On the homotopy types of the independence complexes of grid graphs with cylindrical identification

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Abstract We herein investigate the homotopy types of the independence complexes of square grid graphs $C_{m,n}$ with cylindrical identification for $m=3$ and $n=6,7$. We show that they are all homotopy equivalent to a wedge of spheres.

1. Introduction

Let $G=(V,E)$ be a graph without loops, where V denotes the vertex set of $G=(V,E)$ and E is defined as a subset of $V\times V$ such that $(u,v)\in E$ if u and v are adjacent. A set of vertices $\sigma\subset V$ is said to be *independent* if for all $u,v\in\sigma$ we have $(u,v)\notin E$. The *independence complex* $I(G)$ of G is a simplicial set for which the set of vertices is V and the simplices are all the independent sets of G. For any simplicial set Δ there is a graph G for which the independence complex is homeomorphic to Δ, where we identify a simplicial set with its geometric realization.

We are interested in the homotopy type of the independence complexes of square grid graphs $C_{m,n}$ with cylindrical identification. The vertex set of $C_{m,n}$ is $\{1,2,\dots,m\}\times\mathbb{Z}_n$, and there is an edge between (u_1,u_2) and (v_1,v_2) if $u_1=v_1$ and $u_2=v_2\pm 1$ or if $|u_1-v_1|=1$ and $u_2=v_2$.

When we draw a picture of the graph $C_{m,n}$ and its independent set, we reverse the order of the first and second coordinates to save vertical space.

Figure 1 shows the graph $C_{3,4}$ and its independent set $\{(1,2),(1,4),(2,3),(3,1)\}$, where the edges on both sides indicate that $(i,1)$ and $(i,4)$ are adjacent for $i=1,2,3$.

In 1999, Kozlov [6] determined the homotopy type of $I(C_{1,n})$.

THEOREM 1.1

We have

$$I(C_{1,3k+i})\simeq\begin{cases}\bigvee_2 S^{k-1} & \text{if } i=0,\\ S^{k-1} & \text{if } i=1,\\ S^{k} & \text{if } i=2,\end{cases}$$

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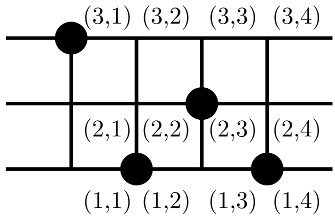

Figure 1. $C_{3,4}$ and its independent set

where $\bigvee_j X$ *denotes a wedge of* j *copies of* X.

Later, Thapper [8] obtained the following results.

THEOREM 1.2

We have

$$I(C_{2,4k+i}) \simeq \begin{cases} \bigvee_3 S^{2k-1} & \text{if } i=0, \\ S^{2k-1} & \text{if } i=1, \\ S^{2k} & \text{if } i=2, \\ S^{2k+1} & \text{if } i=3, \end{cases} \qquad I(C_{2k+i,2}) \simeq \begin{cases} S^{k-1} & \text{if } i=0, \\ S^{k} & \text{if } i=1, \end{cases}$$

$$I(C_{3k+i,3}) \simeq \begin{cases} S^{2k-1} & \text{if } i=0, \\ \bigvee_2 S^{2k} & \text{if } i=1, \\ S^{2k+1} & \text{if } i=2, \end{cases} \qquad I(C_{2k+i,4}) \simeq \begin{cases} \bigvee_{2k+1} S^{2k-1} & \text{if } i=0, \\ \bigvee_{2k} S^{2k} & \text{if } i=1, \end{cases}$$

$$I(C_{2k+i,5}) \simeq \begin{cases} S^{2k-1} & \text{if } i=0, \\ S^{2k+1} & \text{if } i=1. \end{cases}$$

The explicit results of the present study are presented in the following theorems.

THEOREM 1.3

We have

$$I(C_{3,8k+i}) \simeq \begin{cases} \bigvee_5 S^{6k-1} & \text{if } i=0, \\ S^{6k-1} & \text{if } i=1, \\ S^{6k+1} & \text{if } i=2, \\ S^{6k+1} & \text{if } i=3, \\ \bigvee_3 S^{6k+2} & \text{if } i=4, \\ S^{6k+3} & \text{if } i=5, \\ S^{6k+3} & \text{if } i=6, \\ S^{6k+5} & \text{if } i=7. \end{cases}$$

THEOREM 1.4
For $i = 0, 1, 2, 3$, we have

$$I(C_{4k+i,7}) \simeq S^{6k+2i-1}$$

COROLLARY 1.5
We have

$$I(C_{3,2k+1}) \simeq I(C_{k,7}).$$

THEOREM 1.6
We have

$$I(C_{6k+i,6}) \simeq \begin{cases} S^{8k-1} \vee \bigvee_2 (S^{8k} \vee S^{8k+1} \vee \cdots \vee S^{9k-1}) & \text{if } i = 0, \\ \bigvee_2 S^{8k+1} \vee \bigvee_2 (S^{8k+1} \vee S^{8k+2} \vee \cdots \vee S^{9k}) & \text{if } i = 1, \\ S^{8k+2} \vee \bigvee_2 (S^{8k+3} \vee S^{8k+4} \vee \cdots \vee S^{9k+2}) & \text{if } i = 2, \\ S^{8k+3} \vee \bigvee_2 (S^{8k+4} \vee S^{8k+5} \vee \cdots \vee S^{9k+3}) & \text{if } i = 3, \\ \bigvee_2 S^{8k+5} \vee \bigvee_2 (S^{8k+5} \vee S^{8k+6} \vee \cdots \vee S^{9k+5}) & \text{if } i = 4, \\ S^{8k+6} \vee \bigvee_2 (S^{8k+7} \vee S^{8k+8} \vee \cdots \vee S^{9k+6}) & \text{if } i = 5. \end{cases}$$

Here, if $s > t$, then $(S^s \vee S^{s+1} \vee \cdots \vee S^t)$ denotes a point.

Let Δ be a family of subsets of a finite set. We define the *partition function* $Z(\Delta; z)$ of Δ as

$$Z(\Delta; z) = \sum_{\sigma \in \Delta} z^{|\sigma|}$$

and write $Z(\Delta) = Z(\Delta; -1)$, where $|\sigma|$ denotes the cardinality of the set σ. If Δ is a finite simplicial set, where we assume that the empty set $\emptyset$ is in Δ, then we have

$$Z(\Delta) = 1 - \chi(\Delta),$$

where χ denotes the unreduced Euler characteristic.

Jonsson [4] investigated $Z(C_{m.n}) = Z(I(C_{m,n}))$ and conjectured that $Z(C_{m,n}) = 1$ for odd n, unless $\gcd(m-1, n)$ is a multiple of three, in which case $Z(C_{m,n}) = -2$. This conjecture was solved affirmatively by Thapper [8] and Jonsson [5].

For even n, the situation appears to be far more complicated.

COROLLARY 1.7
We have

$$Z(C_{m,4}) = \begin{cases} m+1 & \text{if } m \text{ is even}, \\ -m & \text{if } m \text{ is odd}, \end{cases}$$

$$Z(C_{m,6}) = \begin{cases} -1 & \textit{if } m \equiv 2,5,6,9 \pmod{12}, \\ 1 & \textit{if } m \equiv 0,3,8,11 \pmod{12}, \\ 2 & \textit{if } m \equiv 1,10 \pmod{12}, \\ 4 & \textit{if } m \equiv 4,7 \pmod{12}. \end{cases}$$

The first result of Corollary 1.7 is taken from [8].

