Quantum toroidal $\mathfrak{gl}_1$-algebra: Plane partitions

B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin

Abstract   In this, the third paper of the series, we construct a large family of representations of the quantum toroidal $\mathfrak{gl}_1$-algebra whose bases are parameterized by plane partitions with various boundary conditions and restrictions. We study the corresponding formal characters. As an application we obtain a Gelfand–Zetlin-type basis for a class of irreducible lowest weight $\mathfrak{gl}_\infty$-modules.

Contents

1. Introduction

In this paper we continue our study of representations of an algebra $\mathcal{E} = \mathcal{E}_{q_1,q_2,q_3}$, depending on three complex parameters q_1, q_2, q_3 with $q_1 q_2 q_3 = 1$. It was originally introduced by [Mi] as a two-parameter analogue of the $W_{1+\infty}$-algebra. The basic structure theory of $\mathcal{E}$ and its representations has been established in [Mi]. Its connection with the Macdonald operator and the deformed Virasoro/W-algebras was also revealed there. Some of these results are recalled in Section 2. Essentially the same algebra was rediscovered later on and was given various other names: the Ding–Iohara algebra in [FT] and [FHHS] or the elliptic Hall algebra in [SV1] and [SV2]. Having been unaware of the work of Miki, we have called $\mathcal{E}$ "quantum continuous $\mathfrak{gl}_\infty$" in [FFJMM1] and [FFJMM2]. We are sorry about this oversight.

We have decided to call $\mathcal{E}$ the quantum toroidal $\mathfrak{gl}_1$-algebra for the following reason. Let $\mathfrak{d}_q$ be the algebra generated by the symbols $Z^{\pm 1}, D^{\pm 1}$ satisfying $DZ = qZD$, where $q \in \mathbb{C}^\times$, and regard it as a Lie algebra endowed with the Lie bracket $[a, b] = ab - ba$. The Lie algebra $\mathfrak{d}_q$ has a two-dimensional central extension $\mathfrak{d}_{q,c_1,c_2}$. As mentioned in [Mi], the algebra $\mathcal{E}_{q_1,q_2,q_3}$ is a quantization of the universal enveloping algebra $U(\mathfrak{d}_{q,c_1,c_2})$, where one of the parameters, say q_1, is the quantization parameter and $q_2 = q$. The situation is similar to that of the quantum toroidal algebra $U_q(\mathfrak{sl}_{N,\text{tor}})$, whose classical limit is a central extension of the Lie algebra of $(N \times N)$-matrices x with entries in $\mathfrak{d}_q$, such that $\text{restr}(x) = 0$. (Here $\text{res}(a) = a_{0,0}$ for $a = \sum a_{i,j} Z^i D^j \in \mathfrak{d}_q$.)

Throughout this paper we restrict our considerations to representations of a quotient of $\mathcal{E}$ by a one-dimensional center. The classical limit of the quotient algebra is $\mathfrak{d}_{q,\kappa,0}$ (see Section 2).

The algebra $\mathfrak{d}_q$ is isomorphic to the algebra of q-difference operators. Namely, $\mathfrak{d}_q$ has a faithful representation in the space $V = \mathbb{C}[Z, Z^{-1}]$ such that Z acts as the multiplication operator $f(Z) \mapsto Zf(Z)$, and D as the q-difference operator $f(Z) \mapsto f(qZ)$. This gives rise to a Lie algebra homomorphism $\mathfrak{d}_{q,\kappa,0} \to \mathfrak{gl}_{\infty,\kappa}$, where $\mathfrak{gl}_{\infty,\kappa}$ is the central extension of the Lie algebra of linear transformations $T : V \to V$, $T(Z^j) = \sum_j T_{ij} Z^i$, such that there exists $N \in \mathbb{Z}$ for which $T_{ij} = 0$ whenever $|i - j| > N$.

The Lie algebra $\mathfrak{gl}_{\infty,\kappa}$ has a rich representation theory. Let $\mathfrak{gl}_\infty \subset \mathfrak{gl}_{\infty,\kappa}$ be the Lie subalgebra of linear operators T with finitely many nonzero matrix elements T_{ij}. Let W_θ be the irreducible representation of $\mathfrak{gl}_\infty$ with the lowest weight

$$\theta = (\ldots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \ldots), \qquad E_{i,i} v_\theta = \theta_i v_\theta,$$

where v_θ is a lowest weight vector in W_θ:

$$E_{i,j} v_\theta = 0 \quad \text{if } i > j.$$

If the sequence $\{\theta_i\}$ stabilizes as $i \to \pm\infty$, then W_θ can be extended to the representation of $\mathfrak{gl}_{\infty,\kappa}$. Suppose that $\theta_i = \theta_-$ for $i \ll 0$ and $\theta_i = \theta_+$ for $i \gg 0$. Then the central element κ acts in W_θ by the scalar $\theta_- - \theta_+$.

Conjecturally all such W_θ can be deformed to the representations of $\mathcal{E}$. In this paper we confirm it in several special cases.

If $\theta = (\ldots, 0, 0, 0, 1, 1, 1, \ldots)$, then W_θ is the well-known Fock representation given by semiinfinite wedges. In [FFJMM1] the construction of the semiinfinite wedges was deformed, and as a result we get the Fock representation of $\mathcal{E}$. If the weight θ is antidominant, that is, $\theta_i \in \mathbb{Z}$ and $\theta_i - \theta_{i+1} \le 0$ for all $i \in \mathbb{Z}$, the $\mathcal{E}$-modules corresponding to W_θ were also constructed in [FFJMM1] (see also Section 5).

Note that all these representations of $\mathcal{E}$ are described explicitly. We have a natural basis and an explicit formula for the action on this basis.

In the present paper we continue with the case $\theta(r) = (\ldots, 0, 0, 0, r, r, r, \ldots)$, where $r \in \mathbb{C}$ is generic. The character of $W_{\theta(r)}$ in the principal grading is given by the infinite product $\prod_{i=1}^\infty (1 - q^i)^{-i}$. Incidentally, it coincides with the well-known

MacMahon formula for the generating series of the plane partitions. Recall that a plane partition is a collection of nonnegative integers $\{\mu_i^{(k)}\}_{i,k=1}^{\infty}$ satisfying $\mu_i^{(k)} \geq \mu_i^{(k+1)}$, $\mu_i^{(k)} \geq \mu_{i+1}^{(k)}$ for all i,k and $\mu_i^{(k)} = 0$ for $i+k$ large enough.

We construct a representation of $\mathcal{E}$ which is a deformation of $W_{(r)}$. It depends on a complex parameter $K \neq 0$, which is the value of a central element of $\mathcal{E}$ and is called the level of the representation. It has an additional complex parameter $u \neq 0$, which is related to an automorphism of $\mathcal{E}$. Most importantly, it has a distinguished basis labeled by the plane partitions, and the action of $\mathcal{E}$ is explicit in this basis. We call this $\mathcal{E}$-module the MacMahon representation, and denote it by $\mathcal{M}(u, K)$.

Next, we observe that our construction has the following natural generalization. Given three partitions α, β, γ, we call a collection of numbers $\{\mu_j^{(k)}\}_{i,k=1}^{\infty}$, $\mu_j^{(k)} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a plane partition with the boundary condition α, β, γ, if the following conditions are satisfied:

(i) $\mu_i^{(k)} \geq \mu_i^{(k+1)}$, $\mu_i^{(k)} \geq \mu_{i+1}^{(k)}$ for all i,k,
(ii) $\mu_i^{(k)} = \alpha_k$ for $i \gg 0$,
(iii) $\mu_i^{(k)} = \gamma_i$ for $k \gg 0$,
(iv) $\mu_i^{(k)} = \infty$ if and only if $i \leq \beta_k$.

Let $\mathcal{P}[\alpha, \beta, \gamma]$ be the set of plane partitions with the boundary condition α, β, γ.

The set $\mathcal{P}[\alpha, \beta, \gamma]$ appears in topological field theory as a fixed point set on Hilbert schemes on toric 3-dimensional Calabi–Yau manifolds (see [ORV]).

We show that for generic values of q_1, q_2, K the algebra $\mathcal{E}_{q_1,q_2,q_3}$ has an irreducible representation $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$ depending on an additional arbitrary complex parameter u with a basis labeled by the set $\mathcal{P}[\alpha, \beta, \gamma]$, and we give an explicit formula for the action.

Here the genericity assumption for q_1, q_2 means $q_1^{i_1} q_2^{i_2} q_3^{i_3} \neq 1$ unless $i_1 = i_2 = i_3$, and for K it means $K \neq q_1^{i_1} q_2^{i_2} q_3^{i_3}$ for all integers i_1, i_2, i_3. If $\alpha = \beta = \gamma = \emptyset$ we have $\mathcal{M}_{\alpha,\beta,\gamma}(u, K) = \mathcal{M}(u, K)$.

In the resonance case $K = q_1^{i_1} q_2^{i_2} q_3^{i_3}$ we show that the module $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$ is still well defined but becomes reducible. We describe singular vectors of $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$ and the irreducible quotient generated by the vector corresponding to the minimal partition in $\mathcal{P}[\alpha, \beta, \gamma]$. In the simplest case where $\alpha = \beta = \gamma = \emptyset$ and $K = q_1 q_2^{i_2} q_3^{i_3}$ $(i_2, i_3 \geq 1)$, the irreducible quotient has a basis labeled by plane partitions $\mu_i^{(k)} \in P(0,0,0)$ such that $\mu_{i_3}^{(i_2)} = 0$.

For the case of general α, β, γ the representation $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$ does not have the limit $q_1 \to 1$. But we show that if $\beta = \emptyset$, it does and therefore it is a deformation of a $\mathfrak{gl}_{\infty,\kappa}$-module. Suppose further that for $n, c \in \mathbb{Z}_{\geq 0}$,

$$\gamma = (\underbrace{c, c, \ldots, c}_{n}, 0, 0 \ldots), \qquad K = (q_2 q_3)^n;$$

then the irreducible quotient has the limit $q_1 \to 1$ and the limit is an irreducible $\mathfrak{gl}_{\infty,\kappa}$-module. The lowest weight θ of this module is given in (4.21).

The basis of this irreducible quotient is labeled by the set $P^n(\alpha, c)$ consisting of all $\{\mu_i^{(k)}\}_{i,k=1}^{\infty} \in P(\alpha, \emptyset, \gamma)$ such that

$$(1.1) \qquad\qquad\qquad \mu_{n+1}^{(n+1)} = 0.$$

This basis leads us to find a Gelfand–Zetlin (GZ)-type basis for the $\mathfrak{gl}_{\infty,\kappa}$-module W_θ, where θ is given by (4.21). The action is given explicitly by GZ-type formulas (see (4.3), (4.4), (4.14), (4.19), (4.20)). We expect that similar bases exist for all θ, but we were unable to find them in the literature (except for the standard case of the dominant weights).

Following [KR2], we give an explicit bosonic construction of W_θ. A version of the Schur–Weyl–Howe duality established in [KR2] allows us to write bosonic character formulas for W_θ in the principal grading. Equivalently, our formula computes the generating function for the set $P^n(\alpha, c)$.

We do not have a recipe for how to compute the characters of the $\mathcal{E}$-modules $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$ where $K = q_1^{i_1} q_2^{i_2} q_3^{i_3}$ in general, but this problem seems to have a very intriguing structure.

For example, if $K = q_2^N$, then $\mathcal{E}$ has a big two-sided ideal. After factorization we get a smaller algebra $\mathcal{E}_{q_1,q_2,q_3,K}^{\mathrm{red}}$ which can be identified with the elliptic W-algebra. In particular, an appropriate limit of $\mathcal{E}_{q_1,q_2,q_3,K}^{\mathrm{red}}$ gives us the W-algebra for $\widehat{\mathfrak{gl}}_N$. The representation theory of the W-algebra for $\widehat{\mathfrak{gl}}_N$ is a well-known subject. In particular, the W-algebra has a class of representations appearing in the minimal models. In [FFJMM2] we showed that there exist $\mathcal{E}$-modules which have the same characters as the representations of minimal models of the W-algebra.

In the case $K = q_2^m q_3^n$ the algebra $\mathcal{E}$ also has a big two-sided ideal, and conjecturally the quotient in the appropriate limit gives us the W-algebra for the superalgebra $\widehat{\mathfrak{gl}}(m, n)$. This conjecture allows us to predict some character formulas, which can be checked by a computer for small values of parameters. We give some examples of such formulas at the end of the paper.

The paper is organized as follows. In Section 2 we recall and discuss some known facts about the algebra $\mathcal{E}$ and its representation. In Section 3 we construct and study the MacMahon representations $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$. In Section 4, we study the $\mathfrak{gl}_\infty$-limits of the MacMahon representations. In particular, we describe the GZ-type basis for some $\mathfrak{gl}_{\infty,\kappa}$-modules. In Section 5 we construct the $\mathfrak{gl}_\infty$-modules using Heisenberg algebra and compute their characters by the Schur–Weyl–Howe duality of [KR2]. We finish with some conjectural character formulas.

2. Preliminaries

2.1. Algebra $\mathcal{E}$

Let $q_1, q_2, q_3 \in \mathbb{C}$ be complex parameters satisfying the relation $q_1 q_2 q_3 = 1$. We assume that q_1, q_2, q_3 are not roots of unity. Let

$$g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w).$$

The *quantum toroidal* $\mathfrak{gl}_1$-algebra is an associative algebra $\mathcal{E}$ with generators e_i, f_i, $i \in \mathbb{Z}$, $\psi_r^{\pm}$, $r \in \mathbb{Z}_{>0}$, and invertible elements $\psi_0^{\pm}$, C, satisfying the following

defining relations:

$$C: \text{central},$$

$$\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z),$$

$$g(Cz,w)g(Cw,z)\psi^{+}(z)\psi^{-}(w) = g(z,Cw)g(w,Cz)\psi^{-}(w)\psi^{+}(z),$$

$$g(Cz,w)\psi^{+}(z)e(w) = -g(w,Cz)e(w)\psi^{+}(z),$$

$$g(z,w)\psi^{-}(z)e(w) = -g(w,z)e(w)\psi^{-}(z),$$

$$g(w,z)\psi^{+}(z)f(w) = -g(z,w)f(w)\psi^{+}(z),$$

$$g(w,Cz)\psi^{-}(z)f(w) = -g(Cz,w)f(w)\psi^{-}(z),$$

$$[e(z),f(w)] = \frac{1}{g(1,1)}\big(\delta(Cw/z)\psi^{+}(w) - \delta(Cz/w)\psi^{-}(z)\big),$$

$$g(z,w)e(z)e(w) = -g(w,z)e(w)e(z),$$

$$g(w,z)f(z)f(w) = -g(z,w)f(w)f(z),$$

$$\big[e_0,[e_1,e_{-1}]\big] = \big[f_0,[f_1,f_{-1}]\big] = 0.$$

Here $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ denotes the formal delta function, and the generating series of the generators of $\mathcal{E}$ are given by

$$e(z) = \sum_{i\in\mathbb{Z}} e_i z^{-i}, \qquad f(z) = \sum_{i\in\mathbb{Z}} f_i z^{-i}, \qquad \psi^{\pm}(z) = \sum_{\pm i \geq 0} \psi_i^{\pm} z^{-i}.$$

Note that $\mathcal{E}$ depends on the *unordered* set of parameters $\{q_1, q_2, q_3\}$, as all q_i enter the relations symmetrically through the function $g(z,w)$.

Algebra $\mathcal{E}$ has been introduced and studied by [Mi] under the name "(q,γ)-analogue of the $W_{1+\infty}$-algebra." (To be precise, in [Mi] an additional relation $\psi_0^+ \psi_0^- = 1$ is imposed, and $\mathcal{E}$ is a one-dimensional split central extension of that of [Mi].)

Consider the associative $\mathbb{C}$-algebra with generators $Z^{\pm 1}$, $D^{\pm 1}$ with the relation $DZ = qZD$. Let $\mathfrak{d}_q$ be the same algebra viewed as a Lie algebra by $[a,b] = ab - ba$. Then $\mathfrak{d}_q$ has a two-dimensional central extension (see [KR1]) $\mathfrak{d}_{q,c_1,c_2} = \mathfrak{d}_q \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2$, where c_1, c_2 are central elements and the commutator is given by

$$[Z^{i_1}D^{j_1}, Z^{i_2}D^{j_2}]$$
$$= (q^{j_1 i_2} - q^{j_2 i_1})Z^{i_1+i_2}D^{j_1+j_2} + \delta_{i_1+i_2,0}\delta_{j_1+j_2,0}q^{-i_1 j_1}(i_1 c_1 + j_1 c_2).$$

The element $Z^0 D^0 = 1$ is (split) central in $\mathfrak{d}_{q,c_1,c_2}$. The quantum toroidal $\mathfrak{gl}_1$-algebra $\mathcal{E}$ is a quantization of the universal enveloping algebra $U\mathfrak{d}_{q,c_1,c_2}$ where q_1 is a parameter of the quantization and $q_2 = q^2$. Algebra $\mathcal{E}$ has three central elements, C and $\psi_0^{\pm}$. Among the latter only the ratio $(\psi_0^+)^{-1}\psi_0^-$ is essential. We say that an $\mathcal{E}$-module V has *level* $(x,y) \in \mathbb{C}^2$ if C^2 acts by x and $(\psi_0^+)^{-1}\psi_0^-$ acts by y.

