A few examples of local rings, I

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Abstract In this paper, we first recall and apply the fundamental techniques of constructing bad Noetherian local domains, due to C. Rotthaus, T. Ogoma, R. C. Heitmann, and M. Brodmann and C. Rotthaus, to show several basic examples:

(1) a three-dimensional Nagata normal local domain, which is a complete intersection, whose regular locus is *not* open;

(2) a three-dimensional Henselian Nagata *normal* local domain, which is *not* catenary.

Next we present a unified version of Brodmann and Rotthaus's and Ogoma's methods in order to obtain a particular local domain A with a specified *prime* element x such that the local domain A/xA is the bad Noetherian local domain given above:

(3) a three-dimensional unmixed local domain A that has $xA = \mathfrak{p} \in \operatorname{Spec}(A)$ such that $A/\mathfrak{p}$ is *not* unmixed.

Finally we follow Ogoma's construction of factorial local domains whose completions are designated complete local domains. Then, we gather some examples of bad factorial local domains.

Contents

0. Introduction

This paper is the first part of our study entitled: *A few examples of local rings, I, II, III.* In this part I, we first recall the fundamental techniques of constructing bad Noetherian local domains, due to C. Rotthaus [33], T. Ogoma [26], R. C. Heitmann [12], and M. Brodmann and C. Rotthaus [5]. Then we apply these techniques to show several basic examples. Some of the examples we give here were constructed by Akizuki [1], Nagata [21], and Ferrand and Raynaud [8], and

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others were obtained by the above-mentioned authors to settle long-unsolved questions or conjectures.*

Next we present a unified version of Brodmann and Rotthaus's [6] and Ogoma's [29] method in order to obtain a particular local domain A with a specified *prime* element x such that the local domain A/xA is the bad Noetherian local domain given above. We show some interesting examples, including Valabrega's [34] bad regular local rings.

Finally we follow Ogoma's construction [28] of factorial local domains whose completions are designated complete local domains. Then, we gather some examples of bad factorial local domains.

Thus part I may be regarded as a concise review of the well-known results. However, we should emphasize that, to get factorial local domains whose completion could be *almost all* complete local rings, we need to use freely as common knowledge, without explicitly referencing them, these fundamental ideas and techniques throughout part II. This is the reason that we include part I in our series of articles.

Now let us summarize the contents of this paper. Fixing notation and terminologies, we begin Section 1 with a basic lemma due to Heitmann, which plays a key role throughout our study. Using Heitmann's lemma [12, Proposition 1], we prove Theorem 1.4.

THEOREM 1.4

Let K be a purely transcendental extension field of countably infinite degree over a countable field K_0, let $n, r, m \in \mathbb{N}$ with $m < n$, and let $z_1, \dots, z_n$ be indeterminates over K. Let $R := K[z_1, \dots, z_n]_{(z_1,\dots,z_n)}$, and let $\hat{R}$ denote the completion of R; that is, $\hat{R} = K[[z_1, \dots, z_n]]$. For each j with $1 \leq j \leq r$, let $F_j := F_j(Z_1, \dots, Z_m)$ be a polynomial in m variables with coefficients in K_0 such that $F_j \in (Z_1, \dots, Z_m)K_0[Z_1, \dots, Z_m]$. Then there exist

(1) *elements $\zeta_1, \dots, \zeta_n \in \hat{R}$ that are analytically independent over K such that $K[[\zeta_1, \dots, \zeta_n]] = K[[z_1, \dots, z_n]]$, and*

(2) *a local domain A with $R \overset{\iota}{\subset} A \subset Q(R)$, where $Q(R)$ denotes the field of fractions of R, such that the ring A and the ζ_i satisfy the conditions (1.4.1), (1.4.2) and (1.4.3) given below:*

$$(1.4.1) \qquad \tilde{\imath}\colon K[[\zeta_1, \dots, \zeta_n]]/\big(F_1(\underline{\zeta}), \dots, F_r(\underline{\zeta})\big) = \hat{R}/(f_1, \dots, f_r) \overset{\cong}{\hookrightarrow} \hat{A}.$$

That is, for (1.4.2), if we set the notation: for each j, $f_j := F_j(\underline{\zeta}) = F_j(\zeta_1, \dots, \zeta_m) \in K_0[[\zeta_1, \dots, \zeta_m]] \subset K[[\zeta_1, \dots, \zeta_n]] = \hat{R}$; then the canonical ring homomorphism $\hat{\imath}$ from $\hat{R}$ to $\hat{A}$ induced by $\iota\colon R \hookrightarrow A$ is a surjection with kernel $(f_1, \dots, f_r)$. For convenience, we denote by $\tilde{\imath}$ the associated isomorphism shown in (1.4.1):

$$(1.4.2) \qquad \hat{\mathfrak{p}} := \big(\tilde{\imath}(\zeta_1), \dots, \tilde{\imath}(\zeta_m)\big)\hat{A} \text{ is a prime ideal of } \hat{A} \text{ and } \hat{\mathfrak{p}} \cap A = (0),$$

*The recent article [15] contains additional details and motivation for the construction. Other examples of rings constructed using power series are given in [11].

(1.4.3) *$A/\mathfrak{p}$ is essentially of finite type over K for every $\mathfrak{p} \in \mathrm{Spec}(A) \setminus \{(0)\}$.*

Further, we include Corollary 1.5 as a slight generalization of Theorem 1.4.

Section 2 consists of Examples 2.1–2.15 derived from Theorem 1.4 and/or Corollary 1.5:

Example 2.1: a one-dimensional analytically *ramified* and/or *reducible* local domain of *arbitrary* characteristic;

Example 2.2: a one-dimensional local domain with given embedding dimension and multiplicity, which is δ-simple for a derivation $\delta \in \mathrm{Der}(A, A)$;

Example 2.3: a two-dimensional local domain whose completion has *embedded* associated prime ideal(s);

Example 2.4: a two-dimensional Cohen–Macaulay local domain $(A, \mathfrak{m})$ that has infinitely many non-Noetherian intermediate quasi-local domains between A and its derived normal ring $\bar{A}$;

Example 2.5: a two-dimensional analytically *(ir)reducible* Nagata normal local domain that is *not* analytically normal;

Example 2.6: a two-dimensional quasi-excellent catenary local domain, which is *not* universally catenary;

Example 2.7: a two-dimensional local domain, which is a complete intersection, whose regular (nor normal) locus is *not* open;

Example 2.8: a two-dimensional Gorenstein local domain whose complete intersection locus is *not* open;

Example 2.9: a two-dimensional Cohen–Macaulay local domain whose Gorenstein locus is *not* open;

Example 2.10: a three-dimensional local domain whose Cohen–Macaulay locus is *not* open;

Example 2.11: a three-dimensional Nagata normal local domain, which is a complete intersection, whose regular locus is *not* open;

Example 2.12: a three-dimensional Nagata normal Gorenstein local domain whose complete intersection locus is *not* open;

Example 2.13: a three-dimensional Nagata normal Cohen–Macaulay local domain whose Gorenstein locus is *not* open;

Example 2.14: a four-dimensional Nagata normal local domain that has *nonopen* Cohen–Macaulay locus;

Example 2.15: a three-dimensional Henselian Nagata *normal* local domain, which is *not* catenary.

Next in Section 3, thanks to Brodmann and Rotthaus [6], Ogoma [29], and Brezuleanu and Rotthaus [4], we modify Theorem 1.4 to the following form that makes it possible to specify a prime element.