Through the explicit calculation of the homology groups of the independence complexes for small sizes, Thapper conjectured that, for $j,k \geq 1$,

$$H_*\big(I(C_{j,2k+1})\big) \cong H_*\big(I(C_{k,2j+1})\big).$$

All of the results obtained herein indicate that the following two conjectures hold.

CONJECTURE 1.8
If $n > 1$ $I(C_{m,n})$ is homotopy equivalent to a wedge of spheres.

CONJECTURE 1.9
For $j,k \geq 1$, $I(C_{j,2k+1})$ is homotopy equivalent to $I(C_{k,2j+1})$.

2. Discrete Morse theory and matching tree

In this section we recall the *discrete Morse theory*, which was introduced by Forman [3]. The discrete Morse theory is a method by which to construct a CW-complex with a simpler structure, which is homotopy equivalent to a given simplicial set. We also recall the matching tree in [2] and [8] to define a Morse matching on the independence complex of a graph.

The following explanation of the discrete Morse theory is taken from [7].

DEFINITION 2.1

(1) A partial matching in a poset P is a subset $M \subset P \times P$ such that

(i) $(a,b) \in M$ implies $a \prec b$, that is, $a < b$, and there is no $c \in P$ such that $a < c < b$,

(ii) each $a \in P$ belongs to at most one element in M.

When $(a,b) \in M$, we write $a = d(b)$ and $b = u(a)$.

(2) A partial matching on P is said to be acyclic if there does not exist a cycle

$$b_1 \succ d(b_1) \prec b_2 \succ d(b_2) \prec \cdots \prec b_n \succ d(b_n) \prec b_1,$$

with $n \geq 2$, and all b_i being distinct.

For a simplicial set Δ we define the *face poset* $P(\Delta)$ on Δ as the set of faces in Δ ordered by inclusion. We remark that $P(\Delta)$ has the empty set as the minimum element.

DEFINITION 2.2

A Morse matching M on a simplicial set Δ is a partial matching in the face poset $P(\Delta)$, which is acyclic. If there are no unmatched elements, the Morse matching is perfect.

The main theorem of the discrete Morse theory can be stated as follows, using the notion of Morse matching.

THEOREM 2.3

Let Δ be a simplicial set with a Morse matching M. Assume that for each $i \geq 0$, there are c_i unmatched i-dimensional simplices. Then, Δ is homotopy equivalent to a CW-complex with exactly c_i cells of each positive dimension i, and $c_0 + 1$ cells of dimension 0.

COROLLARY 2.4

Let Δ be a simplicial set with a Morse matching M such that $c_j = 0$ for all j but one i. Then, for this particular i,

$$\Delta \simeq \bigvee_{c_i} S^i.$$

That is, Δ is homotopy equivalent to a wedge of c_i i-dimensional spheres.

COROLLARY 2.5

Let Δ be a simplicial set with a perfect Morse matching M. Then, Δ is contractible.

The *matching tree* in [2] and [8] is a method by which to construct a Morse matching on the independence complex of a graph.

For a graph $G = (V, E)$ and a vertex $a \in V$, we define $N(a) = N_G(a)$ and $N[a] = N_G[a]$ by $N(a) = \{b \in V \mid (a, b) \in E\}$ and $N[a] = N(a) \cup \{a\}$. For a subset A of V, $N(A)$ is defined by $N(A) = \bigcup_{a \in A} N(a)$.

For the independence complex $I(G)$ of a graph G we will construct a finite, plane rooted tree in which each internal node has either one or two children. Each node is a subset of $I(G)$ of the form

$$I(A, B) = I(G)(A, B) = \big\{\sigma \in I(G) \;\big|\; A \subset \sigma \text{ and } B \cap \sigma = \emptyset\big\},$$

where A and B are two subsets of V such that

(1) $\qquad A$ is an independent set, $\quad A \cap B = \emptyset \qquad$ and $\qquad N(A) \subset B$.

The root is $I(\emptyset, \emptyset) = I(G)$, and other nodes will be defined recursively as follows. If the node is the empty set (no unmatched elements), we declare the node a leaf. Otherwise, the node is of the form $I(A, B)$, which is a nonempty set.

If $A \cup B = V$, then $I(A, B) = \{A\}$ is a node with an unmatched element of cardinality $|A|$, and we also declare this node a leaf.

We are left with nodes of the form $I(A, B)$, with $A \cup B \neq V$. Choose a vertex p in $V' = V \setminus (A \cup B)$ and proceed as follows.

(i) If p has at most one neighbor in V', define $\Delta(A,B,p)$ to be the subset of $I(A,B)$ formed of sets that do not intersect $N(p)$:

$$\Delta(A,B,p) = \{\sigma \mid A \subset \sigma \text{ and } B \cap \sigma = \sigma \cap N(p) = \emptyset\}.$$

Then,

$$M(A,B,p) = \{(\sigma, \sigma \cup \{p\}) \mid \sigma \in \Delta(A,B,p) \text{ and } p \notin \sigma\}$$

gives a perfect matching of $\Delta(A,B,p)$. In this case, we call p the *pivot* of this matching. Assign to the node $I(A,B)$ a unique child, namely, the set $U = I(A,B) \setminus M(A,B,p)$ of unmatched elements. This set is empty if p has no neighbor in V'. In this case, we say that p is a *free vertex* of $I(A,B)$. If p has exactly one neighbor v in V', then $U = I(A \cup \{v\}, B \cup N(v))$. We say that the triple (A,B,p) is a *matching site* of the tree.

(ii) Otherwise, node $I(A,B)$ has two children. The left child is $I(A, B \cup \{p\})$, and the right child is $I(A \cup \{p\}, B \cup N(p))$. The union of these sets is $I(A,B)$. Here, (A,B,p) is said to be a *splitting site* of the tree. In this case, we refer to p as the *pivot* of this splitting.

Unless they are empty, the new nodes satisfy condition (2.1).

In a figure showing a matching tree, the vertices of A and B are described in black and white, respectively. The pivots are denoted by triangles (e.g., see Figure 2).

The following theorem was proved in [2] and [8].

THEOREM 2.6

For any graph G and any matching tree of G, the matching of $I(G)$ obtained by taking the union of all partial matchings $M(A,B,p)$ performed at the matching sites is a Morse matching.

3. Matching tree and star clusters

A simplicial set is hereinafter referred to simply as a complex. In this section, we recall the notion of a star cluster introduced by Barmak [1]. The (simplicial) star $\mathrm{st}_K(\sigma)$, or simply $\mathrm{st}(\sigma)$, of a simplex σ in a complex K is the subcomplex of simplices τ such that $\sigma \cup \tau \in K$.