In what follows, we always consider representations of $\mathcal{E}$ on which C acts as identity. In other words, we study representations of the quotient algebra

$\mathcal{E}/\langle C - 1 \rangle$, where the defining relations simplify as follows:

$$(2.1) \qquad\qquad \psi^{\epsilon}(z)\psi^{\epsilon'}(w) = \psi^{\epsilon'}(w)\psi^{\epsilon}(z)(\epsilon, \epsilon' \in \{+, -\}),$$

$$(2.2) \qquad\qquad g(z, w)\psi^{\pm}(z)e(w) = -g(w, z)e(w)\psi^{\pm}(z),$$

$$(2.3) \qquad\qquad g(w, z)\psi^{\pm}(z)f(w) = -g(z, w)f(w)\psi^{\pm}(z),$$

$$(2.4) \qquad\qquad [e(z), f(w)] = \frac{\delta(z/w)}{g(1, 1)}\left(\psi^{+}(w) - \psi^{-}(z)\right),$$

$$(2.5) \qquad\qquad g(z, w)e(z)e(w) = -g(w, z)e(w)e(z),$$

$$(2.6) \qquad\qquad g(w, z)f(z)f(w) = -g(z, w)f(w)f(z),$$

$$(2.7) \qquad\qquad \big[e_0, [e_1, e_{-1}]\big] = \big[f_0, [f_1, f_{-1}]\big] = 0.$$

In the quotient algebra, the subalgebra generated by $\psi^{\pm}_{\pm i}$, $i \in \mathbb{Z}_{\geq 0}$, is commutative. We call an $\mathcal{E}$-module V *tame* if $\psi^{\pm}_{\pm i}$, $i \in \mathbb{Z}_{\geq 0}$, act by diagonalizable operators with simple joint spectrum.

Algebra $\mathcal{E}$ is $\mathbb{Z}^2$-graded with the assignment

$$\deg e_i = (1, i), \qquad \deg f_i = (-1, i), \qquad \deg \psi^{\pm}_i = (0, i), \qquad \deg C = (0, 0).$$

Let $\mathcal{E}_{(j,k)}$ denote the homogeneous component of $\mathcal{E}$ of degree $(j, k) \in \mathbb{Z}^2$. We say that an $\mathcal{E}$-module $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is $\mathbb{Z}$-*graded* if $\mathcal{E}_j V_n \subset V_{n+j}$ where $\mathcal{E}_j = \sum_{k \in \mathbb{Z}} \mathcal{E}_{(j,k)}$. We call it *quasi-finite* if $\dim V_n < \infty$ for all $n \in \mathbb{Z}$. Let $\phi^{\pm}(z) \in \mathbb{C}[[z^{\mp 1}]]$ be formal power series in $z^{\mp 1}$ with nonvanishing constant term. We say that a $\mathbb{Z}$-graded $\mathcal{E}$-module V is a *lowest weight module* with *lowest weight* $(\phi^{+}(z), \phi^{-}(z))$ if it is generated by a nonzero vector v such that

$$f(z)v = 0, \qquad \psi^{\pm}(z)v = \phi^{\pm}(z)v, \qquad Cv = v.$$

The following result due to Miki is an analogue of the classification theorem for finite-dimensional modules of quantum affine algebras.

THEOREM 2.1 (SEE [Mi])
Up to isomorphisms, an irreducible lowest weight module V is uniquely determined by its lowest weight. It is quasi-finite if and only if there exists a rational function $R(z)$, which is regular and nonzero at $z = 0, \infty$, such that $\phi^{\pm}(z)$ is the expansion of $R(z)$ at $z^{\pm 1} = \infty$.

By abuse of language we call $R(z)$ the lowest weight of the representation.

REMARK
In [Mi], highest weight modules are considered. Though this is purely a matter of convention, in this paper we shall deal with lowest weight modules for historical reasons. We note that we have called the same objects "highest weight modules" in [FFJMM1] and [FFJMM2].

Algebra $\mathcal{E}$ has the formal comultiplication

(2.8) $$\triangle e(z) = e(z) \otimes 1 + \psi^-(z) \otimes e(z),$$

(2.9) $$\triangle f(z) = f(z) \otimes \psi^+(z) + 1 \otimes f(z),$$

(2.10) $$\triangle \psi^\pm(z) = \psi^\pm(z) \otimes \psi^\pm(z).$$

These formulas do not define a comultiplication in the usual sense since the right-hand sides contain infinite sums. In [Mi], it is shown that the twisted coproduct by a certain automorphism is well defined on tensor products of a class of modules (called restricted modules). In this paper, we take a slightly different approach and use the original coproduct given by (2.8), (2.9), (2.10) when it makes sense. The arguments for justification can be found, for example, in the proof of [FFJMM1, Proposition 3.1].

2.2. Fock modules

Let $u \in \mathbb{C}$. Let $\mathcal{F}_i(u)$ be the irreducible $\mathcal{E}$-module with the lowest weight given by

$$R(z) = \frac{1 - q_i u/z}{1 - u/z}.$$

Now we describe the explicit construction of $\mathcal{F}(u) = \mathcal{F}_2(u)$.

Let $V(u) = V_1(u)$ be a complex vector space spanned by basis $[u]_i$, $i \in \mathbb{Z}$. Then the formulas

$$(1 - q_1)e(z)[u]_i = \delta(q_1^i u/z)[u]_{i+1},$$

$$-(1 - q_1^{-1})f(z)[u]_i = \delta(q_1^{i-1} u/z)[u]_{i-1},$$

(2.11) $$\psi^+(z)[u]_i = \frac{(1 - q_1^i q_3 u/z)(1 - q_1^i q_2 u/z)}{(1 - q_1^i u/z)(1 - q_1^{i-1} u/z)}[u]_i,$$

(2.12) $$\psi^-(z)[u]_i = \frac{(1 - q_1^{-i} q_3^{-1} z/u)(1 - q_1^{-i} q_2^{-1} z/u)}{(1 - q_1^{-i} z/u)(1 - q_1^{-i+1} z/u)}[u]_i$$

define a structure of an irreducible tame quasi-finite $\mathcal{E}$-module on $V(u)$ of level $(1,1)$. We call the $\mathcal{E}$-module $V(u)$ the *vector representation*. The vector representation; is not a lowest weight representation; it is the counterpart of the $\mathfrak{d}_q$-module $\mathbb{C}[Z, Z^{-1}]$.

Note that q_1 plays a special role in the definition of $V(u)$, while q_2 and q_3 participate symmetrically. Therefore there are two other vector representations $V_2(u), V_3(u)$ which are obtained from $V(u)$ by switching roles of q_1 to q_2, q_3.

We set

$$\psi_i(z) = \psi(q_1^i z), \qquad \psi(z) = \frac{(1 - q_3 z)(1 - q_2 z)}{(1 - z)(1 - q_2 q_3 z)}.$$

By (2.11) and (2.12), we have $\psi^\pm(z)[u]_i = \psi_i(u/z)[u]_i$.

The Fock representation $\mathcal{F}(u) = \mathcal{F}_2(u) = \bigoplus_\lambda \mathbb{C}|\lambda\rangle$ is constructed in the infinite tensor product of the vector representations (see [FFJMM1]):

$$\mathcal{F}(u) \subset V(u) \otimes V(uq_2^{-1}) \otimes V(uq_2^{-2}) \otimes \cdots,$$

(2.13) $$|\lambda\rangle = [u]_{\lambda_1} \otimes [uq_2^{-1}]_{\lambda_2-1} \otimes [uq_2^{-2}]_{\lambda_3-2} \otimes \cdots.$$

Here $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition: $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\lambda_j \in \mathbb{Z}_{\geq 0}$; $\lambda_N = 0$ for a large N. We denote the corresponding Young diagram by Y_λ.

In this tensor product we have two problems. First, we should avoid poles in the substitution. Let us examine the factor in the action of $e(z)$.

We prepare some notation. Let $\lambda \pm 1_i$ denote the partition μ such that $\mu_j = \lambda_j$ if $j \neq i$ and $\mu_i = \lambda_i \pm 1$. We say (i,j) is a concave (resp., convex) corner of λ if and only if

$$\lambda_i = j - 1 < \lambda_{i-1} \quad (\text{resp.,} \ \lambda_i = j > \lambda_{i+1}).$$

Denote by $CC(Y_\lambda)$ (resp., $CV(Y_\lambda)$) the set of concave (resp., convex) corners of λ.

From the comultiplication rule we have

$$e(z)|\lambda\rangle = \sum_{i=1}^\infty \frac{\psi_{\lambda,i}}{1-q_1} \delta(q_1^{\lambda_i} q_3^{i-1} u/z)|\lambda+1_i\rangle,$$

(2.14) $$\psi_{\lambda,i} = \prod_{k=1}^{i-1} \frac{(1-q_1^{\lambda_k-\lambda_i} q_3^{k-i+1})(1-q_1^{\lambda_k-\lambda_i-1} q_3^{k-i-1})}{(1-q_1^{\lambda_k-\lambda_i} q_3^{k-i})(1-q_1^{\lambda_k-\lambda_i-1} q_3^{k-i})}.$$

Note that $\psi_{\lambda,i}$ has no pole. It has a zero when $\lambda_{i-1} = \lambda_i$. This zero prohibits a term $|\mu\rangle$ with $\mu = \lambda + 1_i$, which breaks the condition $\mu_{i-1} \geq \mu_i$, from appearing in the right-hand side. Thus the above sum reduces to a finite sum:

$$e(z)|\lambda\rangle = \sum_{(i,j)\in CC(\lambda)} \frac{\psi_{\lambda,i}}{1-q_1} \delta(q_1^{j-1} q_3^{i-1} u/z)|\lambda+1_i\rangle.$$

Second, when we deal with the semiinfinite tensor product we have to give a meaning to the infinite product which appears in the action of $\psi^\pm(z)$ and $f(z)$. Let us give a meaning to the infinite product which appears in the action of $\psi^\pm(z)$:

$$\psi^\pm(z)|\lambda\rangle = \psi_\lambda(u/z)|\lambda\rangle, \qquad \psi_\lambda(u/z) = \prod_{i=1}^\infty \psi_{\lambda_i-i+1}(uq_2^{-i+1}/z).$$

The product can be written as

(2.15) $$\psi_\lambda(u/z) = \frac{1-q_1^{\lambda_1-1} q_3^{-1} u/z}{1-q_1^{\lambda_1} u/z} \prod_{j=1}^\infty \frac{(1-q_1^{\lambda_j} q_3^j u/z)(1-q_1^{\lambda_{j+1}-1} q_3^{j-1} u/z)}{(1-q_1^{\lambda_{j+1}} q_3^j u/z)(1-q_1^{\lambda_j-1} q_3^{j-1} u/z)},$$

which is convergent because of the boundary condition $\lambda_N = 0$ for large N. We remark that the convergence is valid, in general, if $\lim_{i\to\infty} \lambda_i$ exists. We use (2.15) under that condition later. This formula implies that the level of $\mathcal{F}(u)$ is $(1, q_2)$.
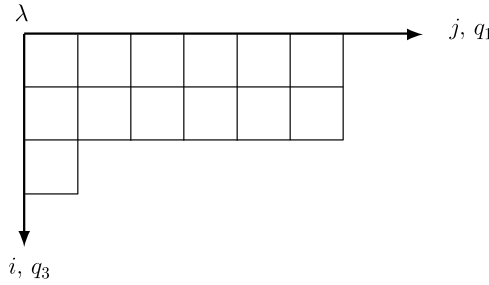
$$CC(\lambda) = \{(1,\,7),\,(3,\,2),\,(4,\,1)\},\ CV(\lambda) = \{(2,\,6),\,(3,\,1)\}$$

Figure 1. Partition

For the vacuum $|\emptyset\rangle$, that is, the empty Young diagram, we have

$$(2.16) \qquad \qquad \psi_\emptyset(u/z) = \frac{1 - q_2 u/z}{1 - u/z}.$$

This is the lowest weight of $\mathcal{F}(u)$. The general formula (2.15) for $|\lambda\rangle$ can be understood as starting from the lowest weight (2.16) for the vacuum, and multiplying the contribution from each box of Y_λ. Namely, set

$$\psi_{i,j}(u/z) = \frac{\psi_{j-i+1}(uq_2^{-i+1}/z)}{\psi_{j-i}(uq_2^{-i+1}/z)}$$

$$= \frac{(1 - q_1^j q_3^i u/z)(1 - q_1^{j-2} q_3^{i-1} u/z)(1 - q_1^{j-1} q_3^{i-2} u/z)}{(1 - q_1^{j-1} q_3^i u/z)(1 - q_1^j q_3^{i-1} u/z)(1 - q_1^{j-2} q_3^{i-2} u/z)}.$$

The rational function $\psi_\lambda(u/z)$ can be determined recursively by

$$(2.17) \qquad \qquad \psi_\lambda(u/z) = \psi_{i,\lambda_i}(u/z)\psi_{\lambda-1_i}(u/z).$$

This formula immediately follows from (2.11), (2.12), and the comultiplication rule. It says that the contribution from the box (i,j) is $\psi_{i,j}(u/z)$. Using (2.17), we can easily see that

$$(2.18) \qquad \psi_\lambda(u/z) = \prod_{(i,j) \in CC(\lambda)} \frac{1 - q_1^{j-2} q_3^{i-2} u/z}{1 - q_1^{j-1} q_3^{i-1} u/z} \prod_{(i,j) \in CV(\lambda)} \frac{1 - q_1^j q_3^i u/z}{1 - q_1^{j-1} q_3^{i-1} u/z}.$$

From this, one can see that the representation is tame.

The formula for the action of $f(z)$ is obtained similarly:

$$f(z)|\lambda\rangle = \sum_{i=1}^{\infty} \frac{q_1 \psi'_{\lambda,i}}{1 - q_1} \delta(q_1^{\lambda_i - 1} q_3^{i-1} u/z)|\lambda - 1_i\rangle,$$

$$(2.19) \quad \psi'_{\lambda,i} = \frac{1 - q_1^{\lambda_{i+1} - \lambda_i}}{1 - q_1^{\lambda_{i+1} - \lambda_i + 1} q_3} \prod_{k=i+1}^{\infty} \frac{(1 - q_1^{\lambda_k - \lambda_i + 1} q_3^{k-i+1})(1 - q_1^{\lambda_{k+1} - \lambda_i} q_3^{k-i})}{(1 - q_1^{\lambda_{k+1} - \lambda_i + 1} q_3^{k-i+1})(1 - q_1^{\lambda_k - \lambda_i} q_3^{k-i})}.$$

Again, $\psi'_{\lambda,i}$ has no pole, and the zero at $\lambda_i = \lambda_{i+1}$ prohibits the appearance of terms $|\mu\rangle$, which breaks the condition $\mu_i \geq \mu_{i+1}$. Thus, the action of $f(z)$ reads

as

$$(2.20) \qquad f(z)|\lambda\rangle = \sum_{(i,j)\in CV(\lambda)} \frac{q_1\psi'_{\lambda,i}}{1-q_1}\delta(q_1^{j-1}q_3^{i-1}u/z)|\lambda-1_i\rangle.$$

If we exchange q_1 with q_3 the representation $\mathcal{F}(u)$ changes. Let us denote it by $\mathcal{F}'(u)$. This representation is realized inside the semiinfinite tensor product $V_3(u)\otimes V_3(uq_2^{-1})\otimes V_3(uq_2^{-2})\otimes\cdots$. Since $\mathcal{F}'(u)$ has the same lowest weight (2.16) as that of $\mathcal{F}(u)$, these two modules are isomorphic by Theorem 2.1. We can construct the isomorphism explicitly. Look at the action of $\psi^\pm(z)$ (see (2.18)). If we exchange $(i,j)\leftrightarrow(j,i)$ and $q_1\leftrightarrow q_3$ simultaneously, the factors are invariant. Therefore, as $\psi_\pm(z)$-modules, for arbitrary nonzero constants c_λ the mapping $|\lambda\rangle\mapsto c_\lambda|\lambda'\rangle$, with λ' being the transpose of λ, is an intertwiner. Since the representations are tame, this is the only way of intertwining these two $\mathcal{E}$-modules. Theorem 2.1 shows the existence of the set of constants c_λ.

In Section 3.5 we use the following realization of $\mathcal{F}_3(u)$:

$$\mathcal{F}_3(u)\subset V(u)\otimes V(uq_3^{-1})\otimes V(uq_3^{-2})\otimes\cdots,$$

which is obtained by a construction similar to that of $\mathcal{F}(u)=\mathcal{F}_2(u)$ with q_2 replaced by q_3.

Now let us discuss the relation (2.4). On the subspace of $V(u)\otimes V(uq_2^{-1})\otimes\cdots\otimes V(uq_2^{-N+1})$ that is spanned by the vector

$$(2.21) \qquad |\lambda\rangle^{(N)} = [u]_{\lambda_1}\otimes[uq_2^{-1}]_{\lambda_2-1}\otimes\cdots\otimes[uq_2^{-N+1}]_{\lambda_N-N+1},$$

where $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N$, the relation (2.4) is valid. The left-hand side is a sum of products of delta functions of the form $c(q_1,q_3)\delta(z/w)\delta(q_1^aq_3^bu/z)$. The right-hand side is a difference of two series $\psi^\pm(z)$: one is obtained from the rational function with simple poles,

$$\psi_\lambda^{(N)}(u/z) = \prod_{j=1}^{N}\psi_{\lambda_j-j+1}(q_2^{-j+1}u/z),$$

by expanding it in z^{-1}, and the other from the same rational function by expanding it in z. The difference is a sum of delta functions. They can be computed from the position of the poles of $\psi_\lambda^{(N)}(u/z)$ and their residues.

Now, consider the semiinfinite action on (2.13), and compare it with the action on (2.21). If N is large enough so that $\lambda_N=0$, we identify (2.13) with (2.21). Then the action of $e(z)$ is the same. The actions of $\psi^\pm(z)$ are slightly different:

$$(2.22) \qquad \psi_\lambda^{(N)}(u/z) = \psi_\lambda(u/z)\frac{1-q_3^Nu/z}{1-q_1^{-1}q_3^{N-1}u/z}.$$

The factor $(1-q_3^Nu/z)/(1-q_1^{-1}q_3^{N-1}u/z)$ is dropped in the action of $\psi^\pm(z)$ on $\mathcal{F}(u)$. We also drop the same factor from the action of $f(z)$. This explains the

factor $\psi'_{\lambda,i}$ in (2.20): for large N, we have

$$(2.23) \qquad \psi'_{\lambda,i} = \prod_{k=i+1}^{N} \psi_{\lambda_k-k+1}(q_2^{-k+1}u/z) \times \left.\frac{1 - q_3^N q_2 u/z}{1 - q_3^N u/z}\right|_{u/z \to q_1^{-\lambda_i+1} q_3^{-i+1}}.$$

Since the tensor product (2.13) does not have the $(N + 1)$st component, the equality (2.4) for (2.21) contains an extra term with the delta function $\delta(q_1^{-1} q_3^{N-1} u/z)$. On the other hand, in the action on $\mathcal{F}(u)$, this term is killed by the zero of $\psi'_{\lambda,N}$ as discussed before; when we go to (2.13) this term is dropped in both sides of (2.4). The effect of the modification (2.22) is the same on each delta function term because it is the multiplication by the same factor. Thus the equality (2.4) is valid on $\mathcal{F}(u)$.