THEOREM 3.4

Let K be a purely transcendental extension field of countably infinite degree over a *countable field K_0, let $n, r, m \in \mathbb{N}$ with $m < n$, and let x, $z_1, \ldots, z_n$ be $n+1$ indeterminates over K. Let $R := K[x, z_1, \ldots, z_n]_{(x,z_1,\ldots,z_n)}$, and let $\hat{R}$ denote the completion of R; that is, $\hat{R} = K[[x, z_1, \ldots, z_n]]$. For each j with $1 \le j \le r$, let $G_j := G_j(X, Z_1, \ldots, Z_m)$ be a polynomial in the $m+1$ variables $X, Z_1, \ldots, Z_m$ with coefficients in K_0 and zero constant term. For convenience, we let $\underline{Z} := (Z_1, \ldots, Z_m)$. Define $F_j := F_j(\underline{Z}) = G_j(0, \underline{Z})$; we consider F_j as an element of $K_0[\underline{Z}]$.*

Further, by taking another variable Q, we let $\tilde{\varphi}$ and φ be the ring surjections fixing $K_0[X, \underline{Z}, Q]$ and $K_0[\underline{Z}, Q]$, respectively, shown below:

$$\tilde{\varphi}: \ K_0[X, \underline{Z}, Q][T_1, \ldots, T_r] \to K_0[X, \underline{Z}, Q][G_1/Q, \ldots, G_r/Q] \quad \text{with } T_j \mapsto G_j/Q,$$

$$\varphi: \ K_0[\underline{Z}, Q][T_1, \ldots, T_r] \to K_0[\underline{Z}, Q][F_1/Q, \ldots, F_r/Q] \quad \text{with } T_j \mapsto F_j/Q.$$

We regard $K_0[\underline{Z}, Q]$ as $K_0[X, \underline{Z}, Q]/XK_0[X, \underline{Z}, Q]$, so that tensoring a $K_0[X, \underline{Z}, Q]$-module with $K_0[\underline{Z}, Q]$ over $K_0[X, \underline{Z}, Q]$ is the same as tensoring over $K_0[X, \underline{Z}, Q]$ with $K_0[X, \underline{Z}, Q]/XK_0[X, \underline{Z}, Q]$, that is, going modulo X or setting $X = 0$. Suppose that we have

$$\operatorname{Ker}\varphi = K_0[\underline{Z}, Q] \otimes_{K_0[X,\underline{Z},Q]} \operatorname{Ker}\tilde{\varphi}. \tag{3.4.0}$$

That is,

$$K_0[\underline{Z}, Q] \otimes_{K_0[X,\underline{Z},Q]} K_0[X, \underline{Z}, Q][G_1/Q, \ldots, G_r/Q] \cong K_0[\underline{Z}, Q][F_1/Q, \ldots, F_r/Q].$$

Then there exist

(1) a local domain $(A, \mathfrak{m})$ (where $R \subset A \subset Q(K[x, z_1, \ldots, z_n])$) with prime element $x \in \mathfrak{m}$ that is transcendental over K,

(2) elements $\zeta_1, \zeta_2, \ldots, \zeta_n \in \hat{R}$ that are analytically independent over $K[x]$ such that $K[[x, \zeta_1, \ldots, \zeta_n]] = K[[x, z_1, \ldots, z_n]]$, and

(3) a natural isomorphism $\tilde{\iota}$ that satisfies the following, where $\bar{\zeta}_i$ denotes the image mod x, $\underline{\zeta}$ abbreviates $\zeta_1, \ldots, \zeta_m$, and $\underline{\bar{\zeta}} := (\bar{\zeta}_1, \ldots, \bar{\zeta}_m)$:

$$\tilde{\iota}\colon K[[x, \zeta_1, \ldots, \zeta_n]]/\big(G_1(x, \underline{\zeta}), \ldots, G_r(x, \underline{\zeta})\big) = \hat{R}/(g_1, \ldots, g_r) \overset{\cong}{\hookrightarrow} \hat{A}, \tag{3.4.1}$$

$$\tilde{\bar{\iota}}\colon K[[\bar{\zeta}_1, \ldots, \bar{\zeta}_n]]/\big(F_1(\underline{\bar{\zeta}}), \ldots, F_r(\underline{\bar{\zeta}})\big) = \widehat{R/xR}/(f_1, \ldots, f_r) \overset{\cong}{\hookrightarrow} \hat{A}/x\hat{A}, \tag{3.4.2}$$

$$\hat{\mathfrak{q}} := \big(\tilde{\iota}(x), \tilde{\iota}(\zeta_1), \ldots, \tilde{\iota}(\zeta_m)\big)\hat{A} \text{ is a prime ideal of } \hat{A} \text{ and } \hat{\mathfrak{q}} \cap A = xA, \tag{3.4.3}$$

$$A/\mathfrak{p} \text{ is essentially of finite type over } K \text{ for every } \mathfrak{p} \in \operatorname{Spec}(A) \setminus \{xA, (0)\}. \tag{3.4.4}$$

We also get Corollary 3.5 as a modification of Theorem 3.4. Needless to say, in applying Theorem 3.4 and/or Corollary 3.5 to get desired examples, we remark that the crucial point is to check the assumption (3.4.0). This is often straightforward but sometimes a bit hard as we see in Examples 4.2–4.7 in Section 4:

Example 4.1: a discrete valuation ring of positive characteristic, which is *not* a Nagata ring;

Example 4.2: a two-dimensional normal local domain whose generic formal fiber is *not* connected;

Example 4.3: a two-dimensional regular local ring of *arbitrary* characteristic, which is *not* a Nagata ring;

Example 4.4: a two-dimensional Nagata regular local ring of characteristic $p > 0$, which is *not* excellent;

Example 4.5: a three-dimensional Nagata regular local ring of arbitrary characteristic, which is *not* excellent;

Example 4.6: a three-dimensional analytically irreducible Nagata normal local domain A that has $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $A_{\mathfrak{p}}$ is analytically *reducible*;

Example 4.7: a three-dimensional unmixed local domain A that has $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $A/\mathfrak{p}$ is *not* unmixed.

Further, following Ogoma's original clever idea, we construct factorial local domains with curious generic formal fiber.*

THEOREM 5.5

Let K be a purely transcendental extension field of countably infinite degree over a countable field K_0, let $n, r, m \in \mathbb{N}$ with $m < n$, and let $z_1, \ldots, z_n$ be indeterminates over K. Let $R := K[z_1, \ldots, z_n]_{(z_1,\ldots,z_n)}$, and let $\hat{R}$ denote the completion of R; that is, $\hat{R} = K[[z_1, \ldots, z_n]]$. For each j with $1 \le j \le r$, let $F_j := F_j(Z_1, \ldots, Z_m)$ be a polynomial in m variables over K_0 with no constant term. Suppose that $F_1(\underline{Z}), \ldots, F_r(\underline{Z})$ satisfy the absolute irreducibility condition:

$$L[Z_1, \ldots, Z_m]/\big(F_1(\underline{Z}), \ldots, F_r(\underline{Z})\big) \text{ is a domain, which is not a field,}$$

for every extension field L of K_0.

Then there exist

(1) *elements $\zeta_1, \zeta_2, \ldots, \zeta_n \in \hat{R}$ that are analytically independent over K such that $K[[\zeta_1, \ldots, \zeta_n]] = K[[z_1, \ldots, z_n]]$,*

(2) *a factorial local domain $(A, \mathfrak{m})$ with $R \overset{\iota}{\subset} A \subset Q(R)$, where $Q(R)$ denotes the field of fractions of R, and*

(3) *a natural isomorphism $\tilde{\iota}$ that satisfies the following:*

$$\tilde{\iota}\colon K[[\zeta_1, \ldots, \zeta_n]]/\big(F_1(\underline{\zeta}), \ldots, F_r(\underline{\zeta})\big) = \hat{R}/(f_1, \ldots, f_r) \overset{\cong}{\hookrightarrow} \hat{A}, \tag{5.5.1}$$

$$\hat{\mathfrak{p}} := \big(\tilde{\iota}(\zeta_1), \ldots, \tilde{\iota}(\zeta_m)\big)\hat{A} \text{ is a prime ideal of } \hat{A} \text{ and } \hat{\mathfrak{p}} \cap A = (0), \tag{5.5.2}$$

(5.5.3) *$A/\mathfrak{p}$ is essentially of finite type over K for every $\mathfrak{p} \in \mathrm{Spec}(A) \setminus \{(0)\}$.*

As above, we also get Corollary 5.6 as a variation of Theorem 5.5. Finally we close this paper by presenting a couple of examples as good demonstrations of Theorem 5.5 and/or Corollary 5.6:

*More details concerning a similar construction are given in [15, Theorem 10].

