

Saito–Kurokawa liftings of level N and practical construction of Jacobi forms

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To the memory of Hiroshi Saito

Abstract Saito–Kurokawa liftings from Jacobi forms of degree one of weight k of level N to Siegel modular forms of degree two of weight k of level N with or without character are explicitly given by describing their Fourier expansions. Their L -functions are also given as well as the action of Hecke operators including some operators at bad primes. Also, a practical way of constructing Jacobi forms of given level is explained. This is explicitly executed for Jacobi forms of level up to 5, and their explicit structure theorems are given.

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1. Introduction

In this paper, we generalize the theory of Saito–Kurokawa lifting in [20] and [8] for arbitrary level with or without character, including the Hecke theory. Then we also give explicit structure theorems on Jacobi forms of index one for levels up to 5. Here we mean by Saito–Kurokawa lifting a lifting from Jacobi forms of level N of index one to Siegel modular forms of degree two of level N belonging to $\Gamma_0^{(2)}(N)$ with or without character. (Such lifts from Jacobi forms are often called Maass lifts by several mathematicians since Hans Maass found this as a link to the lifting from elliptic modular forms to Siegel modular forms (cf. [20]), but in this paper we loosely call all of these Saito–Kurokawa lifts.) The arguments in this paper are largely an imitation of those in [8], but there are several subtle points for generalization, and these are not completely trivial. For example, we

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must give a correct definition of index shift operators on Jacobi forms, we need generators of groups $\Gamma_0^{(2)}(N)$, we must show that cusp forms are lifted to cusp forms, and we must newly calculate the action of Hecke operators, and so on. Also, the lifting depends on N if Jacobi forms are *old* forms, and we can define several different liftings for these. We do not treat here the correspondence between Jacobi forms of index one and modular forms of half-integral weight, or Shimura correspondence between half-integral weight and integral weight, since there are many references for these, for example, [8], [19], [29], [33], [17], [32], and so on. There are already several related works on Saito–Kurokawa lifting of level N , for example, [24] for the trivial character case through Jacobi forms and [18] through integral transformation, but none of them covers the content of this paper. Also note that the definitions in [24] of the index shift operators or lifting operators contain typos, and most of the proofs there are very sketchy. Other references quoting it have the same defect as far as the author knows. As a related thing, Böcherer and Schulze-Pillot studied conditions where the Yoshida liftings for level N starting from pairs of modular forms of one variable do not vanish. The shapes of the L -functions of the Saito–Kurokawa liftings are the same as those of Yoshida liftings from pairs of a cusp form f of weight $2k - 2$ and the Eisenstein series of weight two. It does not seem clear if a Yoshida lifting of this type is exactly equal to the Saito–Kurokawa lifting in our sense, but a Yoshida lifting of this type vanishes when $L(k - 1, f) = 0$ (see [7]). Since our construction has no such restriction anyway, the Yoshida liftings do not cover all the Saito–Kurokawa liftings in general. Finally, we would like to call the reader’s attention to several subjects which are not treated in this paper and remain for a future investigation by researchers:

- (1) direct relations between Yoshida liftings and Saito–Kurokawa liftings (see, e.g., [26]);
- (2) relations between inner products of Jacobi forms and Siegel modular forms obtained by liftings;
- (3) characterization of the image of liftings by relations of Fourier coefficients like the Maass relation.

As for (3), see Section 3.4 for some explanation.

I have heard that H. Aoki obtained results similar to those in this paper independently. V. Gritsenko treated the case of paramodular forms starting from Jacobi forms of higher indices with *level one* in [10], but this is a different story.

2. Definitions and notation

We review here well-known definitions of Siegel modular forms and Jacobi forms in order to fix notation. We denote by \mathfrak{H}_n the Siegel upper half-space of degree n . We denote by $\mathrm{Sp}(n, \mathbb{R})$ the usual symplectic group of matrix size $2n$ and by $G\mathrm{Sp}^+(n, \mathbb{R})$ the group of symplectic similitudes of matrix size $2n$ with positive

multiplicators defined by

$$G\mathrm{Sp}^+(n, \mathbb{R}) = \{g \in M_{2n}(\mathbb{R}); {}^t g J_n g = n(g) J_n, n(g) > 0\},$$

where $J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. The group $G\mathrm{Sp}^+(n, \mathbb{R})$ acts on \mathfrak{H}_n as usual. We put $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z}) = \mathrm{Sp}(n, \mathbb{R}) \cap M_{2n}(\mathbb{Z})$. We define the Hecke-type subgroup of Γ_n of level N by

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n; C \equiv 0 \pmod{N} \right\}.$$

When $n = 1$, we simply write $\Gamma_0^{(1)}(N) = \Gamma_0(N)$. By a Dirichlet character χ modulo N (*not necessarily primitive*), we mean a \mathbb{C} -valued function on \mathbb{Z} such that $\chi(a) = 0$ if $(a, N) \neq 1$ and that for $(a, N) = 1$, $\chi(a)$ induces a character on $(\mathbb{Z}/N\mathbb{Z})^\times$. The smallest natural number $f|N$ such that there exists a Dirichlet character χ^0 modulo f such that $\chi^0(a) = \chi(a)$ for any a with $(a, N) = 1$ is called the conductor of χ , and such a χ^0 is called the primitive character associated with χ . For a Dirichlet character χ modulo N , we define a group character of $\Gamma_0^{(n)}(N)$ by $\chi(\gamma) = \chi(\det(D))$ for $\gamma = \begin{pmatrix} A & B \\ NC & D \end{pmatrix}$, and by abuse of language, we denote this also by χ . For any non-negative integer k , any function F of \mathfrak{H}_n , and any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G\mathrm{Sp}^+(n, \mathbb{R})$, we write

$$(F|_k[g])(Z) = J(g, Z)^{-k} F(gZ),$$

where $gZ = (AZ + B)(CZ + D)^{-1}$ and $J(g, Z) = \det(CZ + D)$. We denote by $A_k(\Gamma_0^{(n)}(N), \chi)$ the vector space of Siegel modular forms of weight k of level N with character χ , that is, holomorphic functions F of \mathfrak{H}_n such that $F|_k[\gamma] = \chi(\gamma)F$ for any $\gamma \in \Gamma_0^{(n)}(N)$ with additional conditions of holomorphy at cusps when $n = 1$. The subspace of cusp forms is denoted by $S_k(\Gamma_0^{(n)}(N), \chi)$. When the conductor of χ is one, we sometimes write $A_k(\Gamma_0^{(n)}(N)) = A_k(\Gamma_0^{(n)}(N), \chi)$ and $S_k(\Gamma_0^{(n)}(N)) = S_k(\Gamma_0^{(n)}(N), \chi)$.

Now we introduce Jacobi groups. For later use, we include the case when GL_2 is the semisimple part, not only SL_2 . For any subfield or subring K of \mathbb{R} , we denote by $\mathrm{GL}_2^+(K)$ the group of invertible elements of $M_2(K)$ with positive determinants. We define the Jacobi group $G^J(K)$ over K by $\mathrm{GL}_2^+(K) \ltimes H(K)$, where $H(K)$ is the Heisenberg group which is $K^2 \times K$ as a set and the multiplication of $G^J(K)$ is defined as follows. For $g_i \in \mathrm{GL}_2^+(K)$ ($i = 1, 2$) and $((\lambda, \mu), \kappa) \in H(K)$, we have

$g_1 g_2 =$ the usual multiplication of the matrices,

$$\begin{aligned} ((\lambda, \mu), \kappa) \times ((\lambda', \mu'), \kappa') &= (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda'), \\ ((\lambda, \mu), \kappa) \times g &= g \times (\det(g)^{-1}(\lambda, \mu)g, \det(g)^{-1}\kappa). \end{aligned}$$

When $\kappa = 0$, we sometimes write $(\lambda, \mu) = ((\lambda, \mu), 0)$. We have a group isomorphism of $G^J(K)$ into $G\mathrm{Sp}^+(2, \mathbb{R})$ given by embedding $(g, ((\lambda, \mu), \kappa))$ to

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & \det(g) & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G\mathrm{Sp}^+(2, \mathbb{R}),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(K)$. For any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ with finite index, we denote by Γ^J the subgroup of $G^J(\mathbb{Q})$ defined by

$$\Gamma^J = \{ (g, ((\lambda, \mu), \kappa)) \in G^J(\mathbb{Q}); g \in \Gamma, \lambda, \mu, \kappa \in \mathbb{Z} \}.$$

In particular, we can regard $\Gamma_0(N)^J$ as a subgroup of $\Gamma_0^{(2)}(N)$ through the above embedding.

For any complex number x and non-negative integer m , we write $e(x) = e^{2\pi ix}$ and $e^m(x) = e(mx)$. Let $f(\tau, z)$ be a function of $\mathfrak{H}_1 \times \mathbb{C}$. For $\omega \in \mathfrak{H}_1$ such that $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathfrak{H}_2$ and any $\tilde{g} = (g, ((\lambda, \mu), \kappa)) \in G^J(\mathbb{R})$ with $\det(g) = l > 0$, \tilde{g} acts on $f(\tau, z)e^m(\omega)$ through the above embedding to $G\mathrm{Sp}^+(2, \mathbb{R})$, and moreover, we can write

$$(f(\tau, z)e^m(\omega))|_k[\tilde{g}] = \tilde{f}(\tau, z)e^{ml}(\omega)$$

for some function \tilde{f} of $\mathfrak{H}_1 \times \mathbb{C}$ depending on m, k, \tilde{g} , and f . If we write

$$\tilde{f} = f|_{k,m}[\tilde{g}],$$

for $\tilde{g} \in G^J(\mathbb{R})$, then this is a group action of $\{\tilde{g} \in G^J(\mathbb{R}); \det(g) = 1\}$ which is a subgroup of $G^J(\mathbb{R})$. More explicitly, for any $g \in \mathrm{GL}_2(\mathbb{R})$ with $\det(g) = l > 0$ and $((\lambda, \mu), \kappa) \in H(\mathbb{R})$, we have

$$(f|_{k,m}[g])(\tau, z) = (c\tau + d)^{-k} e^{ml} \left(-\frac{cz^2}{c\tau + d} \right) f \left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d} \right),$$

$$(f|_m[((\lambda, \mu), \kappa)])(\tau, z) = e^m(\lambda^2\tau + 2\lambda z + \lambda\mu + \kappa) f(\tau, z + \lambda\tau + \mu).$$

Since the second action does not depend on k , we omit k in the suffix. If $g_1, g_2 \in \mathrm{GL}_2^+(\mathbb{Q})$ and $0 < \det(g_1) = l \in \mathbb{Z}$, then we have

$$f|_{k,m}[g_1 g_2] = f|_{k,m}[g_1]|_{k,m}[g_2].$$

Let χ be a Dirichlet character modulo N . We say that a holomorphic function $\phi(\tau, z)$ of $\mathfrak{H}_1 \times \mathbb{C}$ is a Jacobi form of weight k of index m with character χ with respect to $\Gamma_0(N)^J$ if it satisfies the following three conditions:

- (1) $f|_{k,m}[M] = \chi(M)f$ for any $M \in \Gamma_0(N)$;
- (2) $f|_m[(\lambda, \mu)] = f$ for any $\lambda, \mu \in \mathbb{Z}$;
- (3) for any $M \in \mathrm{GL}_2^+(\mathbb{Q})$, $f|_{k,m}[M]$ has the Fourier expansion in the following type:

$$f|_{k,m}[M] = \sum_{n,r} c^M(n, r) q^n \zeta^r,$$

where $q = e(\tau)$, $\zeta = e(z)$, $r \in \mathbb{Q}$, $n \in \mathbb{Q}$, and $c^M(n, r) = 0$ unless $4nm - r^2 \geq 0$.

We say that f is a Jacobi cusp form if it satisfies the condition

(4) $c^M(n, r) = 0$ unless $4nm - r^2 > 0$ for any $M \in \mathrm{GL}_2^+(\mathbb{Q})$ in the above condition (3).

Since $\mathrm{GL}_2(\mathbb{Q})$ is a semidirect product of $\mathrm{SL}_2(\mathbb{Z})$ and the group $P_0(\mathbb{Q})$ of rational upper triangular matrices, and $\mathrm{SL}_2(\mathbb{Z})$ normalizes the subgroup $H(\mathbb{Z}) = \{([\lambda, \mu], \kappa); \lambda, \mu, \kappa \in \mathbb{Z}\}$ of $\Gamma_0(N)^J$, we can replace condition (3) by the following condition.

(3') For any $M \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$f|_{k,m}[M] = \sum_{n \in n_M^{-1}\mathbb{Z}, r \in \mathbb{Z}} c^M(n, r) q^n \zeta^r$$

for some integer n_M depending on M and $c^M(n, r) = 0$ unless $4nm - r^2 \geq 0$.

Here actually we can restrict M to representatives M of the double coset $\Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z}) / (P_0(\mathbb{Q}) \cap \mathrm{SL}_2(\mathbb{Z}))$, which corresponds bijectively to equivalence classes of cusps of $\Gamma_0(N) \backslash \mathfrak{H}_1$. We can change condition (4) for cusp forms in the same way. We denote the space of Jacobi forms defined above by $J_{k,m}(\Gamma_0(N)^J, \chi)$ and Jacobi cusp forms by $J_{k,m}^{\mathrm{cusp}}(\Gamma_0(N)^J, \chi)$. When the conductor of χ is one, we write $J_{k,m}(\Gamma_0(N)^J) = J_{k,m}(\Gamma_0(N)^J, \chi)$, and similarly for cusp forms.

We should note that we are not defining here cusps of Jacobi groups when we define cusp forms. Even if the level is one, there exist more than one Eisenstein series when m is not square free as you see in [8, p. 25]. As for a more representation theoretic characterization of the cuspidality condition, see [6]. Incidentally, we can also see that $f \in J_{k,m}(\Gamma_0(N)^J, \chi)$ is a Jacobi cusp form if and only if the constant term of the Fourier expansion of $f|_{k,m}[M]$ is zero for each $M \in G^J(\mathbb{Q})$. (This fact was shared with the author by S. Böcherer.)

By the way, we note that if the index is one, we have $J_{k,1}(\Gamma_0(N)^J, \chi) = \{0\}$ unless $\chi(-1) = (-1)^k$. This is shown as follows. If $\phi = \sum_{n,r} c(n, r) q^n \zeta^r \in J_{k,1}(\Gamma_0(N), \chi)$, then by the same proof as in [8], we can show that $c(n, r)$ depends only on $4n - r^2$, and in particular, we have $c(n, -r) = c(n, r)$. In addition, by the automorphy with respect to the action of $(-1_2, (0, 0)) \in \Gamma_0(N)^J$, we have $c(n, -r) = \chi(-1)(-1)^k c(n, r)$. So $c(n, r) = 0$ unless $\chi(-1) = (-1)^k$ (see also Section 5 for an alternative proof).

3. Lifting map of level N

In this section, we define the Saito–Kurokawa lifting from $J_{k,1}(\Gamma_0(N)^J, \chi)$ to $A_k(\Gamma_0^{(2)}(N), \chi)$. For that purpose, we first define an operator $V_{l,\chi}$ to change the index of Jacobi forms. For any positive integer l and N , we put

$$\Delta_{N,0}(l) = \left\{ g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}; a, b, c, d \in \mathbb{Z}, \det(g) = l, (a, N) = 1 \right\}$$

and

$$\Delta_{N,0} = \bigcup_{l=1}^{\infty} \Delta_{N,0}(l).$$

Then $\Delta_{N,0}$ is a semigroup. For a Dirichlet character χ with conductor dividing N , we can define a multiplicative function χ of $\Delta_{N,0}$ to \mathbb{C}^\times by $\chi(g) = \chi(a)^{-1}$ for $g = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Delta_{N,0}$.