DEFINITION 3.1

Let σ be a simplex of a complex K. We define the star cluster of σ in K as the subcomplex

$$\mathrm{SC}(\sigma) = \bigcup_{v \in \sigma} \mathrm{st}_K(v).$$

A complex K is said to be a clique complex if for each nonempty set of vertices σ such that $\{v,w\} \in K$ for every $v,w \in \sigma$, we have $\sigma \in K$. By definition the independence complex $I(G)$ of a graph G is a clique complex.

Although, in general, $\mathrm{SC}(\sigma)$ is not contractible, Barmak [1] proved the following.

LEMMA 3.2
Let K be a clique complex, let σ be a simplex of K, and let $\sigma_0, \sigma_1, \ldots, \sigma_r$ be a collection of faces of σ ($r \geq 0$). Then,

$$\bigcup_{i=0}^{r} \bigcap_{v \in \sigma_i} \mathrm{st}_K(v)$$

is contractible. In particular, the star cluster of a simplex in a clique complex is contractible.

Next, using Lemma 3.2, he obtained the following results.

THEOREM 3.3
Let G be a graph, and let v be a nonisolated vertex of G that is not contained in any triangle. Then, $N_G(v)$ is a simplex of $I(G)$, and

$$I(G) \simeq \Sigma\big(\mathrm{st}(v) \cap \mathrm{SC}(N_G(v))\big).$$

Now, using the star cluster, we explain how a matching tree M of a graph G describes the homotopy type of its independence complex.

A simplex σ is said to be *extensible* to v if $\sigma \cap N[v] = \emptyset$.

We assume that $(\emptyset, \emptyset, u_1)$ is a matching site, and u_1 has exactly one neighbor v_1. Then, according to Theorem 3.3, we have

$$I(G) \simeq \Sigma\big(\mathrm{st}(u_1) \cap \mathrm{st}(v_1)\big).$$

The simplices of the complex $\mathrm{st}(u_1) \cap \mathrm{st}(v_1)$ are exactly the independent sets of G that are extensible to both u_1 and v_1. Therefore, $\mathrm{st}(u_1) \cap \mathrm{st}(v_1)$ is the independence complex of subgraph G_1 of G induced by vertices other than v_1 or any of its neighbors, that is, $G_1 = G \setminus N[v_1]$. Thus, we have

$$I(G) \simeq \Sigma I(G_1).$$

The child of the root is $I(v_1, N(v_1))$, and again we assume that $(v_1, N(v_1), u_2)$ is a matching site and that u_2 has exactly one neighbor v_2 in G_1. By the same argument, we have

$$I(G) \simeq \Sigma I(G_1) \simeq \Sigma^2 I(G_2),$$

where $G_2 = G_1 \setminus N_{G_1}[v_2] = G \setminus (N_G[v_1] \cup N_G[v_2])$.

Let $(\{v_1, v_2, \ldots, v_k\}, \bigcup_{i=1}^{k} N(v_i), v)$ be the first splitting site from the root. Then, by an inductive argument, we see that

$$I(G) \simeq \Sigma^k I(G_k),$$

where $G_k = G \setminus \bigcup_{i=1}^{k} N_G[v_i]$. The matching tree M defines the matching tree M_k of the graph G_k as follows. The root of M_k is $I(G_k)(\emptyset, \emptyset)$. To a node $I(G)(A, B)$

in the tree M, which is a successor of the node $I(G)\big(\{v_1, v_2, \dots, v_k\}, \bigcup_{i=1}^{k} N(v_i)\big)$, we associate the node $I(G_k)\big(A \setminus \{v_1, v_2, \dots, v_k\}, B \setminus \bigcup_{i=1}^{k} N(v_i)\big)$ of M_k. Then, M_k is a matching tree of G_k and is isomorphic to the subtree of M consisting of $I(G)\big(\{v_1, v_2, \dots, v_k\}, \bigcup_{i=1}^{k} N(v_i)\big)$ and its successors.

Since we are interested in $I(G_k)$, we use the matching tree M_k.

The root has two children. The left child is

$$I(G_k)(\emptyset, \{v\}),$$

and the right child is

$$I(G_k)\big(\{v\}, N(v)\big).$$

Let $\mathrm{lk}(v)$ be the link of vertex v in $I(G_k)$. Then, $I(\{v\}, N(v)) \cup \mathrm{lk}(v)$ is a cone over the link $\mathrm{lk}(v)$. Thus,

$$\begin{aligned} I(G_k) &= I(\emptyset, \{v\}) \cup I\big(\{v\}, N(v)\big) = I(\emptyset, \{v\}) \cup \big(I(\{v\}, N(v)) \cup \mathrm{lk}(v)\big) \\ &= I(\emptyset, \{v\}) \cup C \ \ \mathrm{lk}(v). \end{aligned}$$

Let $H = G_k \setminus N[v]$. Then, $I(H) = \mathrm{lk}(v)$, and we have a cofiber sequence

$$I(H) \to I(G_k)(\emptyset, \{v\}) \to I(G_k) \to \Sigma I(H),$$

which induces another cofiber sequence

$$\Sigma^k I(H) \to \Sigma^k I(G_k)(\emptyset, \{v\}) \to I(G) \simeq \Sigma^k I(G_k) \to \Sigma^{k+1} I(H).$$

Repeating the same argument as above for the two children, we obtain two cofiber sequences that describe the homotopy types of $I(G_k)(\emptyset, \{v\})$ and $I(H)$. In this way, we can describe the homotopy type of $I(G)$.

4. Independence complex of $C_{3,n}$

We construct a matching tree of the graph $C_{3,n}$ with the following properties.

LEMMA 4.1

There is a matching tree of the graph $C_{3,n}$ with

(i) *five unmatched elements of cardinality* $6k$ *if* $n = 8k$,
(ii) *a unique unmatched element of cardinality* $6k$ *if* $n = 8k + 1$,
(iii) *a unique unmatched element of cardinality* $6k + 2$ *if* $n = 8k + 2$ *or* $8k + 3$,
(iv) *three unmatched elements of cardinality* $6k + 3$ *if* $n = 8k + 4$,
(v) *a unique unmatched element of cardinality* $6k + 4$ *if* $n = 8k + 5$ *or* $8k + 6$,
(vi) *a unique unmatched element of cardinality* $6k + 6$ *if* $n = 8k + 7$.

Then, Theorem 1.3 follows from Lemma 4.1 by Corollary 2.4 and Theorem 2.6.

Proof

Figure 2 shows the splitting sites of the tree that we are going to construct. For simplicity, we show only the case $n = 3$.