Example 6.1: a two-dimensional Cohen–Macaulay factorial excellent local domain with a Gorenstein module, which has *no* dualizing (= canonical) module;

Example 6.2: a three-dimensional excellent factorial Cohen–Macaulay local domain that has *no* Gorenstein module.

Throughout this paper, all rings are commutative with 1. A local ring $(A, \mathfrak{m})$ means a *Noetherian* ring A with a unique maximal ideal $\mathfrak{m}$. We fully use the notation and terminology of EGA [10], Matsumura [19], and Nagata [21]. The set of natural numbers and that of nonnegative integers are denoted, respectively, by $\mathbb{N}$ and $\mathbb{N}_0$.

1. Heitmann's lemma and fundamental construction of bad local domains

In this section, thanks to R. C. Heitmann, we first prove a fundamental lemma that guarantees a *good* enumeration on a countable set $\mathcal{P}$ (for the definition, see (1.0.1)). It is needless to say that this lemma plays a key role throughout our papers. Namely, with the aid of Heitmann's lemma, we get a concise recipe for making bad local domains that was originally obtained by Rotthaus [33] and developed by Ogoma [26], Brodmann and Rotthaus [5], and Heitmann [12].

1.0. Notation and numbering on $\mathcal{P}$

Let K_0 be a countable field; for example, let $\mathbb{Q}$ be the field of rational numbers, let $\mathbb{F}_q$ be the finite field with q elements, or let $\bar{\mathbb{F}}_p$ be the algebraic closure of the prime field of characteristic $p > 0$, and so on, and let K be a purely transcendental extension field of countable degree over K_0, that is, $K = K_0(\{a_{ik}\})$ with transcendental basis $\{a_{ik} \mid i = 1, \ldots, n; k = 1, 2, \ldots\}$, and we express it as

$$K = \bigcup_k K_k, \quad \text{where } K_k = K_{k-1}(a_{1k}, \ldots, a_{nk}) \text{ for } k \in \mathbb{N}.$$

Take n indeterminates $z_1, \ldots, z_n$ over K, and let

$$S_0 = K_0[z_1, \ldots, z_n] \quad \text{with maximal ideal } \mathfrak{N}_0 = (z_1, \ldots, z_n)S_0,$$

$$S_k = S_{k-1}[a_{1k}, \ldots, a_{nk}] \quad \text{with } \mathfrak{N}_k = (z_1, \ldots, z_n)S_k \text{ for } k \in \mathbb{N},$$

$$S = \bigcup_{k \in \mathbb{N}} S_k = K_0[\{a_{ik}\}_{i=1}^n, k \in \mathbb{N}][z_1, \ldots, z_n] \quad \text{with } \mathfrak{N} = (z_1, \ldots, z_n)S.$$

We localize these polynomial rings by the prime ideals above and obtain

$$R_0 = (S_0)_{\mathfrak{N}_0} = K_0[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)} \quad \text{with } \mathfrak{n}_0 = (z_1, \ldots, z_n)R_0,$$

$$R_k = (S_k)_{\mathfrak{N}_k} = K_k[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)} \quad \text{with } \mathfrak{n}_k = (z_1, \ldots, z_n)R_k,$$

$$R = S_{\mathfrak{N}} = K[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)} \quad \text{with } \mathfrak{n} = (z_1, \ldots, z_n)R.$$

Then $R_k = R_{k-1}(a_{1k}, \ldots, a_{nk})$, and $(R, \mathfrak{n})$ is a *countable* regular local ring that satisfies the following:

$$\text{(1.0.0)} \qquad R = K[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)} = \bigcup_k R_k.$$

With the notation and assumptions above, we denote by $\mathcal{P}$ a set of nonzero elements of $\mathfrak{N}$,

$$\mathcal{P} \subset \mathfrak{N} \setminus \{0\}, \tag{1.0.1}$$

that contains enough elements. Namely, for each nonzero $\mathfrak{p} \in \operatorname{Spec}(R)$, there exists at least one $p \in \mathcal{P}$ such that $p \in \mathfrak{p}$. Then $\mathcal{P}$ is a *countable* set, and we may assume that

$$z_1 + \cdots + z_n \in \mathcal{P}$$

and that $\mathcal{P}$ contains an *infinite* number of elements of S_0.

We fix a surjective mapping $\rho\colon \mathbb{N} \to \mathcal{P}$, which we call a *numbering* on $\mathcal{P}$, and set $\rho(i) = p_i$. By the remark above, we may assume that $p_1 = z_1 + \cdots + z_n$ and that ρ satisfies the following:

$$p_k \in S_{k-2} \quad \text{for every } k \geq 2. \tag{1.0.2}$$

Next we take a sequence of strictly increasing natural numbers $\varepsilon_1, \ldots, \varepsilon_k, \ldots$, for example, $\varepsilon_k = k$, and we define

$$p_1 = z_1 + \cdots + z_n, \tag{1.0.3}$$

$$z_{i0} = z_i, \tag{1.0.4}$$

$$q_k = p_1 \cdots p_k, \tag{1.0.5}$$

$$z_{ik} = z_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ik} q_k^{\varepsilon_k} \quad \text{for } k \geq 1. \tag{1.0.6}$$

Then by the definition above, $P_k = (z_{1k}, \ldots, z_{mk})R$ becomes a prime ideal of height m for $k \geq 0$.

Thanks to Rotthaus [33], Ogoma [26], Rotthaus and Brodmann [5], and Heitmann [12], we prove a fundamental lemma.

LEMMA 1.1 (HEITMANN'S NUMBERING)

With the notation above, suppose that $m < n$. Let ρ be a numbering on $\mathcal{P}$ that satisfies (1.0.2)*. Then $(z_{1k}, \ldots, z_{\ell k})S_k$ is a prime ideal generated by an S_k-regular sequence $z_{1k}, \ldots, z_{\ell k}$ for every $\ell = 1, \ldots, m$ and*

$$p_h \notin P_k \quad \textit{whenever } h \leq k+1. \tag{1.1.1}$$

Proof

We prove the lemma by induction on k. The assertions are clear for $k = 0$, because $(z_1, \ldots, z_\ell)S_0$ is a prime ideal generated by an S_0-regular sequence $z_1, \ldots, z_\ell$ and because $p_1 = z_1 + \cdots + z_n \notin (z_1, \ldots, z_m)S_0 = P_0 \cap S_0$.

Let us consider the case $k > 0$ and assume that the assertions are verified for $k-1$. Namely, $(z_{1(k-1)}, \ldots, z_{\ell(k-1)})S_{k-1}$ is a prime ideal generated by an S_{k-1}-regular sequence $z_{1(k-1)}, \ldots, z_{\ell(k-1)}$ for every ℓ $(1 \leq \ell \leq m)$ and

$$q_k \notin P_{k-1} \cap S_{k-1} = (z_{1(k-1)}, \ldots, z_{m(k-1)})S_{k-1}.$$

Hence $z_{1(k-1)}, \dots, z_{m(k-1)}$, q_k forms an S_{k-1}-regular sequence. Here we claim that

(1.1.2) $\qquad q_k^{\varepsilon_k}, z_{1(k-1)}, \dots, z_{m(k-1)}$ is an S_{k-1}-regular sequence, too.

We notice the following elementary fact. Let S be a ring and M an S-module. Then, an M-regular sequence w, q is permutable; that is, q, w also forms an M-regular sequence if and only if q is a nonzero-divisor on M.

In fact, on the S_{k-1}-regular sequence $z_{1(k-1)}, \dots, z_{\ell(k-1)}, q_k$, we can permute $z_{\ell(k-1)}$ and q_k, because q_k is not zero in the domain $S_{k-1}/(z_{1(k-1)}, \dots, z_{(\ell-1)(k-1)})$ for every ℓ $(1 \le \ell \le m)$, and this shows (1.1.2).