The value of ψ_0^- has been changed by the multiplication because the value of this factor at $z = 0$ is q_2. The modification produces the nontrivial level $(1, q_2)$ for the representation $\mathcal{F}(u)$.

3. MacMahon modules

3.1. Vacuum MacMahon modules

Let us construct a level $(1, K)$ representation

$$(3.1) \qquad \mathcal{M}(u, K) \subset \mathcal{F}(u) \otimes \mathcal{F}(uq_2) \otimes \mathcal{F}(uq_2^2) \otimes \cdots$$

with basis

$$\mathcal{M}(u, K) = \bigoplus_{\boldsymbol{\lambda}} \mathbb{C}|\boldsymbol{\lambda}\rangle, \quad \boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots),$$

where $\boldsymbol{\lambda}$ is a plane partition, that is, each $\lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, 0, 0, \ldots)$ is a partition and

$$(3.2) \qquad \lambda_i^{(k)} \geq \lambda_i^{(k+1)}$$

is satisfied. We require $\lambda^{(N)} = \emptyset$ for large N. In particular, we set

$$\emptyset = (\emptyset, \emptyset, \emptyset, \ldots).$$

With each $\boldsymbol{\lambda}$ we associate a subset $Y_{\boldsymbol{\lambda}}$ of $(\mathbb{Z}_{\geq 1})^3$ such that $(i, j, k) \in Y_{\boldsymbol{\lambda}}$ if and only if $j \leq \lambda_i^{(k)}$. This is a finite set.

We call the representation $\mathcal{M}(u, K)$ the vacuum MacMahon representation.

In [FFJMM2, Theorem 3.4], the action of $\mathcal{E}$ was defined on the subspace $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{(n)}$ ($\mathbf{a} = (a_1, \ldots, a_{n-1}), \mathbf{b} = (b_1, \ldots, b_{n-1})$) of $\mathcal{F}(u_1) \otimes \mathcal{F}(u_2) \otimes \cdots \otimes \mathcal{F}(u_n)$ where

$$(3.3) \qquad u_{i+1} = u_i q_1^{-a_i} q_2 q_3^{-b_i}$$

is spanned by the vectors

$$|\boldsymbol{\lambda}^{(n)}\rangle = |\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle$$

satisfying

$$(3.4) \qquad \lambda_i^{(k)} + a_k \geq \lambda_{i+b_k}^{(k+1)}.$$

The tensor product (3.1) with the restriction (3.2) is the limit $n \to \infty$ of this construction in the case $\mathbf{a} = \mathbf{b} = \mathbf{0}$. From the discussion in Section 2.2, the method for constructing the action on the infinite tensor product based on the results in [FFJMM2] is clear. However, we must be careful on the definition of $\psi^{\pm}(z)$ since the level of the representation $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{(n)}$ is $(1, q_2^n)$, and the simple-minded limit $n \to \infty$ is not defined. In fact, this is not a defect, but there is room for introducing an arbitrary parameter for the level.

We define the action of $\psi^{\pm}(z)$ by

$$(3.5) \quad \psi^{\pm}(z)|\boldsymbol{\lambda}\rangle = \psi_{\boldsymbol{\lambda}}(u/z)|\boldsymbol{\lambda}\rangle,$$

$$\psi_{\boldsymbol{\lambda}}(u/z) = \psi_{\boldsymbol{\emptyset}}(u/z) \prod_{(i,j,k) \in Y_{\boldsymbol{\lambda}}} \psi_{i,j,k}(u/z),$$

$$\psi_{\boldsymbol{\emptyset}}(u/z) = \frac{1 - Ku/z}{1 - u/z},$$

$$(3.6)\, \psi_{i,j,k}(u/z) = \frac{(1 - q_1^j q_2^{k-1} q_3^i u/z)(1 - q_1^{j-1} q_2^k q_3^i u/z)(1 - q_1^j q_2^k q_3^{i-1} u/z)}{(1 - q_1^{j-1} q_2^k q_3^{i-1} u/z)(1 - q_1^{j-1} q_2^{k-1} q_3^i u/z)(1 - q_1^j q_2^{k-1} q_3^{i-1} u/z)}.$$

Here K is an arbitrary nonzero parameter. The level of representation is $(1, K)$. It is easy to see that the action of $\psi^{\pm}(z)$ is tame. In fact, the partition $\lambda^{(1)}$ can be read from $\psi_{\boldsymbol{\lambda}}(u/z)$ by identifying its concave and convex corners recursively: the rightmost concave corner $(i_1 + 1, j_1) = (1, \lambda_1^{(1)} + 1)$ can be identified by the pole coming from the factor $1 - q_1^{\lambda_1^{(1)}} u/z = 1 - q_1^{j_1-1} q_3^{i_1} u/z$. Among the factors of the form $(1 - q_1^x u/z)$ in the denominator of $\psi_{\boldsymbol{\lambda}}(u/z)$, the one with $x = \lambda_1^{(1)}$ is the largest in x. Next, the rightmost convex corner $(i_2, j_1 - 1)$ can be identified by the zero at $(1 - q_1^{j_1-1} q_3^{i_2} u/z)$. Among the factors of the form $(1 - q_1^{j_1-1} q_3^x u/z)$ in the numerator of $\psi_{\boldsymbol{\lambda}}(u/z)$, the one with $x = i_2$ is the largest in x. Similarly, one can identify the concave corner $(i_2 + 1, j_2)$, then the convex corner $(i_3, j_2 - 1)$, and so on, from the factors in $\psi_{\boldsymbol{\lambda}}(u/z)$. After identifying $\lambda^{(1)}$, we divide $\psi^{(1)}(u/z) = \psi_{\boldsymbol{\lambda}}(u/z)$ by the factors corresponding to $\lambda^{(1)}$ and obtain new $\psi^{(2)}(u/z)$. Then, one can determine $\lambda^{(2)}$ by the same procedure using this $\psi^{(2)}(u/z)$. Continuing in this way, we can completely determine $\boldsymbol{\lambda}$ from $\psi_{\boldsymbol{\lambda}}(u/z)$.

The rational function $\psi_{\boldsymbol{\lambda}}(u/z)$ can be determined recursively. Denote $\boldsymbol{\mu}$ such that $\mu^{(m)} = \lambda^{(m)}$ if $m \neq k$, and $\mu^{(k)} = \lambda^{(k)} \pm 1_i$ by $\boldsymbol{\lambda} \pm 1_i^{(k)}$. Then we have

$$\psi_{\boldsymbol{\lambda}}(u/z) = \psi_{i,\lambda_i^{(k)},k}(u/z)\psi_{\boldsymbol{\lambda}-1_i^{(k)}}(u/z).$$

Let us compare $\psi_{\boldsymbol{\lambda}}(u/z)$ with

$$(3.7) \qquad \psi_{\boldsymbol{\lambda}}^{(k)}(u/z) = \prod_{m=1}^{k} \psi_{\lambda^{(m)}}(u_m/z), \quad u_m = uq_2^{m-1}.$$

For $N \gg 1$, we have

$$(3.8) \qquad \psi_{\boldsymbol{\lambda}}(u/z) = \psi_{\boldsymbol{\lambda}}^{(N)}(u/z)\frac{1 - Ku/z}{1 - q_2^N u/z}.$$

This is because for large N we have the same recursion

$$\psi_{\boldsymbol{\lambda}}^{(N)}(u/z) = \psi_{i,\lambda^{(k)},k}(u/z)\psi_{\boldsymbol{\lambda}-1_i^{(k)}}^{(N)}(u/z).$$

Note that the structure of poles is the same for $\psi_{\boldsymbol{\lambda}}^{(N)}(u/z)$ and $\psi_{\boldsymbol{\lambda}}(u/z)$ because the former (for large N) has a zero at $1 - q_2^N u/z = 0$ and the latter at $1 - Ku/z = 0$. This is important in the derivation of (2.5). Namely, the position of delta functions appearing in the right-hand side of the equality does not change by changing the rational function from $\psi_{\boldsymbol{\lambda}}^{(N)}(u/z)$ to $\psi_{\boldsymbol{\lambda}}(u/z)$. It is also invariant in the left-hand side because for large N we have $\lambda^{(N)} = \emptyset$ and $|\lambda^{(N)}\rangle = |\emptyset\rangle$ is the lowest weight vector, which is killed by the action of $f(z)$. Thus, we can establish the existence of the representation on $\mathcal{M}(u,K)$ with the level $(1,K)$ and the lowest weight $(1 - Ku/z)/(1 - u/z)$.

For completeness we give the action of $e(z), f(z)$ on $\mathcal{M}(u,K)$.

The action of $e(z)$ on $|\boldsymbol{\lambda}\rangle$ is defined by

$$(3.9) \quad e(z)|\boldsymbol{\lambda}\rangle = \sum_{k=1}^{\infty} \psi_{\boldsymbol{\lambda}}^{(k-1)}(u/z) \sum_{i=1}^{\infty} \psi_{\lambda^{(k)},i} \frac{1}{1-q_1} \delta(q_1^{\lambda_i^{(k)}} q_2^{k-1} q_3^{i-1} u/z)|\boldsymbol{\lambda}+1_i^{(k)}\rangle.$$

It follows from [FFJMM2] that, for the finite tensor product, the delta function does not pick up poles of $\psi_{\boldsymbol{\lambda}}^{(k-1)}(u/z)$ and does pick up a zero if and only if $\boldsymbol{\mu} = \boldsymbol{\lambda} + 1_i^{(k)}$ breaks the condition $\mu_i^{(k-1)} \geq \mu_i^{(k)}$. Here we give a simple proof of these statements using (2.18).

Set $j = \lambda_i^{(k)} + 1$. Suppose that

$$(i,j) \in CC(\lambda^{(k)}).$$

From (2.18) we see that the function $\psi_{\boldsymbol{\lambda}}^{(k-1)}(u/z)$ has a pole at $q_1^{j-1} q_2^{k-1} q_3^{i-1} u/z = 1$ only if for some $m \leq k-1$, there exists a box $(\bar{i},\bar{j})$ such that

$$(\bar{i},\bar{j}) \in CC(\lambda^{(m)}) \sqcup CV(\lambda^{(m)})$$

and

$$q_1^{j-1} q_2^{k-1} q_3^{i-1} = q_1^{\bar{j}-1} q_2^{m-1} q_3^{\bar{i}-1}.$$

The latter implies

$$\bar{i} = i + m - k < i, \qquad \bar{j} = j + m - k < j.$$

This is a contradiction with $\lambda_l^{(m)} \geq \lambda_l^{(k)}$. We have shown the statement about the poles.

Let us show the statement about the zeros. A zero occurs only if either

$$(\bar{i},\bar{j}) \in CC(\lambda^{(m)}) \quad \text{and} \quad q_1^{j-1} q_2^{k-1} q_3^{i-1} = q_1^{\bar{j}-2} q_2^{m-1} q_3^{\bar{i}-2}$$

or

$$(\bar{i},\bar{j}) \in CV(\lambda^{(m)}) \quad \text{and} \quad q_1^{j-1} q_2^{k-1} q_3^{i-1} = q_1^{\bar{j}} q_2^{m-1} q_3^{\bar{i}}.$$

The former case really occurs when the condition $\mu_i^{(k-1)} \geq \mu_i^{(k)}$ is broken, while the latter leads to a contradiction.

A box (i, j, k) is called a concave (resp., convex) corner of $Y_{\boldsymbol{\lambda}}$ if

$$(i, j, k) \notin Y_{\boldsymbol{\lambda}} \quad (\text{resp., } (i, j, k) \in Y_{\boldsymbol{\lambda}})$$

and

$$(i - 1, j, k), (i, j - 1, k), (i, j, k - 1) \in Y_{\boldsymbol{\lambda}}$$
$$(\text{resp., } (i + 1, j, k), (i, j + 1, k), (i, j, k + 1) \notin Y_{\boldsymbol{\lambda}}).$$

We denote by $CC(Y_{\boldsymbol{\lambda}})$ (resp., $CV(Y_{\boldsymbol{\lambda}})$) the set of concave (resp., convex) corners of $\boldsymbol{\lambda}$. They are finite sets. The action of $e(z)$ adds a box at each concave corner (see (2.14), (3.7)):

$$(3.10) \quad e(z)|\boldsymbol{\lambda}\rangle = \sum_{(i,j,k) \in CC(Y_{\boldsymbol{\lambda}})} \psi_{\boldsymbol{\lambda},i,j,k} \psi_{\lambda^{(k)},i} \frac{1}{1 - q_1} \delta(q_1^j q_2^k q_3^i u/z)|\boldsymbol{\lambda} + 1_i^{(k)}\rangle,$$

$$(3.11) \quad \psi_{\boldsymbol{\lambda},i,j,k} = \psi_{\boldsymbol{\lambda}}^{(k-1)}(q_1^{-j} q_2^{-k} q_3^{-i}).$$

Similarly, we have the formula for the action of $f(z)$ (see (2.23)):

$$f(z)|\boldsymbol{\lambda}\rangle = \sum_{(i,j,k) \in CV(Y_{\boldsymbol{\lambda}})} \psi'_{\boldsymbol{\lambda},i,j,k} \psi'_{\lambda^{(k)},i} \frac{q_1}{1 - q_1} \delta(q_1^j q_2^k q_3^i u/z)|\boldsymbol{\lambda} - 1_i^{(k)}\rangle,$$

$$\psi'_{\boldsymbol{\lambda},i,j,k} = \psi'^{(k+1)}_{\boldsymbol{\lambda}}(q_1^{-j} q_2^{-k} q_3^{-i}),$$

$$\psi'^{(k)}_{\boldsymbol{\lambda}}(u/z) = \lim_{N \to \infty} \prod_{m=k}^{N} \psi_{\lambda^{(m)}}(q_2^{m-1} u/z) \times \frac{1 - Ku/z}{1 - q_2^N u/z}.$$

As we discussed, $\psi'_{\lambda^{(k)},i}$ (see (2.19)) has no pole, and it has a zero if and only if $\boldsymbol{\mu} = \boldsymbol{\lambda} - 1_i^{(k)}$ breaks the condition for the plane partitions. The discussion for poles and zeros of $\psi'_{\boldsymbol{\lambda},i,j,k}$ is exactly the same as $\psi_{\boldsymbol{\lambda},i,j,k}$ for $e(z)$.

3.2. MacMahon modules with nontrivial boundary conditions

In this subsection we generalize the MacMahon representation to the case where the plane partitions have nontrivial boundary conditions. We repeat the semiinfinite tensor product construction. We remove the restriction $\mathbf{a} = \mathbf{b} = \mathbf{0}$ in (3.4) and also remove the condition $\lambda^{(N)} = \emptyset$ for large N.

It is convenient to use another notation. Consider a set of three partitions $\alpha = (\alpha_1, \alpha_2, \ldots, 0, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots, 0, \ldots)$, $\gamma = (\gamma_1, \gamma_2, \ldots, 0, \ldots)$. We call a sequence $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$, where $\mu_i \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ is a generalized partition if and only if $\mu_i \geq \mu_{i+1}$ holds for all $i \geq 1$. A sequence of generalized partitions $\boldsymbol{\mu} = \{\mu^{(k)}\}_{k \geq 1}$ is called a plane partition with the boundary conditions (α, β, γ) if and only if the following conditions hold:

$$(3.12) \qquad\qquad \mu_i^{(k)} \geq \mu_i^{(k+1)},$$

$$\lim_{i \to \infty} \mu_i^{(k)} = \alpha_k,$$

$$\mu_i^{(k)} = \infty \quad \text{if } 1 \leq i \leq \beta_k,$$
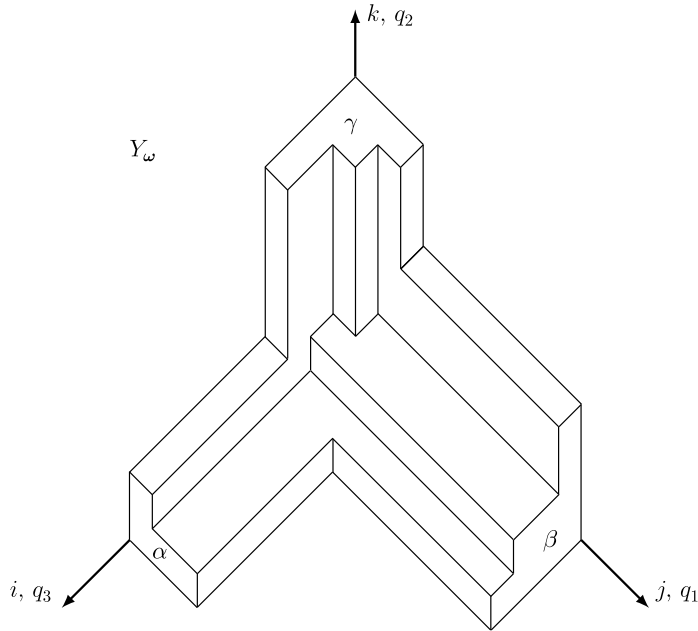
Figure 2. Plane partition with the boundary condition (α, β, γ). The diagram corresponding to the minimal plane partition ω in (3.23) is shown.

$$\lim_{k \to \infty} \mu_i^{(k)} = \gamma_i.$$

We denote by $\mathcal{P}[\alpha, \beta, \gamma]$ the set of $\boldsymbol{\mu}$ satisfying these conditions. For each $\boldsymbol{\mu}$ we define a subset $Y_{\boldsymbol{\mu}} \subset (\mathbb{Z}_{\geq 1})^3$ by

$$(3.13) \qquad\qquad\qquad (i, j, k) \in Y_{\boldsymbol{\mu}} \leftrightarrow j \leq \mu_i^{(k)}.$$

This definition is a generalization of $Y_{\boldsymbol{\lambda}}$ when $\alpha = \beta = \gamma = \emptyset$. A new feature is that $Y_{\boldsymbol{\mu}}$ can be an infinite set. If α is nonzero for $\boldsymbol{\mu}$, then $Y_{\boldsymbol{\mu}}$ has an elevation in the i-axis. Similarly, if β (resp., γ) is nonzero, it has an elevation in the j-axis (resp., k-axis; see Figure 2).

