\Omega(l_1,l_2)$ and $\Omega(o_1,o_2)$. A point p on the curve Γ is called an $\Omega(l,o)$-*vertex* if $\omega_p(l,o)$ hyperosculates at p. We now fix a base point b on Γ such that $b \neq o$ and $b \notin l$. The convex curve Γ is divided into two closed arcs by l and o. We denote by Γ^+ (resp., Γ^-) the one of these two arcs that passes through b (resp., does not pass through b). We call Γ^+ the *future part* and Γ^- the *past part*. By definition, Γ^+ and Γ^- both meet l (resp., o) at one of their boundary points. We then define two subsets $\Omega^+(l,o)$ and $\Omega^-(l,o)$ as follows. A conic $\omega \in \Omega(l,o)$ belongs to $\Omega^+(l,o)$ (resp., $\Omega^-(l,o)$) if one can divide $\omega \in \Omega(l,o)$ into two closed arcs ω^+ and ω^- bounded by l and o such that

$$\Gamma^+ \subset \mathbf{P}^2 \setminus D_\omega, \quad \omega^- \subset \mathbf{P}^2 \setminus D_\Gamma, \qquad (\text{resp., } \Gamma^- \subset \mathbf{P}^2 \setminus D_\omega, \quad \omega^+ \subset \mathbf{P}^2 \setminus D_\Gamma).$$

If we put b on Γ^-, then the roles of Γ^+ and Γ^- are interchanged. In this section, we prove the following theorem.

THEOREM 4.1

Let l and $o(\notin l)$ be a tangent line and a point on a strictly convex curve Γ. Then, there exist four distinct points p_1,p_2,p_3,p_4 on Γ satisfying $p_1 \prec p_2 \prec p_3 \prec p_4$ and the following properties:

(1) *p_i is a clean $\Omega^+(l,o)$-vertex for $i=1,3$,*
(2) *p_j is a clean $\Omega^-(l,o)$-vertex for $j=2,4$.*

Moreover, an analogue of formula (1.1) holds.

To prove the theorem, it is sufficient to show the existence of a compatible pair of intrinsic circle systems. For this purpose, it is sufficient to show that the following proposition holds. For each $\omega \in \Omega^+(l,o)$ (resp., $\omega \in \Omega^-(l,o)$), we denote by $G^+(\omega)$ (resp., $G^-(\omega)$) the subset of Γ^+ (resp., Γ^-) where ω^+ (resp., ω^-) meets Γ^+ (resp., Γ^-) with multiplicity more than one.

PROPOSITION 4.2

For each p on Γ, there exists a unique conic ω_p^1 (resp., ω_p^2) in $\Omega^+(l,o)$ (resp., $\Omega^-(l,o)$) such that

(1) *ω_p^1 (resp., ω_p^2) is the $\Omega(l,o)$-osculating conic, or*
(2) *$\{p\}$ is a proper subset of F_p^+ (resp., F_p^-), where*

$$F_p^+ := G^+(\omega_p^1) \cup G^-(\omega_p^1), \qquad F_p^- := G^+(\omega_p^2) \cup G^-(\omega_p^2).$$

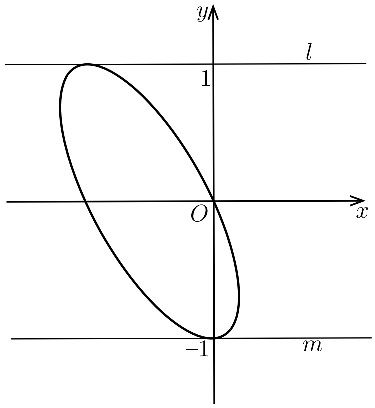

Figure 6. A conic belonging to $\Omega(l, o)$

The uniqueness of ω_p^1 and ω_p^2 is proved using the fact that two conics having more than four points in common must coincide. So it is sufficient to prove the existence of ω_p^1 and ω_p^2.

We consider the case $p \neq o$ and p does not lie in l. (The case $p = o$ or $p \in l$ can be proved by taking a limit $p_n \to p$ as in the proof of Proposition 2.3.) Let m be the tangent line of Γ at p. By a suitable projective transformation, we may assume that Γ lies in the affine plane $(\mathbf{R}^2; x, y)$ such that $o = (0, 0)$, l equal to $y = 1$, and m coincides with the line $y = -1$ such that Γ is tangent to the line m (i.e., $y = -1$) at $(0, -1)$. We fix a conic ω_0 passing through $(0, 0)$ which is tangent to $y = 1$ and is tangent to $y = -1$ at $(0, -1)$ (see Fig. 6). Let $c(t) = (x(t), y(t))$ be a parameterization of the conic ω_0 such that $c(0) = p$ and $c(t + 1) = c(t)$ for $t \in \mathbf{R}$. We set $c_\lambda = (\lambda x(t), y(t))$ $(\lambda \in \mathbf{R})$, and we denote its image by C_λ. Now, Proposition 4.2 can be proved by applying the following lemma as in the proof of Proposition 3.3.