Now the assumption $q_k \in S_{k-1}$ (cf. (1.0.2)) shows that

$$z_{ik} = \Big(z_i + \sum_{j=1}^{k-1} a_{ij} q_j^{\varepsilon_j}\Big) + a_{ik} q_k^{\varepsilon_k} = z_{i(k-1)} + a_{ik} q_k^{\varepsilon_k}$$

is a linear polynomial in a_{ik} with coefficients contained in S_{k-1}. Thus $(z_{1k}, \dots, z_{\ell k})S_k$ is a prime ideal for every ℓ $(1 \le \ell \le m)$ generated by an S_k-regular sequence $z_{1k}, \dots, z_{\ell k}$ and

$$(z_{1k}, \dots, z_{mk})S_k \cap S_{k-1} = (0).$$

Indeed, the following is well known. Let S be a ring, and let T be an indeterminate. Suppose that $q, w_1, \dots, w_\ell$ is an S-regular sequence. Then, $(qT - w_1)A[T]$ is the kernel of an S-algebra homomorphism $\phi\colon S[T] \to S[w_1/q] = S'$, mapping T to w_1/q, and q, $w_2, \dots, w_\ell$ becomes an S'-regular sequence.

Hence $(z_{1k}, \dots, z_{\ell k})S_k$ is the kernel of an S_{k-1}-algebra homomorphism,

$$\phi_\ell\colon S_{k-1}[a_{nk}, \dots, a_{1k}] \to S_{k-1}[a_{nk}, \dots, a_{(\ell+1)k}]\Big[\frac{z_{1(k-1)}}{q_k^{\varepsilon_k}}, \dots, \frac{z_{\ell(k-1)}}{q_k^{\varepsilon_k}}\Big],$$

mapping a_{ik} to $-z_{i(k-1)}/q_k^{\varepsilon_k}$ for $1 \le i \le \ell$ $(\le m)$, and this proves the assertions. Therefore, $(z_{1k}, \dots, z_{\ell k})S_k$ is a prime ideal generated by an S_k-regular sequence $z_{1k}, \dots, z_{\ell k}$ and (1.1.1) holds for k. □

REMARK

We remark here that if, in place of (1.0.2), we assume that

(1.1.3) $\quad p_k \in S_{k-1}$ for every $k \ge 1 \qquad$ and $\qquad p_k \notin (z_{1(k-1)}, \dots, z_{\ell(k-1)})S_{k-1}$,

then the proof above shows that $(z_{1k}, \dots, z_{\ell k})S_k$ is a prime ideal generated by an S_k-regular sequence $z_{1k}, \dots, z_{\ell k}$ and that (1.1.1) holds for k (cf. (5.1.2), [24, Lemma 1.9]).

1.2. Relations

Let $n, r, m \in \mathbb{N}$ with $m < n$. For each j with $1 \le j \le r$, let $F_j := F_j(Z_1, \dots, Z_m)$ be a polynomial in m variables with coefficients in K_0 such that

$$F_j \in (Z_1, \dots, Z_m)K_0[Z_1, \dots, Z_m]$$

and a sequence of strictly increasing natural numbers $\nu_1, \ldots, \nu_k, \ldots$, for example, $\nu_k = k$ such that $\nu_k \leq \varepsilon_k$ for every k, and set

(1.2.1) $$\alpha_{jk} := \frac{1}{q_k^{\nu_k}} F_j(z_{1k}, \ldots, z_{mk}) \in Q(R)$$

for $j = 1, \ldots, r$, where $Q(R) = K(z_1, \ldots, z_n)$ is the field of fractions of R (cf. (1.0.0)). Then

$$\begin{aligned}\alpha_{j(k+1)} &= \frac{1}{q_{k+1}^{\nu_{k+1}}} F_j(z_{1(k+1)}, \ldots, z_{m(k+1)}) \\ &= \frac{1}{q_{k+1}^{\nu_{k+1}}} F_j(z_{1k} + a_{1(k+1)} q_{k+1}^{\varepsilon_{k+1}}, \ldots, z_{mk} + a_{m(k+1)} q_{k+1}^{\varepsilon_{k+1}}).\end{aligned}$$

Thus we have the following relation between α_{jk} and $\alpha_{j(k+1)}$,

(1.2.2) $$\alpha_{jk} = \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{j(k+1)} + \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} s_{jk} \quad \text{with } s_{jk} \in S_{k+1}.$$

Let

(1.2.3) $$B := \bigcup_{k \in \mathbb{N}} R[\alpha_{1k}, \ldots, \alpha_{rk}] \subset Q(R).$$

Then we have the following.

LEMMA 1.3

With the notation above, let $M = (z_1, \ldots, z_n)B$. Then M is a maximal ideal of B.

Proof

Let $\iota\colon R \to B$ be the canonical inclusion. Because $\alpha_{jk} \in M$ for every j and k by (1.2.2), we have a canonical surjection $\bar{\iota}\colon R/\mathfrak{n} \to B/M$. To get the assertion, it suffices to show

(1.3.0) $$M \neq B.$$

Indeed, assume the contrary, that is, $M = B$. Then we find elements $\beta_1, \ldots, \beta_n \in B$ that satisfy

$$\beta_1 \cdot z_1 + \cdots + \beta_n \cdot z_n = 1.$$

We may assume that $\beta_1, \ldots, \beta_n \in R[\alpha_{1k}, \ldots, \alpha_{rk}]$ for sufficiently large k. Thus there exist $r_1, \ldots, r_n \in R$ and $\nu \in \mathbb{N}$ such that $q_k^{\nu}(\beta_1 - r_1), \ldots, q_k^{\nu}(\beta_n - r_n) \in P_k$. Hence $q_k^{\nu}(r_1 \cdot z_1 + \cdots + r_n \cdot z_n - 1) \in P_k$. Therefore $q_k \in P_k$, because $r_1 \cdot z_1 + \cdots + r_n \cdot z_n - 1$ is a unit in R. This is a contradiction. □

We define

(1.3.1) $$A := B_M \subset Q(R).$$

Then A is a quasi-local domain with its maximal ideal $\mathfrak{m} = MA$. In addition, we define

$$(1.3.2)\qquad \zeta_i := z_i + a_{i1}q_1^{\varepsilon_1} + \cdots + a_{ik}q_k^{\varepsilon_k} + \cdots = z_i + \sum_{k=1}^{\infty} a_{ik}q_k^{\varepsilon_k},$$

$$(1.3.3)\qquad f_j := F_j(\underline{\zeta}) = F_j(\zeta_1,\dots,\zeta_m) \in K_0[[\zeta_1,\dots,\zeta_m]] \subset K[[\zeta_1,\dots,\zeta_n]] = \hat{R}$$

for $i = 1,\dots,n$ and for $j = 1,\dots,r$.

THEOREM 1.4

Let K be a purely transcendental extension field of countably infinite degree over a countable field K_0. Take polynomials $F_j := F_j(Z_1,\dots,Z_m)$ with $1 \le j \le r$, in m variables over K_0 without constant term. Then, for every $n > m$, the quasi-local domain $(A,\mathfrak{m})$ defined in (1.3.1) is Noetherian and satisfies the following:

$$(1.4.1)\qquad \tilde{\imath}\colon K[[\zeta_1,\dots,\zeta_n]]/\big(F_1(\underline{\zeta}),\dots,F_r(\underline{\zeta})\big) = \hat{R}/(f_1,\dots,f_r) \overset{\cong}{\hookrightarrow} \hat{A},$$

$$(1.4.2)\qquad \hat{\mathfrak{p}} := \big(\tilde{\imath}(\zeta_1),\dots,\tilde{\imath}(\zeta_m)\big)\hat{A} \textit{ is a prime ideal of } \hat{A} \textit{ and } \hat{\mathfrak{p}} \cap A = (0),$$

$$(1.4.3)\quad A/\mathfrak{p} \textit{ is essentially of finite type over } K \quad \textit{for every } \mathfrak{p} \in \operatorname{Spec}(A) \setminus \{(0)\}.$$

Here $\tilde{\imath}$ is a map induced by the inclusion $R := K[z_1,\dots,z_n]_{(z_1,\dots,z_n)} \hookrightarrow A$, the ζ_i are defined in (1.3.2), $\underline{\zeta}$ abbreviates $\zeta_1,\dots,\zeta_m$, and each f_j is as defined in (1.3.3).