We define an operator $V_{l,\chi}$ on $J_{k,m}(\Gamma_0(N)^J, \chi)$ by

$$\begin{aligned} & (\phi|_{k,m}V_{l,\chi})(\tau, z) \\ &= l^{k-1} \sum_{g \in \Gamma_0(N) \setminus \Delta_{N,0}(l)} \chi(g)^{-1} \phi|_{k,m}[g] \\ &= l^{k-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \setminus \Delta_{N,0}(l)} \chi(a)(c\tau + d)^{-k} e^{lm} \left(-\frac{cz^2}{c\tau + d} \right) \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d} \right). \end{aligned}$$

LEMMA 3.1

If $\phi \in J_{k,m}(\Gamma_0(N)^J, \chi)$, then $\phi|_{k,m}V_{l,\chi} \in J_{k,ml}(\Gamma_0(N)^J, \chi)$. If $\phi \in J_{k,m}^{\text{cusp}}(\Gamma_0(N)^J, \chi)$, then $\phi|_{k,m}V_{l,\chi} \in J_{k,ml}^{\text{cusp}}(\Gamma_0(N)^J, \chi)$.

Proof

The first half follows from the facts that $\Delta_{N,0}(l)$ is a left- and right-invariant set of $\Gamma_0(N)$ and that $\chi(a_1)^{-1} = \chi(d_1) = \chi(\gamma)$ for any $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_0(N)$. The second assertion is obvious since if ϕ satisfies the condition (4) of the definition of Jacobi cusp forms, then $\phi|_{k,m}[g]$ also satisfies (4) for any $g \in \text{GL}_2^+(\mathbb{Q})$. \square

3.1. Cusp forms

We assume that $\chi(-1) = (-1)^k$. For any $\phi \in J_{k,1}^{\text{cusp}}(\Gamma_0(N)^J, \chi)$, we define a function of $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathfrak{H}_2$ by

$$(L_{N,\chi}\phi)(Z) = \sum_{l=1}^{\infty} (\phi|_{k,1}V_{l,\chi})(\tau, z) e^l(\omega).$$

This is called the Saito–Kurokawa lifting of ϕ . This converges absolutely uniformly on $\{Z = X + iY \in \mathfrak{H}_2; Y \geq c1_2\}$ for any positive constant c , where $Y > c1_2$ means that $Y - c1_2$ is positive definite. The proof of the convergence is sketched as follows. The Fourier coefficients of Jacobi forms of weight k have a bound $|c(n, r)| = |c(4n - r^2)| \leq C(4n - r^2)^{k-1/2}$ for some positive constant C for $4n - r^2 \neq 0$, and hence the Fourier coefficient $A(T)$ of $L_{N,\chi}\phi$, which is given explicitly below, is also bounded by a constant times some power of $\det(T)$ or of a diagonal component of T when $\det(T) = 0$. Then the convergence of the above series is reduced to the convergence of $\sum_{n=1}^{\infty} n^{c_1} e^{-c_2n}$ for any positive constant c_1 and c_2 . We omit the details since this is a standard analysis.

Now we would like to write down the Fourier expansion of $L_{N,\chi}\phi$. Since we have

$$\Gamma_0(N)\backslash\Delta_{N,0}(l) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; a, b, d \in \mathbb{Z}, (a, N) = 1, ad = l, b = 0, \dots, d-1 \right\}$$

(see, e.g., [30]), we have

$$(L_{N,\chi}\phi)(Z) = \sum_{l=1}^{\infty} \sum_{\substack{n,r \in \mathbb{Z} \\ 4nl-r^2 > 0}} l^{k-1} \sum_{\substack{ad=l \\ (a,N)=1}} \sum_{b=0}^{d-1} \chi(a) d^{-k} \phi\left(\frac{a\tau+b}{d}, az\right) e^l(\omega).$$

(In the above summation, we can omit the condition $(a, N) = 1$ since $\chi(a) = 0$ if not by the definition of χ , but sometimes we write this to avoid any confusion.) So, if we write

$$\phi = \sum_{\substack{n,r \in \mathbb{Z} \\ 4n-r^2 > 0}} c(n,r) q^n \zeta^r,$$

where $q = e(\tau)$, $\zeta = e(z)$, then

$$L_{N,\chi}\phi = \sum_{l=1}^{\infty} \sum_{\substack{n,r \in \mathbb{Z} \\ 4nl-r^2 > 0}} \sum_{a|(n,l,r)} a^{k-1} \chi(a) c\left(\frac{nl}{a^2}, \frac{r}{a}\right) q^n \zeta^r e^l(\omega).$$

Since we can show that $c(n,r)$ depends only on $4n-r^2$ for Jacobi forms of index one by the same proof as in [8], we may write $c(n,r) = c(4n-r^2)$. So we have

$$L_{N,\chi}\phi = \sum_{l=1}^{\infty} \sum_{\substack{n,r \in \mathbb{Z} \\ 4nl-r^2 > 0}} \sum_{a|(n,l,r)} \chi(a) a^{k-1} c\left(\frac{4nl-r^2}{a^2}\right) q^n \zeta^r e^l(\omega).$$

THEOREM 3.2

We assume that $\chi(-1) = (-1)^k$. Then the above map $L_{N,\chi}$ gives an injective linear map from $J_{k,1}^{\text{cusp}}(\Gamma_0(N)^J, \chi)$ to $S_k(\Gamma_0^{(2)}(N), \chi)$.

To prove this theorem, we prepare two lemmas. One is on generators of $\Gamma_0^{(2)}(N)$, and the other is on cusps of $\Gamma_0^{(2)}(N)$.

We put

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, $x \in \mathbb{R}$, and $S = {}^t S \in M_2(\mathbb{R})$, we put

$$u(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \iota(g) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u(S) = \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}.$$

LEMMA 3.3 (AOKI AND IBUKIYAMA [3, LEMMA 6.2, P. 265])

For any natural number N , the group $\Gamma_0^{(2)}(N)$ is generated by R , $u(x)$, $u(S)$, and $\iota(M)$, where x , S , or M runs over $x \in \mathbb{Z}$, $S = {}^tS \in M_2(\mathbb{Z})$, or $M \in \Gamma_0(N)$, respectively.

We define the standard maximal parabolic subgroup P_1 of $\mathrm{Sp}(2, \mathbb{Q})$ corresponding to one-dimensional cusps by

$$P_1(\mathbb{Q}) = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Q}) \right\},$$

where $*$ runs over the rational numbers.

LEMMA 3.4

The representatives of the double cosets $\Gamma_0^{(2)}(N) \backslash \mathrm{Sp}(2, \mathbb{Q}) / P_1(\mathbb{Q})$ are chosen from the elements in $P_1(\mathbb{Q})R$, where R is defined as above.

Proof

We put $P'_1 = R^{-1}P_1(\mathbb{Q})R = RP_1(\mathbb{Q})R$. It is sufficient to prove that the representatives of $\Gamma_0^{(2)}(N) \backslash \mathrm{Sp}(2, \mathbb{Q}) / P'_1$ are taken in $P_1(\mathbb{Q})$, since if $\mathrm{Sp}(2, \mathbb{Q}) = \bigsqcup_i \Gamma_0^{(2)}(N) \times g_i P'_1$ for some $g_i \in P_1(\mathbb{Q})$, then $\mathrm{Sp}(2, \mathbb{Q}) = \mathrm{Sp}(2, \mathbb{Q})R = \bigsqcup_i \Gamma_0^{(2)}(N) g_i R P_1(\mathbb{Q})$ and $g_i R \in P_1(\mathbb{Q})R$. Now for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Q})$, we write $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $D = (d_{ij})$ with $1 \leq i, j \leq 2$ and try to change g in the same double coset. First, we can assume that c_{11} or c_{21} is nonzero. Indeed if $c_{11} = c_{21} = 0$, then a_{11} or a_{21} is nonzero since $\det(g) \neq 0$, so multiplying g by $\begin{pmatrix} 12 & 0 \\ N12 & 12 \end{pmatrix} \in \Gamma_0^{(2)}(N)$ from the left, we see that c_{11} or c_{21} is nonzero. For any $x, y \in \mathbb{Q}$, we have $U^t(x, y) = {}^t(z, 0)$ for some $U \in \mathrm{SL}_2(\mathbb{Z})$, and here $z \neq 0$ if $(x, y) \neq (0, 0)$. So multiplying g from the left by $\begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}$ for some $U \in \mathrm{SL}_2(\mathbb{Z})$, we can assume that $c_{21} = 0$ and $c_{11} \neq 0$. We can take $V = (v_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$ so that $(c_{22}, d_{22})V = (0, *)$; so multiplying $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_{11} & 0 & v_{12} \\ 0 & 0 & 1 & 0 \\ 0 & v_{21} & 0 & v_{22} \end{pmatrix} \in P'_1$ from the right, we also assume that $c_{22} = 0$.

Multiplying $\begin{pmatrix} 1 & -c_{11}^{-1}c_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c_{11}^{-1}c_{12} & 1 \end{pmatrix} \in P'_1$ from the right, we can make $c_{12} = 0$, so

that we now have $C = \begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}$ with $c_{11} \neq 0$. Since $C^t D$ and ${}^t A C$ are symmetric, we have $d_{21} = a_{12} = 0$. This means that $g \in P_1(\mathbb{Q})$. So the lemma is proved. \square

Proof of Theorem 3.2

We first show that $L_{N, \chi} \phi \in A_k(\Gamma_0^{(2)}(N), \chi)$. By the Fourier expansion, $L_{N, \chi} \phi$ is invariant by translation by integers, so it is invariant by $u(S)$ with $S = {}^tS \in M_2(\mathbb{Z})$. Also, it is invariant by exchange of l and n , which means that it is invariant by R . Since $\phi|_{k, 1} V_{l, \chi} \in J_{k, l}(\Gamma_0(N))^J, \chi$ by Lemma 3.1, for $\gamma \in \Gamma_0(N)$

we have

$$\begin{aligned} (L_{N,\chi}\phi)|_k[\iota(\gamma)] &= \sum_{l=1}^{\infty} ((\phi|_{k,1}V_{l,\chi})(\tau, z)e^l(\omega))|_k[\iota(\gamma)] \\ &= \sum_{l=1}^{\infty} ((\phi|_{k,1}V_{l,\chi})|_{k,l}[\gamma])(\tau, z)e^l(\omega) \\ &= \sum_{l=1}^{\infty} (\phi|_{k,1}V_{l,\chi})(\tau, z)e^l(\omega)\chi(\gamma) = \chi(\gamma)(L_{N,\chi}\phi). \end{aligned}$$

Since $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-k}\chi(R) = (-1)^k\chi(-1) = 1$, $\det(1_2)^{-k}\chi(u(S)) = 1$, and $\chi(\iota(\gamma)) = \chi(d) = \chi(\gamma)$, we have $(L_{N,\chi}\phi)|_k[g] = \chi(g)L_{N,\chi}\phi$ for all the generators g of $\Gamma_0^{(2)}(N)$, and hence $L_{N,\chi}\phi \in A_k(\Gamma_0^{(2)}(N), \chi)$. Now we prove that $L_{N,\chi}\phi$ is a cusp form. By Lemma 3.1, we have $\phi|_{k,1}V_{l,\chi} \in J_{k,l}^{\text{cusp}}(\Gamma_0(N)^J, \chi)$. By virtue of Lemma 3.4, for any $g \in \text{Sp}(2, \mathbb{Q})$, we have $g = \gamma p_1 R p_2$ for some $p_i \in P_1(\mathbb{Q})$ ($i = 1, 2$), and $\gamma \in \Gamma_0^{(2)}(N)$. So, any $F \in A_k(\Gamma_0^{(2)}(N), \chi)$ is a cusp form if and only if $\Phi(F|_k[p_1 R]) = 0$ where Φ is the Siegel Φ -operator, in other words, if for the Fourier expansion $F|_k[p_1 R] = \sum_{n,r,m} c(n, r, m)q^n \zeta^r e^m(\omega)$ we have $c(n, r, m) = 0$ unless $4nm - r^2 > 0$. This is shown for $L_{N,\chi}\phi$ as follows. We write

$$p_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $p_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Q})$ and $x \in \mathbb{Q}^\times$, $\lambda, \mu, \kappa \in \mathbb{Q}$. We put $p_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 3.1, $V_{l,\chi}\phi$ is a Jacobi cusp form of index l and

$$(\phi|_{k,1}V_{l,\chi})(\tau, z)e^l(\omega)|_k \iota(p_0) = (\phi|_{k,1}V_{l,\chi}|_{k,l}[p_0])(\tau, z)e^l(\omega),$$

so if we write

$$(\phi|_{k,1}V_{l,\chi}|_{k,l}[p_0]) = \sum_{n,r} c_{l,p_0}(n, r)q^n \zeta^r,$$

then $c_{l,p_0}(n, r) = 0$ unless $4nl - r^2 > 0$. Then by calculating the action, we see that

$$\begin{aligned} (L_{N,\chi}\phi)|_k[p_0] &= x^k \sum_{l=1}^{\infty} \sum_{n,r} e(rx\mu)e^l(x^2(\mu\lambda + \kappa)) \\ &\quad \times c_{l,p_0}(n, r)q^{n+\lambda rx+lx^2\lambda^2} \zeta^{rx+2l\lambda x^2} e^l(x^2\omega). \end{aligned}$$

Since

$$4(n + \lambda rx + lx^2\lambda^2)lx^2 - (rx + 2l\lambda x^2)^2 = (4nl - r^2)x^2,$$

this is positive if and only if $4nl - r^2 > 0$. Since the action of R only exchanges τ and ω , we see that the Fourier coefficient of $F|_k[p_1 R]$ at a matrix T vanishes unless T is positive definite. So $L_{N,\chi}\phi$ is a cusp form. \square

3.2. Noncusp forms

Now assume that $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$ where ϕ is not necessarily a Jacobi cusp form. We explain how to modify our definition of $L_{N,\chi}$. If we define $L_{N,\chi}\phi$ as above, then again we have $(L_{N,\chi}\phi)|_k[g] = \chi(g)L_{N,\chi}\phi$, $(L_{N,\chi}\phi)|_k u(S) = L_{N,\chi}\phi$, and $(L_{N,\chi}\phi)|_k u(x) = L_{N,\chi}\phi$ for $g \in \Gamma_0(N)$, $S = {}^t S \in M_2(\mathbb{Z})$, $x \in \mathbb{Z}$. But we cannot say $(L_{N,\chi}\phi)|_k R = \chi(R)L_{N,\chi}\phi$ in general. This is because $c(0)q^0\zeta^0 e^l(\omega)$ is not zero if $c(0) \neq 0$, but there is no $c(0)q^l\zeta^0 e^0(\omega)$ in the series by the previous definition. So to correct this, at least we must add to $L_{N,\chi}\phi$ the term $c(0)\sum_{n=1}^{\infty}\sum_{a|n, (a,N)=1}\chi(a)a^{k-1}q^n$. But to make $L_{N,\chi}\phi$ invariant by $\iota(\gamma)$ for $\gamma \in \Gamma_0(N)$, we must add a modular form in $A_k(\Gamma_0(N), \chi)$. In fact, the series supplied above becomes a modular form if we add a constant term $c(0)L(1-k, \chi)/2$. This seems more or less well known, but there might not exist a good reference for the case when χ is not necessarily primitive, so we give here some details for the convenience of the readers. Let f be a conductor of χ , and let χ^0 be the primitive character modulo f associated with χ . For a natural number k , we put

$$A^*(k, \chi^0) = \frac{(-2\pi i)^k W(\chi^0)}{f^k (k-1)! L(k, \chi^0)}$$

where $W(\chi^0)$ is the Gauss sum associated with χ^0 and $L(s, \chi^0)$ is the Dirichlet L -function. We note that we always have $L(k, \chi^0) \neq 0$ under our assumption. This is clear by the Euler product if $k > 1$, and if $k = 1$, then by our assumption $\chi(-1) = (-1)^k = -1$, χ^0 is not the principal character and $L(1, \chi^0) \neq 0$ by Dirichlet. For any natural number M and a Dirichlet character ψ , we put $\sigma_{k-1, \psi}(n) = \sum_{d|n} \psi(d)d^{k-1}$ and $\sigma_{k-1, \psi}^M(n) = \sum_{d|n, (d,M)=1} \psi(d)d^{k-1}$. When we say that χ is a Dirichlet character modulo N , we understand that $\chi(d) = 0$ if $(d, N) \neq 1$, so we have $\sigma_{k-1, \chi^0}^N(n) = \sigma_{k-1, \chi}(n)$. Let N_0 be the greatest divisor of N which is coprime to f . We put

$$c_{k, \chi}(N) = A^*(k, \overline{\chi^0})^{-1} \sum_{t|N_0} t^{k-1} \mu(t) \chi^0(t),$$

where μ is the Möbius function. By the well-known functional equation of the Dirichlet L -function $L(s, \chi^0)$, we have $A^*(k, \overline{\chi^0}) = (1/2)L(1-k, \chi^0)$, so we have

$$c_{k, \chi}(N) = \frac{1}{2} L(1-k, \chi).$$

Here we understand that the series $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ is continued analytically to the whole s -plane by the relation

$$L(s, \chi) = L(s, \chi^0) \sum_{t|N_0} t^{-s} \mu(t) \chi^0(t) = L(s, \chi^0) \prod_{p|N, p \nmid f} (1 - \chi^0(p)p^{-s}),$$

and we have $L(1-k, \chi) = L(1-k, \chi^0) \prod_{p|N, p \nmid f} (1 - \chi^0(p)p^{k-1})$.