LEMMA 4.3

The family $\{C_\lambda\}_{\lambda \in (0,\infty)}$ *satisfies the following properties.*

(a) *The velocity vector of* c_λ *at* $(0, 0)$ *tends to be horizontal if* $\lambda \to \infty$ *and to be vertical if* $\lambda \to 0$.

(b) *There exist positive constants* ε *and* δ *such that* $c_\lambda([-\varepsilon, \varepsilon])$ *lies in* $\bar{D}_\Gamma$ *if* $\lambda < 1/\delta$ *and* $c_\lambda([-\varepsilon, \varepsilon])$ *lies in* $\mathbf{P}^2 \setminus D_\Gamma$ *if* $\lambda > \delta$.

(c) *If* λ *is sufficiently large,* C_λ *belongs to* $\Omega^-(l, o)$.

(d) *If* λ *is sufficiently small,* C_λ *belongs to* $\Omega^+(l, o)$.

Proof

Assertion (a) follows from the fact that the tangent line of ω_0 at $(0, 0)$ is not vertical. Assertion (b) follows from the fact that the (Euclidean) curvature of C_λ is equal to $\lambda^{-2}\kappa_0$, where κ_0 is the curvature of ω_0 at $(0, -1)$, as in the proof of (a) of Lemma 2.4.

Next, we prove (c) (resp., (d)). It is obvious that C^+_λ and $C^+_{1/\lambda}$ do not meet Γ^- for sufficiently large λ. So if (c) (resp., (d)) fails, for each positive integer n, there exist a positive number λ_n and a point $q_n(\neq p)$ on C^+_λ (resp., $C^-_{1/\lambda}$) such that $q_n \in \Gamma^+$ and $\{\lambda_n\}$ diverges to ∞. Since Γ^+ is compact, we may assume that the sequence $\{q_n\}$ converges to a point $q_\infty \in \Gamma^+$. Since the x-component of $c_{\lambda_n}(t)$ ($t \notin \mathbf{Z}$) (resp., $c_{1/\lambda_n}(t)$) diverges to ∞ (resp., converges to 0) when $n \to \infty$, we can conclude that $q_\infty = p$ or $q_\infty = o$. However, this contradicts (a) or (b). □

Appendix: A criterion for multiplicity

The following lemma is needed to prove Proposition 1.2.

LEMMA A.4
Let $\gamma_i(t)$ $(i = 1, 2)$ be a pair of regular curves satisfying $\gamma_1(0) = \gamma_2(0)(=: p)$ and $\dot\gamma_1(0) = \dot\gamma_2(0)$. We denote by $\kappa_i(t)$ $(i = 1, 2)$ the Euclidean curvature function of $\gamma_i(t)$. Then γ_1 meets γ_2 at p with multiplicity at least $n + 2$ if

$$(*)_j \qquad \frac{d^j\kappa_1(0)}{dt^j} = \frac{d^j\kappa_2(0)}{dt^j}$$

holds for $j = 0, 1, \ldots, n-1$ and $n \geq 1$.

Proof
We introduce arc-length parameter s on γ_i $(i = 1, 2)$ so that $s = 0$ corresponds to p. We prove the assertion by induction. If $n = 0$, $(*)_0$ implies that the first two derivatives of $\gamma_1(s)$ and $\gamma_2(s)$ at $s = 0$ coincide. The chain rule implies that $d^k\kappa_i/ds^k$ $(i = 1, 2)$ can be expressed in terms of $d^j\kappa_i/dt^j$ $(0 \leq j \leq k)$ and the first k derivatives of $\gamma_i(t)$. We now suppose that $(*)_j$ $(j = 0, \ldots, k-1)$ implies the first $k+1$ derivatives of $\gamma_1(s)$ at $s = 0$ coincide with those of $\gamma_2(s)$. Then, as a consequence, $d^k\kappa_1(0)/ds^k$ is equal to $d^k\kappa_2(0)/ds^k$, which implies that the first $k + 2$ derivatives of $\gamma_1(s)$ and $\gamma_2(s)$ coincide for $s = 0$. This proves the assertion. □

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Thorbergsson: Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany; gthorbergsson@mi.uni-koeln.de

Umehara: Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1-W8-34, O-okayama Meguro-ku, Tokyo 152-8552, Japan; umehara@is.titech.ac.jp