Proof

With the notation above, we first show that A is Noetherian. By a theorem of Cohen (cf. [21, (3.4)]), it is enough to see that every nonzero prime ideal $\mathfrak{p}$ of A is finitely generated. Take a nonzero prime ideal $\mathfrak{p}$ of A. Then $\mathfrak{p} \cap R \ne (0)$, because R and A have the same field of fractions. Thus there exists $\ell \in \mathbb{N}$ such that $p_\ell \in \mathfrak{p} \cap \mathcal{P}$. Then $\alpha_{jk} \in R + p_\ell A$ for every $j = 1,\dots,r$ and for every $k = 1,2,\dots$ by (1.2.2). Hence we have a canonical surjection $\iota_\ell\colon R \to A/p_\ell A$, and $A/p_\ell A$ is essentially of finite type over K. Consequently $\mathfrak{p}$ is finitely generated and satisfies (1.4.3). Further, ι_ℓ induces the canonical surjection $\hat{\imath}\colon \hat{R} \to \hat{A}$.

We determine $\operatorname{Ker}\hat{\imath}$, verifying (1.4.2) at the same time. We have

$$(1.4.4)\qquad f_j - q_k^{\nu_k}\alpha_{jk} = F_j(\zeta_1,\dots,\zeta_m) - F_j(z_{1k},\dots,z_{mk}) = q_{k+1}^{\varepsilon_{k+1}}\eta_{jk}$$

with $\eta_{jk} \in \hat{R}$ for $j = 1,\dots,r$, because $\zeta_i - z_{ik} \in q_{k+1}^{\varepsilon_{k+1}}\hat{R}$ for $i = 1,\dots,n$ (cf. (1.2.1), (1.3.3)). Thus $\hat{\imath}(f_j) \in q_k^{\nu_k}\hat{A}$ for every $k \in \mathbb{N}$. Then $f_j \in \operatorname{Ker}\hat{\imath}$.

Set $\hat{P} := (\zeta_1,\dots,\zeta_m)\hat{R}$, a prime ideal of height m. We claim that

$$(1.4.5)\qquad \hat{P} \cap R = (0).$$

Assume the contrary, that is, $\hat{P} \cap R \ne (0)$. Then we find $h \in \mathbb{N}$ such that $p_h \in \hat{P} \cap R$, by the condition on $\mathcal{P}$ (cf. (1.0.1)). Hence $p_h, z_{1(h-1)},\dots,z_{m(h-1)} \in \hat{P}$. This is a contradiction, because $(p_h, z_{1(h-1)},\dots,z_{m(h-1)})\hat{R}$ has height $m+1$ (cf. (1.1.1)). Thus our claim is completed.

Consequently $\hat{R}/\underline{f}\hat{R}$ is R-torsion-free, where $\underline{f}$ abbreviates $f_1, \ldots, f_r$, because, for every $\hat{Q} \in \operatorname{Ass}_{\hat{R}}(\hat{R}/\underline{f}\hat{R})$, we have $\hat{Q} \subset \hat{P}$. Further, the canonical homomorphism $\pi\colon R \to \hat{R}/\underline{f}\hat{R}$ induces an R-algebra homomorphism $\psi\colon A \to Q(R) \otimes_R \hat{R}/\underline{f}\hat{R}$, mapping α_{jk} to $\alpha_{jk} \otimes 1$, and $\alpha_{jk} \otimes 1 = 1 \otimes (-q_k^{\varepsilon_{k+1}-\nu_k} p_{k+1}^{\varepsilon_{k+1}} \eta_{jk}) \in Q(R) \otimes_R \hat{R}/\underline{f}\hat{R}$ by (1.4.4). Thus ψ factors through $\hat{R}/\underline{f}\hat{R}$, which is R-torsion-free. We then have the following commutative diagram:

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\hat{\psi}} & \hat{R}/\underline{f}\hat{R} \\ {\scriptstyle \hat{\iota}}\uparrow & & \uparrow{\scriptstyle \hat{\pi}} \\ \hat{R} & = & \hat{R} \end{array}$$

where $\hat{\iota}$, $\hat{\pi}$, and $\hat{\psi}$ are canonical homomorphisms. Therefore $\operatorname{Ker} \hat{\iota} \subset (f_1, \ldots, f_r)$. This gives (1.4.1), and we get $\hat{\mathfrak{p}} \cong \hat{P}/\underline{f}\hat{R}$. Thus $\hat{\mathfrak{p}}$ is a prime ideal of $\hat{A}$, and (1.4.5) implies that $\hat{\mathfrak{p}} \cap A = (0)$. □

We end this section with the following result, which is a corollary to the proof of Theorem 1.4. The additional hypotheses enable us to bypass some parts of the proof and thus obtain a slight generalization of the theorem, so that $n = m$.

COROLLARY 1.5

We use the notation above, except that $n = m$. Let $F_1(\underline{Z}), \ldots, F_r(\underline{Z})$ be polynomials in the variables $\underline{Z} := (Z_1, \ldots, Z_n)$ over K_0 with zero constant term. Let $f_{jk} = F_j(z_{1k}, \ldots, z_{nk})$ and $I_k = (f_{1k}, \ldots, f_{rk})R$. Suppose that

(1.5.0) $\quad p_h \notin \sqrt{I_k}$ *whenever $h \le k$ for every sufficiently large k*

and

$$\hat{R}/\big(F_1(\underline{\zeta}), \ldots, F_r(\underline{\zeta})\big)\hat{R} \text{ is } R\text{-torsion-free,}$$

where $\underline{\zeta}$ abbreviates $\zeta_1, \ldots, \zeta_n$. Then $(A, \mathfrak{m})$, the quasi-local domain defined in (1.3.1), is Noetherian and satisfies the following:

(1.5.1) $\quad \tilde{\iota}\colon K[[\zeta_1, \ldots, \zeta_n]]/\big(F_1(\underline{\zeta}), \ldots, F_r(\underline{\zeta})\big) = \hat{R}/(f_1, \ldots, f_r) \overset{\cong}{\hookrightarrow} \hat{A};$

that is, the homomorphism $\tilde{\iota}$ induced by the containment $R \overset{\iota}{\hookrightarrow} B \hookrightarrow A$ from the map ι of the proof of Lemma 1.3 is an isomorphism;

(1.5.2) $\quad A/\mathfrak{p}$ *is essentially of finite type over K for every $\mathfrak{p} \in \operatorname{Spec}(A) \setminus \{(0)\}$.*

Proof

To see this, observe that the notation can be set with $m = n$, and the proof still holds as in Lemma 1.3, to show $M \neq B$ (1.3.0), where $p_h \notin \sqrt{I_k}$ comes in. Then item (1.4.5) requires that $n > m$. However, the condition that $\hat{R}/\underline{f}\hat{R} = \hat{R}/(F_1(\underline{\zeta}), \ldots, F_r(\underline{\zeta}))\hat{R}$ is R-torsion-free permits the next step. □

2. Examples

As applications of Theorem 1.4 and/or Corollary 1.5, we obtain the following examples of local domains.

EXAMPLE 2.1 ([1], [21, EXAMPLE 3, P. 205])

A one-dimensional analytically *ramified* and/or *reducible* local domain of *arbitrary* characteristic.

CONSTRUCTION

With notation as in Corollary 1.5, let K_0 be a countable field of arbitrary characteristic, and let $n=2$. For $b_1,\dots,b_d \in K_0$ and $c_1,\dots,c_d \in \mathbb{N}$, let

$$F(Z_1,Z_2)=(Z_1-b_1Z_2)^{c_1}\cdots(Z_1-b_dZ_2)^{c_d}.$$

Then by Corollary 1.5, we get a one-dimensional local domain $(A,\mathfrak{m})$ such that

$$\hat{A}\cong K[[\zeta_1,\zeta_2]]/\big((\zeta_1-b_1\zeta_2)^{c_1}\cdots(\zeta_1-b_d\zeta_2)^{c_d}\big).$$

To see that this setup satisfies the hypothesis (1.5.0) of Corollary 1.5, notice that with $n=m=2$, we do have $p_h \notin \sqrt{I_k}=(z_{1k}-b_1z_{2k})\cdots(z_{1k}-b_dz_{2k})R$ whenever $h\le k$ as in the proof of Lemma 1.1 and $(\zeta_1-b\zeta_2)\hat{R}\cap R=(0)$ as used in (1.4.5) to show that $\hat{R}/\underline{f}\hat{R}$ is R-torsion-free.