Plane partitions $\boldsymbol{\mu}$ with the boundary conditions (α, β, γ) are in one-to-one correspondence with sets of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots)$.

Set

$$(3.14) \qquad\qquad a_k = \alpha_k - \alpha_{k+1}, \qquad b_k = \beta_k - \beta_{k+1}, \qquad c_k = \gamma_k,$$

$$(3.15) \qquad\qquad \lambda_i^{(k)} = \mu_{i+\beta_k}^{(k)} - \alpha_k.$$

The condition (3.12) for $\boldsymbol{\mu}$ and the condition (3.4) for $\boldsymbol{\lambda}$ are equivalent through (3.14) and (3.15).

We fix the parameter u_i as

$$u_i = u q_1^{\alpha_i} q_2^{i-1} q_3^{\beta_i},$$

which implies (3.3). When we discuss the tensor product we use

$$|\boldsymbol{\lambda}\rangle = |\lambda^{(1)}\rangle \otimes |\lambda^{(2)}\rangle \otimes |\lambda^{(3)}\rangle \otimes \cdots \subset \mathcal{F}(u_1) \otimes \mathcal{F}(u_2) \otimes \mathcal{F}(u_3) \otimes \cdots,$$

and when we discuss the plane partition we use $Y_{\boldsymbol{\mu}}$. We show this correspondence by denoting $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{\mu}}$ when it is necessary.

Consider a linear subspace of the semiinfinite tensor product

$$(3.16) \qquad \mathcal{M}_{\alpha,\beta,\gamma} \subset \mathcal{F}(u_1) \otimes \mathcal{F}(u_2) \otimes \cdots.$$

By definition the space $\mathcal{M}_{\alpha,\beta,\gamma}$ is spanned by $|\boldsymbol{\lambda}\rangle$ where $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots)$ is a sequence of partitions satisfying (3.4) and the boundary condition.

$$(3.17) \qquad \lim_{k\to\infty} \lambda_i^{(k)} = \gamma_i.$$

The construction of a representation with basis $|\boldsymbol{\lambda}\rangle$ can be done using the result on the finite tensor product, [FFJMM2, Theorem 3.4].

We consider $|\lambda^{(i)}\rangle$ as an element of $\mathcal{F}(u_i)$ and identify $|\boldsymbol{\lambda}\rangle$ with

$$|\lambda^{(1)}\rangle \otimes |\lambda^{(2)}\rangle \otimes \cdots \otimes |\lambda^{(N)}\rangle \in \mathcal{F}(u_1) \otimes \mathcal{F}(u_2) \otimes \cdots \otimes \mathcal{F}(u_N)$$

for large enough N. Then, the action of $e(z)$ on $\mathcal{M}_{\alpha,\beta,\gamma}$ is the same as in the finite tensor product. Below let us describe the action of $\psi^\pm(z)$ and $f(z)$.

We describe the action of $e(z)$. The action of $e(z)$ adds a box at each concave corner as before (see (3.10), (3.11)):

$$e(z)|\boldsymbol{\lambda}\rangle = \sum_{(i,j,k)\in CC(Y_{\boldsymbol{\mu}})} \psi_{\boldsymbol{\lambda},i,j,k}\psi_{\lambda^{(k)},i-\beta_k} \frac{1}{1-q_1} \delta(q_1^j q_2^k q_3^i u/z)|\boldsymbol{\lambda} + 1_{i-\beta_k}^{(k)}\rangle.$$

Let us discuss the well-definedness of this action. This point was discussed in Section 3.1 in the case $\alpha = \beta = \gamma = \emptyset$. The argument is the same in the general case, but $\boldsymbol{\mu}$ must be used instead of $\boldsymbol{\lambda}$ because the structure of plane partitions is respected by $\boldsymbol{\mu}$, not by $\boldsymbol{\lambda}$ (see (3.13)).

Recall (2.15), and change (3.7) to

$$\psi_{\boldsymbol{\lambda}}^{(k)}(u/z) = \prod_{m=1}^k \psi_{\lambda^{(m)}}(u_m/z), \quad u_m = q_1^{\alpha_m} q_2^{m-1} q_3^{\beta_m} u.$$

Then using (3.15) we obtain

$$(3.18) \qquad \psi_{\lambda^{(m)}}(u_m/z) = \psi_{\mu^{(m)}}(q_2^{m-1}u/z),$$

where we understand $q_1^\infty = 0$. Thus, the action of $e(z)$ takes the same form as in the vacuum case wherein $\boldsymbol{\lambda}$ is replaced by $\boldsymbol{\mu}$.

Define the action of $\psi^\pm(z)$ on $\mathcal{M}_{\alpha,\beta,\gamma} = \mathcal{M}_{\alpha,\beta,\gamma}(u,K)$ by setting

$$(3.19) \qquad \psi_{\boldsymbol{\lambda}}(u/z) = \frac{1-Ku/z}{1-q_1^{\lambda_1^{(1)}}u_1/z} \prod_{i=1}^\infty \frac{1-q_1^{\lambda_1^{(i)}}q_2 u_i/z}{1-q_1^{\lambda_1^{(i+1)}}u_{i+1}/z} \prod_{j=1}^\infty \frac{1-q_1^{\lambda_j^{(1)}}q_3^j u_1/z}{1-q_1^{\lambda_{j+1}^{(1)}}q_3^j u_1/z}$$

$$\times \prod_{i=1}^\infty \prod_{j=1}^\infty \frac{(1-q_1^{\lambda_j^{(i+1)}}q_3^j u_{i+1}/z)(1-q_1^{\lambda_{j+1}^{(i)}}q_2 q_3^j u_i/z)}{(1-q_1^{\lambda_{j+1}^{(i+1)}}q_3^j u_{i+1}/z)(1-q_1^{\lambda_j^{(i)}}q_2 q_3^j u_i/z)}$$

in the formula (3.5). This is a finite product. This expression follows from the formal infinite product

$$\psi_{\boldsymbol{\lambda}}(u) = \prod_{i=1}^{\infty} \psi_{\lambda^{(i)}}(u_i/z)$$

by substituting (2.15) and modifying it as we did in (2.15) and (3.8).

The function $\psi_{\boldsymbol{\lambda}}(u/z)$, in general, can be better understood in terms of $Y_{\boldsymbol{\mu}}$. Let $(i,j,k) \in CC(Y_{\boldsymbol{\mu}})$ be a concave corner. Then, adding one box at (i,j,k) to $Y_{\boldsymbol{\mu}}$ corresponds to changing $\boldsymbol{\lambda}$ to $\boldsymbol{\lambda} + 1_{i-\beta_k}^{(k)}$, where $\lambda_{i-\beta_k}^{(k)} + \alpha_k + 1 = j$. From (3.15), this relation can be rewritten as

$$\mu_i^{(k)} + 1 = j.$$

Using (3.6) and (3.19) we obtain

$$\psi_{\boldsymbol{\lambda}+1_{i-\beta_k}^{(k)}}(u/z) = \psi_{i,j,k}(u/z)\psi_{\boldsymbol{\lambda}}(u/z).$$

Using (3.18), for $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{\mu}}$, we have

$$(3.20) \qquad \psi_{\boldsymbol{\lambda}}(u/z) = \frac{1 - Ku/z}{1 - u/z} \prod_{(i,j,k) \in Y_{\boldsymbol{\mu}}} \psi_{i,j,k}(u/z).$$

Here the infinite product is defined as follows. Set

$$Y_{\boldsymbol{\mu}}^{(N)} = \left\{ (i,j,k) \in Y_{\boldsymbol{\mu}} \mid i,j,k \leq N \right\},$$

and define $\psi_{\boldsymbol{\lambda}}^{(N)}(u/z)$ by (3.20) with $Y_{\boldsymbol{\mu}}$ replaced by $Y_{\boldsymbol{\mu}}^{(N)}$. For large N the difference between $\psi_{\boldsymbol{\lambda}}^{(N)}(u/z)$ and $\psi_{\boldsymbol{\lambda}}^{(N+1)}(u/z)$ consists only of N dependent factors which come from the $\psi_{i,j,k}(u/z)$ such that $i \sim N$ or $j \sim N$ or $k \sim N$. We define the infinite product by removing these factors from $\psi_{\boldsymbol{\lambda}}^{(N)}(u/z)$. By the definition it is independent of N. Once we define the infinite product in this way, the equality (3.20) is clear from (3.18) for $\gamma = \emptyset$. We discuss the case $\gamma \neq \emptyset$ at the end of this subsection.

In fact, it is possible to rewrite the infinite product as a finite product. We will do it later in Section 3.3. Here we remark that each cube in $Y_{\boldsymbol{\mu}}$ contributes to poles and zeros through eight corners of the cube: poles from $(i,j,k), (i-1,j-1,k), (i-1,j,k-1), (i,j-1,k-1)$ and zeros from $(i-1,j-1,k-1), (i-1,j,k), (i,j-1,k), (i,j,k-1)$; two of them, (i,j,k) and $(i-1,j-1,k-1)$, cancel each other because of the restriction $q_1 q_2 q_3 = 1$. From this it follows that the $\psi^{\pm}(z)$-action enjoys the $\mathfrak{S}_3$-symmetry

$$(3.21) \qquad (i,j,k) \leftrightarrow (j,i,k) \Leftrightarrow (q_1,q_2,q_3) \leftrightarrow (q_3,q_2,q_1),$$

$$(3.22) \qquad (i,j,k) \leftrightarrow (k,j,i) \Leftrightarrow (q_1,q_2,q_3) \leftrightarrow (q_1,q_3,q_2).$$

The first line means the following. If we transform $Y_{\boldsymbol{\mu}}$ where $\boldsymbol{\mu} \in \mathcal{P}[\alpha,\beta,\gamma]$, by the involution $(i,j,k) \leftrightarrow (j,i,k)$, we obtain $Y_{\tilde{\boldsymbol{\mu}}}$ where $\tilde{\boldsymbol{\mu}} \in \mathcal{P}[\beta,\alpha,\gamma']$. Set $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{\mu}}$, and set $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}_{\tilde{\boldsymbol{\mu}}}$. Then, we have the equality

$$\psi_{\boldsymbol{\lambda}}(u/z)|_{(q_1,q_2,q_3) \to (q_3,q_2,q_1)} = \psi_{\tilde{\boldsymbol{\lambda}}}(u/z).$$

Similarly, from the involution $(i, j, k) \leftrightarrow (j, i, k)$, we have

$$\psi_\lambda(u/z)|_{(q_1, q_2, q_3) \to (q_1, q_3, q_2)} = \psi_{\tilde{\lambda}}(u/z),$$

where $\tilde{\tilde{\lambda}} = \lambda_{\tilde{\tilde{\mu}}}$ and $\tilde{\tilde{\mu}} \in \mathcal{P}[\gamma, \beta', \alpha]$.

Define a plane partition $\boldsymbol{\omega} = \{\omega^{(k)}\}_{k \geq 1}$ with the boundary condition (α, β, γ) by

(3.23)
$$\omega_i^{(k)} = \begin{cases} \infty & \text{if } i \leq \beta_k, \\ \max(\gamma_i, \alpha_k) & \text{otherwise.} \end{cases}$$

Then we have

$$\omega_i^{(k)} = \begin{cases} \omega_{i+1}^{(k)} & \text{if } \omega_i^{(k)} = \alpha_k, \\ \omega_i^{(k+1)} & \text{if } \omega_i^{(k)} = \gamma_i. \end{cases}$$

Among all $\boldsymbol{\mu} \in \mathcal{P}[\alpha, \beta, \gamma]$, $Y_{\boldsymbol{\omega}} \subset Y_{\boldsymbol{\mu}}$ is the minimum. There is no convex corner in $Y_{\boldsymbol{\omega}}$ (see Figure 2). The set of partitions $\lambda = \lambda_{\boldsymbol{\omega}}$ associated with $\boldsymbol{\omega}$ is given by

$$\lambda_i^{(k)} = \max(\gamma_{i+\beta_k}, \alpha_k) - \alpha_k.$$

Let us compute a few examples of the eigenvalues (3.19) for $\lambda = \lambda_{\boldsymbol{\omega}}$.

α	β	γ	$\psi_\lambda(u/z)$
$\emptyset$	$\emptyset$	$\emptyset$	$\frac{1-Ku/z}{1-u/z}$
$\{1\}$	$\emptyset$	$\emptyset$	$\frac{(1-Ku/z)(1-q_1q_2u/z)}{(1-q_1u/z)(1-q_2u/z)}$
$\{2\}$	$\emptyset$	$\emptyset$	$\frac{(1-Ku/z)(1-q_1^2q_2u/z)}{(1-q_1^2u/z)(1-q_2u/z)}$
$\{1\}$	$\{1\}$	$\emptyset$	$\frac{(1-Ku/z)(1-q_1q_2q_3u/z)}{(1-q_2u/z)(1-q_1q_3u/z)}$
$\{1\}$	$\{1\}$	$\{1\}$	$\frac{(1-Ku/z)(1-q_1q_2q_3u/z)^2}{(1-q_1q_2u/z)(1-q_1q_3u/z)(1-q_2q_3u/z)}$

Finally, we give the action of $f(z)$ on $\mathcal{M}_{\alpha, \beta, \gamma}(u, K)$:

$$f(z)|\lambda\rangle = \sum_{(i,j,k) \in CV(Y_{\boldsymbol{\mu}})} \psi'_{\lambda, i, j, k} \psi'_{\lambda^{(k)}, i-\beta_k} \frac{q_1}{1-q_1} \delta(q_1^j q_2^k q_3^i u/z) |\lambda - 1_{i-\beta_k}^{(k)}\rangle,$$

$$\psi'_{\lambda, i, j, k} = \psi'^{(k+1)}_{\lambda}(q_1^{-j} q_2^{-k} q_3^{-i}),$$

$$\psi'^{(k)}_{\lambda}(u/z) = \lim_{N \to \infty} \prod_{m=k}^N \psi_{\lambda^{(m)}}(q_2^{m-1} u/z)$$

(3.24)
$$\cdot \frac{1-Ku/z}{1-q_1^{\gamma_1} q_2^N u/z} \prod_{j=1}^\infty \frac{1-q_1^{\gamma_j} q_2^N q_3^j u/z}{1-q_1^{\gamma_{j+1}} q_2^N q_3^j u/z}.$$

We do not repeat the argument which assures the well-definedness of this action. However, we note that the multiplication of the last infinite product is in fact a finite product corresponding to the convex corners of γ, and it removes the extra poles and zeros in the finite tensor product which do not occur in the semiinfinite product. It is also important to notice that

(3.25)
$$\psi_\lambda(u/z) = \psi'^{(1)}_\lambda(u/z),$$

where the left-hand side is given by (3.19) and the right-hand side by (3.24). From this the equality (3.20) for nontrivial γ follows.

We summarize the result in this and the previous subsections as the following.

THEOREM 3.1

There is an action of the algebra $\mathcal{E}$ on $\mathcal{M}_{\alpha,\beta,\gamma}(u,K)$ induced from the infinite tensor product (3.16). This is an irreducible, quasi-finite, and tame representation. The level is $(1,K)$ with a generic parameter K. The lowest weight is given by (3.19) (see also (3.27) below) with $\boldsymbol{\mu} = \boldsymbol{\omega}$ given by (3.23). The representations thus obtained admit the $\mathfrak{S}_3$-symmetry (3.21), (3.22).

3.3. Shell formula for the action of $\psi^{\pm}(z)$

For the Fock representation the rational function $\psi_\lambda(u/z)$ is factorized into the contribution from the concave and convex corners. Let us derive the three-dimensional version of this statement for the MacMahon representations. It follows from (3.20).

Define an auxiliary object

$$\Psi_{\boldsymbol{\lambda}}(u/z) = \prod_{(i,j,k) \in Y_{\boldsymbol{\mu}}} \psi_{i,j,k}(u/z),$$

where we consider q_1, q_2, q_3 as free, that is, we do not require $q_1 q_2 q_3 = 1$. Once we obtain $\Psi_{\boldsymbol{\lambda}}(u/z)$ as a finite product we get $\psi_{\boldsymbol{\lambda}}(u/z)$ by

(3.26) $$\psi_{\boldsymbol{\lambda}}(u/z) = \frac{1 - Ku/z}{1 - u/z} \Psi_{\boldsymbol{\lambda}}(u/z),$$

where $q_1 q_2 q_3 = 1$ is imposed. Let us define the shell of $Y_{\boldsymbol{\mu}}$ by

$$\mathcal{S}_{\boldsymbol{\mu}} = \big\{ (i,j,k) \in \mathbb{Z}^3 \, \big| \, i,j,k \geq 0, (i+1,j+1,k+1) \notin Y_{\boldsymbol{\mu}},$$

$$\big\{ (i,j,k), (i+1,j,k), (i,j+1,k), (i,j,k+1),$$

$$(i+1,j+1,k), (i+1,j,k+1), (i,j+1,k+1) \big\} \cap Y_{\boldsymbol{\mu}} \neq \emptyset \big\}.$$

For example,

$Y_{\boldsymbol{\mu}}$	$\mathcal{S}_{\boldsymbol{\mu}}$
$\{\}$	$\{\}$
$\{(1,1,1)\}$	$\{(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$.