LEMMA 3.5

We assume that $\chi(-1) = (-1)^k$ as before.

(1) We have

$$\sum_{d|(N_0, n)} \mu(d) \chi^0(d) d^{k-1} \sigma_{k-1, \chi^0} \left(\frac{n}{d} \right) = \sigma_{k-1, \chi}(n) = \sigma_{k-1, \chi^0}^N(n).$$

(2) Unless $(k, f, N) = (2, 1, 1)$, the function

$$f_{k, \chi}(\tau) = c_{k, \chi}(N) + \sum_{n=1}^{\infty} \sigma_{k-1, \chi}(n) q^n$$

is a modular form in $A_k(\Gamma_0(N), \chi)$.

Proof

First we prove (1). We put

$$\begin{aligned} C(\chi) &= \sum_{d|(N_0, n)} \mu(d) d^{k-1} \chi^0(d) \sigma_{k-1, \chi^0} \left(\frac{n}{d} \right) \\ &= \sum_{d|(N_0, n)} \mu(d) \chi^0(d) d^{k-1} \sum_{a|(n/d)} a^{k-1} \chi^0(a). \end{aligned}$$

Since $d|N_0$ is coprime to f , $\chi^0(ad) = 0$ if and only if $\chi^0(a) = 0$, so $\chi^0(a) = \chi^0(ad)/\chi^0(d)$. So writing $l = ad$, we have

$$\begin{aligned} C(\chi) &= \sum_{d|(N_0, n), d|l, l|n} \chi^0(d) \mu(d) l^{k-1} \chi^0(l/d) \\ &= \sum_{l|n} \chi^0(l) l^{k-1} \sum_{d|(l, N_0, n)} \mu(d) = \sum_{l|n, (N_0, n, l)=1} \chi^0(l) l^{k-1} \\ &= \sum_{l|n, (N_0, l)=1} \chi^0(l) l^{k-1}. \end{aligned}$$

If $(l, N/N_0) \neq 1$, then $(l, f) \neq 1$ and $\chi^0(l) = 0$. So we have $C(\chi) = \sum_{l|n, (l, N)=1} \chi^0(l) l^{k-1} = \sigma_{k-1, \chi^0}^N(n) = \sigma_{k-1, \chi}(n)$.

Next we prove (2). This seems more or less well known, but for completeness we sketch the proof here. We put

$$E_{k, f}^*(\tau, s, \chi^0) = \frac{1}{2} \sum_{\substack{c \equiv 0 \pmod{f} \\ (c, d)=1}} \chi^0(d) (c\tau + d)^{-k} |c\tau + d|^{-2s}.$$

If $k > 0$, then $E_{k, f}^*(\tau, s; \chi^0)$ is holomorphic at $s = 0$ as a function of s (cf. Miyake [22, Corollary 7.2.10]). We put $E_{k, f}^*(\tau, \chi^0) = E_{k, f}^*(\tau, 0; \chi^0)$. It is obvious that $E_{k, f}^*(\gamma\tau, \chi^0) = \overline{\chi^0}(d) (c\tau + d)^k E_{k, f}^*(\tau, \chi^0)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$, where $\overline{\chi^0}$ is the complex conjugation. By [22, Theorems 7.1.3, 7.1.12, 7.2.13, 7.2.62], we have

$$E_{k, f}^*(\tau, \chi^0) = 1 + D^*(k, \chi^0) y^{-1} + A^*(k, \chi^0) \sum_{m=1}^{\infty} \sigma_{k-1, \overline{\chi^0}}(m) q^m,$$

where

$$D^*(k, \chi^0) = \begin{cases} -\frac{\pi}{2L(k, \chi^0)} & \text{if } k = 2 \text{ and } f = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We put

$$f_{k, \chi}(\tau) = A^*(k, \bar{\chi}^0)^{-1} \sum_{t|N_0} \mu(t) \chi^0(t) t^{k-1} E_{k, f}^*(t\tau, \bar{\chi}^0).$$

Unless $(k, f) = (2, 1)$, this is obviously holomorphic. If $k = 2$, $f = 1$, and $N \neq 1$, then $N_0 = N$, $\chi^0(t) = 1$ for any $t|N$, and since $\text{Im}(\tau) = ty$ and $\sum_{t|N} ((t\mu(t)\chi^0(t))/ty) = y^{-1} \sum_{t|N} \mu(t) = 0$, the term containing y^{-1} disappears and $f_{k, \chi}(\tau)$ is holomorphic also in this case. Since $E_{k, f}^*(\gamma\tau, \bar{\chi}^0) = (c\tau + d)^k \chi^0(d) E_{k, f}^*(\tau)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$, we have

$$E_{k, f}^*(t\gamma\tau, \bar{\chi}^0) = (c\tau + d)^k \chi^0(d) E^*(t\tau, \bar{\chi}^0)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f) \subset \Gamma_0(N)$ with $t|N_0$. So we have $f_{k, \chi}(\tau) \in A_k(\Gamma_0(N), \chi)$. On the other hand, if we calculate the expansion using Lemma 3.5(1), we have

$$f_{k, \chi}(\tau) = A^*(k, \bar{\chi}^0)^{-1} \sum_{t|N_0} \chi^0(t) t^{k-1} \mu(t) + \sum_{n=1}^{\infty} \sigma_{k-1, \chi}(n) q^n.$$

So if we put $c_{k, \chi}(N) = L(1 - k, \chi)/2$, then

$$c_{k, \chi}(N) + \sum_{n=1}^{\infty} \sigma_{k-1, \chi}(n) q^n$$

is a modular form in $A_k(\Gamma_0(N), \chi)$. This proves Lemma 3.5. \square

Now for any natural number k , any Dirichlet character χ modulo N with $\chi(-1) = (-1)^k$, and any $\phi \in J_{k, 1}(\Gamma_0(N)^J, \chi)$ with constant term $c(0)$, we define

$$L_{N, \chi} \phi = c(0) f_{k, \chi}(\tau) + \sum_{l=1}^{\infty} (\phi|_{k, 1} V_{l, \chi})(\tau, z) e^l(\omega).$$

Here we note that $J_{2, 1}(\text{SL}_2(\mathbb{Z})^J) = \{0\}$, and we need not consider the case $N = f = 1$ and $k = 2$ for our purpose. Then we have the following.

THEOREM 3.6

The mapping $L_{N, \chi}$ gives a linear injection from $J_{k, 1}(\Gamma_0(N)^J, \chi)$ into $A_k(\Gamma_0^{(2)}(N), \chi)$.

The injectivity is obvious since the image $L_{N, \chi} \phi$ determines all the Fourier coefficients of ϕ . The rest has already been explained.

3.3. Comparison between different levels

Let N_1 and N be natural numbers such that $N_1|N$. For a Dirichlet character χ_1 modulo N_1 (not necessarily primitive), let χ be the Dirichlet character χ modulo

N defined by $\chi(a) = \chi_1(a)$ if $(a, N) = 1$ and $\chi(a) = 0$ if $(a, N) \neq 1$. Then we have $J_{k,1}(\Gamma_0(N_1)^J, \chi_1) \subset J_{k,1}(\Gamma_0(N)^J, \chi)$. So for $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi_1)$, we can define two maps $L_{N_1, \chi_1} \phi$ and $L_{N, \chi} \phi$. We compare these.

PROPOSITION 3.7

We have

$$(L_{N, \chi} \phi)(Z) = \sum_{d|N, (d, N_1)=1} \mu(d) d^{k-1} \chi_1(d) (L_{N_1, \chi_1} \phi)(dZ) \quad (Z \in \mathfrak{H}_2).$$

Proof

First we compare the positive index part of the Fourier–Jacobi expansion. The positive index part of the right-hand side is given by

$$\begin{aligned} & \sum_{d|N, (d, N_1)=1} \mu(d) d^{k-1} \chi_1(d) \sum_{a|(n, l, r), (a, N_1)=1} \chi_1(a) a^{k-1} \\ & \quad \times c((4ln - r^2)/a^2) q^{dn} \zeta^{dr} e^{dl}(\omega) \\ &= \sum_{d|N, (d, N_1)=1} \mu(d) \sum_{a|(n, l, r), (a, N_1)=1} \chi_1(ad) (ad)^{k-1} \\ & \quad \times c((4nl - r^2)/a^2) q^{dn} \zeta^{dr} e^{dl}(\omega) \\ &= \sum_{m|(n_1, l_1, r_1), (m, N_1)=1} \chi_1(m) m^{k-1} \sum_{d|N, d|m, (d, N_1)=1} \mu(d) c\left(\frac{4n_1 l_1 - r_1^2}{m^2}\right) \\ & \quad \times q^{n_1} \zeta^{r_1} e^{l_1}(\omega). \end{aligned}$$

If $d|m$, then $(d, N_1) = 1$ since $(m, N_1) = 1$. So we have $\sum_{d|N, d|m, (d, N_1)=1} \mu(d) = \sum_{d|(m, N)} \mu(d) = 0$ if $(m, N) \neq 1$ and this sum equal to 1 if $(m, N) = 1$. By the definition of χ , we have

$$\sum_{m|(n_1, l_1, r_1), (m, N)=1} \chi_1(m) m^{d-1} = \sum_{m|(n_1, l_1, r_1)} \chi(m) m^{d-1}.$$

So the right-hand side is nothing but the positive index part of $L_{N, \chi} \phi$. As for the index-zero part, except for the constant term, the proof is essentially the same as above. As for the constant term, we must compare $c_{k, \chi}(N)$ and $c_{k, \chi_1}(N_1)$. We denote by χ^0 the primitive character of the conductor f associated with χ . Then this is also the primitive character associated with χ_1 . Since $A^*(k, \overline{\chi^0})$ depends only on k and χ^0 , the difference between $c_{k, \chi}(N)$ and $c_{k, \chi_1}(N_1)$ comes only from the remaining part. If we denote by N_{10} the greatest divisor of N_1 which is coprime to f , we have

$$\begin{aligned} & \sum_{d|N, (d, N_1)=1} \mu(d) \chi_1(d) d^{k-1} c_{k, \chi_1}(N_1) \\ &= A^*(k, \overline{\chi^0})^{-1} \sum_{d|N, (d, N_1)=1} \mu(d) \chi^0(d) d^{k-1} \sum_{t|N_{10}} \mu(t) \chi^0(t) t^{k-1} \end{aligned}$$

$$\begin{aligned}
 &= A^*(k, \overline{\chi^0})^{-1} \prod_{p|N, p \nmid N_1} (1 - \chi^0(p)p^{k-1}) \prod_{p|N_1, p \nmid f} (1 - \chi^0(p)p^{k-1}) \\
 &= A^*(k, \overline{\chi^0})^{-1} \prod_{p|N, p \nmid f} (1 - \chi(p)p^{k-1}) \\
 &= c_{k, \chi}(N).
 \end{aligned}$$

So the proposition is proved. □

REMARK

When N_1 and N have the same prime divisors, Proposition 3.7 means just $L_{N, \chi} \phi = L_{N_1, \chi_1} \phi \in A_k(\Gamma_0(N_1), \chi_1)$. Of course we still have $(L_{N_1, \chi_1} \phi)(dZ) \in A_k(\Gamma_0(N), \chi)$ for any $d|(N/N_1)$.

EXAMPLE

Assume that $N = p$ is a prime, $N_1 = 1$, and χ_1 is the principal character χ_0 modulo 1. Then for $\phi \in J_{k, 1}(\mathrm{SL}_2(\mathbb{Z})^J)$, we have

$$\begin{aligned}
 (L_{p, \chi} \phi)(Z) &= (L_{1, \chi_0} \phi)(Z) + \mu(p)p^{k-1} \chi_{1, \chi_0}(p) L_{1, \chi_0}(pZ) \\
 &= (L_{1, \chi_0} \phi)(Z) - p^{k-1} (L_{1, \chi_0} \phi)(pZ),
 \end{aligned}$$

where $L_{1, \chi_0} \phi$ is nothing but the usual Saito–Kurokawa lifting of level 1 in [8]. So we can define *two old forms* of level p . Actually there are *three old forms* from level one to p (e.g., Roberts and Schmidt [25] or my thesis [14]).

3.4. Relations between Fourier coefficients

When the level is one, the image of the Saito–Kurokawa lifting is characterized as the space of Siegel modular forms whose Fourier coefficients satisfy certain relations called Maass relations. The fact that the lifted forms satisfy these relations follows directly from the definition of the lifting (see [8]). The surjectivity of the lifting to this space was proved by Andrianov [2] by using the converse theorem. But in this paper, about this direction, we content ourselves with only short remarks here. For any $F \in A_k(\Gamma_0^{(2)}(N), \chi)$, we write the Fourier expansion of F as

$$F(Z) = \sum_T A(T; F) \exp(\mathrm{Tr}(TZ)),$$

where T runs over semipositive definite half-integral symmetric matrices. We consider the following relations between Fourier coefficients:

$$(1) \quad A\left(\begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix}, F\right) = \sum_{\substack{a|(n, l, m) \\ (a, N)=1}} \chi(a)a^{k-1} A\left(\begin{pmatrix} 1 & l/2a \\ l/2a & mn/a^2 \end{pmatrix}, F\right).$$

We denote by $\mathfrak{M}_k(\Gamma_0^{(2)}(N), \chi)$ the linear subspace of forms $F \in A_k(\Gamma_0^{(2)}(N), \chi)$ such that coefficients $A(T, F)$ satisfy the above relations. For a Jacobi form $\phi =$

$\sum_{n,r} c(4n - r^2)q^n \zeta^r \in J_{k,1}(\Gamma_0(N)^J, \chi)$, by definition we have

$$A\left(\begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix}, L_{N,\chi}(\phi)\right) = \sum_{a|(n,l,m)} \chi(a)a^{k-1}c\left(\frac{4nm - l^2}{a^2}\right).$$

In particular, if $(n, l, m) = 1$, then this is equal to $c(4nm - l^2)$, so we have

$$c\left(\frac{4nm - l^2}{a^2}\right) = A\left(\begin{pmatrix} 1 & l/2a \\ l/2a & nm/a^2 \end{pmatrix}, L_{N,\chi}(\phi)\right).$$

This gives the following proposition.

PROPOSITION 3.8

Elements in the image of the Saito–Kurokawa lifting satisfy the relation (1); that is, we have

$$L_{N,\chi}(J_{k,1}(\Gamma_0(N)^J, \chi)) \subset \mathfrak{M}_k(\Gamma_0^{(2)}(N), \chi).$$

The relation (1) is similar to the Maass relation of the Saito–Kurokawa lift of level 1, but there is one big difference when $N > 1$. When $p|N$, the above relation (1) does not say anything about the relation between $A(pT, L_{N,\chi}\phi)$ and $A(T, L_{N,\chi}\phi)$. This relation depends on the action of the Hecke operator $U_J(p)$ at $p|N$ which is defined in Section 4. This action essentially comes from the action of a Hecke operator at the bad prime p on $S_{2k-2}(\Gamma_0(N), \chi^2)$, and candidates of the eigenvalues (if it is an eigenform) are fairly restricted, as we see for example in [22, Theorem 4.6.17, p. 170], but there are several cases. Also, it is well described only for new forms. So if we want to show any surjective of the Saito–Kurokawa lifting to a space of Siegel modular forms characterized by a sort of Maass relation, then maybe we should describe them for each space of new forms. Also, a generalization of the arguments in [2] using the converse theorems for general cases would be more complicated than the level-one case. So we do not want to go further here, and we would like to leave the characterization of the images by relations of Fourier coefficients as an open problem.