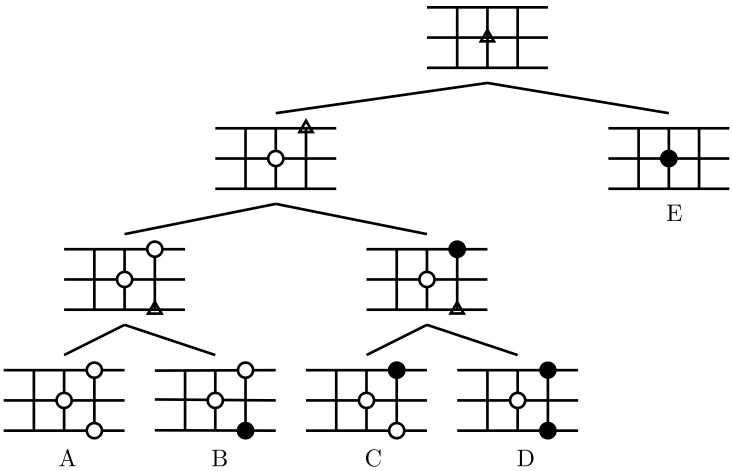

Figure 2. Splitting sites of the tree

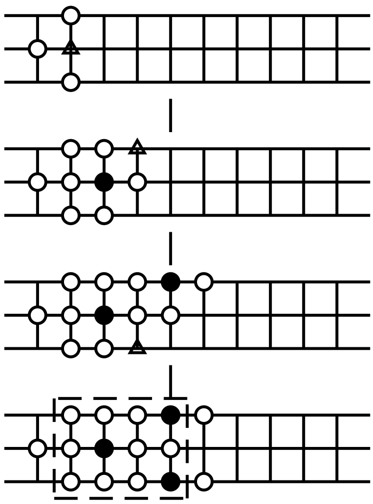

Figure 3. Subtree (A)

A remaining subtree of (A) is shown in Figure 3, which shows that

(i) there is a unique unmatched element of cardinality $3k$ if $n = 4k$ or $4k+1$,
(ii) there is no unmatched element if $n = 4k+2$ or $4k+3$.

In the following, to shorten the tree, we take two or more pivots at the same time for a matching site if doing so causes no confusion.

A remaining subtree of (B) is shown in Figure 4, which shows that

(i) there is a unique unmatched element of cardinality $6k$ if $n = 8k$,

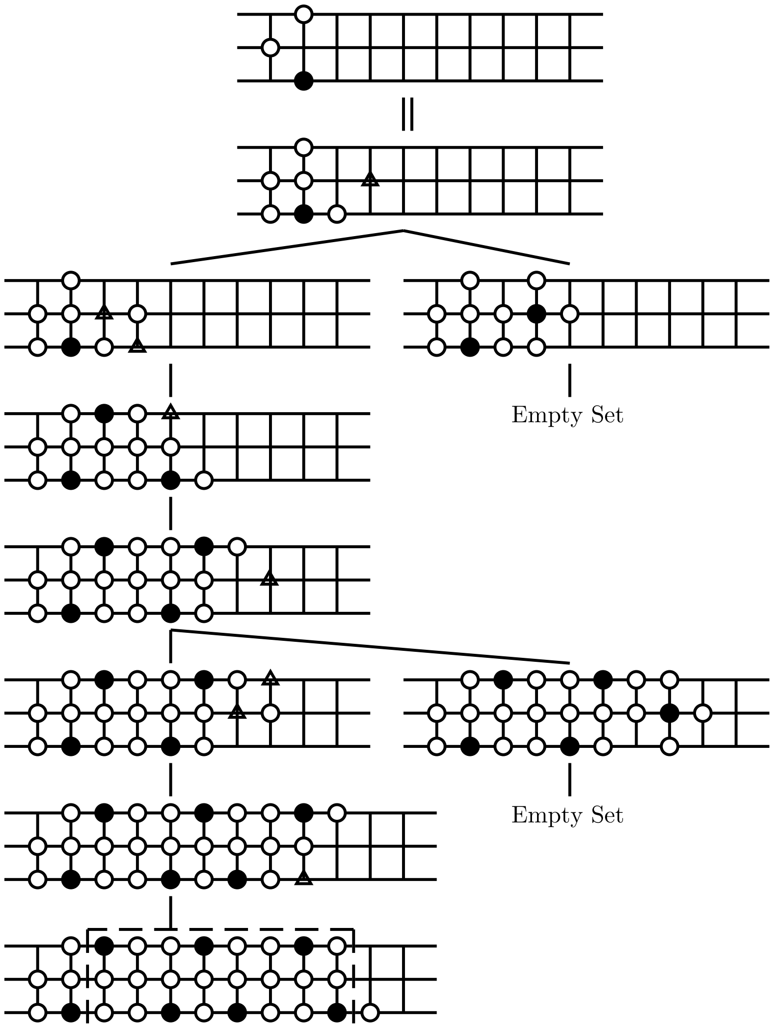

Figure 4. Subtree (B)

(ii) there is no unmatched element if $n = 8k+1, 8k+2, 8k+4$, or $8k+7$,

(iii) there is a unique unmatched element of cardinality $6k+2$ if $n = 8k+3$,

(iv) there is a unique unmatched element of cardinality $6k+4$ if $n = 8k+5$ or $8k+6$.

Since node (C) is isomorphic to node (B), there is a remaining subtree of node (C) that is isomorphic to that of node (B).

A remaining subtree of (D) is shown in Figure 5, which shows that

(i) there is a unique unmatched element of cardinality $3k$ if $n = 4k$,

(ii) there is no unmatched element if $n = 2k+1$,

(iii) there is a unique unmatched element of cardinality $3k+2$ if $n = 4k+2$.

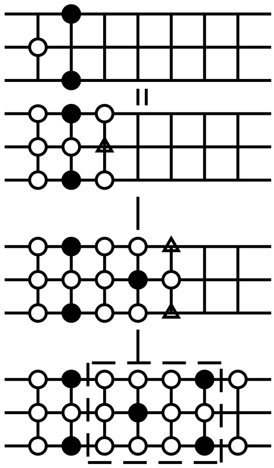

Figure 5. Subtree (D)

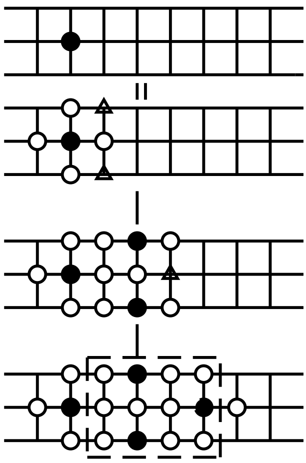

Figure 6. Subtree (E)

A remaining subtree of (E) is shown in Figure 6, which shows that

(i) there is a unique unmatched element of cardinality $3k$ if $n = 4k$,
(ii) there is no unmatched element if $n = 4k+1$ or $4k+2$,
(iii) there is a unique unmatched element of cardinality $3k+3$ if $n = 4k+3$.