The rational function $\Psi_{\boldsymbol{\lambda}}(u/z)$ has neither a pole nor a zero at $1 - q_1^j q_2^k q_3^i u/z = 0$ unless $(i,j,k) \in \mathcal{S}_{\boldsymbol{\mu}}$. It is also worth noting that for a fixed (i,j,k) the intersection of $\{i+n, j+n, k+n) \, | \, n \in \mathbb{Z}\}$ with $\mathcal{S}_{\boldsymbol{\mu}}$ is at most one point.

We classify the points in the shell $\mathcal{S}_{\boldsymbol{\mu}}$ into $\mathcal{S}_{\boldsymbol{\mu}}^{(n)}$ ($-1 \leq n \leq 2$), where

$$\mathcal{S}_{\boldsymbol{\mu}}^{(n)} = \big\{ (i,j,k) \in \mathcal{S}_{\boldsymbol{\mu}} \, \big| \, \Psi_{\boldsymbol{\lambda}}(u/z) \text{ has a zero of order } n \text{ at } 1 - q_1^j q_2^k q_3^i u/z = 0 \big\}.$$

For $(i,j,k) \in \mathcal{S}_{\boldsymbol{\mu}}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 = 0, 1$, we define

$$A_{i,j,k}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{cases} 1 & \text{if } (i+\varepsilon_1, j+\varepsilon_2, k+\varepsilon_3) \in Y_{\boldsymbol{\mu}}, \\ 0 & \text{otherwise.} \end{cases}$$

According to the set of values given in the form of two matrices

$$T_{i,j,k} = \begin{pmatrix} A_{i,j,k}(0,1,0) & A_{i,j,k}(1,1,0) \\ A_{i,j,k}(0,1,1) & A_{i,j,k}(1,1,1) \end{pmatrix},$$

$$B_{i,j,k} = \begin{pmatrix} A_{i,j,k}(0,0,0) & A_{i,j,k}(1,0,0) \\ A_{i,j,k}(0,0,1) & A_{i,j,k}(1,0,1) \end{pmatrix},$$

the order of zero at $1 - q_1^j q_2^k q_3^i u/z = 0$ of the rational function $\Psi_{\boldsymbol{\lambda}}(u/z)$ is determined

$T_{i,j,k}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$B_{i,j,k}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
order of zero	-1	1	1	1	2

$T_{i,j,k}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$B_{i,j,k}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
order of zero	1	1	1	-1

The cases not listed in this table are neither poles nor zeros.

For any α, β, γ and $\boldsymbol{\mu} \in \mathcal{P}[\alpha, \beta, \gamma]$ the union of $\mathcal{S}_{\boldsymbol{\mu}}^{(a)}$ ($a = -1, 1, 2$) is finite. Thus, the formula (3.26) becomes a finite product

$$(3.27) \qquad \psi_{\boldsymbol{\lambda}}(u/z) = (1 - Ku/z) \prod_{a=-1,1,2} \prod_{(i,j,k) \in \mathcal{S}_{\boldsymbol{\mu}}^{(a)}} (1 - q_1^j q_2^k q_3^i u/z)^a.$$

3.4. Resonance and submodules

Now we utilize the factor $1 - Ku/z$ (see (3.24)) in the action of $f(z)$. Consider the specialization of the level

$$(3.28) \qquad\qquad\qquad K = q_1^b q_2^c q_3^a = q_2^m q_3^n.$$

Note that

$$m = c - b, \qquad n = a - b.$$

At this point, the $\mathcal{E}$-module $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$ is reducible. We denote it by $\mathcal{M}_{\alpha,\beta,\gamma}^{m,n}(u)$. In fact, the module $\mathcal{M}_{\alpha,\beta,\gamma}^{m,n}(u)$ contains an infinite sequence of submodules. Let us describe these submodules.

Let $\boldsymbol{\omega}$ be the minimum configuration in $\mathcal{P}[\alpha, \beta, \gamma]$ (see (3.23)). Recall $q_1 q_2 q_3 = 1$. For each triple (α, β, γ) and K of the form (3.28), we determine a unique (a, b, c) with $a, b, c \geq 1$, satisfying (3.28) and

$$(a, b, c) \notin Y_{\boldsymbol{\omega}}, \qquad (a-1, b-1, c-1) \in Y_{\boldsymbol{\omega}}.$$

The action of $f(z)$ on $\mathcal{M}^{m,n}_{\alpha,\beta,\gamma}(u)$ is such that removing a box at

$$(3.29) \qquad\qquad (i,j,k) = (a+t, b+t, c+t) \quad (t \in \mathbb{Z}_{\geq 0})$$

is prohibited. This is because the coefficient of $|\boldsymbol{\lambda} - 1_i^{(k)}\rangle$ in $f(z)|\boldsymbol{\lambda}\rangle$, where $(i,j,k) \in CV(Y_{\boldsymbol{\mu}})$ $(\boldsymbol{\lambda} = \boldsymbol{\lambda_{\mu}})$ and $\lambda_i^{(k)} = j$, contains the factor $(1 - Ku/z) \times \delta(q_1^j q_2^k q_3^i u/z)$ but does not contain poles at $q_1^j q_2^k q_3^i u/z = 1$. The poles may appear only if for some $s \geq 0$, $(i+1+s, j+s, k+s) \in Y_{\boldsymbol{\mu}}$ or $(i+s, j+1+s, k+s) \in Y_{\boldsymbol{\mu}}$ or $(i+s, j+s, k+1+s) \in Y_{\boldsymbol{\mu}}$. However, this is not possible if $(i,j,k) \in CV(Y_{\boldsymbol{\mu}})$. Therefore if the position of the box is of the form (3.29) the coefficient vanishes when K is specialized as (3.28).

The lowest weight vector $|\boldsymbol{\lambda_{\omega}}\rangle$ is still cyclic in $\mathcal{M}^{m,n}_{\alpha,\beta,\gamma}(u)$. There is no K in the action of $e(z)$. The above consideration tells us that once a box is added at (i,j,k) of the form (3.29), one cannot remove it by the action of $f(z)$. In fact, we show that the module $\mathcal{M}^{m,n}_{\alpha,\beta,\gamma}(u)$ contains an infinite series of singular vectors.

Define $\boldsymbol{\omega}_t = (\omega_t^{(1)}, \omega_t^{(2)}, \ldots) \in \mathcal{P}[\alpha,\beta,\gamma]$ $(t \in \mathbb{Z}_{\geq 0})$ by

$$\omega_{t,i}^{(k)} = \begin{cases} \max(b+t-1, \omega_i^{(k)}) & \text{if } i \leq a+t-1 \text{ and } k \leq c+t-1, \\ \omega_i^{(k)} & \text{otherwise.} \end{cases}$$

This is the minimal configuration among $\boldsymbol{\mu} \in \mathcal{P}[\alpha,\beta,\gamma]$ such that $(a+t-1, b+t-1, c+t-1) \in Y_{\boldsymbol{\mu}}$. Note that $\boldsymbol{\omega}_0 = \boldsymbol{\omega}$ and

$$Y_{\boldsymbol{\omega}_0} \subset Y_{\boldsymbol{\omega}_1} \subset Y_{\boldsymbol{\omega}_2} \subset \cdots.$$

For $t \geq 1$ we have

$$CV(\boldsymbol{\omega}_t) = \{(a+t-1, b+t-1, c+t-1)\}.$$

Set

$$\mathcal{M}^{m,n,t}_{\alpha,\beta,\gamma}(u) = \bigoplus_{Y_{\boldsymbol{\mu}} \supset Y_{\boldsymbol{\omega}_t}} \mathbb{C}|\boldsymbol{\lambda_{\mu}}\rangle.$$

This is a submodule of $\mathcal{M}^{m,n}_{\alpha,\beta,\gamma}(u)$ with the lowest vector $|\boldsymbol{\lambda_{\omega_t}}\rangle$ satisfying

$$f(z)|\boldsymbol{\lambda_{\omega_t}}\rangle = 0.$$

We have the inclusions

$$\mathcal{M}^{m,n}_{\alpha,\beta,\gamma}(u) = \mathcal{M}^{m,n,0}_{\alpha,\beta,\gamma}(u) \supset \mathcal{M}^{m,n,1}_{\alpha,\beta,\gamma}(u) \supset \mathcal{M}^{m,n,2}_{\alpha,\beta,\gamma}(u) \supset \cdots.$$

In this subsection we study the quotient

$$\mathcal{N}^{m,n}_{\alpha,\beta,\gamma}(u) = \mathcal{M}^{m,n}_{\alpha,\beta,\gamma}(u)/\mathcal{M}^{m,n,1}_{\alpha,\beta,\gamma}(u).$$

As we explained (a,b,c) is uniquely determined once $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and m, n are fixed.

PROPOSITION 3.2

The module $\mathcal{N}^{m,n}_{\alpha,\beta,\gamma}(u)$ is an irreducible, quasi-finite, tame $\mathcal{E}$-module of level $K = q_2^m q_3^n$. It has a basis parameterized by the set

$$P^{m,n}_{\alpha,\beta,\gamma} = \{\boldsymbol{\mu} \in P_{\alpha,\beta,\gamma}, (a,b,c) \notin Y_{\boldsymbol{\mu}}\}.$$

The lowest weight is given by (3.19) or (3.27), with $\boldsymbol{\mu} = \boldsymbol{\omega}$ given by (3.23).

3.5. The case of tensor product of the Fock spaces
Set
$$\overline{Y}_\omega = \big\{(i,j,k) \in \mathbb{Z}^3 \,\big|\, i \le 0 \text{ or } j \le 0 \text{ or } k \le 0\big\} \sqcup Y_\omega.$$

In this subsection we consider the case where the following condition is satisfied for some a,b,c:

$$(3.30) \qquad (a-1,b-1,s),(a-1,s,c-1),(s,b-1,c-1) \in \overline{Y}_\omega \quad (s \in \mathbb{Z}_{>0}).$$

Then, each of the Young diagrams $Y_\alpha, Y_\beta, Y_\gamma$ contains the following rectangle:
$$Y_\alpha \supset C_\alpha = \big\{(k,j) \,\big|\, 1 \le k \le c-1, 1 \le j \le b-1\big\},$$
$$Y_\beta \supset C_\beta = \big\{(k,i) \,\big|\, 1 \le k \le c-1, 1 \le i \le a-1\big\},$$
$$Y_\gamma \supset C_\gamma = \big\{(i,j) \,\big|\, 1 \le i \le a-1, 1 \le j \le b-1\big\},$$

and each of α,β,γ splits into three parts, core, arms, and legs. We define partitions which determine arms and legs of α,β,γ as follows:
$$\alpha_{\text{arms}} = (\alpha_1 - b + 1, \ldots, \alpha_{c-1} - b + 1),$$
$$\alpha_{\text{legs}} = (\alpha_1' - c + 1, \ldots, \alpha_{b-1}' - c + 1),$$
$$\beta_{\text{arms}} = (\beta_1 - a + 1, \ldots, \beta_{c-1} - a + 1),$$
$$\beta_{\text{legs}} = (\beta_1' - c + 1, \ldots, \beta_{a-1}' - c + 1),$$
$$\gamma_{\text{arms}} = (\gamma_1 - b + 1, \ldots, \gamma_{a-1} - b + 1),$$
$$\gamma_{\text{legs}} = (\gamma_1' - a + 1, \ldots, \gamma_{b-1}' - a + 1).$$

Introduce the notation
$$\mathcal{M}^{2,(n)}_{\alpha,\beta}(u) = \mathcal{M}^{(n)}_{\mathbf{a,b}}(u) = \mathcal{M}^{(n)}_{\mathbf{a,b}}(u; q_1, q_2, q_3),$$
$$\mathcal{M}^{1,(n)}_{\alpha,\beta}(u) = \mathcal{M}^{(n)}_{\mathbf{a,b}}(u; q_3, q_1, q_2),$$
$$\mathcal{M}^{3,(n)}_{\alpha,\beta}(u) = \mathcal{M}^{(n)}_{\mathbf{a,b}}(u; q_2, q_3, q_1).$$

Here α,β are related to $\mathbf{a,b}$ by (3.14). For example,
$$\mathcal{M}^{1,(1)}_{r,s}(u) \simeq \mathcal{F}_2(q_1^r q_3^s u), \qquad \mathcal{M}^{3,(1)}_{r,s}(q_3 u) \simeq \mathcal{F}_3(q_1^s q_2^r q_3 u).$$

Let us consider a few examples. The simplest case is $(a,b,c) = (1,1,1)$ and $(\alpha,\beta,\gamma) = (\emptyset,\emptyset,\emptyset)$. In this case $\mathcal{N}^{0,0}_{\emptyset,\emptyset,\emptyset}(u)$ is the trivial one-dimensional module,
$$\mathcal{N}^{0,0}_{\emptyset,\emptyset,\emptyset}(u) \simeq \mathbb{C}, \qquad K = 1, \qquad \psi_{\boldsymbol{\lambda}_\omega}(u/z) = 1.$$

We have other specializations for the same $(\alpha,\beta,\gamma) = (\emptyset,\emptyset,\emptyset)$:
$$\mathcal{N}^{r,0}_{\emptyset,\emptyset,\emptyset}(u) \simeq \mathcal{M}^{2,(r)}_{\mathbf{0,0}}(u) \subset \mathcal{F}_2(u) \otimes \mathcal{F}_2(q_2 u) \otimes \cdots \otimes \mathcal{F}_2(q_2^{r-1} u),$$
$$K = q_2^r, \qquad \psi_{\boldsymbol{\lambda}_\omega}(u/z) = \frac{1 - q_2^r u/z}{1 - u/z}, \qquad (a,b,c) = (1,1,r+1) \quad (r \ge 1).$$

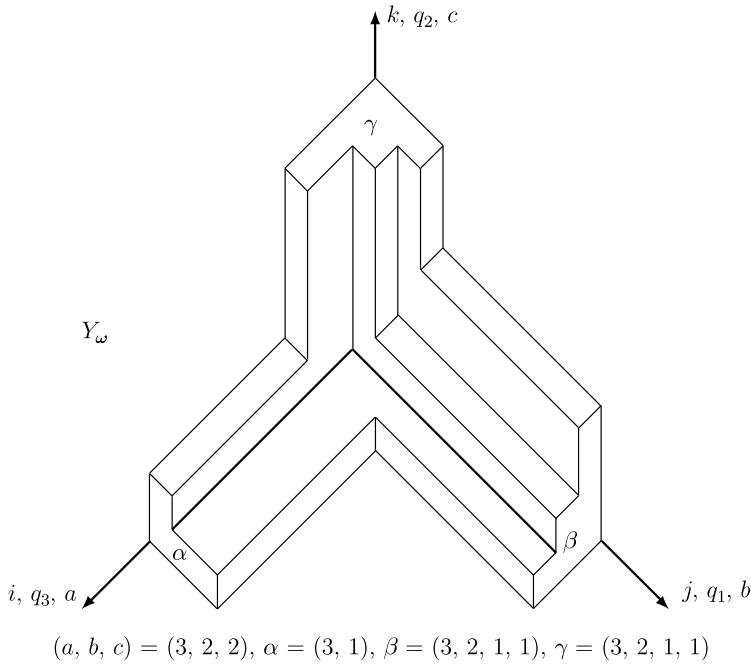$(a,\,b,\,c) = (3,\,2,\,2),\ \alpha = (3,\,1),\ \beta = (3,\,2,\,1,\,1),\ \gamma = (3,\,2,\,1,\,1)$

Figure 3. The case of tensor product

If $(\alpha, \beta, \gamma) = (\emptyset, (1), \emptyset)$, then we have the following cases; we consider the cases up to the symmetry:

$$\mathcal{N}^{r,0}_{\emptyset,(1),\emptyset}(u) \simeq \mathcal{M}^{2,(r)}_{\mathbf{0},(1,0,\ldots,0)}(u) \subset \mathcal{F}_2(q_3 u) \otimes \mathcal{F}_2(q_2 u) \otimes \cdots \otimes \mathcal{F}_2(q_2^{r-1} u),$$

$$K = q_2^r, \qquad \psi_{\boldsymbol{\lambda}_{\boldsymbol{\omega}}}(u/z) = \frac{1 - q_2^r u/z}{1 - q_2 u/z} \cdot \frac{1 - q_2 q_3 u/z}{1 - q_3 u_z},$$

$$(a,b,c) = (1,1,r+1) \quad (r \geq 1);$$

$$\mathcal{N}^{1,1}_{\emptyset,(1),\emptyset}(u) \simeq \mathcal{F}_2(q_3 u) \otimes \mathcal{F}_3(q_2 u),$$

$$K = q_2 q_3, \qquad \psi_{\boldsymbol{\lambda}_{\boldsymbol{\omega}}}(u/z) = \frac{(1 - q_2 q_3 u/z)^2}{(1 - q_2 u/z)(1 - q_3 u/z)},$$

$$(a,b,c) = (2,1,2).$$

If $(\alpha, \beta, \gamma) = ((1), (1), \emptyset)$, then we have the following cases up to the symmetry:

$$\mathcal{N}^{r,0}_{(1),(1),\emptyset}(u) \simeq \mathcal{M}^{(r)}_{(1,0,\ldots,0),(1,0,\ldots,0)}(u) \subset \mathcal{F}_2(q_1 q_3 u) \otimes \mathcal{F}_2(q_2 u) \otimes \cdots \otimes \mathcal{F}_2(q_2^{r-1} u),$$

$$K = q_2^r, \qquad \psi_{\boldsymbol{\lambda}_{\boldsymbol{\omega}}}(u/z) = \frac{1 - q_2^r u/z}{1 - q_2 u/z} \cdot \frac{1 - q_1 q_2 q_3 u/z}{1 - q_1 q_3 u/z},$$

$$(a,b,c) = (1,1,r+1) \quad (r \geq 1),$$

$$\mathcal{N}^{1,1}_{(1),(1),\emptyset}(u) \simeq \mathcal{F}_2(q_1 q_3 u) \otimes \mathcal{F}_3(q_2 u),$$

$$K = q_2 q_3, \qquad \psi_{\boldsymbol{\lambda_\omega}}(u/z) = \frac{1 - q_2 q_3 u/z}{1 - q_2 u/z}\frac{1 - q_1 q_2 q_3 u/z}{1 - q_1 q_3 u/z},$$

$$(a, b, c) = (2, 1, 2).$$

If $(\alpha, \beta, \gamma) = ((1), (1), (1))$, then we have the following cases up to the symmetry:

$$\mathcal{N}^{0,0}_{(1),(1),(1)}(u) \simeq \mathcal{F}_2(q_1 q_3 u) \otimes \mathcal{F}_3(q_1 q_2 u) \otimes \mathcal{F}_1(q_2 q_3 u),$$

$$K = 1, \qquad \psi_{\boldsymbol{\lambda_\omega}}(u/z) = \frac{(1 - q_1 q_2 q_3 u/z)^3}{(1 - q_1 q_2 u/z)(1 - q_1 q_3 u/z)(1 - q_2 q_3 u/z)},$$

$$(a, b, c) = (2, 2, 2),$$

$$\mathcal{N}^{0,0}_{(1),(1),(1)}(u) \simeq \mathcal{F}_2(q_1 q_3 u) \otimes \mathcal{F}_3(q_1 q_2 u),$$

$$K = q_2 q_3, \qquad \psi_{\boldsymbol{\lambda_\omega}}(u/z) = \frac{(1 - q_1 q_2 q_3 u/z)^2}{(1 - q_1 q_3 u/z)(1 - q_1 q_2 u/z)},$$

$$(a, b, c) = (2, 1, 2).$$

Finally we give the general statement.