EXAMPLE 2.2 ([9, EXAMPLE D])

A one-dimensional local domain with given embedding dimension and multiplicity, which is δ-simple for a derivation $\delta\in \operatorname{Der}(A,A)$.

CONSTRUCTION

Let $K_0=\mathbb{Q}$. For every natural numbers m and t, let

$$F_{11}(Z_1,\dots,Z_m)=Z_1^{t+1} \qquad \text{and} \qquad F_{ij}(Z_1,\dots,Z_m)=Z_iZ_j$$

with $i\le j$ for $i=1,\dots,m$ and for $j=2,\dots,m$. Then by Theorem 1.4, we obtain a one-dimensional local domain $(A,\mathfrak{m})$ that satisfies the following:

$$\hat{A}\cong K[[\zeta_1,\dots,\zeta_{m+1}]]/(\zeta_1^{t+1},\zeta_1\zeta_2,\dots,\zeta_m^2).$$

We see that the embedding dimension of A is $\dim_K \mathfrak{m}/\mathfrak{m}^2=m+1$ and that A has multiplicity $e_{\mathfrak{m}}(A)=m+t$. Here we define a derivation $\delta\in\operatorname{Der}(A,A)$ that makes A δ-simple. Firstly, let

$$\delta(z_1+\cdots+z_{m+1})=\delta(q_1)=1 \qquad \text{and} \qquad \delta(z_i)=-\varepsilon_1 a_{i1}q_1^{\varepsilon_1-1}$$
$$\text{for } i=1,\dots,m \text{ (cf. (1.0.6))}.$$

Next we determine the values of $\delta(a_{ik})$ as follows:

$$\delta(a_{ik})=-\frac{1}{q_k^{\varepsilon_k}}\big(\varepsilon_{k+1}a_{i(k+1)}q_{k+1}^{\varepsilon_{k+1}-1}\delta(q_{k+1})\big)\in S_{k+1}$$
$$\text{for } i=1,\dots,m \text{ and}$$

$$\delta(a_{(m+1)k}) = 0.$$

Then we see that $\delta(\alpha_{jk}) \in S_{k+1}[\alpha_{j(k+1)}]$, because

$$\delta(\alpha_{jk}) = \frac{1}{q_k^{2\nu_k}}\big(\delta(F_j(z_{1k},\ldots,z_{mk}))q_k^{\nu_k} - F_j(z_{1k},\ldots,z_{mk})\delta(q_k^{\nu_k})\big),$$

$$\delta(z_{ik}) = q_k^{\varepsilon_k}\delta(a_{ik}) = -\varepsilon_{k+1}a_{i(k+1)}q_{k+1}^{\varepsilon_{k+1}-1}\delta(q_{k+1}).$$

We get a desired derivation $\delta \in \operatorname{Der}(A,A)$.

EXAMPLE 2.3 ([8, PROPOSITION 3.3, P. 304], [5])

A two-dimensional local domain whose completion has *embedded* associated prime ideal(s).

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 3$. Let

$$F_1(Z_1,Z_2,Z_3) = Z_1^3, \qquad F_2(Z_1,Z_2,Z_3) = Z_1^2Z_3, \qquad F_3(Z_1,Z_2,Z_3) = Z_1Z_2Z_3^2.$$

Then we get a two-dimensional local domain $(A,\mathfrak{m})$ such that

$$\begin{aligned}\hat{A} &\cong K[[\zeta_1,\zeta_2,\zeta_3]]/(\zeta_1^3,\zeta_1^2\zeta_3,\zeta_1\zeta_2\zeta_3^2)\\ &= K[[\zeta_1,\zeta_2,\zeta_3]]/(\zeta_1)\cap(\zeta_1,\zeta_2)^2\cap(\zeta_1,\zeta_3)^3.\end{aligned}$$

Hence $(A,\mathfrak{m})$ is a universally catenary local domain, which is *not* unmixed, with multiplicity 1 (cf. [21, (40.6)]). Further, for every height-one prime $P \in \operatorname{Spec}(A)$, A_P is a discrete valuation ring (cf. (1.5.0)). The derived normal ring $\bar{A} = A(\mathfrak{m})$, which is the total transform of A, and every intermediate ring B between A and $\bar{A}$ are Noetherian (cf. [8, Proposition 1.1], [18]).

In fact, our construction shows that the derived normal ring $\bar{A}$ is

$$\bar{A} = \bigcup_k R[\beta_{1k}] \quad \text{where } \beta_{1k} = \frac{1}{q_k^{\nu_k}}z_{1k} \text{ (cf. Section 1.2).} \tag{2.3.1}$$

Consequently, we have a canonical *sur*jection (cf. (1.2.1)):

$$\hat{A} \cong K[[\zeta_1,\zeta_2,\zeta_3]]/(\zeta_1^3,\zeta_1^2\zeta_3,\zeta_1\zeta_2\zeta_3^2) \longrightarrow K[[\zeta_1,\zeta_2,\zeta_3]]/(\zeta_1) \cong (\bar{A})^\wedge.$$

This shows that $\bar{A}$ is regular (cf. [8, Proposition 3.3]).

REMARK

Let K_0 be a countable field, and let $n = m+1$. Let $F_{ij}(Z_1,\ldots,Z_m) := Z_iZ_j$ for $1 \le i,j \le m$. Then we get a one-dimensional local domain $(A,\mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1,\ldots,\zeta_{m+1}]]/(\zeta_1,\ldots,\zeta_m)^2.$$

When $m = 1$, $(A,\mathfrak{m})$ is a complete intersection, which is *not* Japanese. However, when $m \ge 2$, $\hat{A} \otimes Q(A)$ is *not* Gorenstein (cf. (1.4.2), [8, Proposition 3.1]). As above, we have a canonical surjection:

$$\hat{A} \longrightarrow (\bar{A})^\wedge \cong K[[\zeta_1,\ldots,\zeta_{m+1}]]/(\zeta_1,\ldots,\zeta_m) \text{ (cf. (2.3.1)).}$$

EXAMPLE 2.4 ([21, EXAMPLE 4, P. 207], [27], [22, (5.8)])

A two-dimensional Cohen–Macaulay local domain $(A, \mathfrak{m})$ that has infinitely many non-Noetherian intermediate quasi-local domains between A and its derived normal ring $\bar{A}$.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 3$. Let

$$F(Z_1) = Z_1^c \quad \text{with } c \geq 2.$$

Then we get a two-dimensional local domain $(A, \mathfrak{m})$ with its completion:

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1^c).$$

For every nonzero element $a \in \mathfrak{m}$, let $C = \bar{A} \cap A[1/a]$, the integral closure of A in $A[1/a]$. We claim that C is *not* Noetherian.

Indeed, assume that C is Noetherian. Then, because $a^\nu \bar{A} \cap C = a^\nu C$, we have canonical injections $C/a^\nu C \hookrightarrow \bar{A}/a^\nu \bar{A}$ for every ν and $C^* \hookrightarrow \bar{A}^*$, where C^* and $\bar{A}^*$ represent the aC-adic completion of C and $a\bar{A}$-adic completion of $\bar{A}$, respectively. Hence C^* of C is reduced. Further, for every prime ideal $\mathfrak{q} \in \operatorname{Spec}(C/aC)$, $C/\mathfrak{q}$ is a Nagata ring (cf. (1.4.3), [21, (33.10), (36.5)]). Thus $\hat{C}$ is reduced by a theorem of Marot [17] (cf. [21, (36.4)]), and $C_\mathfrak{p} (= A_\mathfrak{p})$ is analytically unramified for every $\mathfrak{p} \in \operatorname{Spec}(C[1/a])$ $(= \operatorname{Spec}(A[1/a]))$ (cf. [21, (36.8)]). Then, for every $b \in \mathfrak{m}$ such that a, b is a system of parameters of A, the bA-adic completion A^* of A is reduced. Consequently $\hat{A}$ should be reduced, because $A/\mathfrak{p}$ is a Nagata ring for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(A/bA)$ by (1.4.3), a contradiction.