In the matching tree that we have constructed, there are

(i) five unmatched elements of cardinality $6k$ if $n = 8k$,
(ii) a unique unmatched element of cardinality $6k$ if $n = 8k+1$,
(iii) a unique unmatched element of cardinality $6k+2$ if $n = 8k+2$,

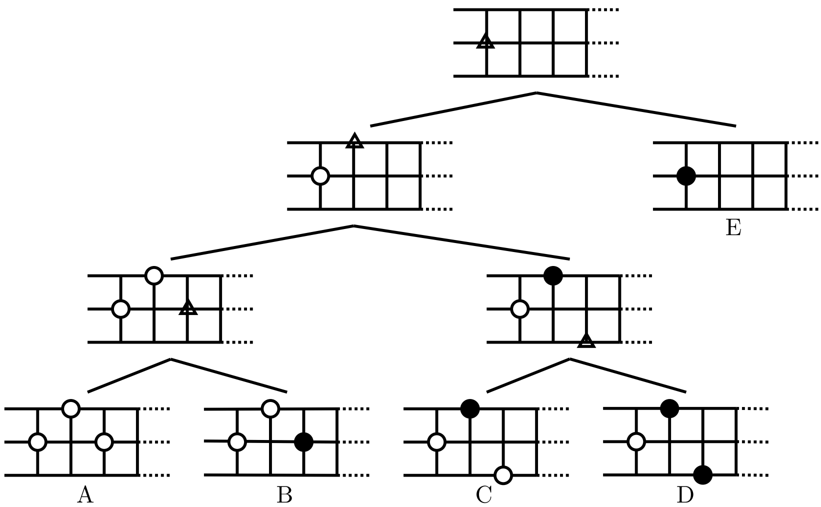

Figure 7. Splitting sites for $n = 8k + 5$ or $8k + 6$

(iv) two unmatched elements of cardinality $6k+2$ and an unmatched element of cardinality $6k + 3$ if $n = 8k + 3$,

(v) three unmatched elements of cardinality $6k + 3$ if $n = 8k + 4$,

(vi) an unmatched element of cardinality $6k+3$ and two unmatched elements of cardinality $6k + 4$ if $n = 8k + 5$,

(vii) two unmatched elements of cardinality $6k + 4$ and an unmatched element of cardinality $6k + 5$ if $n = 8k + 6$,

(viii) a unique unmatched element of cardinality $6k + 6$ if $n = 8k + 7$.

Hence, unless $n = 8k + 3$, $8k + 5$, or $8k + 6$, the above-described matching tree is a desired matching tree for Lemma 4.1. For the case of $n = 8k + 3$, $8k + 5$, or $8k + 6$, we just construct a new matching tree to prove Lemma 4.1.

For $n = 8k+5$ or $8k+6$, we use the following matching tree shown in Figure 7.

For $n = 8k + 5$, as before, we can construct a matching tree such that there is a unique unmatched element of cardinality $6k + 4$ in the remaining subtree of (D) and there are no unmatched elements in other subtrees.

Similarly, for $n = 8k + 6$, we can construct a matching tree such that there is a unique unmatched element of cardinality $6k + 4$ in the remaining subtree of (A) and there are no unmatched elements in other subtrees.

Thus, Lemma 4.1 is proven for these cases.

For $n = 8k + 3 > 3$, we use the matching tree shown in Figure 8, and we can construct a matching tree such that there is a unique unmatched element of cardinality $6k + 2$ in the remaining subtree of (A) and there are no unmatched elements in other subtrees.

For $n = 3$ it is easy to construct a matching tree with a unique unmatched element of cardinality 2.

Thus, we complete the proof of Lemma 4.1. $\square$

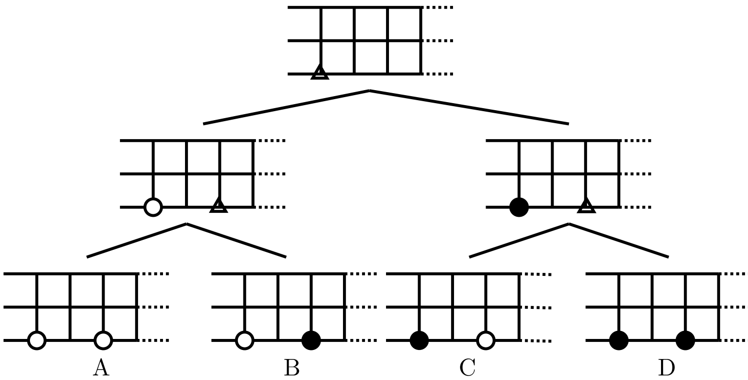

Figure 8. Splitting sites for $n = 8k + 3$

5. Independence complex of $C_{m,7}$

We prove Theorem 1.4 by using the same method used to prove Theorem 1.3.

LEMMA 5.1
There is a matching tree of the graph $C_{m,7}$ with a unique unmatched element of cardinality $6k + 2i$ if $m = 4k + i$ for $0 \leq i < 4$.

First, we split both on $u = (1, 2)$ and $v = (1, 4)$.

LEMMA 5.2
There are perfect matchings in $I(C_{m,7})(\{v\}, \{u\})$ and $I(C_{m,7})(\{u\}, \{v\})$.

Proof
To simplify the description, a part of the matching tree shown in Figure 9 is drawn simply as shown in Figure 10.

Figure 11 shows that if $I(C_{m-2,7})(\{v\}, \{u\})$ can be perfectly matched, so can $I(C_{m,7})(\{v\}, \{u\})$. For $m = 1, 2$, it is easy to see that $I(C_{m,7})(\{v\}, \{u\})$ can be perfectly matched. Thus, $I(C_{m,7})(\{v\}, \{u\})$ can be perfectly matched for any m.

Since $I(C_{m,7})(\{u\}, \{v\})$ is isomorphic to $I(C_{m,7})(\{v\}, \{u\})$, the proof is completed. $\square$

Proof of Lemma 5.1
It is easy to construct a matching tree of $I(C_{m,7})(\emptyset, \{u, v\})$ for $m \leq 4$ with a unique unmatched element of cardinality 2 if $m = 1$, no unmatched element if $m = 2, 3$, and a unique unmatched element of cardinality 6 if $m = 4$.

In Figure 12, the right-hand node is isomorphic to $I(C_{m,7})(\{v\}, \{u\})$ and therefore is perfectly matched.

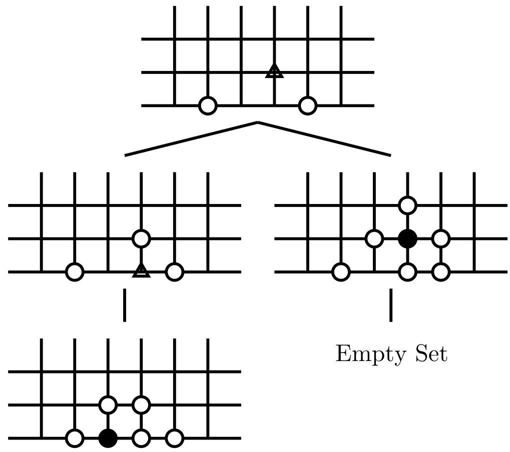

Figure 9

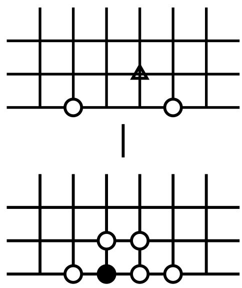

Figure 10

Then, Figure 12 shows that there is a matching tree for $I(C_{m,7})(\emptyset, \{u, v\})$ with a unique unmatched element of cardinality $6k + 2$ if $m = 4k + 1$, no unmatched element for $m \equiv 2, 3 \pmod 4$, and a unique unmatched element of cardinality $6k + 6$ if $m = 4k + 4$.

Next, Figure 13 shows that there is a matching tree for $I(C_{m,7})(\{u, v\}, \emptyset)$ with a unique unmatched element of cardinality $6k + 4$ if $m = 4k + 2$, a unique unmatched element of cardinality $6k+6$ if $m = 4k+3$, and no unmatched element if $m \equiv 0, 1 \pmod 4$.