PROPOSITION 3.3

Under condition (3.30), we have

$$\mathcal{N}^{c-b,a-b}_{\alpha,\beta,\gamma}(u) \simeq \mathcal{M}^{2,(c-1)}_{\alpha_{\mathrm{arms}},\beta_{\mathrm{arms}}}(q_1^{b-1} q_3^{a-1} u)$$
$$\otimes \mathcal{M}^{3,(a-1)}_{\beta_{\mathrm{legs}},\gamma_{\mathrm{arms}}}(q_1^{b-1} q_2^{c-1} u) \otimes \mathcal{M}^{1,(b-1)}_{\alpha_{\mathrm{legs}},\gamma_{\mathrm{legs}}}(q_2^{c-1} q_3^{a-1} u).$$

4. $\mathfrak{gl}_\infty$-modules and Gelfand–Zetlin basis

4.1. Algebra $\mathfrak{gl}_\infty$

In this section, we introduce a family of $\mathfrak{gl}_\infty$-modules which arises as a limit of $\mathcal{E}$-modules considered in the previous section.

We fix the notation as follows. By definition, $\mathfrak{gl}_\infty$ is the complex Lie algebra with basis $\{E_{i,j}\}_{i,j\in\mathbb{Z}}$ and the commutation relations $[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{l,i}E_{k,j}$. We set

$$E_i = E_{i,i+1}, \qquad F_i = E_{i+1,i}, \qquad H_i = E_{i,i} - E_{i+1,i+1}.$$

We shall consider also the following Lie subalgebras of $\mathfrak{gl}_\infty$:

$$\mathfrak{gl}^+_{\infty/2} = \mathrm{span}\{E_{i,j} \mid i, j \geq 1\},$$
$$\mathfrak{gl}^-_{\infty/2} = \mathrm{span}\{E_{i,j} \mid i, j \leq 0\},$$
$$\mathfrak{g}_{r,s} = \mathrm{span}\{E_{i,j} \mid r \leq i, j \leq s\} \simeq \mathfrak{gl}_{s-r+1},$$

where $r, s \in \mathbb{Z}$, $r < s$.

For a sequence of complex numbers $\theta = (\theta_i)_{i\in\mathbb{Z}}$, we denote by $\mathcal{W}_\theta$ the unique irreducible $\mathfrak{gl}_\infty$-module generated by a vector v such that

$$E_{i,j}v = 0 \quad (i > j), \qquad E_{i,i}v = \theta_i v \quad (i \in \mathbb{Z}).$$

The θ is called the lowest weight and the vector v is called the lowest weight vector.

4.2. GZ basis

Let N be a positive integer. We recall the GZ basis for irreducible representations of $\mathfrak{g}_{-N+1,0} \simeq \mathfrak{gl}_N$.

A GZ pattern for $\mathfrak{gl}_N$ is an array of integers

$$
(4.1) \qquad \mu = \begin{matrix}
\mu_1^{(1)} & & & \\
\mu_1^{(2)} & \mu_2^{(2)} & & \\
\vdots & \ddots & \ddots & \\
\mu_1^{(N)} & \mu_2^{(N)} & \cdots & \mu_N^{(N)}
\end{matrix}
$$

such that

$$
(4.2) \qquad \mu_j^{(i)} \geq \mu_{j+1}^{(i)}, \qquad \mu_j^{(i)} \geq \mu_j^{(i+1)} \quad \text{for all } i, j.
$$

Quite generally, we shall denote by $\mu \pm 1_j^{(i)}$ the GZ pattern obtained by changing $\mu_j^{(i)}$ to $\mu_j^{(i)} \pm 1$ while keeping the rest of the entries unchanged.

Given a set of integers $\eta = (\eta_1, \ldots, \eta_N)$, $\eta_1 \geq \cdots \geq \eta_N$, let L_η be the vector space with basis $\{|\mu\rangle_{(N)}\}$, where μ runs over all GZ patterns for $\mathfrak{gl}_N$ satisfying

$$
\mu_i^{(i)} = \eta_i \quad (i = 1, \ldots, N).
$$

We set $|\mu\rangle_{(N)} = 0$ if the condition (4.2) is violated.

Notation being as above, the following formulas define an action of $\mathfrak{g}_{-N+1,0}$ on L_η:

$$
(4.3) \qquad
\begin{aligned}
E_{-i,-i+1}|\mu\rangle_{(N)} &= \sum_{j=1}^{N-i} |\mu + 1_j^{(i+j)}\rangle_{(N)} \\
&\times \frac{\prod_{k=1}^{N-i+1}(\ell_j^{(i+j)} - \ell_k^{(i-1+k)})}{\prod_{1 \leq k(\neq j) \leq N-i}(\ell_j^{(i+j)} - \ell_k^{(i+k)})} \quad (1 \leq i \leq N-1),
\end{aligned}
$$

$$
(4.4) \qquad
\begin{aligned}
E_{-i+1,-i}|\mu\rangle_{(N)} &= -\sum_{j=1}^{N-i} |\mu - 1_j^{(i+j)}\rangle_{(N)} \\
&\times \frac{\prod_{k=1}^{N-i-1}(\ell_j^{(i+j)} - \ell_k^{(i+1+k)})}{\prod_{1 \leq k(\neq j) \leq N-i}(\ell_j^{(i+j)} - \ell_k^{(i+k)})} \quad (1 \leq i \leq N-1),
\end{aligned}
$$

$$
(4.5) \quad E_{-i,-i}|\mu\rangle_{(N)} = \left(\sum_{j=1}^{N-i} \mu_j^{(i+j)} - \sum_{j=1}^{N-i-1} \mu_j^{(i+1+j)} \right) |\mu\rangle_{(N)} \quad (0 \leq i \leq N-1),
$$

where

$$
(4.6) \qquad \ell_j^{(i+j)} = \mu_j^{(i+j)} - j + 1.
$$

The representation L_η is irreducible. The highest weight is $(\theta_{-N+1}, \theta_{-N+2}, \ldots,$ $\theta_0) = (\eta_1, \eta_2, \ldots, \eta_N)$, and the lowest weight is $(\eta_N, \eta_{N-1}, \ldots, \eta_1)$, the corresponding highest (resp., lowest) weight vector being given by the GZ pattern with $\mu_j^{(i)} = \eta_j$ (resp., $\mu_j^{(i)} = \eta_i$) for all i, j.

Now we extend this construction to the case of $\mathfrak{gl}^-_{\infty/2}$. In the following we fix a positive integer n. Consider an infinite GZ pattern of width n,

$$
\text{(4.7)} \qquad \mu =
\begin{matrix}
\mu_1^{(1)} & & & & & \\
\vdots & \ddots & & & & \\
\mu_1^{(n)} & \cdots & \mu_n^{(n)} & & & \\
\mu_1^{(n+1)} & \cdots & \mu_n^{(n+1)} & 0 & & \\
\mu_1^{(n+2)} & \cdots & \mu_n^{(n+2)} & 0 & 0 & \\
\vdots & \cdots & \vdots & 0 & 0 & 0 \cdots,
\end{matrix}
$$

that is, an array of integers $\mu = (\mu_j^{(i)})_{i \geq j \geq 1}$ satisfying (4.2) and

$$
\text{(4.8)} \qquad \mu_j^{(i)} = 0 \quad \text{if } j > n.
$$

Let $\eta = (\eta_1, \ldots, \eta_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ be partitions such that $\eta_i \geq \gamma_i$, $i = 1, \ldots, n$. Let $\mathcal{Y}^-_{\eta, \gamma}$ be the vector space with basis $\{|\mu\rangle\}$, where $\mu = (\mu_j^{(i)})_{i \geq j \geq 1}$ runs over GZ patterns (4.7) of width n satisfying the conditions

$$
\text{(4.9)} \qquad \mu_i^{(i)} = \eta_i \quad (i = 1, \ldots, n),
$$

$$
\text{(4.10)} \qquad \mu_j^{(i)} = \gamma_j \quad (i \gg 1, j = 1, \ldots, n).
$$

PROPOSITION 4.1

The following formulas define a representation of $\mathfrak{gl}^-_{\infty/2}$ on $\mathcal{Y}^-_{\eta, \gamma}$:

$$
\text{(4.11)} \qquad
\begin{aligned}
E_{-i,-i+1}|\mu\rangle &= \sum_{j=1}^{n} |\mu + 1_j^{(i+j)}\rangle c_{i+j,j}^+(\mu), \\
E_{-i+1,-i}|\mu\rangle &= \sum_{j=1}^{n} |\mu - 1_j^{(i+j)}\rangle c_{i+j,j}^-(\mu) \quad (i \geq 1),
\end{aligned}
$$

$$
\text{(4.12)} \qquad E_{-i,-i}|\mu\rangle = \sum_{j=1}^{n} (\mu_j^{(i+j)} - \mu_j^{(i+1+j)})|\mu\rangle \quad (i \geq 0),
$$

where

$$
c_{i+j,j}^\pm(\mu) = \pm \frac{\prod_{k=1}^{n}(\ell_j^{(i+j)} - \ell_k^{(i\mp 1+k)})}{\prod_{1 \leq k(\neq j) \leq n}(\ell_j^{(i+j)} - \ell_k^{(i+k)})},
$$

and $\ell_j^{(i+j)}$ is defined by (4.6).

Proof
Clearly the operators (4.11) and (4.12) preserve the space $\mathcal{Y}^-_{\eta, \gamma}$. Given i, take N so that $N > n + i + 1$. Then, under the condition (4.8), the formulas (4.3)–

(4.5) reduce to (4.11) and (4.12) after making a base change of the form $|\mu\rangle = f(\mu)|\mu\rangle_{(N)}$. Consider (4.3). The range of summation $1 \le j \le N - i$ reduces to the fact that in (4.11) $1 \le j \le n$ because $\mu_j^{(i+j)} = 0$ is unchanged. We see also that the coefficient $f(\mu)$ can be used to satisfy $f(\mu + 1_j^{(i+j)}) = f(\mu)(\ell_j^{(i+j)} + N - i)$, which can be solved easily. Hence the commutation relations of the generators are obviously satisfied. $\qquad\square$

PROPOSITION 4.2

If $\gamma_1 = \cdots = \gamma_n$, then $\mathcal{Y}_{\eta,\gamma}^-$ is an irreducible $\mathfrak{gl}_{\infty/2}^-$-module.

Proof
For $N > n$, consider the subspace of $\mathcal{Y}_{\eta,\gamma}^-$,

$$W_N = \text{span}\big\{|\mu\rangle \in \mathcal{Y}_{\eta,\gamma}^- \mid \mu_j^{(i)} = \gamma_j \ (i > N, j = 1, \dots, n)\big\}.$$

Because of the restriction $\gamma_1 = \cdots = \gamma_n$, the subspace W_N is invariant under the action of $\mathfrak{g}_{-N+1,0}$. The vector $|\mu\rangle \in W_N$ defined by $\mu_j^{(i)} = \eta_i$ $(1 \le i \le n)$ and $\mu_j^{(i)} = \gamma_1$ $(i > n)$ is a $\mathfrak{g}_{-N+1,0}$-singular vector with the lowest weight $(\theta_{-N+1}, \dots, \theta_0) = (0, \dots, 0, \eta_n - \gamma_1, \dots, \eta_1 - \gamma_1)$. Moreover, W_N has the same dimension as that of the irreducible lowest weight module of the same lowest weight. To see this one can rearrange the table as the usual GT pattern:

$$
\begin{array}{ccccccccc}
\eta_1 & & \cdots & \eta_n & & \gamma_1 & & \cdots & \gamma_1 \\
& \mu_1^{(2)} & & & \mu_n^{(n+1)} & & \ddots & & \\
& & \ddots & & & \ddots & & \gamma_1 & \\
& & & & & & \mu_n^{(N)} & & \\
& & & \ddots & & & & & \\
& & & & \mu_1^{(N)} & & & &
\end{array}
$$

Hence W_N is $\mathfrak{g}_{-N+1,0}$-irreducible.

By the definition we have $\mathcal{Y}_{\eta,\gamma}^- = \bigcup_{N>n} W_N$. The irreducibility follows from this. $\qquad\square$

By applying the involutive automorphism $\sigma(E_{i,j}) = -E_{1-j,1-i}$ of $\mathfrak{gl}_\infty$, we obtain representations of the subalgebra $\mathfrak{gl}_{\infty/2}^+ = \sigma(\mathfrak{gl}_{\infty/2}^-)$. For later reference let us write the relevant formulas for $\mathfrak{gl}_{\infty/2}^+$.

For $\mathfrak{g}_{\infty/2}^+$, we use the transposed GZ patterns $\mu = (\mu_j^{(i)})_{1 \le i \le j}$ of depth n,

$$
(4.13) \qquad \mu =
\begin{array}{ccccc}
\mu_1^{(1)} & \cdots & \mu_n^{(1)} & \mu_{n+1}^{(1)} & \mu_{n+2}^{(1)} & \cdots \\
& \ddots & & \vdots & \vdots & \\
& & \mu_n^{(n)} & \mu_{n+1}^{(n)} & \mu_{n+2}^{(n)} & \cdots \\
& & & 0 & 0 & \cdots \\
& & & & 0 & \cdots
\end{array}
$$

Let $\eta = (\eta_1, \ldots, \eta_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ be partitions such that $\eta_i \geq \alpha_i$, $i = 1, \ldots, n$. We set $\alpha_i = 0$ for $i > n$. Let $\mathcal{Y}_{\eta,\alpha}^+$ be the vector space with basis $\{|\mu\rangle\}$ where the $\mu = (\mu_j^{(i)})_{1 \leq i \leq j}$ satisfy $\mu_j^{(i)} = 0$ if $i > n$ and

$$\mu_i^{(i)} = \eta_i \quad (i = 1, \ldots, n),$$
$$\mu_j^{(i)} = \alpha_i \quad (i = 1, \ldots, n, j \gg 1).$$

PROPOSITION 4.3

The following formulas define a representation of $\mathfrak{gl}_{\infty/2}^+$ on $\mathcal{Y}_{\eta,a}^+$:

$$(4.14) \qquad E_{i,i+1}|\mu\rangle = \sum_{j=1}^n |\mu + 1_{i+j}^{(j)}\rangle c_{j,i+j}^+(\mu),$$

$$E_{i+1,i}|\mu\rangle = \sum_{j=1}^n |\mu - 1_{i+j}^{(j)}\rangle c_{j,i+j}^-(\mu),$$

$$(4.15) \qquad E_{i,i}|\mu\rangle = \left(\sum_{j=1}^n \mu_{i+j}^{(j)} - \sum_{j=1}^n \mu_{i-1+j}^{(j)} - n \right)|\mu\rangle,$$

where $i \geq 1$, and

$$c_{j,i+j}^\pm(\mu) = \mp \frac{\prod_{k=1}^n (\ell_{i+j}^{(j)} - \ell_{i\mp1+k}^{(k)})}{\prod_{1 \leq k(\neq j) \leq n} (\ell_{i+j}^{(j)} - \ell_{i+k}^{(k)})}, \quad \ell_{i+j}^{(j)} = \mu_{i+j}^{(j)} - j + 1.$$

We note that it is always possible to twist a given representation by changing $E_{i,j}$ to $E_{i,j} + x\delta_{i,j} \cdot \mathrm{id}$ for some $x \in \mathbb{C}$. Utilizing this freedom we have chosen $x = -n$ in (4.15), which will be convenient in the next subsection.

4.3. Representations of $\mathfrak{gl}_\infty$

In this subsection, we glue together the representations of $\mathfrak{gl}_{\infty/2}^\pm$ to define representations of the full algebra $\mathfrak{gl}_\infty$. Consider now a GZ pattern $\mu = (\mu_j^{(i)})_{i,j \geq 1}$ such that $\mu_j^{(i)} = 0$ if $i > n$ and $j > n$, that is,

$$(4.16) \qquad \mu = \begin{matrix}
\mu_1^{(1)} & \cdots & \mu_n^{(1)} & \mu_{n+1}^{(1)} & \mu_{n+2}^{(1)} & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \\
\mu_1^{(n)} & \cdots & \mu_n^{(n)} & \mu_{n+1}^{(n)} & \mu_{n+2}^{(n)} & \cdots \\
\mu_1^{(n+1)} & \cdots & \mu_n^{(n+1)} & 0 & 0 & \cdots \\
\mu_1^{(n+2)} & \cdots & \mu_n^{(n+2)} & 0 & 0 & \cdots \\
\vdots & \cdots & \vdots & 0 & 0 & \cdots
\end{matrix}$$

We say that μ has a hook shape of width n. We assume further that

$$(4.17) \qquad \mu_j^{(i)} = \alpha_i \quad \text{for } i = 1, \ldots, n, \ j \gg 1,$$

$$(4.18) \qquad \mu_j^{(i)} = \gamma_j \quad \text{for } j = 1, \ldots, n, \ i \gg 1.$$

We are guided by the condition (1.1) for the basis in the case of a quotient of the MacMahon $\mathcal{E}$-module $\mathcal{M}_{\alpha,\emptyset,\gamma}(u,(q_2 q_3)^n)$.