3.5. Old forms coming from higher indices

Since $J_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$ has a correspondence with a certain subspace of $M_{2k-2}(\Gamma_0(m))$ (see [31]) and $J_{k,1}(\Gamma_0(N)^J)$ has a correspondence to a subspace of $M_{2k-2}(\Gamma_0(N))$, we can expect that there is a mapping from $J_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J)$ to $J_{k,1}(\Gamma_0(m)^J)$. We give a simple remark when $m = p$ is a prime for this direction, though we do not go further for a possible generalization.

For any function $f(\tau, z)$ on $\mathfrak{H}_1 \times \mathbb{C}$, we define a function $R_p f$ on $\mathfrak{H}_1 \times \mathbb{C}$ by $(R_p f)(\tau, z) = f(p\tau, z)$. If $f \in J_{k,p}(\mathrm{SL}_2(\mathbb{Z})^J)$, then we have $R_p f|_{k,1}[\gamma] = R_p f$ for any $\gamma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma_0(p)$, since

$$\begin{aligned} f\left(\frac{p(a\tau + b)}{pc\tau + d}, \frac{z}{pc\tau + d}\right) &= f\left(\frac{a(p\tau) + pb}{c(p\tau) + d}, \frac{z}{c(p\tau) + d}\right) \\ &= (c(p\tau) + d)^{-k} e^p (cz^2 / (c(p\tau) + d)) f(p\tau, z). \end{aligned}$$

On the other hand, as for the action of $(\lambda, \mu) \in \mathbb{Z}^2$, we have

$$(R_p f)|_1(\lambda, \mu) = e(\lambda^2 \tau + 2\lambda z) f(p\tau, z + \lambda\tau + \mu).$$

This is equal to $R_p f$ if $\lambda \in p\mathbb{Z}$, but it might not be equal for the other $\lambda \in \mathbb{Z}$. So $R_p f$ might not belong to $J_{k,1}(\Gamma_0(p)^J)$. To make this invariant also by the Heisenberg part, we take an average. For any f on $\mathfrak{H}_1 \times \mathbb{C}$ such that $f|_1[(p\lambda, 0)] = f$ for any $\lambda \in \mathbb{Z}$, we define

$$(S_p f)(\tau, z) = \sum_{\lambda=0}^{p-1} f|_1[(\lambda, 0)] = \sum_{\lambda=0}^{p-1} e(\lambda^2 \tau + 2\lambda z) f(\tau, z + \lambda\tau).$$

PROPOSITION 3.9

We assume that k is even and p is an odd prime. For any $f \in J_{k,p}(\mathrm{SL}_2(\mathbb{Z})^J)$, we have $S_p R_p f \in J_{k,1}(\Gamma_0(p)^J)$. Besides, this mapping $S_p R_p$ is injective.

Proof

It is clear that $S_p R_p f$ is invariant by \mathbb{Z}^2 . We have $(\lambda, 0)\gamma = \gamma \times ((\lambda, 0)\gamma)$ for $\gamma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma_0(p)$. Since $(\lambda, 0)g = (a\lambda, b\lambda)$ and $R_p f|_1(a\lambda, b\lambda) = R_p f|_1(a\lambda, 0)$ and $a\lambda$ runs over the representatives modulo p when λ runs over the same set, we see that $S_p R_p f$ is invariant by $\Gamma_0(p)$. The injectivity is proved by seeing the action on the Fourier expansion. We write

$$f(\tau, z) = \sum_{n,r} c(n, r) q^n \zeta^r$$

as usual. Since $f \in J_{k,p}(\mathrm{SL}_2(\mathbb{Z})^J)$, we have $c(n_1, r_1) = c(n_2, r_2)$ if $4n_1p - r_1^2 = 4n_2p - r_2^2$ and $r_1 \equiv r_2 \pmod{2p}$. We also have $c(n, -r) = (-1)^k c(n, r) = c(n, r)$ by the action of -1_2 . We have

$$(S_p R_p f)(\tau, z) = \sum_{\lambda=0}^{p-1} c(n, r) q^{np+\lambda^2+\lambda r} \zeta^{r+2\lambda}.$$

Now fix n_0 and r_0 , and find n, r , and λ with $0 \leq \lambda \leq p-1$ such that $n_0 = np + \lambda^2 + \lambda r$ and $r_0 = r + 2\lambda$. Since $n = (n_0 - \lambda r_0 + \lambda^2)/p$ and $r = r_0 - 2\lambda$, the existence of the integer n is assured only when $((r_0^2 - 4n_0)/p) \neq -1$. If there is only one such λ modulo p , then n, r are unique and the Fourier coefficient is $c(n, r)$. If there are λ_1, λ_2 which satisfy the condition, then we put $r_i = r_0 - 2\lambda_i$, $n_i = (n_0 - \lambda_i r_0 + \lambda_i^2)/p$ for $i = 1, 2$. Since $4n_1p - r_1^2 = 4n_2p - r_2^2 = 4n_0 - r_0^2$, we have $r_1 \equiv \pm r_2 \pmod{2p}$. If $r_1 \equiv r_2 \pmod{2p}$, then we have $c(n_1, r_1) = c(n_2, r_2)$. Even if $r_1 \equiv -r_2 \pmod{p}$, we have $c(n_1, r_1) = c(n_2, -r_2) = c(n_2, r_2)$ since we assume that k is even. So the coefficient of $q^{n_1} \zeta^{r_1}$ is $2c(n_1, r_1)$ in this case. In any case, if $S_p R_p f = 0$, then we have $c(n, r) = 0$ for all n, r , so $f = 0$. □

4. Hecke operators

4.1. Definitions and results

The Hecke theory of the Saito–Kurokawa lifting can be developed mostly in the same way as in [8], but there are two differences. One is the existence of bad

primes. In general, the local Hecke algebra with respect to $\Gamma_0^{(n)}(N)$ at a prime $p|N$ is noncommutative even if $n = 1$ and is complicated. But here we define Hecke operators at bad primes only to the extent that we can define a good L -function for $n = 2$, and in that sense this is not so complicated. Another is the point that, in Eichler and Zagier [8], it had been known that the space consisting of lifted forms is invariant by Hecke operators by Andrianov [2]. We have no such results for level N , so we must do this complicated part. This makes our proof a little longer than the one in [8]. For completeness, we give almost all the proofs again here.

First we define Hecke operators on $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$. For any prime $p \nmid N$, we put

$$(2) \quad \phi|_{k,\chi} T_J(p) = p^{k-4} \sum_{g_\nu} \sum_{(\lambda, \mu) \in (\mathbb{Z}/p\mathbb{Z})^2} \chi(a_\nu) \phi|_{k,1}[g_\nu/p]_1[\lambda, \mu],$$

where a_ν is the $(1, 1)$ -component of g_ν and g_ν runs over a complete set of representatives of

$$(3) \quad \Gamma_0(N) \setminus \{g \in \Delta_{0,N}(p^2); \gcd(g) = \square\}.$$

Here $\gcd(g)$ means the greatest common divisor of the components of g and \square means a square integer. The only difference from the case of $N = 1$ is the condition $(a_\nu, N) = 1$, and the representatives are given, for example, by

$$(4) \quad \left\{ \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} (b = 0, \dots, p^2 - 1), \begin{pmatrix} p & b \\ 0 & p \end{pmatrix} (b = 1, \dots, p - 1), \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

For $p|N$, we define an operator $U_J(p)$ by the same expression as (2), but here we let g_ν run over the set of representatives of

$$(5) \quad \Gamma_0(N) \setminus \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \Gamma_0(N) = \bigcup_{b \bmod p^2} \Gamma_0(N) \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix}.$$

Incidentally, in [8], there is a definition of Hecke operators for primes which divide the index. These operators are more complicated than those you see there. Since we are assuming that the index is one in our context, we do not treat them here.

The Hecke operators of Siegel modular forms are defined as usual. Assume that, for $g \in G\mathrm{Sp}^+(2, \mathbb{Q}) \cap M_4(\mathbb{Z})$ with $gJ^t g = n(g)J$, the double coset has the following coset decomposition:

$$T = \Gamma_0^{(2)}(N)g\Gamma_0^{(2)}(N) = \bigcup_{\nu} \Gamma_0^{(2)}(N) \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix}.$$

Here we may assume that each $\det(A_\nu)$ is coprime to N in all the cases we treat below. Then for any $F \in A_k(\Gamma_0^{(2)}(N), \chi)$, we define an action of T by

$$F|_{k,\chi} T = n(g)^{2k-3} \sum_{\nu} \chi(\det(A_\nu)) \det(C_\nu Z + D_\nu)^{-k} F((A_\nu Z + B_\nu)(C_\nu Z + D_\nu)^{-1}).$$

When $p \nmid N$, for any diagonal matrix $\text{diag}(p^a, p^b, p^c, p^d)$ with $a + c = b + d$ and $p \nmid N$, we put

$$T_S(p^a, p^b, p^c, p^d) = \Gamma_0^{(2)}(N) \begin{pmatrix} p^a & 0 & 0 & 0 \\ 0 & p^b & 0 & 0 \\ 0 & 0 & p^c & 0 \\ 0 & 0 & 0 & p^d \end{pmatrix} \Gamma_0^{(2)}(N).$$

We put $T_S(p) = T_S(1, 1, p, p)$ and $T_S(p^2) = T_S(1, p, p^2, p) + T_S(1, 1, p^2, p^2) + T_S(p, p, p, p)$. We put $T'_S(p) = T_S(p)^2 - T_S(p^2)$. Then we have

$$T'_S(p) = pT_S(1, p, p^2, p) + p(1 + p + p^2)T_S(p, p, p, p).$$

Since this is simpler than $T_S(p^2)$, we use this below. When $p|N$, we put $U_S(p) = \Gamma_0^{(2)}(N)\text{diag}(1, 1, p, p)\Gamma_0^{(2)}(N)$.

THEOREM 4.1

For any $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$, if $p \nmid N$, then we have

$$\begin{aligned} (L_{N,\chi}\phi)|_{k,\chi}T_S(p) &= L_{N,\chi}((\phi|_{k,1}T_J(p)) + \chi(p)(p^{k-2} + p^{k-1})\phi), \\ (L_{N,\chi}\phi)|_{k,\chi}T'_S(p) &= L_{N,\chi}(\chi(p)(p^{k-2} + p^{k-1})(\phi|_{k,\chi}T_J(p)) \\ &\quad + \chi(p)^2(2p^{2k-3} + p^{2k-4})\phi), \end{aligned}$$

and if $p|N$, then we have

$$(L_{N,\chi}\phi)|_{k,\chi}U_S(p) = L_{N,\chi}(\phi|_{k,1}U_J(p)).$$

REMARK

Let $F \in A_k(\Gamma_0^{(2)}(N), \chi)$ be a common eigenform of all $T_S(p)$, $T_S(1, p, p^2, p)$ ($p \nmid N$), and $U_S(p)$ ($p|N$). Denote by $\lambda(p)$, $\omega(p^2)$, or $\mu(p)$ each eigenvalue corresponding to each of these operators. Then the L -function of $F \in A_k(\Gamma_0(N), \chi)$ including bad primes is defined, for example, in [21] by

$$L(s, F) = \prod_{p|N} (1 - \mu(p)p^{-s})^{-1} \prod_{p \nmid N} Q_p(p^{-s})^{-1},$$

where

$$\begin{aligned} Q_p(p^{-s}) &= 1 - \lambda(p)p^{-s} + (p\omega(p^2) + (p^2 + 1)\chi(p)^2p^{2k-5})p^{-2s} \\ &\quad - \lambda(p)\chi(p)^2p^{2k-3-3s} + \chi(p)^4p^{4k-6-4s}. \end{aligned}$$

On the other hand, if we denote by $\lambda_J(p)$ the eigenvalues of $T_J(p)$ or $U_J(p)$ of a common eigenfunction $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$, the above theorem means that

$$\begin{aligned} L(s, L_{N,\chi}\phi) &= L(s - k + 1, \chi)L(s - k + 2, \chi) \\ &\quad \times \prod_{p|N} (1 - \lambda_J(p)p^{-s})^{-1} \prod_{p \nmid N} (1 - \lambda_J(p)p^{-s} + \chi(p)^2p^{2k-3-2s})^{-1} \end{aligned}$$

where $L(s, \chi)$ is the Dirichlet L -function. In particular, if ϕ has a nonzero constant term, then we have

$$L(s, L_{N, \chi} \phi) = \zeta(s) L(s - 2k + 3, \chi^2) L(s - k + 1, \chi) L(s - k + 2, \chi),$$

where $\zeta(s)$ is the Riemann zeta function.

Since we see that $J_{k,1}(\Gamma_0(N)^J, \chi) \cong S_{k-1/2}^+(\Gamma_0(N), \chi \psi^k)$ (Kohnen’s plus subspace, see [19] or [4]) and $S_{k-1/2}^+(\Gamma_0(N), \chi)$ corresponds with $S_{2k-2}(\Gamma_0(N), \chi^2)$ by Shimura correspondence, the part coming from ϕ is essentially the L -function of modular forms of integral weight $2k - 2$. We do not go in this direction here since we restrict ourselves in this paper to correspondence between Jacobi forms and Siegel modular forms.

4.2. Proof

We denote the Fourier coefficients of any $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$ by

$$\phi(\tau, z) = \sum_{n,r,4n-r^2 \geq 0} c(n, r; \phi) q^n \zeta^r.$$

Now we fix ϕ and write $c(4n - r^2) = c(n, r; \phi)$. We calculate the action of Hecke operators on the Fourier coefficients of ϕ and $L_{N, \chi} \phi$. For an odd prime p and any integer a , we define a Dirichlet character ψ_p by $\psi_p(a) = (a/p)$, which is the usual quadratic residue symbol. When $p = 2$, we define $\psi_2(a) = 1, -1$, or 0 if $a \equiv 1 \pmod 8, 5 \pmod 8$, or otherwise, respectively. Then the Fourier coefficient of $\phi|_{k, \chi} T_J(p)$ is given by

$$\begin{aligned} & c(n, r; \phi|_{k, \chi} T_J(p)) \\ &= c(p^2(4n - r^2)) + p^{k-2} \chi(p) \psi_p(r^2 - 4n) c(4n - r^2) + p^{2k-3} \chi(p)^2 c\left(\frac{4n - r^2}{p^2}\right) \end{aligned}$$

where we understand that $c(x) = 0$ if $x \notin \mathbb{Z}$ (and $c(x) = 0$ unless $x \equiv 0$ or $3 \pmod 4$ by definition). For $p|N$, we have

$$\phi|U_J(p) = \sum_{n,r} c(p^2(4n - r^2)) q^n \zeta^r.$$

Actually this is the same formula as that of $\phi|_{k, \chi} T_J(p)$ above since we are assuming that $\chi(p) = 0$ for $p|N$. For each p with $p \nmid N$ or $p|N$, these formulas are easily shown by using explicit representatives of (4) for (3), or (5), and the details of the proof are omitted here.