Thus, we have proven Lemma 5.1, which completes the proof of Theorem 1.4. $\square$

6. Independence complex of $C_{m,6}$

Let $s = (1, 1)$, $t = (1, 2)$, $u = (1, 3)$, and $v = (1, 5)$. We first prove the following lemma.

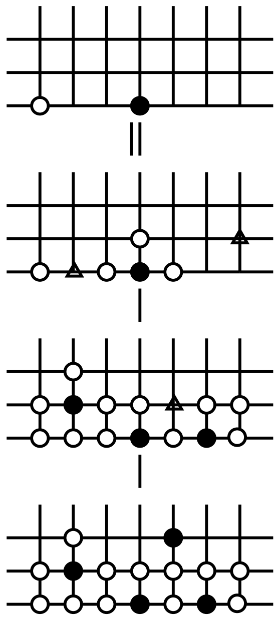

Figure 11. A matching tree for $I(C_{m,7})(\{v\},\{u\})$

LEMMA 6.1

We have

$$I(C_{6k+i,6}\setminus\{s,u\})\simeq\begin{cases} S^{8k-1}\vee S^{8k}\vee\cdots\vee S^{9k-1} & \text{if } i=0,\\ S^{8k+1}\vee S^{8k+1}\vee\cdots\vee S^{9k} & \text{if } i=1,\\ S^{8k+i}\vee S^{8k+i+1}\vee\cdots\vee S^{9k+i} & \text{if } i=2,3,\\ S^{8k+i+1}\vee S^{8k+i+2}\vee\cdots\vee S^{9k+i+1} & \text{if } i=4,5. \end{cases}$$

Proof

By induction on $m=6k+i$, we prove the lemma. It is easy to show that the lemma holds for $m\leq 4$. Let $m>4$, and consider the following cofiber sequence:

$$I\big(C_{m,6}\setminus(\{s,u\}\cup N[v])\big)\to I(C_{m,6}\setminus\{s,u,v\})\to I(C_{m,6}\setminus\{s,u\}).$$

Figure 14 shows that

$$\begin{aligned} I(C_{m,6}\setminus\{s,u,v\}) &\simeq I(C_{m-2,6}\setminus\{s,u,v\})\\ &\simeq\begin{cases} S^{(3m/2)-1} & m\equiv 0 \pmod 2,\\ * & m\equiv 1 \pmod 2, \end{cases} \end{aligned}$$

and that

$$I\big(C_{m,6}\setminus(\{s,u\}\cup N[v])\big)\simeq\Sigma^3 I(C_{m-3,6}\setminus\{s,u\}).$$

In the cofiber sequence

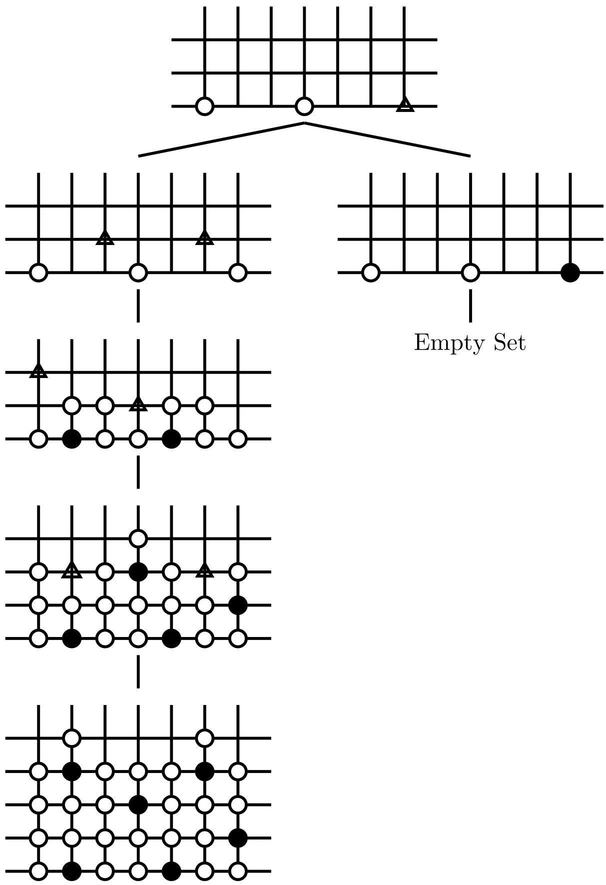

Figure 12. A matching tree for $I(C_{m,7})(\emptyset,\{u,v\})$

$$\Sigma^3 I(C_{m-3,6}\setminus\{s,u\}) \to I(C_{m,6}\setminus\{s,u,v\}) \to I(C_{m,6}\setminus\{s,u\}),$$

the first inclusion map is null-homotopic by the following table, in which we know the homotopy types of $\Sigma^3 I(C_{m-3,6}\setminus\{s,u\})$ by induction:

m	$\Sigma^3 I(C_{m-3,6}\setminus\{s,u\})$	$I(C_{m,6}\setminus\{s,u,v\})$
$2k+1$	a wedge of spheres	$*$
$6k$	$S^{8k-2}\vee S^{8k-1}\vee\cdots\vee S^{9k-3}$	S^{9k-1}
$6k+2$	$S^{8k+1}\vee S^{8k+2}\vee\cdots\vee S^{9k}$	S^{9k+2}
$6k+4$	$S^{8k+4}\vee S^{8k+5}\vee\cdots\vee S^{9k+3}$	S^{9k+5}

Thus, we have

$$I(C_{m,6}\setminus\{s,u\}) \simeq \Sigma^4 I(C_{m-3,6}\setminus\{s,u\}) \vee I(C_{m,6}\setminus\{s,u,v\}),$$