REMARK

We have that $\bar{A}$ above is Noetherian (see [21, (33.12)]), and, as in Example 2.3, we have a canonical surjection:

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1^c) \longrightarrow K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1) \cong (\bar{A})^\wedge.$$

Hence $\bar{A}$ is regular. However, the fact that C is non-Noetherian for every $a \in \mathfrak{m} \setminus \{0\}$ shows that the normal locus $\operatorname{Nor}(A) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid A_\mathfrak{p} \text{ is normal}\}$ of A contains no nonempty open subset (cf. Example 2.7).

EXAMPLE 2.5 ([32])

A two-dimensional analytically *(ir)reducible* Nagata normal local domain that is *not* analytically normal.

CONSTRUCTION

Let $K_0 = \mathbb{Q}$, and let $n = 3$. Take

$$F(Z_1, Z_2) = Z_1^2 - Z_2^3 \qquad \text{or} \qquad F(Z_1, Z_2) = Z_1 Z_2.$$

Then we obtain a two-dimensional local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1^2 - \zeta_2^3) \qquad \text{or} \qquad K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1 \zeta_2).$$

Because $\operatorname{Sing}(\hat{A}) = V((\zeta_1, \zeta_2))$, the regular locus $\operatorname{Reg}(A)$ of A is $\operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$ (cf. (1.4.2)). Thus A is a normal Nagata local domain, which is *not* analytically normal (cf. (1.4.3)).

REMARK

When K_0 is a countable field of characteristic $p > 2$ and $n = 3$, let

$$F(Z_1, Z_2) = Z_1^2 - Z_2^p.$$

Then we get a two-dimensional local domain $(A, \mathfrak{m})$ with its completion:

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1^2 - \zeta_2^p).$$

Because $\operatorname{Sing}(\hat{A}) = V((\zeta_1))$, A satisfies Serre's condition (R_1). Thus A is normal and A is a Nagata local domain whenever $\varepsilon_k \equiv 0 \pmod p$ (cf. (4.4.1)).

EXAMPLE 2.6 ([21, EXAMPLE 2, P. 203])

A two-dimensional quasi-excellent catenary local domain, which is *not* universally catenary.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 3$. Take

$$F_1(Z_1, Z_2, Z_3) = Z_1 Z_2 \qquad \text{and} \qquad F_2(Z_1, Z_2, Z_3) = Z_1 Z_3.$$

Then by Corollary 1.5, we get a two-dimensional local domain $(A, \mathfrak{m})$ as follows:

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1\zeta_2, \zeta_1\zeta_3) = K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1) \cap (\zeta_2, \zeta_3).$$

Thus A is a catenary quasi-excellent local domain but *not* universally catenary (cf. [30], [20, Theorem 31.7]).

REMARK

For every $n = m + 1 \geq 3$, take

$$F_1(Z_1, \ldots, Z_n) = Z_1 Z_2, \ldots, F_m(Z_1, \ldots, Z_n) = Z_1 Z_n.$$

Then as above, we get an m-dimensional local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \ldots, \zeta_n]]/(\zeta_1) \cap (\zeta_2, \ldots, \zeta_n).$$

Hence A is also a catenary quasi-excellent local domain but *not* universally catenary (cf. [31], [25]). We remark, however, that these examples are *not* normal.

EXAMPLE 2.7 ([5, PROPOSITION 21, P. 393])

A two-dimensional local domain, which is a complete intersection, whose regular (nor normal) locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 3$. Take

$$F(Z_1) = Z_1^c \quad \text{with } c \geq 2.$$

Then as in Example 2.4, we get a two-dimensional local domain $(A, \mathfrak{m})$ with its completion:

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3]]/(\zeta_1^c).$$

Thus $\mathrm{Nor}(A) = \{(0)\}$. Hence the normal locus (nor the regular locus) of A contains no nonempty open subset.

EXAMPLE 2.8 ([5, PROPOSITION 21, P. 393] CF. [7, P. 480])

A two-dimensional Gorenstein local domain whose complete intersection locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 5$. Let

$$F_1(Z_1, Z_2, Z_3) = Z_2^2, \qquad F_2(Z_1, Z_2, Z_3) = Z_1 Z_3,$$
$$F_3(Z_1, Z_2, Z_3) = Z_1 Z_2 + Z_3^2, \qquad F_4(Z_1, Z_2, Z_3) = Z_2 Z_3,$$
$$F_5(Z_1, Z_2, Z_3) = Z_1^2.$$

Then we obtain a two-dimensional Gorenstein local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]]/(\zeta_2^2, \zeta_1\zeta_3, \zeta_1\zeta_2 + \zeta_3^2, \zeta_2\zeta_3, \zeta_1^2).$$

Hence $\mathrm{CI}(A) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}} \text{ is a complete intersection}\} = \{(0)\}$. Namely, the complete intersection locus of A contains no nonempty open subset.

EXAMPLE 2.9 ([5, PROPOSITION 21, P. 393])

A two-dimensional Cohen–Macaulay local domain whose Gorenstein locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 4$. Take

$$F_1(Z_1, Z_2) = Z_1^2, \qquad F_2(Z_1, Z_2) = Z_1 Z_2 \qquad \text{and} \qquad F_3(Z_1, Z_2) = Z_2^2.$$

Then we get a two-dimensional Cohen–Macaulay local domain $(A, \mathfrak{m})$ with

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1, \zeta_2)^2.$$

Hence $\mathrm{Gor}(A) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}} \text{ is Gorenstein}\} = \{(0)\}$ (cf. remark of Example 2.3). Namely, the Gorenstein locus of A contains no nonempty open subset.

EXAMPLE 2.10 ([5, PROPOSITION 21, P. 393], CF. [8, PROPOSITION 3.5])

A three-dimensional local domain whose Cohen–Macaulay locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic with $n = 4$. Take

$$F_1(Z_1, Z_2) = Z_1^2 \qquad \text{and} \qquad F_2(Z_1, Z_2) = Z_1 Z_2.$$

Then, we obtain a three-dimensional local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1) \cap (\zeta_1, \zeta_2)^2.$$

We show that the Cohen–Macaulay locus of A contains no nonempty open subset $D(a) = \{\mathfrak{q} \in \operatorname{Spec}(A) \mid \mathfrak{q} \not\ni a\}$ for every nonzero $a \in \mathfrak{m}$.

Indeed, we find $\hat{\mathfrak{q}} \in \hat{D}(a) := \{\hat{\mathfrak{q}} \in \operatorname{Spec}(\hat{A}) \mid \hat{\mathfrak{q}} \not\ni a\}$ such that $\hat{A}_{\hat{\mathfrak{q}}}$ is *not* Cohen–Macaulay and that $\hat{\mathfrak{q}} \cap A = \mathfrak{q} \in D(a) \setminus \{(0)\}$, because $\hat{A}_{\hat{\mathfrak{p}}}$ is not Cohen–Macaulay and because $\dim \hat{A}/\hat{\mathfrak{p}} = 2$ (cf. (1.4.2)). Because $A/\mathfrak{q}$ is excellent by (1.4.3), $A_{\mathfrak{q}}$ is not Cohen–Macaulay.

EXAMPLE 2.11 ([5, PROPOSITION 21, P. 393])

A three-dimensional Nagata normal local domain, which is a complete intersection, whose regular locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of characteristic *zero*, and let $n = 4$. Take

$$F(Z_1, Z_2) = Z_1^2 - Z_2^3.$$

Then we get a three-dimensional normal local domain $(A, \mathfrak{m})$ with its completion:

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1^2 - \zeta_2^3).$$

Thus A is a Nagata ring by (1.4.3), and the same reasoning as in Example 2.10 shows that $\operatorname{Reg}(A)$ contains *no* nonempty open subset.