Let $\mathcal{Y}_{\alpha,\gamma}$ be the vector space with basis $\{|\mu\rangle\}$, where μ runs over hook-shaped GZ patterns of width n, satisfying (4.17) and (4.18).

PROPOSITION 4.4

Define

$$(4.19) \quad E_{0,1}|\mu\rangle = \sum_{j=1}^{n} |\mu + 1_j^{(j)}\rangle c_{j,j}^+(\mu), \qquad E_{1,0}|\mu\rangle = \sum_{j=1}^{n} |\mu - 1_j^{(j)}\rangle c_{j,j}^-(\mu),$$

$$(4.20) \quad \begin{aligned} c_{j,j}^+(\mu) &= \frac{1}{\prod_{k(\neq j)}(\ell_j^{(j)} - \ell_k^{(k)})}, \\ c_{j,j}^-(\mu) &= -\frac{\prod_{k=1}^{n}(\ell_j^{(j)} - \ell_k^{(k+1)})(\ell_j^{(j)} - \ell_{k+1}^{(k)})}{\prod_{1\le k(\neq j)\le n}(\ell_j^{(j)} - \ell_k^{(k)})}, \end{aligned}$$

with

$$\ell_k^{(k)} = \mu_k^{(k)} - k + 1.$$

Then the above formulas along with (4.11), (4.12), (4.14), and (4.15) give a representation of $\mathfrak{gl}_\infty$ on $\mathcal{Y}_{\alpha,\gamma}$.

Proof

The only nontrivial relations to check are $[E_i, F_j] = \delta_{i,j} H_i$ for $i = 0$ or $j = 0$, and the Serre relations involving them. First we check the former. For the relations $[E_0, F_i] = [E_i, F_0] = 0$ to hold for $i \neq 0$, we must have

$$\frac{c_{j,j}^+(\mu + 1_{1+k}^{(k)})}{c_{j,j}^+(\mu)} = \frac{\ell_{1+k}^{(k)} - \ell_j^{(j)} + 1}{\ell_{1+k}^{(k)} - \ell_j^{(j)}},$$

$$\frac{c_{j,j}^+(\mu + 1_k^{(1+k)})}{c_{j,j}^+(\mu)} = \frac{\ell_k^{(1+k)} - \ell_j^{(j)} + 1}{\ell_k^{(1+k)} - \ell_j^{(j)}}$$

and have $c_{j,j}^\pm(\mu \pm 1_{i+k}^{(k)}) = c_{j,j}^\pm(\mu \pm 1_k^{(i+k)}) = c_{j,j}^\pm(\mu)$ in all other cases. These relations can be verified using (4.20).

A similar calculation shows that, in $[E_0, F_0]|\mu\rangle$, all terms cancel except

$$\sum_{j=1}^{n} \left(c_{j,j}^-(\lambda) c_{j,j}^+(\mu + 1_j^{(j)}) - c_{j,j}^+(\mu) c_{j,j}^-(\mu - 1_j^{(j)}) \right) |\mu\rangle.$$

Substituting (4.20) we find that the coefficient in front of $|\mu\rangle$ can be written as

$$-\sum_{j=1}^{n} \left(\operatorname*{res}_{z=\ell_j^{(j)}} + \operatorname*{res}_{z=\ell_j^{(j)}-1} \right) \prod_{k=1}^{n} \frac{z - \ell_{k+1}^{(k)} + 1}{z - \ell_k^{(k)} + 1} \frac{z - \ell_k^{(k+1)} + 1}{z - \ell_k^{(k)}}$$

$$= \sum_{k=1}^{n} (\ell_{k+1}^{(k)} + \ell_k^{(k+1)} - 2\ell_k^{(k)}) - n.$$

Comparing this with $H_0 = E_{0,0} - E_{1,1}$ we obtain $[E_0, F_0]|\mu\rangle = H_0|\mu\rangle$.

Finally the Serre relations involving E_0 or F_0 can be checked by a tedious but straightforward calculation. $\qquad\square$

PROPOSITION 4.5

If $\gamma_1 = \cdots = \gamma_n = c$, $\mathcal{Y}_{\alpha,\gamma}$ is an irreducible $\mathfrak{gl}_\infty$-module with the lowest weight vector $|\mu^{(n)}(\alpha,c)\rangle$, where

$$\mu^{(n)}(\alpha,c)_j^{(i)} = \begin{cases} \max(\alpha_i, c) & (1 \le i, j \le n), \\ \alpha_i & (1 \le i \le n, j > n), \\ c & (i > n, 1 \le j \le n). \end{cases}$$

If $\alpha_1 \ge \cdots \ge \alpha_k \ge c \ge \alpha_{k+1} \ge \cdots \ge \alpha_n$, then $\mathcal{Y}_{\alpha,\gamma}$ is isomorphic to the irreducible lowest weight $\mathfrak{gl}_\infty$-module $\mathcal{W}_{\theta^{(n)}(\alpha,c)}$, with the lowest weight

$$(4.21) \qquad \theta^{(n)}(\alpha,c)_i = \begin{cases} 0 & (i \le -k), \\ \alpha_{-i+1} - c & (-k+1 \le i \le 0), \\ \alpha_{n-i+1} - c - n & (1 \le i \le n-k), \\ -n & (i \ge n-k+1). \end{cases}$$

Proof

For $N \ge 0$, consider the Lie subalgebra $\mathfrak{a}_N = \mathfrak{g}_{-\infty,N}$ spanned by $E_{i,j}$ with $i,j \le N$. For each partition $\eta = (\eta_1, \ldots, \eta_n)$ such that $\eta_i \ge c$ if $i + N \le n$, we consider the $\mathfrak{a}_N$-module $\mathfrak{X}_{N,\eta}$ given as follows. As a linear space, it is spanned by vectors $|\mu\rangle$ where $\mu = (\mu_j^{(i)})_{j \le i+N}$ runs over GZ patterns of width n such that

$$\mu_{N+j}^{(j)} = \eta_j \quad \text{for } j = 1, \ldots, n,$$

$$\mu_j^{(i)} = c \quad \text{for } 1 \le j \le n, \ i \gg 1.$$

The action of the generators of $\mathfrak{a}_N$ is defined by the same formulas as (4.3)–(4.15). We show that $\mathfrak{X}_{N,\eta}$ is an irreducible $\mathfrak{a}_N$-module for all N and η. The irreducibility of $\mathcal{Y}_{\alpha,\gamma}$ is a simple consequence of this assertion.

For $N = 0$ we have $\mathfrak{a}_0 = \mathfrak{gl}_{\infty/2}^-$, and $\mathfrak{X}_{0,\eta} = \mathcal{Y}_{\eta,c}^-$ is an irreducible $\mathfrak{a}_0$-module by Proposition 4.2. Assume by induction that each $\mathfrak{X}_{N-1,\xi}$ is $\mathfrak{a}_{N-1}$-irreducible for $N > 0$. By the definition, we have a direct sum decomposition into subspaces $\mathfrak{X}_{N,\eta} = \bigoplus_\xi \mathfrak{X}_{N-1,\xi}$, where $\xi_1 \ge \eta_1 \ge \xi_2 \ge \eta_2 \ge \cdots \ge \xi_n \ge \eta_n$. Each $\mathfrak{X}_{N-1,\xi}$ is an irreducible $\mathfrak{a}_{N-1}$-module, which is mutually inequivalent. Therefore, if $W \subset \mathfrak{X}_{N,\eta}$ is a nonzero $\mathfrak{a}_N$-submodule, then we have $W = \bigoplus_\xi W \cap \mathfrak{X}_{N-1,\xi}$ as an $\mathfrak{a}_{N-1}$-module. If $W \cap \mathfrak{X}_{N-1,\xi} \ne 0$, then acting with E_{N-1}, F_{N-1} we obtain that $W \cap \mathfrak{X}_{N-1,(\xi_1,\ldots,\xi_i \pm 1,\ldots,\xi_n)} \ne 0$ for each i, as long as the condition $\eta_{i-1} \ge \xi_i \pm 1 \ge \eta_i$ is not violated. It is now easy to see that $W = \mathfrak{X}_{N,\eta}$. The proof is over. $\qquad\square$

Note that the lowest weight $\theta^{(n)}(\alpha,c)_i$ is increasing, that is, "antidominant," except

$$\theta^{(n)}(\alpha,c)_0 = \alpha_1 - c \geq 0 > -n \geq \theta^{(n)}(\alpha,c)_1 = \alpha_n - c - n.$$

4.4. Degeneration of the algebra $\mathcal{E}$ and $\mathfrak{gl}_\infty$

In this subsection we examine the degeneration of the algebra $\mathcal{E}$ and its modules when one of the parameters q_i tends to 1.

In order to discuss the limit, it is convenient to introduce the elements $h_m \in \mathcal{E}_{q_1,q_2,q_3}$ $(m \neq 0)$ via

$$\psi^\pm(z) = \psi_0^\pm \exp\Big(\mp \sum_{\pm m > 0} \frac{\gamma_m}{m} h_m z^{-m}\Big), \quad \gamma_m = \prod_{i=1}^3 (1 - q_i^m).$$

We have $[h_m, e(z)] = z^m e(z)$, $[h_m, f(z)] = -z^m f(z)$. Further set $\psi_0^+ = q_1^{\kappa_+}$, $\psi_0^- = q_1^{\kappa_-}$. In the limit

(4.22) $$q_1 \to 1, \qquad q_2 \to q, \qquad q_3 \to q^{-1} \quad (q \in \mathbb{C}^\times),$$

the algebra $\mathcal{E}_{q_1,q_2,q_3}$ reduces to the Lie algebra $\mathfrak{d}_{q,\kappa,0}$, which has been mentioned already. The algebra $\mathfrak{d}_{q,\kappa,0}$ is the associative algebra (viewed as a Lie algebra) generated by $Z^{\pm 1}, D^{\pm 1}$ with $DZ = qZD$, extended by a central element κ:

$$[Z^{i_1}D^{j_1}, Z^{i_2}D^{j_2}] = (q^{j_1 i_2} - q^{j_2 i_1})Z^{i_1+i_2}D^{j_1+j_2} + i_1 q^{-i_1 j_1}\delta_{i_1+i_2,0}\delta_{j_1+j_2,0} \cdot \kappa.$$

Writing the limit of the generators e_m, f_m, h_m with bars, we have the identification

(4.23) $$(1-q)\bar{e}_m = D^m Z, \qquad -(1-q^{-1})\bar{f}_m = Z^{-1}D^m,$$

(4.24) $$(1-q^{-m})\bar{h}_m = D^m \quad (m \neq 0), \qquad \kappa_+ - \kappa_- = \kappa.$$

Let $\mathfrak{gl}_{\infty,\kappa}$ be the Lie algebra defined by the symbols $: E_{i,j} :$ and a central element κ, with relations

$$\mathfrak{gl}_{\infty,\kappa} = \Big\{ \sum_{i,j \in \mathbb{Z}} a_{i,j} : E_{i,j} : \Big|\ \exists N > 0,\ a_{i,j} = 0 \text{ for } |i-j| > N \Big\} \oplus \mathbb{C}\kappa,$$

$$\Big[\sum_{i,j} a_{i,j} : E_{i,j} :, \sum_{k,l} b_{k,l} : E_{k,l} :\Big] = \sum \Big(\sum_k a_{i,k}b_{k,j} - \sum_k b_{i,k}a_{k,j}\Big) : E_{i,j} :$$

$$+ \Big(\sum_{i \leq 0 < j} a_{i,j}b_{j,i} - \sum_{i > 0 \geq j} a_{i,j}b_{j,i}\Big)\kappa.$$

It is straightforward to verify that the map

(4.25) $$D^m Z \mapsto \sum_{i \in \mathbb{Z}} : E_{i,i+1} : u^m q^{-im}, \qquad Z^{-1}D^m \mapsto \sum_{i \in \mathbb{Z}} : E_{i+1,i} : u^m q^{-im},$$

(4.26) $$D^m \mapsto \sum_{i \in \mathbb{Z}} : E_{i,i} : u^m q^{-im} + \frac{u^m}{1 - q^m}\kappa \quad (m \neq 0), \qquad \kappa \mapsto \kappa$$

gives an embedding of Lie algebras

$$\iota_u \ : \ \mathfrak{d}_{q,\kappa} \longrightarrow \mathfrak{gl}_{\infty,\kappa}.$$

Here u is an arbitrary nonzero complex number.

We view $\mathfrak{gl}_\infty$ as a subalgebra of $\mathfrak{gl}_{\infty,\kappa}$ by $E_{i,j} \mapsto: E_{i,j} : -\delta_{i,j}\theta(i > 0)\kappa$, where $\theta(i > 0) = 1$ if $i > 0$ and 0 otherwise. The action of $\mathfrak{gl}_\infty$ on $\mathcal{Y}_{\alpha,\gamma}$ can be extended to that of $\mathfrak{gl}_{\infty,\kappa}$, since when acting with the latter on GZ patterns of hook shape only finitely many terms are produced.

Now let us turn to the MacMahon module $\mathcal{M}_{\alpha,\beta,\gamma}(u, K)$. In the limit (4.22), the eigenvalues (3.19) of $\psi^\pm(z)$ tend to

$$\frac{1 - K_1 u/z}{1 - q^{-\beta_1} u/z} \prod_{i=1}^\infty \frac{1 - q^{i-\beta_i} u/z}{1 - q^{i-\beta_{i+1}} u/z},$$

where K_1 denotes the limiting value of K. In order for this limit to be 1, we are forced to take $\beta_i = 0$ for all i and $K_1 = 1$. Assuming this, consider the action of $e(z)$, $f(z)$, which we write in the form

$$e(z)|\boldsymbol{\lambda}\rangle = \sum_{i,k=1}^\infty C_{i,k}^+(\boldsymbol{\lambda})|\boldsymbol{\lambda} + 1_i^{(k)}\rangle, \qquad f(z)|\boldsymbol{\lambda}\rangle = \sum_{i,k=1}^\infty C_{i,k}^-(\boldsymbol{\lambda})|\boldsymbol{\lambda} - 1_i^{(k)}\rangle.$$

Let us compute the action of $e(z)$ in the limit $q_1 \to 1$. We use (3.9) in the form

$$e(z)|\boldsymbol{\lambda}\rangle = \sum_{i,k=1}^\infty \frac{1}{1 - q_1} \psi_{\lambda^{(k)},i} \psi_{\boldsymbol{\lambda}}^{(k-1)}(u/z)\delta(q_1^{\lambda_i^{(k)}} q_3^{i-1} u_k/z)|\boldsymbol{\lambda} + 1_i^{(k)}\rangle,$$

where $\lambda_i^{(k)} = \mu_i^{(k)} - \alpha_k$ and $u_k = u q_1^{\alpha_k} q_2^{k-1}$. We have for $q_1 \to 1$,

$$\frac{1}{1 - q_1} \psi_{\lambda^{(k)},i} = \begin{cases} O(1) & (i \neq 1), \\ O\!\left(\frac{1}{1-q_1}\right) & (i = 1), \end{cases}$$

$$\psi_{\boldsymbol{\lambda}}^{(k-1)}(u/z)\delta(q_1^{\lambda_i^{(k)}} q_3^{i-1} u_k/z) = \begin{cases} O(1) & (i \neq 1), \\ O(1 - q_1) & (i = 1), \end{cases}$$

so that

$$C_{i,k}^+(\boldsymbol{\lambda}) = O(1).$$

Similarly, using

$$f(z)|\boldsymbol{\lambda}\rangle = \sum_{i,k=1}^\infty \frac{q_1}{1 - q_1} \psi_{\lambda'^{(k)},i} \psi_{\boldsymbol{\lambda}}^{'(k+1)}(u/z)\delta(q_1^{\lambda_i^{(k)}} q_3^{i-1} u_k/z)|\boldsymbol{\lambda} + 1_i^{(k)}\rangle,$$

we find

$$C_{i,k}^-(\boldsymbol{\lambda}) = O(1).$$

Hence, the matrix coefficients for $e(z)$, $f(z)$ have well-defined limits. Clearly the same is true about the quotient module $\mathcal{N}_{\alpha,\emptyset,\gamma}^{n,n}(u)$.

Thus we have shown the first part of the following.

PROPOSITION 4.6

(i) If $\beta = \emptyset$, then the MacMahon module $\mathcal{M}_{\alpha,\emptyset,\gamma}(u, q_2^n q_3^n)$ and its irreducible quotient $\mathcal{N}_{\alpha,\emptyset,\gamma}^{n,n}(u)$ have well-defined limits.

(ii) *Assume* $\gamma_1 = \cdots = \gamma_n = c$, $\kappa = n$. *Then as* $\mathfrak{d}_{q,\kappa,0}$-*modules, the limits of* $\mathcal{N}_{\alpha,\emptyset,\gamma}^{n,n}(u)$ *are isomorphic to the pullback* $\iota_u^*(\mathcal{W}_{\theta^{(n)}(\alpha,c)})$ *of the* $\mathfrak{gl}_{\infty,\kappa}$-*module given in (4.21).*

Proof

Let us show (ii). By construction, the limits of $\mathcal{N}_{\alpha,\emptyset,\gamma}^{n,n}(u)$ and $\iota_u^*(\mathcal{W}_{\theta^{(n)}(\alpha,c)})$ both have bases labeled by the same combinatorial set, the GZ pattern of hook type. It is easy to see that $\iota_u^*(\mathcal{W}_{\theta^{(n)}(\alpha,c)})$ is an irreducible $\mathfrak{d}_{q,\kappa,0}$-module. Hence it is sufficient to check that the lowest weights are the same.