Next we give the Fourier coefficients of $F|_{k, \chi} T_S(p)$ and $F|_{k, \chi} T_S(1, p, p^2, p)$ for $F \in A_k(\Gamma_0(p), \chi)$. We write $F(Z) = \sum_T A(T, F) e(Tr(TZ))$ where T runs over positive semidefinite half-integral matrices. Sometimes we write $A(T) = A(T, F)$ for short. For any p , we put

$$R(p) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in \mathbb{Z}, 0 \leq x \leq p - 1 \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

For $T = \begin{pmatrix} n & r/2 \\ r/2 & l \end{pmatrix}$, we define the content of T by $\text{Cont}(T) = \gcd(l, r, n)$, and for $U \in R(p)$, we write $UT^tU = \begin{pmatrix} n_U & r_U/2 \\ r_U/2 & l_U \end{pmatrix}$. Then it is well known that for $p \nmid N$, the Fourier coefficient of $F|_{k,\chi} T_S(p)$ at T is given by

$$A(pT) + \chi(p)p^{k-2} \sum_{U \in R(p)} A \begin{pmatrix} n_U/p & r_U/2 \\ r_U/2 & p l_U \end{pmatrix} + \chi(p)^2 p^{2k-3} A(T/p),$$

where we understand that $A(*) = 0$ if $*$ is not a half-integral matrix (see [1]). For $p|N$, the Fourier coefficient of $F|_{k,\chi} U_S(p)$ is given by $A(pT)$, and this is the same expression as above since $\chi(p) = 0$ for $p|N$. On the other hand, the representatives of $\Gamma_0^{(2)}(N) \backslash T_S(1, p, p^2, p)$ are given by

$$\left\{ \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}; U \in R(p) \right\} \\ \cup \left\{ \begin{pmatrix} p1_2 & A \\ 0 & p1_2 \end{pmatrix}; A = {}^tA \text{ with } \text{rank}(A \bmod p) = 1, A \text{ runs over rep. mod. } p \right\} \\ \cup \left\{ \begin{pmatrix} p & 0 & 0 & pb_1 \\ 0 & 1 & b_1 & b_2 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}; b_1 \bmod p, b_2 \bmod p^2, U \in R(p) \right\}.$$

If we put $\zeta_p = \exp(2\pi i/p)$, then by a standard calculation of exponential sums including $p = 2$, we have

$$\sum_{\substack{xz=y^2, \\ (x,y,z) \in \mathbb{F}_p^3 - \{(0,0,0)\}}} \zeta_p^{xn+yr+zl} = \begin{cases} p^2 - 1 & \text{if } p|(l, r, n), \\ p\psi_p(r^2 - 4nl) - 1 & \text{otherwise.} \end{cases}$$

Using this formula, the Fourier coefficient at T of $F|_{k,\chi} T_S(1, p, p^2, p)$ is calculated and given by the sum of the following (6)–(8):

$$(6) \quad \sum_{U \in R(p)} p^{3k-6} \chi(p)^3 A \begin{pmatrix} n_U/p^2 & r_U/2p \\ r_U/2p & l_U \end{pmatrix},$$

$$(7) \quad p^{2k-6} \chi(p^2) A(T) \times \begin{cases} p\psi_p(r^2 - 4nl) - 1 & \text{if } p \nmid \text{Cont}(T), \\ p^2 - 1 & \text{if } p | \text{Cont}(T), \end{cases}$$

$$(8) \quad p^{k-3} \chi(p) \sum_{U \in R(p)} A \begin{pmatrix} n_U & r_U p/2 \\ r_U p/2 & l_U p^2 \end{pmatrix},$$

where $\text{Cont}(T) = \gcd(n, r, l)$. If $F = L_{N,\chi} \phi$ for $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$, then we have

$$A(T) = \sum_{a | \text{Cont}(T)} \chi(a) a^{k-1} c(\det(2T)/a^2),$$

which depends only on $\det(2T)$ and $\text{Cont}(T) = \gcd(l, n, r)$. So it is important to obtain the contents of the matrices which appear in the action. We denote by t the greatest integer such that $p^t \mid \text{Cont}(T)$. We write $T_0 = p^{-t}T = \begin{pmatrix} n_0 & r_0/2 \\ r_0/2 & l_0 \end{pmatrix}$. If we write $\text{Cont}(T) = p^t C_0$, then C_0 is coprime to p and $\text{Cont}(T_0) = C_0$. For simplicity, for any $T = \begin{pmatrix} n & r/2 \\ r/2 & l \end{pmatrix}$, we put

$$R_p(T) = \left\{ \begin{pmatrix} n_U & pr_U/2 \\ pr_U/2 & p^2 l_U \end{pmatrix}; U \in R(p) \right\}.$$

For odd p , we put $\mathbb{Z}(p) = \mathbb{Z}$, and for $p = 2$, we define $\mathbb{Z}(2)$ by the set of integers x such that $x \equiv 0$ or $3 \pmod{4}$.

PROPOSITION 4.2

For any prime p , the content of any matrix in the set $R_p(T)$ is either $p^t C_0$, $p^{t+1} C_0$, or $p^{t+2} C_0$. The number of matrices in $R_p(T)$ with each content is given by

$$\begin{cases} p^t C_0, & p - \psi_p(-\det(2T_0)), \\ p^{t+1} C_0, & 1 + \psi_p(-\det(2T_0)) \text{ if } \det(2T_0)/p^2 \notin \mathbb{Z}(p), \text{ and } 0 \text{ otherwise,} \\ p^{t+2} C_0, & 1 + \psi_p(-\det(2T_0)) = 1 \text{ if } \det(2T_0)/p^2 \in \mathbb{Z}(p), \text{ and } 0 \text{ otherwise.} \end{cases}$$

Proof

It is enough to prove this for T_0 and $t = 0$. For the sake of simplicity, we denote each element of $R_p(T_0)$ by

$$T_0(x) = \begin{pmatrix} l_0 x^2 + r_0 x + n_0 & p(2l_0 x + r_0)/2 \\ (2l_0 x + r_0)/2 & p^2 l_0 \end{pmatrix} \quad (0 \leq x \leq p-1),$$

$$T'_0 = \begin{pmatrix} l_0 & -pr_0/2 \\ -pr_0/2 & p^2 n_0 \end{pmatrix}.$$

First we prove the case $p = 2$. We assume that r_0 is odd. Then $2l_0 x + r_0$ is odd and $\text{ord}_2(\text{Cont}(T_0(x))) \leq 1$. It is $2C_0$ for x such that $l_0 x^2 + r_0 x + n_0 \equiv 0 \pmod{2}$, and C_0 otherwise. For $x = 0$, it is n_0 . For $x = 1$, it is $l_0 + r_0 + n_0$. On the other hand, $\text{Cont}(T'_0) = 2C_0$ if l_0 is even and C_0 if l_0 is odd. So if both l_0 and n_0 are even, then the number of matrices in $R_p(T_0)$ with a given content is 2 for $2C_0$ and 1 for C_0 . If one of l_0 and n_0 is odd and one is even, then again it is 2 for $2C_0$ and 1 for C_0 . In all these cases, we have $r_0^2 - 4l_0 n_0 \equiv 1 \pmod{8}$, so $\psi(r_0^2 - 4l_0 n_0) = 1$. If both l_0 and n_0 are odd, then we have 3 for C_0 and $\psi(r_0^2 - 4l_0 n_0) = -1$ since $r_0^2 - 4l_0 n_0 \equiv 1 - 4 \equiv 5 \pmod{8}$. So $2 - \psi(r_0^2 - 4l_0 n_0)$ for C_0 and $1 + \psi(r_0^2 - 4l_0 n_0)$ for $2C_0$. So the case when r_0 is odd is as the claim. If r_0 is even, we always have $\psi(r_0^2 - 4l_0 n_0) = 0$. Then $\text{Cont}(T_0(x)) = C_0, 2C_0$, or $4C_0$ according to whether $l_0 x^2 + r_0 x + n_0$ is odd, $\equiv 2 \pmod{4}$, or $0 \pmod{4}$. When $x = 0$, this amounts to saying that n_0 is odd, $\equiv 2 \pmod{4}$, or that $n_0 \equiv 0 \pmod{4}$. When $x = 1$, this amounts to saying that $l_0 + n_0$ is odd, $l_0 + r_0 + n_0 \equiv 2 \pmod{4}$, or $l_0 + r_0 + n_0 \equiv 0 \pmod{4}$. As for T'_0 , the content is $C_0, 2C_0$, or $4C_0$ according to whether l_0 is odd, $l_0 \equiv 2 \pmod{4}$, or $l_0 \equiv 0 \pmod{4}$. (The fact that the content is not divisible by 8 is proved by the assumption that $\gcd(l_0, n_0, r_0)$ is odd.) So

the content is $2C_0$ if $n_0 \equiv 2 \pmod 4$ or $l_0 + r_0 + n_0 \equiv 2 \pmod 4$ or $l_0 \equiv 2 \pmod 4$. By the assumption that $\gcd(n_0, l_0, r_0)$ is odd, only one of these can occur. Besides, we have

$$\frac{4l_0n_0 - r_0^2}{4} = l_0n_0 - \left(\frac{r_0}{2}\right)^2.$$

In each case above, we see that $l_0n_0 \equiv 2 \pmod 4$, so $l_0n_0 - (r_0/2)^2 \equiv 2$ or $1 \pmod 4$, and this does not belong to $\mathbb{Z}(2)$. The content is $4C_0$ if $l_0 \equiv 0 \pmod 4$, $n_0 \equiv 0 \pmod 4$, or $l_0 + r_0 + n_0 \equiv 0 \pmod 4$. Again by the assumption that $\gcd(l_0, n_0, r_0)$ is odd, only one of these can occur. In the first two cases, we have

$$\frac{4l_0n_0 - r_0^2}{4} = l_0n_0 - \left(\frac{r_0}{2}\right)^2 \equiv -(r_0/2)^2 \equiv 0 \text{ or } 3 \pmod 4.$$

If $l_0 + r_0 + n_0 \equiv 0 \pmod 4$, then of course $l_0 + n_0$ is even, and besides, if $r_0/2$ is odd, then $(l_0 + n_0)/2$ is odd, so $l_0 = 2l_1$, $n_0 = 2n_1$ with $l_1 + n_1$ odd, or $l_0 = 4l_1 + 1$, $n_0 = 4n_1 + 1$, or $l_0 = 4n_1 + 3$, $n_0 = 4n_1 + 3$. In the latter case $l_0n_0 \equiv 1 \pmod 4$. So anyway, we have $l_0n_0 - (r_0/2)^2 \equiv 0$ or $3 \pmod 4$. If $r_0/2$ is even, then $(l_0 + n_0)/2$ is even, so $l_0 \equiv -n_0 \pmod 4$ and $l_0n_0 \equiv -l_0^2 \pmod 4$, so $l_0n_0 - (r_0/2)^2 \equiv -l_0^2 \equiv 0$ or $-1 \pmod 4$, so this is in $\mathbb{Z}(2)$. So there is one $4C_0$ or $2C_0$ according to $(4n_0l_0 - r_0^2)/4 \in \mathbb{Z}(2)$ or not. The other two have content C_0 . So we prove the case $p = 2$.

Next we consider the case when p is odd. First we assume that $p \nmid l_0$. Then $\text{Cont}(T'_0) = C_0$ and $\text{Cont}(T_0(x)) = C_0$ for $p - (1 + \psi(r_0^2 - 4l_0n_0))$ numbers of x . So the number of matrices whose content is C_0 is $p - \psi(-\det(2T_0))$. We have $l_0x^2 + r_0x + n_0 \equiv 0 \pmod{p^2}$ if and only if $4l_0(l_0x^2 + r_0x + n_0) = (2l_0x + r_0)^2 + 4l_0n_0 - r_0^2 \equiv 0 \pmod{p^2}$. So $\text{Cont}(T(x)) = p^2C_0$ if and only if $2l_0x + r_0 \equiv 0 \pmod p$ and $4l_0n_0 - r_0^2 \equiv 0 \pmod{p^2}$. So if $4l_0n_0 - r_0^2 \equiv 0 \pmod{p^2}$, then there is exactly one matrix whose content is p^2C_0 . Now assume that $4l_0n_0 - r_0^2 \not\equiv 0 \pmod{p^2}$. If $r_0^2 - 4l_0n_0 \not\equiv 0 \pmod p$ and $l_0x^2 + r_0x + n_0 \equiv 0 \pmod p$, then $2l_0x + r_0 \not\equiv 0 \pmod p$ and the content is pC_0 . If $r_0^2 - 4n_0l_0 \equiv 0 \pmod p$ and $\not\equiv 0 \pmod{p^2}$, and in addition $l_0x^2 + r_0x + n_0 \equiv 0 \pmod p$, then $2l_0x + r_0 \equiv 0 \pmod p$ and $l_0x^2 + r_0x + n_0 \not\equiv 0 \pmod{p^2}$. So there exist $1 + \psi(r_0^2 - 4l_0n_0)$ matrices whose content is pC_0 . So we prove the claim. Now assume that $p|l_0$. If in addition $p|r_0$, then $p \nmid n_0$ since $\gcd(l_0, n_0, r_0)$ is coprime to p , so the content of $T_0(x)$ is C_0 for any x . There are $p - \psi(r_0^2 - 4n_0l_0)$ such x . If the content of $T_0(x)$ is divisible by pC_0 , then $p \nmid r_0$. This means that $2l_0x + r_0 \not\equiv 0 \pmod p$, so the content is exactly pC_0 . Also, we have $\psi(r_0^2 - 4l_0n_0) = 1$ and $l_0x^2 + r_0x + n_0 \equiv r_0x + n_0 \equiv 0 \pmod p$ only for one x . For the rest of x , the content is C_0 and the number is $p - 1 = p - \psi(r_0^2 - 4l_0n_0)$. The content of T'_0 is p^2C_0 if $p^2|l_0$ and $p|r_0$. If $p \nmid r_0$ or $p \nmid l_0$, then the content is pC_0 . These are distinguished by $\det(2T_0)$ as follows. If $p^2|l_0$ and $r_0 \nmid p$, then $\det(2T_0) \equiv 0 \pmod{p^2}$. If r_0 is odd, then of course $r_0^2 - 4l_0n_0 \not\equiv 0 \pmod{p^2}$, and the content is pC_0 . The number of such matrices is, together with $T_0(x)$, given by $2 = 1 + \psi(r_0^2 - 4l_0n_0)$. If $p|r_0$ and $p \nmid l_0$, then $r_0^2 - 4l_0n_0 \equiv 0 \pmod p$ but $\not\equiv 0 \pmod{p^2}$, and the content is pC_0 . The number of such matrices is in this case given by $1 = 1 + \psi(r_0^2 - 4n_0l_0)$. Hence we prove Proposition 4.2 in all the cases. \square

Now we can use this proposition to compare the action of the Hecke operators at T , since the complicated terms in the Hecke operators in question can be calculated by the contents of the matrices in $R_p(p^{-1}T)$, $R_p(T)$ or $R_p(pT)$. Notation being as above, we write $M = \det(2T)$ and $M_0 = \det(2T_0)$ for the sake of simplicity.