which proves the lemma. $\square$

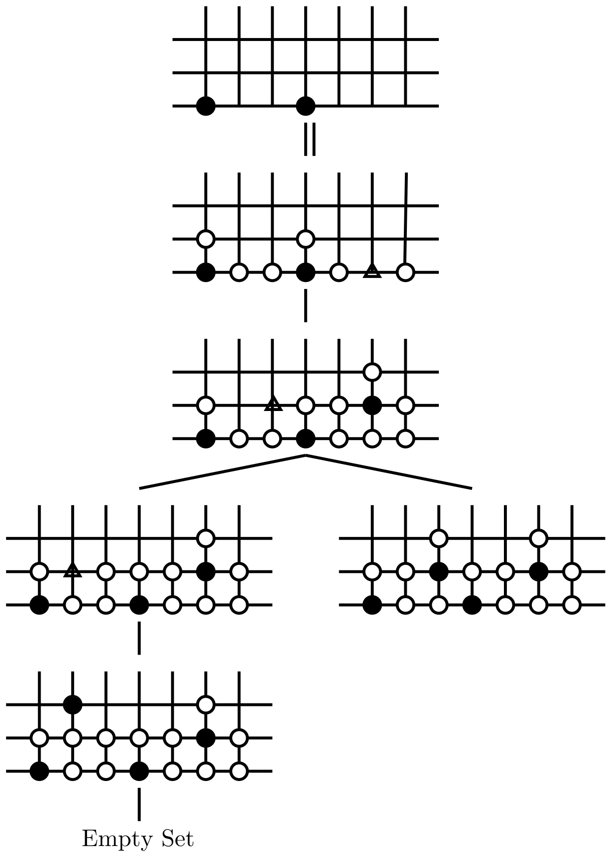

Figure 13. A matching tree for $I(C_{m,7})(\{u,v\},\emptyset)$

LEMMA 6.2
We have

$$I(C_{6k+i,6}\setminus\{s\})\simeq\begin{cases} S^{8k-1}\vee S^{8k}\vee\cdots\vee S^{9k-1} & \textit{if } i=0,\\ S^{8k+1}\vee\big(S^{8k+1}\vee S^{8k+2}\vee\cdots\vee S^{9k}\big) & \textit{if } i=1,\\ S^{8k+i}\vee S^{8k+i+1}\vee\cdots\vee S^{9k+i} & \textit{if } i=2,3,\\ S^{8k+5}\vee\big(S^{8k+5}\vee S^{8k+6}\vee\cdots\vee S^{9k+5}\big) & \textit{if } i=4,\\ S^{8k+6}\vee S^{8k+7}\vee\cdots\vee S^{9k+6} & \textit{if } i=5.\end{cases}$$

Proof
We consider the cofiber sequence

$$I\big(C_{m,6}\setminus(\{s\}\cup N[u])\big)\to I(C_{m,6}\setminus\{s,u\})\to I(C_{m,6}\setminus\{s\}).$$

Figure 15 shows that

$$I\big(C_{m,6}\setminus(\{s\}\cup N[u])\big)\simeq\Sigma^3 I\big(C_{m-3,6}\setminus(\{s\}\cup N[u])\big)$$

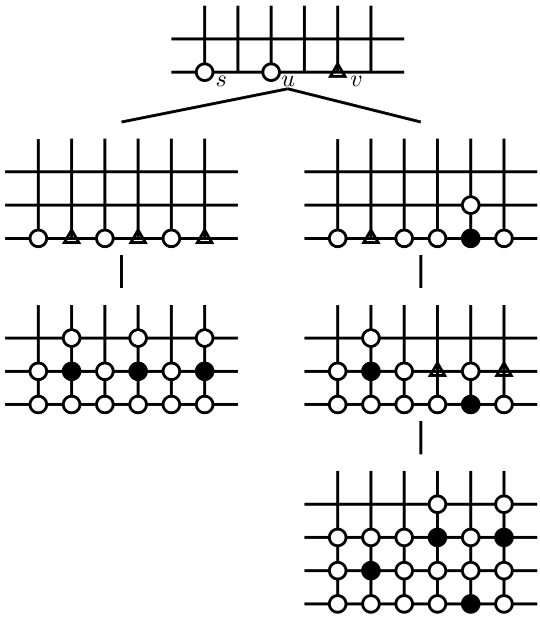

Figure 14. A matching tree for $I(C_{m,6} \setminus \{s,u\})$

$$\simeq \begin{cases} S^{(4(m-1))/3} & m \equiv 1 \pmod 3, \\ * & m \not\equiv 1 \pmod 3. \end{cases}$$

Since, from the dimension, the inclusion map

$$I\bigl(C_{m,6} \setminus (\{s\} \cup N[u])\bigr) \to I(C_{m,6} \setminus \{s,u\})$$

is null-homotopic, we have the homotopy equivalence

$$I(C_{m,6} \setminus \{s\}) \simeq \Sigma I\bigl(C_{m,6} \setminus (\{s\} \cup N[u])\bigr) \vee I(C_{m,6} \setminus \{s,u\}).$$

Then, by the same argument used in the proof of Lemma 6.1, we obtain the lemma. □

LEMMA 6.3

We have

$$I(C_{3k+i,6} \setminus \{s,t\}) \simeq \begin{cases} S^{4k-1} & \textit{if } i = 0, \\ * & \textit{if } i = 1, \\ S^{4k+2} & \textit{if } i = 2. \end{cases}$$

Proof

In Figure 16, the last remaining graph of the left-hand node is isomorphic to $I(C_{m-2,6} \setminus (\{s\} \cup N[u]))$. Thus, as is proven in the proof of Lemma 6.2, there is a matching tree for $I(\emptyset, \{s,t,v\})$ with a unique unmatched element of cardinality $4k$ if $m = 3k$ and no unmatched element if $m \not\equiv 0 \pmod 3$. Similarly, there is a

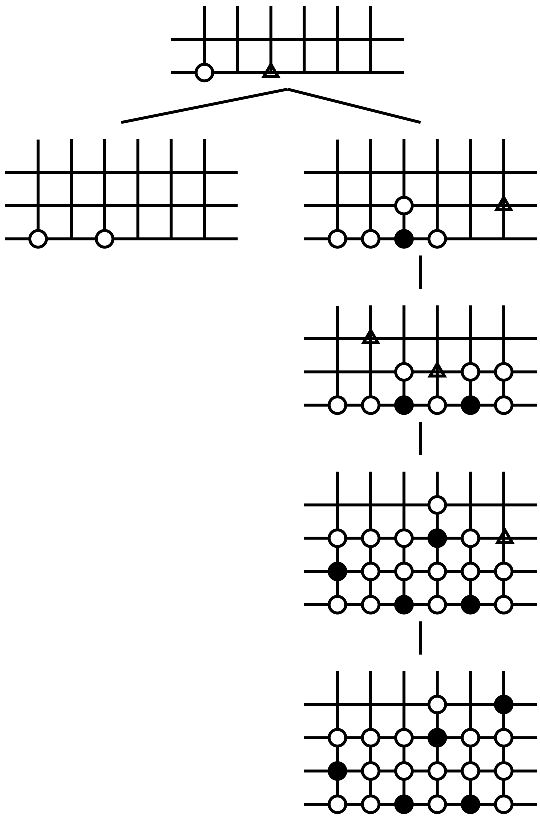

Figure 15. A matching tree for $I(C_{m,6} \setminus \{s\})$

matching tree for $I(\{v\},\{s,t\})$ with a unique unmatched element of cardinality $4k+3$ if $m = 3k+2$ and no unmatched element if $m \not\equiv 2 \pmod 3$. Thus, we have proven the lemma. □

Proof of Theorem 1.6
Since

$$I(C_{m,6}) = I(C_{m,6} \setminus \{s\}) \cup I(C_{m,6} \setminus \{t\}),$$

we have the following pushout diagram:

$$\begin{array}{ccc} I(C_{m,6} \setminus \{s,t\}) & \longrightarrow & I(C_{m,6} \setminus \{t\}) \\ \downarrow & & \downarrow \\ I(C_{m,6} \setminus \{s\}) & \longrightarrow & I(C_{m,6}) \end{array}$$