The limit of $(\psi^+(z) - \psi^-(z))/(1 - q_1)$ gives the eigenvalues of $\bar{h}_m$, which in turn gives those of : $E_{i,i}$: via (4.25), (4.26) and (4.23), (4.24). Denoting by θ_i the eigenvalues of $E_{i,i}$ we find

$$(4.27) \qquad \theta_i = \begin{cases} d_{1,-i+1} - \sum_{j=1}^{\infty}(d_{j,j-i+1} - d_{j+1,j-i+1}) & (i \leq 0), \\ \kappa_- - \sum_{j=1}^{\infty}(d_{j+i-1,j} - d_{j+i,j}) & (i > 0), \end{cases}$$

where $d_{i,j} = \max(\gamma_i, \alpha_j)$ and $K = q_1^{\kappa_-}$. In the case $\kappa_- = -n$, $\alpha_j = \gamma_j = 0$ $(j > n)$, if $\gamma_1 = \cdots = \gamma_n = c$ and $\alpha_1 \geq \cdots \geq \alpha_k \geq c \geq \alpha_{k+1} \geq \cdots \geq \alpha_n$, this reduces to formula (4.21). $\qquad\square$

5. Characters

All $\mathcal{E}$-modules in this paper are graded by the convention $\deg e_i = -\deg f_i = 1$, $\deg \psi_i^{\pm} = 0$. Computation of the characters of $\mathcal{N}_{\alpha,\beta,\gamma}^{m,n}$ is a very interesting and challenging problem. It appears that in a lot of cases there are many seemingly unrelated highly nontrivial formulae.

In this section we compute the characters of $\mathcal{N}_{\alpha,\emptyset,\emptyset}^{n,n}(u)$. Note that $\gamma = \emptyset$. In this case, by Proposition 4.6, our problem is equivalent to computing the characters of $\mathfrak{gl}_{\infty}$-modules $W_{\theta^{(n)}(\alpha,0)}$ with the degree defined by $\deg E_{ij} = j - i$. We also present several conjectures at the end.

5.1. Bosonic construction

In this subsection we follow [KR2]. The main tool is the bosonic construction of $\mathfrak{gl}_{\infty}$-modules. We omit proofs when they are available in [KR2].

Let H be the algebra generated by generators d_i, d_i^*, $i \in \mathbb{Z}$ with defining relations

$$[d_i, d_j] = [d_i^*, d_j^*] = 0, \qquad [d_i^*, d_j] = \delta_{i,-j}.$$

Let U be the cyclic representation of the algebra H with the cyclic vector v satisfying

$$d_{i+1}v = d_i^* v = 0, \quad i \in \mathbb{Z}_{\geq 0}.$$

The following lemma is clear.

LEMMA 5.1

The module U is an irreducible H-module.

Introduce the notation

$$: d_i d^*_{-i} := \begin{cases} d_i d^*_{-i} & (i \le 0), \\ d^*_{-i} d_i & (i > 0). \end{cases}$$

Define an action of $\mathfrak{gl}_1 = \mathbb{C} \cdot e_{11}$ in U by

$$e_{11} = \sum_{i \in \mathbb{Z}} : d_i d^*_{-i} : .$$

Define an action of $\mathfrak{gl}_\infty$ in U by letting the generator E_{ij} act as

$$E_{ij} = d_i d^*_{-j}.$$

PROPOSITION 5.2

The actions of $\mathfrak{gl}_1$ and $\mathfrak{gl}_\infty$ in U commute. We have the decomposition of $\mathfrak{gl}_\infty$-modules

$$U = \bigoplus_{k \in \mathbb{Z}} W_{\theta^{(1)}(k,0)}.$$

Moreover, $W_{\theta^{(1)}(k,0)} = \{v \in U \mid e_{11}v = kv\}$. The module $W_{\theta^{(1)}(k,0)}$ is the irreducible lowest weight $\mathfrak{gl}_\infty$-module with the lowest weight $\theta^{(1)}(k,0)$ given by (4.21) and with the lowest weight vector $d_0^k v$ if $k > 0$ and $(d^*_{-1})^{-k} v$ if $k \le 0$.

Let $H_n = H^{\otimes n}$. We denote the generators of H_n by $d_i^{(k)}, d_i^{(k)*}$, $i \in \mathbb{Z}, k \in \{1, \dots, n\}$. We have

$$[d_i^{(k)}, d_j^{(l)}] = [d_i^{(k)*}, d_j^{(l)*}] = 0, \qquad [d_i^{(k)}, d_j^{(l)}] = \delta_{i,-j} \delta_{k,l}.$$

Then $U_n = U^{\otimes n}$ is naturally an H_n-module. By Lemma 5.1, U_n is an irreducible H_n-module. We set $v_n = v^{\otimes n}$.

Define an action of $\mathfrak{gl}_n$ in U_n by letting the matrix units e_{kl} act as

$$e_{kl} = \sum_{i \in \mathbb{Z}} : d_i^{(k)} d_{-i}^{(l)*} :, \quad k,l = 1, \dots, n.$$

Define an action of $\mathfrak{gl}_\infty$ in U_n by letting the matrix units E_{ij} act as

$$E_{ij} = \sum_{k=1}^n d_i^{(k)} d_{-j}^{(k)*}, \quad i, j \in \mathbb{Z}.$$

Proposition 5.4 below generalizes Proposition 5.2 to the case where $\mathfrak{gl}_1$ is replaced with $\mathfrak{gl}_n$. It was proved in [KR2] (see also [W]). It is a $\mathfrak{gl}_\infty$-version of the Schur–Weyl–Howe duality. The latter states the following.

PROPOSITION 5.3

Let N be an integer such that $N \ge n$. In the above setting consider the subspace

$$U_{n,N} = \mathbb{C}[d_i^{(k)}; 1 \le k \le n, -N+1 \le i \le 0]v_n \subset U_n.$$

We have mutually commutative actions of $\mathfrak{gl}_n$ and $\mathfrak{gl}_N \simeq \mathfrak{gl}_{-N+1,0}$ on $U_{n,N}$, and with respect to these actions, we have the decomposition

$$U_{n,N} = \bigoplus_{\alpha} L_\alpha \otimes \tilde{L}_\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$, $\alpha_i \in \mathbb{Z}$ such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$, and L_α (resp., $\tilde{L}_\alpha$) is the irreducible $\mathfrak{gl}_n$- (resp., $\mathfrak{gl}_N$) module with the highest weight $(\alpha_1, \ldots, \alpha_n)$ (resp., the lowest weight $(0, \ldots, 0, \alpha_n, \ldots, \alpha_1)$). The component $L_\alpha \otimes \tilde{L}_\alpha$ is generated by the cyclic vector

$$v_\alpha^{(n,N)} = \prod_{i=1}^{n-1} (D_i)^{\alpha_i - \alpha_{i+1}} (D_n)^{\alpha_n},$$

where D_i are given by

$$D_i = \det(d_{-j+1}^{(l)})_{j,l=1,\ldots,i}.$$

Similarly we define

$$D_i^* = \det(d_{-j}^{(n+1-l)*})_{j,l=1,\ldots,i}.$$

Now we give the duality statement for $\mathfrak{gl}_n$ and $\mathfrak{gl}_\infty$.

PROPOSITION 5.4

We have the decomposition

$$U_n = \bigoplus_{\alpha} (L_\alpha \otimes W_{\theta^{(n)}(\alpha,0)}),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ $(\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n, \alpha_i \in \mathbb{Z})$, and L_α is the irreducible $\mathfrak{gl}_n$-module with highest weight α and $W_{\theta^{(n)}(\alpha,0)}$ is the irreducible lowest weight $\mathfrak{gl}_\infty$-module with lowest weight $\theta^{(n)}(\alpha,0)$ given by (4.21). Moreover, $L_\alpha \otimes W_{\theta^{(n)}(\alpha,0)}$ is generated by

(5.1)
$$v_\alpha = \prod_{i=1}^{k(\alpha)-1} (D_i)^{\alpha_i - \alpha_{i+1}} (D_{k(\alpha)})^{\alpha_{k(\alpha)}}$$
$$\times \prod_{i=1}^{s(\alpha)-1} (D_i^*)^{\alpha_{n-i} - \alpha_{n-i+1}} (D_{s(\alpha)}^*)^{-\alpha_{n-s(\alpha)+1}} v_n,$$

where $k(\alpha)$, $s(\alpha)$ are the numbers of positive and negative parts of α, respectively.

5.2. Characters of $W_{\theta^{(1)}(k,0)}$

Consider the set

$$C_a = \{(\lambda, \mu) \mid \lambda, \mu\text{-partitions}, \mu_1 + a \geq \lambda_1\}.$$

Let

$$\bar{\chi}_a = \sum_{(\lambda,\mu) \in C_a} q^{|\lambda| + |\mu|}$$

be the corresponding formal character.

Set

$$(q)_\infty = \prod_{i=1}^{\infty}(1-q^i).$$

LEMMA 5.5

For $k \in \mathbb{Z}_{\geq 0}$ we have the recursive relation

$$\bar{\chi}_k(q) + q^{k+1}\bar{\chi}_{k+1}(q) = \frac{1}{(q)_\infty^2}.$$

Proof
We construct a map

$$\iota_k : C_k \sqcup C_{k+1} \to C_\infty := \big\{(\lambda,\mu) \,\big|\, \lambda,\mu\text{-partitions}\big\}$$

as follows. For $(\lambda,\mu) \in C_k$ we set $\iota_k(\lambda,\mu) = (\mu,\lambda)$. For $(\lambda,\mu) \in C_{k+1}$ we set $\iota_k(\lambda,\mu) = (\tilde{\mu},\tilde{\lambda})$, where

$$\tilde{\lambda} = (\mu_1 + k + 1, \lambda_1, \lambda_2, \lambda_3 \dots), \qquad \tilde{\mu} = (\mu_2, \mu_3, \mu_4, \dots).$$

Clearly, ι_k is a bijection. The lemma follows.   □

COROLLARY 5.6

For $k \in \mathbb{Z}_{\geq 0}$ we have

$$\bar{\chi}_k = \frac{1}{(q)_\infty^2} \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2+jk}.$$

Proof
Repeating the use of Lemma 5.5, we compute

$$\bar{\chi}_k(q) = \frac{1}{(q)_\infty^2} - q^{k+1}\bar{\chi}_{k+1}(q) = \frac{1}{(q)_\infty^2} - q^{k+1}\Big(\frac{1}{(q)_\infty^2} - q^{k+2}\bar{\chi}_{k+2}(q)\Big) = \cdots.$$

Continuing, we obtain the corollary.   □

For $k \in \mathbb{Z}_{\geq 0}$, set

$$\chi_k = \bar{\chi}_k, \qquad \chi_{-k} = q^k \bar{\chi}_k.$$

We set $\deg d_i = \deg d_i^* = -i$, $\deg v = 0$.

COROLLARY 5.7

For $k \in \mathbb{Z}$, we have

$$\chi(W_{\theta^{(1)}(k,0)}) = \chi_k.$$

Proof
By Proposition 4.5, for $k \in \mathbb{Z}_{\geq 0}$ the modules $W_{\theta^{(1)}(k,0)}$ and $W_{\theta^{(1)}(-k,0)}$ have bases parameterized by the set C_k; therefore the corollary follows.   □

5.3. Characters of $W_{\theta^{(n)}(\alpha,0)}$

Set $\deg d_i^{(j)} = \deg d_i^{(j)*} = -i$, $\deg v_n = 0$. Proposition 5.4 allows us to compute the character of $W_{\theta(\alpha,0)}$ in terms of χ_k.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ $(\alpha_1 \geq \cdots \geq \alpha_n, \alpha_i \in \mathbb{Z})$, let $k(\alpha)$ be the number of positive parts of α. We set

$$p(\alpha) = \sum_{i=1}^{k(\alpha)} (i-1)\alpha_i + \sum_{i=1}^{n-k(\alpha)} i\alpha_{n-i+1}.$$

Then $p(\alpha)$ is the degree of the singular vector v_α given by (5.1).

Recall that the $\mathfrak{gl}_n$-weight ρ is given by

$$\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2} \right)$$

and that the symmetric group S_n acts on the $\mathfrak{gl}_n$ weights by simply permuting the indexes.

THEOREM 5.8

We have

$$q^{p(\alpha)} \chi(W_{\theta(\alpha,0)}) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \chi_{(\sigma(\alpha+\rho)-\rho)_i}.$$

Proof

The character of the subspace of vectors in U_n of the $\mathfrak{gl}_n$-weight $\mu = (\mu_1, \ldots, \mu_n)$ is obviously given by $\prod_{i=1}^n \chi_{\mu_i}$. By Proposition 5.4, the space $W_{\theta(\alpha,0)}$ is identified with the space of $\mathfrak{gl}_n$-singular vectors of weight α in U_n with the shift of the degree given by $p(\alpha)$. Moreover, Proposition 5.4 asserts that U_n is a direct sum of finite-dimensional $\mathfrak{gl}_n$-modules. Then the character of the space of $\mathfrak{gl}_n$-singular vectors of weight α is computed as the alternating sum of the characters of the weight subspaces. $\qquad\square$

5.4. Other character formulas

We finish with some conjectures which we checked for the small values of parameters.

CONJECTURE 5.9

The character of $\mathcal{N}_{0,0,0}^{(1,m)}$ is given by

$$\chi(\mathcal{N}_{0,0,0}^{(1,m)}) = \frac{\prod_{i=1}^{m-2}(1-q^i)^{m-i-1}}{(q)_\infty^{m+1}} \sum_{j=0}^{\infty} (-1)^j q^{(1/2)j(j+1)} \prod_{i=1}^{m-1}(1-q^{i+j}).$$

CONJECTURE 5.10

The character of $\mathcal{N}_{0,0,0}^{(n,m)}$, where $n \geq m$, is given by

$$\chi(\mathcal{N}_{0,0,0}^{(n,m)}) = \frac{1}{(q)_\infty^{m+n}} \sum_{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0} (-1)^{\sum_{i=1}^m \lambda_i} q^{(1/2)\sum_{i=1}^m (\lambda_i^2 + (2i-1)\lambda_i)}$$

$$\times \prod_{1\leq i<j\leq m}(1-q^{\lambda_i-\lambda_j+j-i}) \prod_{1\leq i<j\leq n}(1-q^{\lambda_i-\lambda_j+j-i}).$$

Here we set $\lambda_j = 0$ if $j > m$.

Acknowledgments. The present work was carried out during the visits of Feigin and Mukhin to Kyoto University. They wish to thank the university for hospitality.

References

[FFJMM1]   B. Feigin, E. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Quantum continuous* $\mathfrak{gl}_\infty$*: Semiinfinite construction of representations*, Kyoto J. Math. **51** (2011), 337–364.

[FFJMM2]   ———, *Quantum continuous* $\mathfrak{gl}_\infty$*: Tensor products of Fock modules and* $\mathcal{W}_n$*-characters*, Kyoto J. Math. **51** (2011), 365–392.

[FHHS]   B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, and S. Yanagida, *A commutative algebra on degenerate* $\mathbb{C}P^1$ *and Macdonald polynomials*, J. Math. Phys. **50** (2009), no. 095215.

[FO]   B. Feigin and A. Odesskii, "Vector bundles on an elliptic curve and Sklyanin algebras" in *Topics in Quantum Groups and Finite-Type Invariants*, Amer. Math. Soc. Transl. Ser. 2 **185**, Amer. Math. Soc., Providence, 1998, 65–84.

[FT]   B. Feigin and A. Tsymbaliuk, *Heisenberg action in the equivariant K-theory of Hilbert schemes via shuffle algebra*, Kyoto J. Math. **51** (2011), 831–854.

[KR1]   V. Kac and A. Radul, *Quasifinite highest weight modules over the Lie algebra of differential operators on the circle*, Comm. Math. Phys. **157** (1993), 429–457.

[KR2]   ———, *Representation theory of the vertex algebra* $\mathcal{W}_{1+\infty}$, Transform. Groups **1** (1996), 41–70.

[Mi]   K. Miki, *A* (q,γ) *analog of the* $W_{1+\infty}$ *algebra*, J. Math. Phys. **48** (2007), no. 123520.

[S]   O. Schiffmann, *Drinfeld realization of the elliptic Hall algebra*, J. Algebraic Combin. **35** (2012) 237–262.

[SV1]   O. Schiffmann and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compos. Math. **147** (2011), 188–234.

[SV2]   ———, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of* $\mathbb{A}^2$, preprint, arXiv:0905.2555v3 [math.QA]

[ORV]   A. Okounkov, N. Reshetikhin, and C. Vafa, "Quantum Calabi-Yau and classical crystals" in *The Unity of Mathematics*, Progr. Math. **244**, Birkhäuser, Boston, 2006, 597–618.

[W]   W. Wang, *Duality in infinite dimensional Fock representations*,
          Commun. Contemp. Math. **1** (1999), 155–199.

Feigin: Landau Institute for Theoretical Physics, Chernogolovka, 142432, Russia
Higher School of Economics, Moscow, 101000, Russia and Independent University of
Moscow, Moscow, 119002, Russia; bfeigin@gmail.com

Jimbo: Department of Mathematics, Rikkyo University, Toshima-ku, Tokyo 171-8501,
Japan; jimbomm@rikkyo.ac.jp

Miwa: Department of Mathematics, Graduate School of Science, Kyoto University,
Kyoto 606-8502, Japan; tmiwa@kje.biglobe.ne.jp

Mukhin: Department of Mathematics, Indiana University-Purdue University
Indianapolis, Indianapolis, Indiana 46202, USA; mukhin@math.iupui.edu