Proof of Theorem 4.1

We compare the Fourier coefficients. First we consider $T_S(p)$. It is easy to see from the definitions that the Fourier coefficient of $L_{N,\chi}(\phi|_{k,\chi}T_J(p))$ at T is given by

$$(9) \quad \sum_{a| \text{Cont}(T)} \chi(a)a^{k-1}c\left(\frac{p^2M}{a^2}\right)$$

$$(10) \quad + \chi(p)p^{k-2} \sum_{a| \text{Cont}(T)} \chi(a)a^{k-1}\psi\left(-\frac{M}{a^2}\right)c\left(\frac{M}{a^2}\right)$$

$$(11) \quad + \chi(p)^2p^{2k-3} \sum_{a| \text{Cont}(T)} \chi(a)a^{k-1}c\left(\frac{M}{a^2p^2}\right).$$

Since $\psi(M/a^2) = 0$ if M/a^2 is divisible by p , the term (10) is rewritten as

$$\chi(p)p^{k-2} \sum_{a_0|C_0} \chi(a_0)a_0^{k-1}\psi\left(-\frac{M_0}{a_0^2}\right)c\left(\frac{M_0}{a_0^2}\right).$$

Now we compare this with the coefficients of $(L_{N,\chi}(\phi)|_{k,\chi}T_S(p))$. We write $\text{Cont}(T) = p^tC_0$ with $p \nmid C_0$ and $T_0 = p^{-t}T$ as above. In the second term containing $U \in R(p)$ in the formula of the Fourier coefficient of $F|_{k,\chi}T_S(p)$, by Proposition 4.2, the number of matrices UT^tU with a given content is $p - \psi(-\det(2T_0))$ for $\text{Cont}(UT^tU) = p^{t-1}C_0$ (or zero if $t = 0$), or $1 + \psi(-\det(2T_0))$ for $\text{Cont}(UT^tU) = p^tC_0$ or $p^{t+1}C_0$, respectively, in the case $\det(2T_0)/p^2 \notin \mathbb{Z}(p)$ or $\in \mathbb{Z}(p)$. In particular if $C(UT^tU) = p^{t+1}C_0$, then we may assume that $\psi(-\det(2T_0)) = 0$. Now adding the terms containing $\psi(-\det(2T_0))$ coming from this part of the coefficients, we have

$$\begin{aligned} & \psi(-\det(2T_0)) \left(- \sum_{a|p^{t-1}C_0} \chi(a)a^{k-1}c\left(\frac{M}{a^2}\right) + \sum_{a|p^tC_0} \chi(a)a^{k-1}c\left(\frac{M}{a^2}\right) \right) \\ &= \psi(-\det(2T_0)) \sum_{a=p^t a_0, a_0|C_0} \chi(a)a^{k-1}c(M/a^2) \\ &= \sum_{a|p^tC_0} \chi(a)a^{k-1}\psi(-M/a^2)c(M/a^2), \end{aligned}$$

since if $a = p^u a_0$ with $a_0|C_0$ and $u < t$, then $\psi(-M/a^2) = 0$. This coincides with (10). Next, we have

$$A(pT) = \sum_{a|p^tC_0} \chi(a)a^{k-1}c\left(\frac{p^2M}{a^2}\right) + \sum_{a=p^{t+1}a_0, a_0|C_0} \chi(a)a^{k-1}c\left(\frac{p^2M}{a^2}\right).$$

Here the first term is equal to (9). As for the second term, we take the sum with the second term considered above, that is, the term containing p in $p - \psi(-\det(2T_0))$ for $\text{Cont}(UT^tU) = p^{t-1}C_0$, and we have

$$\begin{aligned} & \chi(p)p^{k-1} \sum_{a_1=p^t a_0, a_0|C_0} \chi(a_1)a_1^{k-1}c\left(\frac{M}{a_1^2}\right) + p\chi(p)p^{k-2} \sum_{a|p^{t-1}C_0} \chi(a)a^{k-1}c(M/a^2) \\ &= \chi(p)p^{k-1} \sum_{a|p^t C_0} \chi(a)a^{k-1}c(M/a^2) = A(T, F|_{k,\chi}F). \end{aligned}$$

We still have terms coming from the second term of $A(T, F|_{k,\chi})$ which does not contain ψ in the case $\text{Cont}(UT^tU) = p^t C_0$ or $p^{t+1}C_0$. If $M_0/p^2 \in \mathbb{Z}(p)$, then this term is written as $\chi(p)p^{k-2} \sum_{a|p^{t+1}C_0} \chi(a)a^{k-1}c(M/a^2)$, and even if $M_0/p^2 \notin \mathbb{Z}(p)$ and $\text{Cont}(UT^tU) = p^t C_0$, this term is written uniformly in the same way as above since $c(M/p^{2t+2}a_0^2) = 0$ if $M_0/p^2 \notin \mathbb{Z}(p)$. This term is neatly summed up with the term coming from $A(T/p)$. Indeed, the sum of the above and $\chi(p)^2 p^{2k-3} A(T/p)$ is given by

$$\begin{aligned} & \chi(p)^2 p^{2k-3} \sum_{a|p^{t-1}C_0} \chi(a)a^{k-1}c(M/a^2 p^2) + \chi(p)p^{k-2} \sum_{a|p^{t+1}C_0} \chi(a)a^{k-1}c\left(\frac{M}{a^2}\right) \\ &= \chi(p)^2 p^{2k-3} \sum_{a|p^{t-1}C_0} \chi(a)a^{k-1}c(M/p^2 a^2) + \chi(p)p^{k-2} \\ & \quad \times \sum_{a|p^t C_0} \chi(a)a^{k-1}c(M/a^2) \\ & \quad + \chi(p)p^{k-2} \sum_{a=p^{t+1}a_0, a_0|C_0} \chi(a)a^{k-1}c(M/p^2(p^t a_0)^2) \\ &= \chi(p)^2 p^{2k-3} \sum_{a|p^t C_0} \chi(a)a^{k-1}c(M/p^2 a^2) + \chi(p)p^{k-2} \sum_{a|p^t C_0} \chi(a)a^{k-1}c(M/a^2). \end{aligned}$$

The first term of this is (11), and the second term is $\chi(p)p^{k-2} A(T, F|_{k,\chi}T_S(p))$. Hence, as a whole, the coefficients of $(L_{N,\chi}\phi)|_{k,\chi}T_S(p)$ at T are equal to those of $L_{N,\chi}(\phi|_{k,\chi}T_J(p) + \chi(p)(p^{k-2} + p^{k-1})\phi)$. The comparison between $U_S(p)$ and $U_J(p)$ for $p|N$ is similarly proved, noting that $\chi(a) = 0$ for a with $p|a$. We omit the details.

Next we compare $T'_S(p)$ with $T_J(p)$ for $p \nmid N$. We calculate the coefficient of $(L_{N,\chi}\phi)|_{k,\chi}T'_S(p)$. In the formula of the coefficients of $F|_{k,\chi}T_S(1, p, p^2, p)$, $R_p(T)$ and $R_p(p^{-2}T)$ appear, and the contents are $p^t C_0$, $p^{t+1}C_0$, $p^{t+2}C_0$, or $p^{t-2}C_0$, $p^{t-1}C_0$, $p^t C_0$. Of course the term for $p^{-1}C_0$ or $p^{-2}C_0$ is regarded as zero. Calculating in each case when $M_0/p^2 \in \mathbb{Z}(p)$ or not, we can show that the contribution for (6) of the formula of $T_S(1, p, p^2, p)$ is given by

$$\begin{aligned} & p^{2k-5} \chi(p)^2 \sum_{a|p^t C_0} \chi(a)a^{k-1} \psi(-M/a^2) c(M/a^2) \\ & \quad + \chi(p)^3 p^{3k-6} \left(p \sum_{a|p^{t-2}C_0} \chi(a)a^{k-1}c(M/p^2 a^2) + \sum_{a|p^t C_0} \chi(a)a^{k-1}c(M/p^2 a^2) \right) \end{aligned}$$

if $t > 0$ and by

$$\chi(p)^3 p^{3k-6} \sum_{a|C_0} \chi(a) a^{k-1} c(M/p^2 a^2)$$

if $t = 0$. (Note that the contribution for $t = 0$ is not regarded as a special case of the expression for $t > 0$.) The contribution of (7) is given by

$$p^{2k-6} \chi(p)^2 \sum_{a|p^t C_0} \chi(a) a^{k-1} c(M/a^2) \times \begin{cases} p^2 - 1 & \text{if } t \geq 1, \\ p\psi(-M) - 1 & \text{if } t = 0. \end{cases}$$

If $a|C_0$, then we have $\psi(-M) = \psi(-M/a^2)$. So in each case when $t = 0$ or $t > 0$, the terms in (6) and (7) containing $\psi(-M/a^2)$ are given by the same expression

$$p^{2k-5} \chi(p)^2 \sum_{a|p^t C_0} \chi(a) a^{k-1} \psi(-M/a^2) c(M/a^2).$$

Next we consider (8). Here the determinant of the matrices is $p^2 M$, and the contribution is given by

$$\begin{aligned} & p^{2k-4} \chi(p)^2 \sum_{a|p^t C_0} \psi(M/a^2) \chi(a) a^{k-1} c(M/a^2) \\ & + p^{k-3} (p+1) \chi(p) \sum_{a|p^t C_0} \chi(a) a^{k-1} c(p^2 M/a^2) \\ & + \chi(p)^3 p^{3k-5} \sum_{a=p^{t-1} a_1, a_1|pC_0} \chi(a) a^{k-1} c(M/p^2 a^2) \end{aligned}$$

for $t > 0$, and for $t = 0$ the last term in the above should be replaced by

$$\begin{aligned} \chi(p)^2 p^{2k-4} \sum_{a_2|pC_0} \chi(a) a^{k-1} c(M/a^2) &= \chi(p)^2 p^{2k-4} \sum_{a|C_0} \chi(a) a^{k-1} c(M/a^2) \\ &+ \chi(p)^3 p^{3k-5} \sum_{a|C_0} \chi(a) a^{k-1} c(M/p^2 a^2). \end{aligned}$$

Since we have

$$\{a \in \mathbb{Z}_{>0}; a|p^{t-2} C_0\} \cup \{a \in \mathbb{Z}_{>0}; a = p^{t-1} a_1, a_1|pC_0\} = \{a \in \mathbb{Z}_{>0}; a|p^t C_0\},$$

taking the sum of the corresponding terms in (6) and (8), we have

$$\begin{aligned} & \chi(p)^3 p^{3k-5} \left(\sum_{a|p^{t-2} C_0} \chi(a) a^{k-1} c(M/p^2 a^2) + \sum_{a=p^{t-1} a_1, a_1|pC_0} \chi(a) a^{k-1} c(M/p^2 a^2) \right) \\ &= \chi(p)^3 p^{3k-5} \sum_{a|p^t C_0} \chi(a) a^{k-1} c(M/p^2 a^2) \end{aligned}$$

for $t > 0$. So together with the last term of (6), we have

$$\chi(p)^3 (p^{3k-5} + p^{3k-6}) \sum_{a|p^t C_0} \chi(a) a^{k-1} c(M/p^2 a^2).$$

Comparing all these with the coefficients of $L_{N,\chi}(\phi|_{k,\chi} T_J(p))$, we obtain the equality between the image of $L_{N,\chi}(\phi)$ by the action of $pT_S(1, p, p^2, p) + (p +$

$p^2 + p^3)\chi(p)^2p^{2k-6}$ and the image of $(p^{k-1} + p^{k-2})\chi(p)(\phi|_{k,\chi}T_J(p)) + (2p^{2k-3} + p^{2k-4})\chi(p)^2\phi$ by the mapping $L_{N,\chi}$.

In the above calculation, we actually tacitly assume that $T \neq 0$ since $\text{Cont}(0) = \infty$. But for the case $T = 0$ (which does not vanish only in the case when $c(0) \neq 0$), the action of $T_S(p)$ or $T_S(1, p, p^2, p)$ is given by

$$(1 + \chi(p)p^{k-2}(1 + p) + \chi(p)^2p^{2k-3})A(0)$$

or

$$(p^{3k-6}\chi(p)^3(p + 1) + p^{2k-4}\chi(p)^2 + p^{k-3}\chi(p))A(0).$$

On the other hand, the constant term of $\phi|_{k,1}U_J(p)$ is given by

$$(1 + p^{2k-3}\chi(p)^2)c(0).$$

The constant term of $L_{N,\chi}(\phi|_{k,1}U_J(p))$ is given by the multiple of this by the constant depending on k, N , and χ . So by comparing these, we have the equality of the action also in this case. \square

5. Structures of Jacobi forms of small levels

To obtain Siegel modular forms by Saito–Kurokawa lifting, we need explicit Jacobi forms of index one. When $N = 1$, the structure of such forms is known in [8]. We give the same sort of results when $2 \leq N \leq 5$. Kramer [18] gave a formula for $\dim J_{k,1}(\Gamma_0(p)^J)$ and characterized $J_{k,m}(\Gamma_0(N)^J)$ in general by some data of modular forms of one variable. His method there is very useful also for practical construction of Jacobi forms if we combine this with the Atkin–Lehner involution. So in the first subsection we explain a general method based on his result and give explicit results in later sections.

5.1. Taylor expansion and theta expansion

Most of the results in this section are a review of [8] and [19], but we insert this for the reader's convenience. It is classically well known that any holomorphic function $\phi(\tau, z)$ of $\mathfrak{H}_1 \times \mathbb{C}$ which satisfies

$$(12) \quad \phi(\tau, z + \lambda\tau + \mu) = e^m(-\lambda^2\tau - 2\lambda z)\phi(\tau, z)$$

for any $\lambda, \mu \in \mathbb{Z}$ is a linear combination of some standard theta functions as a function of z . Combining this with the Taylor expansion along $z = 0$, we can get a general way to construct Jacobi forms. For the sake of simplicity, we explain here the case $m = 1$, though the other cases are treated similarly (see [19]). For $\nu = 0$ or 1 , we put

$$\vartheta_\nu(\tau, z) = \sum_{n \in \mathbb{Z}} q^{(n+\nu/2)^2} \zeta^{2n+\nu}.$$

Then when $\phi(\tau, z)$ satisfies the above condition (12) for $m = 1$, we have

$$\phi(\tau, z) = c_0(\tau)\vartheta_0(\tau, z) + c_1(\tau)\vartheta_1(\tau, z)$$

for some holomorphic functions $c_0(\tau)$ and $c_1(\tau)$ of \mathfrak{H}_1 . We call this for short a theta expansion of ϕ . By the definition of ϑ_ν , this is an even function with respect to z . Now assume that $\phi(\tau, z) \in J_{k,1}(\Gamma_0(N)^J, \chi)$. By considering the action of -1_2 , we have

$$\phi(\tau, -z) = \chi(-1)(-1)^k \phi(\tau, z),$$

and since this should be an even function with respect to z , we have $\chi(-1) = (-1)^k$ unless $\phi = 0$. So we assume $\chi(-1) = (-1)^k$ from now on. We consider the Taylor expansion of $\phi(\tau, z)$ along $z = 0$. Then we have

$$\phi(\tau, z) = \chi_0(\tau) + z^2 \chi_1(\tau) + O(z^4).$$

It is known by [8] that ϕ is determined only by χ_0 and χ_1 since ϕ has at most two zeros in the fundamental parallelootope of z (for a direct proof, see below). By invariance of ϕ with respect to $\Gamma_0(N)$, we see that

$$\begin{aligned} \chi_0(\tau) &= f_0(\tau), \\ \chi_1(\tau) &= -\frac{\pi^2}{k} f_1(\tau) + \frac{2\pi i}{k} f_0'(\tau) \end{aligned}$$

for some holomorphic functions $f_0(\tau)$, $f_1(\tau)$ on \mathfrak{H}_1 (and the derivative f_0' of f_0 with respect to τ) which satisfy $f_\nu(\gamma\tau) = \chi(\gamma)(cN\tau + d)^{k+2\nu} f_\nu(\tau)$ for any $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$ for each $\nu = 0$ or 1 . In order to show that these are modular forms, we must show that they are holomorphic also at cusps. To show this, we consider the action of $M \in \mathrm{SL}_2(\mathbb{Z})$. We have

$$\begin{aligned} \phi|_{k,1}[M] &= (\chi_0|_k[M])(\tau) + z^2(-2\pi i c(c\tau + d)^{-1} \chi_0|_k[M] + \chi_1|_{k+2}[M]) + O(z^4) \\ &= f_0|_k[M] + z^2 \left(-\frac{\pi^2}{k} f_1|_{k+2}[M] + \frac{2\pi i}{k} (f_0|_k[M])' \right) + O(z^4). \end{aligned}$$