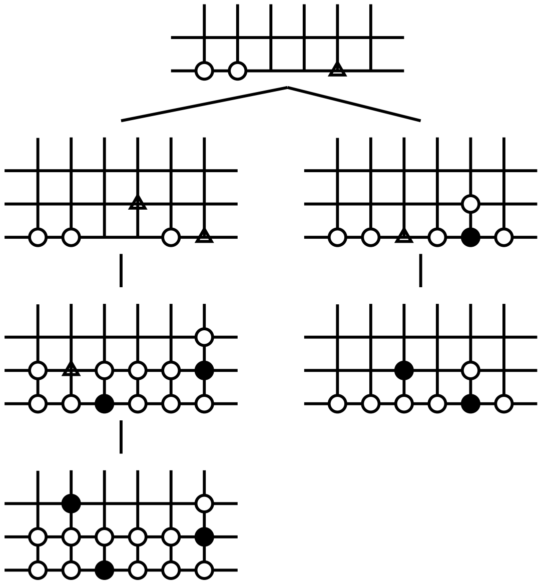

Figure 16. A matching tree for $I(C_{m,6} \setminus \{s,t\})$

We have the following table according to Lemmas 6.1–6.3:

m	$I(C_{m,6} \setminus \{s,t\})$	$I(C_{m,6} \setminus \{s\}) \simeq I(C_{m,6} \setminus \{t\})$
$6k$	S^{8k-1}	$S^{8k-1} \vee S^{8k} \vee \cdots \vee S^{9k-1}$
$6k+1$	$*$	$S^{8k+1} \vee (S^{8k+1} \vee S^{8k+2} \vee \cdots \vee S^{9k})$
$6k+2$	S^{8k+2}	$S^{8k+2} \vee S^{8k+3} \vee \cdots \vee S^{9k+2}$
$6k+3$	S^{8k+3}	$S^{8k+3} \vee S^{8k+4} \vee \cdots \vee S^{9k+3}$
$6k+4$	$*$	$S^{8k+5} \vee (S^{8k+5} \vee S^{8k+6} \vee \cdots \vee S^{9k+5})$
$6k+5$	S^{8k+6}	$S^{8k+6} \vee S^{8k+7} \vee \cdots \vee S^{9k+6}$

If $m \equiv 1 \pmod 3$, the above table shows that

$$I(C_{m,6}) \simeq I(C_{m,6} \setminus \{s\}) \vee I(C_{m,6} \setminus \{t\}),$$

which implies the theorem in this case.

To prove the other cases, we need the following lemma.

LEMMA 6.4

There is a matching tree for $C_{m,6}$ with

(i) *an unmatched element of cardinality $8k$ and two unmatched elements of cardinality $8k+j$ for $j = 1, 2, \ldots, k$ if $m = 6k$,*

(ii) *an unmatched element of cardinality $8k+3$ and two unmatched elements of cardinality $8k+3+j$ for $j = 1, 2, \ldots, k$ if $m = 6k+2$,*

(iii) *an unmatched element of cardinality $8k+4$ and two unmatched elements of cardinality $8k+4+j$ for $j = 1, 2, \ldots, k$ if $m = 6k+3$,*

(iv) *an unmatched element of cardinality* $8k+7$ *and two unmatched elements of cardinality* $8k+7+j$ *for* $j=1,2,\ldots,k$ *if* $m=6k+5$.

Proof
In the proof of Lemma 6.2, we constructed a matching tree for $I(\emptyset,\{s\})$ with

(i) an unmatched element of cardinality $8k+j$ for $j=0,1,2,\ldots,k$ if $m=6k$,
(ii) an unmatched element of cardinality $8k+3+j$ for $j=0,1,2,\ldots,k$ if $m=6k+2$,
(iii) an unmatched element of cardinality $8k+4+j$ for $j=0,1,2,\ldots,k$ if $m=6k+3$,
(iv) an unmatched element of cardinality $8k+7+j$ for $j=0,1,2,\ldots,k$ if $m=6k+5$.

In the proof of Lemma 6.1, we constructed a matching tree for $I(\emptyset,\{s,u\})$ with

(i) an unmatched element of cardinality $8k+2+j$ for $j=0,1,2,\ldots,k-1$ if $m=6k+1$,
(ii) an unmatched element of cardinality $8k+3+j$ for $j=0,1,2,\ldots,k$ if $m=6k+2$,
(iii) an unmatched element of cardinality $8k+6+j$ for $j=0,1,2,\ldots,k$ if $m=6k+4$,
(iv) an unmatched element of cardinality $8k+7+j$ for $j=0,1,2,\ldots,k$ if $m=6k+5$.

Thus, Figure 17 shows that there is a matching tree for $I(\{s\},\emptyset)$ with

(i) an unmatched element of cardinality $8k+j$ for $j=1,2,\ldots,k$ if $m=6k$,
(ii) an unmatched element of cardinality $8k+3+j$ for $j=1,2,\ldots,k$ if $m=6k+2$,
(iii) an unmatched element of cardinality $8k+4+j$ for $j=1,2,\ldots,k$ if $m=6k+3$,
(iv) an unmatched element of cardinality $8k+7+j$ for $j=1,2,\ldots,k$ if $m=6k+5$.

Thus, we have proven the lemma. $\square$

Now, we consider the case in which $m=6k$. In this case, we have the following pushout diagram:

$$\begin{tikzcd} S^{8k-1} \arrow[r] \arrow[d] & S^{8k-1}\vee S^{8k}\vee\cdots\vee S^{9k-1} \arrow[d] \\ S^{8k-1}\vee S^{8k}\vee\cdots\vee S^{9k-1} \arrow[r] & I(C_{m,6}) \end{tikzcd}$$

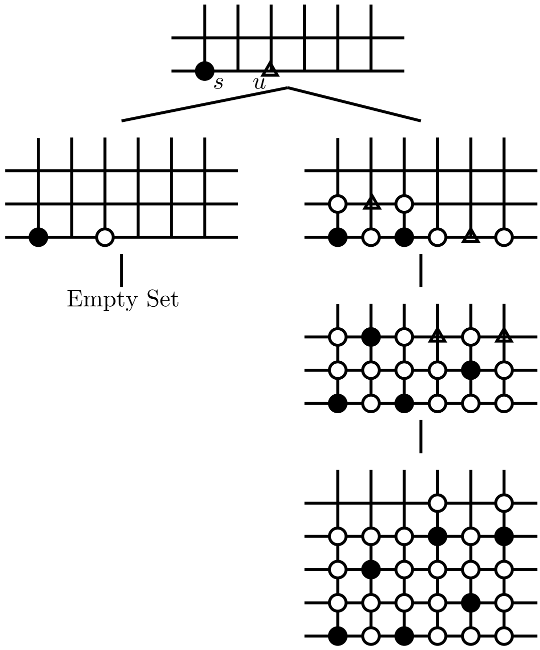

Figure 17. A matching tree of $I(C_{m,6})(\{s\},\emptyset)$ for $m \not\equiv 1 \pmod 3$

Thus,

$$I(C_{m,6}) \simeq X \vee \bigvee_2 (S^{8k} \vee S^{8k+1} \vee \cdots \vee S^{9k-1}),$$

where $X = (S^{8k-1} \vee S^{8k-1}) \cup e^{8k}$.

On the other hand, by Theorem 2.3 and Lemma 6.4, $I(C_{m,6})$ is homotopy equivalent to a CW-complex with one cell in dimension $8k-1$, two cells in dimension $8k, \ldots$, and two cells in dimension $9k-1$. Thus, X must be homotopy equivalent to S^{8k-1}, and thus we obtain the theorem.

The above arguments also prove the theorem for other cases. □

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