Since the Fourier expansion of $\phi|_{k,1}[M]$ has no negative power of q by the definition of Jacobi forms, we see that each $f_\nu|_{k+2\nu}[M]$ for $\nu = 0$ or 1 is holomorphic at $i\infty$. Hence $f_\nu \in A_{k+2\nu}(\Gamma_0(N), \chi)$. Now our problem is to construct Jacobi forms from f_0 and f_1 . The mapping σ from $\phi \in J_{k,1}(\Gamma_0(N)^J, \chi)$ to $\sigma(\phi) = (f_0, f_1) \in A_k(\Gamma_0(N), \chi) \times A_{k+2}(\Gamma_0(N), \chi)$ is injective but not surjective in general. To determine genuine Jacobi forms starting from (f_0, f_1) , we now investigate conditions when the pair (f_0, f_1) is contained in the image of σ . We put $\vartheta_\nu(\tau) = \vartheta_\nu(\tau, 0)$ and $\vartheta'_\nu(\tau) = \frac{d}{d\tau} \vartheta_\nu(\tau)$. If $c_\nu(\tau)$ ($\nu = 0, 1$) are coefficients of a theta expansion of Jacobi forms whose image by σ is (f_0, f_1) , then we have

$$\begin{aligned} c_0(\tau)\vartheta_0(\tau) + c_1(\tau)\vartheta_1(\tau) &= f_0(\tau), \\ c_0(\tau)\vartheta'_0(\tau) + c_1(\tau)\vartheta'_1(\tau) &= \frac{1}{2k} f_0'(\tau) + \frac{\pi i}{4k} f_1(\tau), \end{aligned}$$

since $\vartheta'_\nu(\tau) = (1/(4(2\pi i))) \frac{d^2}{dz^2} \vartheta_\nu(\tau, z)|_{z=0}$. Now instead of assuming that $c_\nu(\tau)$ are coefficients of a Jacobi form, we assume that $f_\nu \in A_{k+2\nu}(\Gamma_0(N), \chi)$ are given for $\nu = 0$ and 1 and regard the above relation as a simultaneous equation for unknown functions $c_\nu(\tau)$. Since we have $\vartheta_0(\tau)\vartheta'_1(\tau) - \vartheta_1(\tau)\vartheta'_0(\tau) = (\pi i)\eta(\tau)^6$

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function which does not vanish anywhere on \mathfrak{H}_1 , we always have a solution $(c_0(\tau), c_1(\tau))$ holomorphic on \mathfrak{H}_1 . Now we put $f(\tau, z) = c_0(\tau)\vartheta_0(\tau, z) + c_1(\tau)\vartheta_1(\tau, z)$ and ask if $f \in J_{k,1}(\Gamma_0(N))^J, \chi$. It is clear that this satisfies (12), and so it is invariant by the Heisenberg part \mathbb{Z}^2 . In $G^J(\mathbb{R})$, we have $M \cdot (\lambda, \mu) = ((\lambda, \mu)M) \cdot M$, and if $M \in \text{SL}_2(\mathbb{Z})$, then $\mathbb{Z}^2 M = \mathbb{Z}^2$, so the function $\phi|_{k,1}[M]$ also satisfies the relation (12) for $m = 1$. So we have

$$f|_{k,1}[M] = c_0^M(\tau)\vartheta_0(\tau, z) + c_1^M(\tau)\vartheta_1(\tau, z)$$

for some holomorphic function $c_0^M(\tau)$ and $c_1^M(\tau)$ on \mathfrak{H}_1 and we get a similar simultaneous equation for $c_\nu^M(\tau)$ given by

$$(13) \quad \begin{cases} c_0^M(\tau)\vartheta_0(\tau) + c_1^M(\tau)\vartheta_1(\tau) = (f_0|_k[M])(\tau), \\ c_0^M(\tau)\vartheta'_0(\tau) + c_1^M(\tau)\vartheta'_1(\tau) \\ = (1/2k)(f_0|_k[M])'(\tau) + ((\pi i)/(4k))(f_1|_{k+2}[M])(\tau). \end{cases}$$

The functions $c_\nu^M(\tau)$ are uniquely determined by this equation. If $M \in \Gamma_0(N)$, then by our assumption we have $f_\nu|_{k+2\nu}[M] = \chi(M)f_\nu$ for $M \in \Gamma_0(N)$, so we have $\chi(M)c_\nu(\tau) = c_\nu^M(\tau)$ for $\nu = 0, 1$ for any $M \in \Gamma_0(N)$. This means that $\phi|_{k,1}[M] = \chi(M)\phi$. So the automorphy with respect to $\Gamma_0(N)$ is always satisfied. So the only remaining condition to assure that f is a genuine Jacobi form is the condition on the Fourier expansion at cusps. This is equivalent to saying that $c_\nu^M(\tau)$ is holomorphic at $i\infty$ for any $M \in \text{SL}_2(\mathbb{Z})$; that is, the Fourier expansion of $c_\nu^M(\tau)$ has no negative power of q . For each $M \in \text{SL}_2(\mathbb{Z})$, there exists the smallest natural number n_M such that $M \begin{pmatrix} 1 & n_M \\ 0 & 1 \end{pmatrix} M^{-1} \in \Gamma_0(N)$, and we have $(f_\nu|_{k+2\nu}[M])(\tau + n_M) = (f_\nu|_{k+2\nu}[M])(\tau)$. Since $\vartheta_\nu(\tau + 1, z) = e(\nu^2/4)\vartheta_\nu(\tau, z)$ and $c_\nu^M(\tau)$ are uniquely determined by $f_\nu|_{k+\nu}M$, we have $c_\nu^M(\tau + n_M) = e(-\nu^2 n_M/4)c_\nu^M(\tau)$. So we may write

$$c_\nu^M(\tau) = \sum_{n \in \mathbb{Z}} c_\nu^M(n)q^{n/n_M - \nu^2/4}.$$

Since there is no negative power of q on the right-hand side of the simultaneous equation (13), by comparing the q -expansions carefully, we see that $c_\nu^M(n) = 0$ if $n < 0$. But the condition of the Fourier expansion of Jacobi forms is

$$c_\nu^M(n) = 0 \quad \text{unless} \quad \frac{n}{n_M} - \frac{\nu^2}{4} \geq 0.$$

We would like to describe this condition as a condition on f_0 and f_1 . For $\nu = 0$ or 1 , we write

$$f_\nu|_{k+2\nu}[M] = \sum_{n=0}^{\infty} b_\nu^M(n)q^{n/n_M}.$$

By comparing the Fourier coefficients at q^{n/n_M} for $n/n_M < 1/4$ of both sides of the above simultaneous equation, we have

$$c_0^M(n) + 2c_1^M(n) = b_0^M(n), \quad c_1^M(n) = \frac{n}{kn_M}b_0^M(n) + \frac{1}{4k}b_1^M(n),$$

by using the expansion

$$\begin{aligned} \vartheta_0(\tau) &= 1 + 2(q + q^4 + \cdots), \\ \vartheta_1(\tau) &= 2q^{1/4}(1 + q^2 + \cdots), \\ \vartheta'_0(\tau) &= (4\pi i)(q + 4q^4 + \cdots), \\ \vartheta'_1(\tau) &= (\pi i)q^{1/4}(1 + 9q^2 + \cdots). \end{aligned}$$

The condition for $c_0^M(n)$ is that this should vanish for $n < 0$ and this does not give conditions on f_i . The condition for $c_1^M(n)$ is that this should vanish for $0 \leq n < n_M/4$. The equivalent condition for this is

$$\frac{n}{n_M}b_0^M(n) + \frac{1}{4}b_1^M(n) = 0$$

for $0 \leq n < n_M/4$ (cf. [19, Bemerkung 4.2, p. 294]). In particular, $b_1^M(0) = 0$ for any $M \in \text{SL}_2(\mathbb{Z})$, so f_1 should be a cusp form. Moreover, if $n_M \leq 4$, then there is no other condition. So if $n_M \leq 4$ for any M , then the image is exactly $A_k(\Gamma_0(N), \chi) \times S_{k+2}(\Gamma_0(N), \chi)$. When $N = p$ is a prime, then the representative of a cusp is given by $i\infty$ or zero and corresponds with 1_2 or $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have $n_{1_2} = 1$ and $n_J = p$, and the conditions are $f_0 \in A_k(\Gamma_0(N), \chi)$, $f_1 \in S_{k+2}(\Gamma_0(N), \chi)$, and

$$b_2^J(n) = -\frac{4n}{p}b_0^J(n)$$

for any $n < p/4$.

Now we add something that is not written in [19]. We take three *weak* Jacobi forms $\phi_{-2,1}$, $\phi_{0,1}$, $\phi_{-1,2}$ with respect to $\text{SL}_2(\mathbb{Z})$ of weight k of index m with $(k, m) = (-2, 1)$, $(0, 1)$, or $(-1, 2)$ as in [8]. Then as shown in [3, Proposition 6.1], the ring of *weak* Jacobi forms with respect to a subgroup Γ of $\text{SL}_2(\mathbb{Z})$ of finite index is generated by these three over $A(\Gamma) = \bigoplus_k A_k(\Gamma)$. In particular, we have

$$J_{k,1}(\Gamma_0(N)^J, \chi) \subset A_k(\Gamma_0(N), \chi)\phi_{-2,1} \oplus A_k(\Gamma_0(N), \chi)\phi_{0,1}.$$

So we would like to write down Jacobi forms of index one corresponding to (f_0, f_2) with $f_\nu \in A_{k+\nu}(\Gamma_0(N), \chi)$ ($\nu = 0, 2$) by linear combinations of the right-hand side above. We put

$$\begin{aligned} P &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) = 1 - 24(q + 3q^2 + 4q^3 + \cdots), \\ E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \\ E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n. \end{aligned}$$

Here E_k is the Eisenstein series of weight k with respect to $\text{SL}_2(\mathbb{Z})$. The function P is the one defined in [27], and this is not a modular form but a holomorphic part of a real analytic modular form of weight 2 which is nearly holomorphic in

the sense of Shimura. This is sometimes called a quasi-modular form. Then we have

$$\begin{aligned}\phi_{-2,1}(\tau, z) &= (2\pi i)^2 z^2 + O(z^4), \\ \phi_{0,1}(\tau, z) &= 12 + (2\pi i)^2 P z^2 + O(z^4).\end{aligned}$$

If we write a Jacobi form f as $f(\tau, z) = g_0(\tau)\phi_{0,1}(\tau, z) + g_1(\tau)\phi_{-2,1}(\tau, z)$, then the Taylor coefficients of f are given by

$$\begin{aligned}\chi_0(\tau) &= 12g_0(\tau), \\ \chi_1(\tau) &= (2\pi i)^2 (P(\tau)g_0(\tau) + g_1(\tau)),\end{aligned}$$

so we have

$$\begin{aligned}g_0(\tau) &= \frac{1}{12}f_0(\tau), \\ g_1(\tau) &= \frac{1}{4k}f_1(\tau) + \frac{1}{2\pi i k}f'_0(\tau) - \frac{1}{12}P(\tau)f_0(\tau).\end{aligned}$$

As in Zagier [34], we have $P = (3E'_4/(2\pi i) + E_6)/E_4 = (2E'_6/(2\pi i) + E_4^2)/E_6$. For $f \in A_k(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in A_l(\Gamma_0(N), \chi)$, we write $\{f, g\}_1 = (kfg' - lgf')/(2\pi i)$. Then this belongs to $A_{k+l+2}(\Gamma_0(N), \chi)$. We have

$$\begin{aligned}g_1(\tau) &= \frac{f_1(\tau)}{4k} + \frac{1}{4kE_4}\{E_4, f_0\}_1 - \frac{E_6 f_0(\tau)}{12E_4} \\ &= \frac{f_1(\tau)}{4k} + \frac{1}{6kE_6}\{E_6, f_0\}_1 - \frac{E_4^2 f_0(\tau)}{12E_6}.\end{aligned}$$

Since E_4 and E_6 have no common zero, this is always holomorphic for any f_0 and f_1 (even at cusps), and we have $g_1 \in A_{k+2}(\Gamma_0(N), \chi)$. Of course $f(\tau, z)$ is a Jacobi form only for (f_0, f_1) satisfying the conditions we describe above.

5.2. Examples for $N \leq 5$

We write $A(N) = \bigoplus_{k=0}^{\infty} A_k(\Gamma_0(N))$ and denote by $S(N)$ the ideal of cusp forms in $A(N)$. We also write $J(N, \chi) = \bigoplus_{k=0}^{\infty} J_{k,1}(\Gamma_0(N)^J, \chi)$ where χ is a Dirichlet character modulo N . When the conductor of χ is 1, we write $J(N) = J(N, \chi)$. Since $J(N, \chi)$ is an $A(N)$ -module, we are interested in the structure of $J(N, \chi)$ as an $A(N)$ -module. Note that $J_{k,1}(\Gamma_0(N)^J, \chi) = 0$ unless $\chi(-1) = (-1)^k$ as we explain above. We also have $J_{0,1}(\Gamma_0(N), \chi) = 0$. To check the conditions at cusps of Jacobi forms, we need the Fourier expansions of $f_\nu|_{k+\nu}[M]$ for the representatives M of cusps of $\Gamma_0(N)$. When $N = p$ is a prime, if we put $\eta_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix} = J_1 \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1/\sqrt{p} \end{pmatrix}$, then η_p normalizes $\Gamma_0(N)$ and the element η_p gives the Atkin-Lehner involution. Since we have $(f_\nu|_{k+2\nu}[\eta_p])(\tau) = p^{k/2+\nu}(f_\nu|_{k+2\nu}[J_1])(p\tau) \in A_{k+2\nu}(\Gamma_0(N), \bar{\chi})$, it is not so difficult to write down the Fourier expansion of $(f|_k[J_1])(\tau)$ for concrete examples. If $N = 4$, there are three cusps, but these are represented by 1_2 , $\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, and these normalize $\Gamma_0(4)$ too. So also in this case, $f|_k[M]$ is easily obtained for each cusp. By using these, we can determine $J_{k,1}(\Gamma_0(N)^J, \chi)$. But actually, when $N \leq 4$, we always have $n_M \leq$

4 for any $M \in \mathrm{SL}_2(\mathbb{Z})$, so the mapping $\sigma : J_{k,1}(\Gamma_0(N)^J, \chi) \rightarrow A_k(\Gamma_0(N), \chi) \times S_{k+2}(\Gamma_0(N), \chi)$ is a bijection by the reason we explain in Section 5.1, so the description of Jacobi forms is much easier. Here we explain the case $N = 5$ for trivial character in detail. As for $N \leq 4$, we give only the results and omit the proofs.

5.2.1. Level 5

First we give $A(5)$ explicitly. By the well-known dimension formula (see, e.g., [30]), we have

$$\sum_{k=0}^{\infty} \dim A_k(\Gamma_0(5))t^k = \frac{1+t^4}{(1-t^2)(1-t^4)}.$$

We put

$$E_2(\tau) = 1 + 6 \sum_{n=1}^{\infty} \sigma_1^5(n)q^n,$$

where $\sigma_1^5(n) = \sum_{d|n, 5 \nmid d} d$. Then this is the Eisenstein series in $A_2(\Gamma_0(5))$, and this is also the theta function $\sum_{x \in \mathbb{Z}^4} e^{(t x S x \tau / 2)}$ associated with the even symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 10 & 5 \\ 0 & 1 & 5 & 4 \end{pmatrix}$$

of level 5 and discriminant 5^2 . We have $E_2|_2[\eta_5] = -E_2$ since $\dim A_2(\Gamma_0(5)) = 1$ and $\dim A_2(\Gamma_0^*(5)) = 0$ by the well-known formula, where $\Gamma_0^*(5)$ is the normalizer of $\Gamma_0(5)$ generated by $\Gamma_0(5)$ and η_5 . We put

$$\begin{aligned} \chi_4(\tau) &= (26E_2^2(\tau) - (E_4(\tau) + 25E_4(5\tau))) / 72 \\ &= q - 4q^2 + 2q^3 + 8q^4 - 5q^5 + \dots, \end{aligned}$$

where E_4 is the Eisenstein series of weight 4 of $\mathrm{SL}_2(\mathbb{Z})$ as before. Since $E_4|_4[\eta_5] = 5^{-2}\tau^{-4}E_4(-1/5\tau) = 5^2E_4(5\tau)$ and $(E_4(5\tau))|_4[\eta_5] = 5^{-2}E_4(\tau)$, we have $\chi_4|_4[\eta_5] = \chi_4$, so this is a cusp form and this spans the one-dimensional space $S_4(\Gamma_0(5))$. We see that two forms $E_2(\tau)$ and χ_4 are algebraically independent since E_2 does not vanish at $q = 0$ while χ_4 does. We put $f_4(\tau) = E_4(\tau) - 25E_4(5\tau)$. Then we have $f_4|_4[\eta_5] = -f_4$. We show that

$$\begin{aligned} A(\Gamma_0(5)) &= \mathbb{C}[E_2, \chi_4] \oplus f_4\mathbb{C}[E_2, \chi_4], \\ S(\Gamma_0(5)) &= \chi_4 A(\Gamma_0(5)), \end{aligned}$$

where \oplus means a direct sum as a module. Indeed, we obviously have $\mathbb{C}[E_2, \chi_4] = \mathbb{C}[E_2^2, \chi_4] \oplus E_2\mathbb{C}[E_2^2, \chi_4]$. If $P_1 + E_2P_2 + f_4P_3 + E_2f_4P_4 = 0$ for some $P_i \in \mathbb{C}[E_2^2, \chi_4]$, then by applying η_5 , we see that $P_1 - E_2P_2 - f_4P_3 + E_2f_4P_4 = 0$, and so $P_1 + E_2f_4P_4 = E_2P_2 + f_4P_3 = 0$. This means that $P_i = 0$ for all i since the weights of P_i are multiples of 4. Hence the assertion is proved. By the way, comparing the

Fourier coefficients, we can see that

$$\begin{aligned} E_4 &= 13E_2^2 - 36\chi_4 + f_4/2, \\ E_6 &= -62E_2^3 + 864E_2\chi_4 - 63E_2f_4/24, \\ f_4^2 &= 576(E_2^4 - 44E_2^2\chi_4 - 16\chi_4^2). \end{aligned}$$

Next we study $J(5)$. By [19], we have

$$\sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_0(5)^J) t^k = \sum_{k=2}^{\infty} A_{2k-2}(\Gamma_0(5)) t^k = \frac{t^2(1+t^2)}{(1-t^2)^2}.$$

We take $f_0 \in A_k(\Gamma_0(5))$ and $f_1 \in A_{k+2}(\Gamma_0(5))$ and assume that $\sigma(\phi) = (f_0, f_1)$ for $\phi \in J_{k,1}(\Gamma_0(5)^J)$. As we explain in the last subsection, the condition that (f_0, f_1) is in the image of σ is described by the conditions on the Fourier expansion at each cusp. In this case, the condition at the cusp $i\infty$ is that f_1 vanishes at $i\infty$. We see the condition at the cusp 0. We write $5^{k/2+\nu}(f_\nu|_{k+2\nu}[J_1])(5\tau) = f_\nu|_{k+2\nu}[\eta_5] = \sum_{n=0}^{\infty} b_\nu(n)q^n$. Then the condition at the cusp 0 is given by the conditions $b_1(l) + 4lb_0(l) = 0$ for $0 \leq l < 5/4$, that is, $b_1(0) = 0$ and $b_1(1) + 4b_0(1) = 0$. So f_1 should be a cusp form. Now we see the case $k = 2$. Then $f_1 = c\chi_4$ for some constant c . If $f_0 = 0$, then $b_0(1) = 0$ and $c = 0$, so $f_1 = 0$, and $\phi = 0$ in this case since σ is injective. So we assume that $f_0 = E_2$. Since $E_2|_2[\eta_5] = -E_2$, we have $b_0(1) = -6$, so $b_1(1) = 24$. Since $\chi_4|_4[\eta_5] = \chi_4$, this means that $c = 24$. We denote by $f_{2,1}$ the element of $J_{2,1}(\Gamma_0(5)^J)$ such that $\sigma(f_{2,1}) = (E_2, 24\chi_4)$. Then we have

$$f_{2,1}(\tau, z) = \frac{1}{12}E_2(\tau)\phi_{0,1}(\tau, z) + \left(3\chi_4(\tau) + \frac{1}{6}E_2(\tau)^2 + \frac{1}{96}f_4\right)\phi_{-2,1}(\tau, z).$$

Next we construct Jacobi forms of weight 4. If we put $(f_0, f_1) = (\chi_4, 4E_2\chi_4)$, then since $\chi_4|_4[\eta_5] = \chi_4$ and $E_2\chi_4|_4[\eta_5] = -E_2\chi_4$, we have $b_0(1) = 1$ and $b_1(1) = -4$, and this pair satisfies the condition. We denote by $\chi_{4,1}$ the corresponding Jacobi form. Then this is given by

$$\chi_{4,1} = \frac{1}{12}(\chi_4\phi_{0,1} + 5E_2\chi_4\phi_{-2,1}).$$

This is a Jacobi cusp form. If we put $(f_0, f_1) = (E_4, 0)$, then since $E_4|_4[\eta_5] = 25E_4(5\tau)$, we have $b_0(1) = 0$, so this pair also satisfies the condition. The corresponding Jacobi form is obviously a Jacobi form belonging to $J_{4,1}(\mathrm{SL}_2(\mathbb{Z})^J)$ and given by $(E_4\phi_{0,1} - E_6\phi_{-2,1})/12$. This is nothing but the Jacobi–Eisenstein series $E_{4,1}$ of weight 4 with respect to $\mathrm{SL}_2(\mathbb{Z})^J$ given in [8]. So we have three forms $f_{2,1}, \chi_{4,1}, E_{4,1}$. Now we put $B = \mathbb{C}[E_2, \chi_4]$, and we show the following.

THEOREM 5.1

We have

$$(14) \quad J(5) = A(5)f_{2,1} \oplus B\chi_{4,1} \oplus BE_{4,1},$$

where \oplus means a direct sum as modules.

Proof

First we show that this is a direct sum. Assume that

$$(h_1 + f_4 h_2)(96f_{2,1}) + h_3(12\chi_{4,1}) + h_4(24E_{4,1}) = 0$$

for some $h_i \in B$ with $1 \leq i \leq 4$. Since $\phi_{0,1}$ and $\phi_{-2,1}$ are linearly independent as a function of z , we compare the coefficients of $\phi_{0,1}$ and $\phi_{-2,1}$. Then we have

$$\begin{aligned} 8E_2(h_1 + f_4 h_2) + \chi_4 h_3 + (312E_2^2 - 864\chi_4 + 12f_4)h_4 &= 0, \\ (288\chi_4 + 16E_2^2 + f_4)(h_1 + f_4 h_2) + \\ 5E_2\chi_4 h_3 + (1488E_2^3 - 20736E_2\chi_4 + 63E_2 f_4)h_4 &= 0. \end{aligned}$$

We have $f_4^2 \in B$ as explained above. Since 1 and f_4 are linearly independent over B , we obtain four equations from the above equations which are given by $Ah = 0$ where $h = {}^t(h_1, h_2, h_3, h_4)$ and

$$A = \begin{pmatrix} 8E_2 & 0 & \chi_4 & 312E_2^2 - 864\chi_4 \\ 0 & 8E_2 & 0 & 12 \\ 288\chi_4 + 16E_2^2 & f_4^2 & 5E_2\chi_4 & 1488E_2^3 - 20738E_2\chi_4 \\ 1 & 288\chi_4 + 16E_2^2 & 0 & 63E_2 \end{pmatrix}.$$

We have $\det(A) = 16\chi_4^2(-E_2^2 + 69120\chi_4)$, which is not identically zero. So we have $h_i = 0$ for all i . Since $\sum_{k=0}^{\infty} \dim(B \cap A_k(\Gamma_0(5)))t^k = 1/(1-t^2)(1-t^4)$, the generating function of the dimensions of the right-hand side of (14) is

$$\frac{t^2(1+t^4)}{(1-t^2)(1-t^4)} + \frac{2t^4}{(1-t^2)(1-t^4)} = \frac{t^2(1+t^2)}{(1-t^2)^2},$$

which is equal to the generating function of $\dim J_{k,1}(\Gamma_0(5))^J$. So we prove Theorem 5.1. □

5.2.2. Level 4

We can proceed almost in the same way as in the level 5 case. For any (not necessarily primitive) Dirichlet character χ modulo 4, the mapping $\sigma : J_{k,1}(\Gamma_0(4))^J, \chi \rightarrow A_k(\Gamma_0(4)) \oplus S_{k+2}(\Gamma_0(4), \chi)$ is bijective as we saw above. It is well known (and easy to see) that $A(4) = \mathbb{C}[\vartheta_0^4, \vartheta_1^4]$ where $\vartheta_\nu = \vartheta_\nu(\tau) = \vartheta_\nu(\tau, 0)$. We denote by ψ_4 the primitive Dirichlet character modulo 4 given by $\psi_4(a) = (-4/a) = (-1)^{(a-1)/2}$ for any odd a . We put

$$\begin{aligned} f_{1,1}(\tau, z) &= \vartheta_0(\tau)\vartheta_0(\tau, z), \\ f_{2,1}(\tau, z) &= \vartheta_0(\tau)^3\vartheta_0(\tau, z) + \vartheta_1(\tau)^3\vartheta_1(\tau, z), \\ f_{3,1}(\tau, z) &= \vartheta_0(\tau)^2 f_{2,1}(\tau, z). \end{aligned}$$

Then by the usual theta transformation formula (see, e.g., [15] or [8]), we can show that $f_{1,1} \in J_{1,1}(\Gamma_0(4))^J, \psi_4$, $f_{3,1} \in J_{3,1}(\Gamma_0(4))^J, \psi_4$, $f_{2,1} \in J_{2,1}(\Gamma_0(2))^J$. Here we have another expression of $f_{2,1}$. Denote by O the maximal order of the definite quaternion algebra B over \mathbb{Q} with discriminant 2∞ ; then we can see that $f_{2,1}(\tau, z) = \sum_{x \in O} q^{n(x)} \zeta^{\text{tr}(x)}$ where $n(x)$ or $\text{tr}(x)$ is the reduced norm or trace, respectively.

THEOREM 5.2

We have

$$J(4) = A(\Gamma_0(4))\vartheta_0^2(\tau)f_{1,1}(\tau, z) \oplus A(\Gamma_0(4))f_{2,1}(\tau, z),$$

$$J(4, \psi_4) = A(\Gamma_0(4))f_{1,1}(\tau, z) \oplus A(\Gamma_0(4))f_{3,1},$$

where \oplus means the direct sum as modules.

The proof is omitted here. By the way, for degree two, the graded ring $\bigoplus_{k=0}^{\infty} (A_k(\Gamma_0^{(2)}(4)) \oplus A_k(\Gamma_0^{(2)}(4), \psi_4))$ of Siegel modular forms is generated by 5 modular forms of weight 1, 2, 2, 3, and 11 (see [3]). Here the form of weight 1 or 3 is with character ψ_4 , and each form of weight 2 or 11 is without character. By the parity condition on character, we cannot get the form of weight 11 by our Saito–Kurokawa lifting. The other four forms are obtained by $(L_{4, \psi_4} f_{1,1})(Z)$, $(L_{4, \chi_0} f_{2,1})(Z)$, $(L_{4, \chi_0} f_{2,1})(2Z)$, $L_{4, \psi_4}((\vartheta_0(\tau)^4 - \vartheta_1(\tau)^4)f_{1,1})$ where χ_0 is the Dirichlet character modulo 4 with conductor 1.

5.2.3. Levels 2 and 3

The content of this section was partly given in [16]. In each case when $N = 2$ or 3, the mapping $\sigma : J_{k,1}(\Gamma_0(N)^J, \chi) \rightarrow A_k(\Gamma_0(N), \chi) \times S_{k+2}(\Gamma_0(N), \chi)$ is bijective.

First we study the level 2 case. We put $\alpha(\tau) = \vartheta_0^4(\tau) + \vartheta_1^4(\tau)$ and $\beta(\tau) = \vartheta_0^4(\tau)\vartheta_1^4(\tau)/16$. Then $\alpha \in A_2(\Gamma_0(2))$, $\beta \in A_4(\Gamma_0(2))$, and we have $\mathbb{C}[\alpha(\tau), \beta(\tau)]$. We put

$$f_{2,1}(\tau, z) = \frac{1}{12}\alpha(\tau)\phi_{0,1}(\tau, z) + \left(-\frac{\alpha(\tau)^2}{12} + 16\beta(\tau)\right)\phi_{-2,1}(\tau, z).$$

This is the same $f_{2,1}$ defined for level 4. Then we have the following.

THEOREM 5.3

We have

$$J(2) = A(2)f_{2,1} \oplus A(2)E_{4,1},$$

where $f_{2,1}$ is as above and $E_{4,1}$ is the Jacobi–Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})^J$ of weight 4.

The proof is omitted here. Next we study the level 3 case with trivial character. We put $g_2 = 1 + 12 \sum_{n=1}^{\infty} \sigma_1^3(n)q^n$. This is the Eisenstein series of weight two in $A_2(\Gamma_0(3))$. We denote by $\chi_6 = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + \cdots$ the unique normalized cusp form in $A_6(\Gamma_0(3))$. Then g_2 and χ_6 are algebraically independent, and we have $A(3) = B(3) \oplus E_4B(3)$ where we put $B(3) = \mathbb{C}[g_2, \chi_6]$. By the way, we have $E_4^2 = -9g_2^4 + 10g_2^2E_4 - 1728g_2\chi_6$ (cf. [12, p. 24]; there 1729 is a typo for 1728). We consider the Jacobi form of weight 2, 4, 4, 6 whose image of σ is $(24g_2, 0)$, $(24E_4, 0)$, $(0, 16\chi_6)$, or $(12\chi_6, 0)$. These are given by

$$g_{2,1} = 2g_2\phi_{0,1} - (3g_2^2 - E_4)\phi_{-2,1},$$

$$g_{4,1} = 2E_4\phi_{0,1} - 2E_6\phi_{-2,1} = 24E_{4,1},$$

$$c_{4,1} = \chi_6 \phi_{-2,1},$$

$$c_{6,1} = \chi_6 \phi_{0,1} + g_2 \chi_6 \phi_{-2,1}.$$

Then we have $g_{2,1} \in J_{2,1}(\Gamma_0(3)^J)$, $c_{4,1}, g_{4,1} \in J_{4,1}(\Gamma_0(3)^J)$, and $c_{6,1} \in J_{6,1}(\Gamma_0(3)^J)$.

THEOREM 5.4

We have

$$J(3) = B(3)g_{2,1} \oplus \mathbb{C}[\chi_6]c_{6,1} \oplus B(3)E_{4,1} \oplus A(3)c_{4,1},$$

where \oplus means a direct sum as modules.

The proof is omitted here. Now we treat the case when $N = 3$ with character $\psi_3(a) = (-3/a)$. We put $A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and

$$\theta_A(\tau) = \sum_{x,y \in \mathbb{Z}} q^{x^2+xy+y^2}, \quad g_3(\tau) = \frac{\eta(3\tau)^9}{\eta(\tau)^3}.$$

Then we have $\theta_A(\tau) \in A_1(\Gamma_0(3), \psi_3)$, $g_3 \in A_3(\Gamma_0(3), \psi_3)$, and

$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_0(3), \psi_3) = \mathbb{C}[g_2, \chi_6]\theta_A \oplus \mathbb{C}[g_2, \chi_6]g_3.$$

By the way, we have

$$\chi_6 = \frac{1}{216}(g_2 E_4 - g_3^3) - 27g_3^2.$$

We also have

$$\begin{aligned} \sum_{k=0}^{\infty} \dim A_k(\Gamma_0(3), \psi_3) t^k &= \frac{t + t^3}{(1-t^2)(1-t^6)}, \\ \sum_{k=0}^{\infty} \dim S_k(\Gamma_0(3), \psi_3) t^k &= \frac{t^7 + t^9}{(1-t^2)(1-t^6)}. \end{aligned}$$

There exists a Jacobi form whose image by σ is $(12\theta_A, 0)$ or $(12g_3, 0)$. Each is given by

$$g_{1,1} = \theta_A \phi_{0,1} + (-g_2 \theta_A + 108g_3) \phi_{-2,1},$$

$$g_{3,1} = g_3 \phi_{0,1} + 3g_2 g_3 \phi_{-2,1},$$

where $g_2 \in A_2(\Gamma_0(3))$ is defined as above.

THEOREM 5.5

We have

$$J(3, \psi_3) = A(3)g_{1,1} \oplus A(3)g_{3,1}.$$

The proof is omitted here.

Also in the case of levels 2 and 3, we can construct generators of the graded ring of Siegel modular forms of degree two given in [3] essentially by Saito–Kurokawa liftings except for weight 19 of level 2 and weight 14 of level 3, which have bad parity conditions. We omit the details.

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