EXAMPLE 2.12 ([5, PROPOSITION 21, P. 393]; CF. [13, EXAMPLE A, P. 192])

A three-dimensional Nagata normal Gorenstein local domain whose complete intersection locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of characteristic *zero* with $n = 6$. Let

$$F_1(Z_1, Z_2, Z_3, Z_4) = Z_1Z_3 - Z_2^2, \qquad F_2(Z_1, Z_2, Z_3, Z_4) = Z_1Z_4 - Z_2Z_3,$$
$$F_3(Z_1, Z_2, Z_3, Z_4) = Z_2Z_4 - Z_3^2, \qquad F_4(Z_1, Z_2, Z_3, Z_4) = Z_1^3 - Z_3Z_4,$$
$$F_5(Z_1, Z_2, Z_3, Z_4) = Z_1^2Z_2 - Z_4^2.$$

Then, we obtain a three-dimensional normal Gorenstein local domain $(A, \mathfrak{m})$ with

$$\begin{aligned}\hat{A} \cong K[[&\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6]] \\ &/(\zeta_1\zeta_3 - \zeta_2^2, \zeta_1\zeta_4 - \zeta_2\zeta_3, \\ &\quad \zeta_2\zeta_4 - \zeta_3^2, \zeta_1^3 - \zeta_3\zeta_4, \zeta_1^2\zeta_2 - \zeta_4^2).\end{aligned}$$

Hence A is a Nagata ring, and as above, $\operatorname{CI}(A)$ contains *no* nonempty open subset.

EXAMPLE 2.13 ([5, PROPOSITION 21, P. 393]; CF. [13, EXAMPLE, P. 180])

A three-dimensional Nagata normal Cohen–Macaulay local domain whose Gorenstein locus is *not* open.

CONSTRUCTION

Let K_0 be a countable field of characteristic *zero* with $n = 5$. Take

$$F_1(Z_1, Z_2, Z_3) = Z_1^3 - Z_2Z_3, \qquad F_2(Z_1, Z_2, Z_3) = Z_1^2Z_2 - Z_3^2,$$
$$F_3(Z_1, Z_2, Z_3) = Z_1Z_3 - Z_2^2.$$

Then we get a three-dimensional normal Cohen–Macaulay local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]]/(\zeta_1^3 - \zeta_2\zeta_3, \zeta_1^2\zeta_2 - \zeta_3^2, \zeta_1\zeta_3 - \zeta_2^2).$$

Thus A is a Nagata ring, and $\mathrm{Gor}(A)$ contains *no* nonempty open subset.

EXAMPLE 2.14 ([5, PROPOSITION 21, P. 393], CF. [14, P. 61])

A four-dimensional Nagata normal local domain that has *non*open Cohen–Macaulay locus.

CONSTRUCTION

Let K_0 be a countable field of characteristic *zero* with $n = 6$. Let

$$F_1(Z_1, Z_2, Z_3, Z_4) = Z_2^3 - Z_3^2, \qquad F_2(Z_1, Z_2, Z_3, Z_4) = Z_2Z_4^2 - Z_1^2,$$
$$F_3(Z_1, Z_2, Z_3, Z_4) = Z_2Z_1 - Z_4Z_3, \qquad F_4(Z_1, Z_2, Z_3, Z_4) = Z_2^2Z_4 - Z_3Z_1.$$

Then we obtain a four-dimensional normal local domain $(A, \mathfrak{m})$ that satisfies

$$\hat{A} \cong K[[\zeta_1, \ldots, \zeta_6]]/(\zeta_2^3 - \zeta_3^2, \zeta_2\zeta_4^2 - \zeta_1^2, \zeta_2\zeta_1 - \zeta_4\zeta_3, \zeta_2^2\zeta_4 - \zeta_3\zeta_1).$$

As above, A is a Nagata ring and $\mathrm{CM}(A)$ contains *no* nonempty open subset.

EXAMPLE 2.15 ([26], [12])

A three-dimensional Henselian Nagata *normal* local domain that is *not* catenary.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 4$. Take

$$F_1(Z_1, Z_2, Z_3) = Z_1Z_2 \qquad \text{and} \qquad F_2(Z_1, Z_2, Z_3) = Z_1Z_3.$$

Then we get a three-dimensional local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1) \cap (\zeta_2, \zeta_3);$$

because $\mathrm{Sing}(\hat{A}) = V((\zeta_1, \zeta_2, \zeta_3))$ and because $\operatorname{depth} A = 2$, A is a *non*catenary Nagata normal domain (cf. (1.4.2), (1.4.3)). When K_0 is a countable field of characteristic $p > 0$, by the same reasoning as in Example 4.4, A is a Nagata ring whenever $\varepsilon_k \equiv 0 \pmod{p}$. Hence, by taking the Henselization of A, we get a desired local domain.

3. Construction of bad local domains with a specified prime element

In this section, we first make a minor change of the notation of Section 1 and have a specified Heitmann's lemma that gives a *good* enumeration on a countable set $\mathcal{P}^*$ (for the definition see (3.0.1)). Then, modifying the construction given in Section 1, we get bad local domains with a peculiar *prime* element, originally due to Brodmann and Rotthaus [6], Ogoma [29], and Brezuleanu and Rotthaus [4].

3.0. Notation and numbering on $\mathcal{P}^*$

Let K_0, K, and K_k be as in Section 1.0. Take $n+1$ indeterminates $x, z_1, \dots, z_n$ over K, and set

$$S_0 = K_0[x, z_1, \dots, z_n] \quad \text{with maximal ideal } \mathfrak{N}_0 = (x, z_1, \dots, z_n)S_0,$$

$$S_k = S_{k-1}[a_{1k}, \dots, a_{nk}] \quad \text{with } \mathfrak{N}_k = (x, z_1, \dots, z_n)S_k \text{ for } k \in \mathbb{N},$$

$$S = \bigcup_{k\in\mathbb{N}} S_k = K_0[\{a_{ik}\}_{i=1}^n, k \in \mathbb{N}][x, z_1, \dots, z_n] \quad \text{with } \mathfrak{N} = (x, z_1, \dots, z_n)S.$$

We localize these polynomial rings by the prime ideals above and obtain

$$R_0 = (S_0)_{\mathfrak{N}_0} = K_0[x, z_1, \dots, z_n]_{(x,z_1,\dots,z_n)} \quad \text{with } \mathfrak{n}_0 = (x, z_1, \dots, z_n)R_0,$$

$$R_k = (S_k)_{\mathfrak{N}_k} = K_k[x, z_1, \dots, z_n]_{(x,z_1,\dots,z_n)} \quad \text{with } \mathfrak{n}_k = (x, z_1, \dots, z_n)R_k,$$

$$R = S_{\mathfrak{N}} = K[x, z_1, \dots, z_n]_{(x,z_1,\dots,z_n)} \quad \text{with } \mathfrak{n} = (x, z_1, \dots, z_n)R.$$

Then, $R_k = R_{k-1}(a_{1k}, \dots, a_{nk})$, and $(R, \mathfrak{n})$ is a *countable* regular local ring that satisfies the following:

$$\text{(3.0.0)} \qquad R = K[x, z_1, \dots, z_n]_{(x,z_1,\dots,z_n)} = \bigcup_k R_k.$$

With the notation and assumptions above, we denote by $\mathcal{P}^*$ a set of elements of $\mathfrak{N} \setminus xS$,

$$\text{(3.0.1)} \qquad \mathcal{P}^* \subset \mathfrak{N} \setminus xS,$$

that contains enough elements. Namely, for each $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{(0), xR\}$, there exists at least one $p \in \mathcal{P}^*$ such that $p \in \mathfrak{p}$. Then $\mathcal{P}^*$ is a *countable* set, and we may assume that

$$z_1 + \cdots + z_n \in \mathcal{P}^*$$

and that $\mathcal{P}^*$ contains an *infinite* number of elements of S_0.

In the following, we denote by $\bar{s}$ the image of $s \in S$ in $\bar{S} = S/xS$ (or in $\bar{R} = R/xR$). Then $\bar{\mathcal{P}} := \{\bar{p} \in \bar{S} \mid p \in \mathcal{P}^*\}$ satisfies the same condition as $\mathcal{P}$ in (1.0.1). Namely, $\bar{\mathcal{P}}$ is a set of nonzero elements of $\bar{\mathfrak{N}} = \mathfrak{N}/xS$,

$$\text{(3.0.2)} \qquad \bar{\mathcal{P}} \subset \bar{\mathfrak{N}} \setminus \{\bar{0}\},$$

that contains enough elements. That is, for each nonzero $\bar{\mathfrak{p}} \in \operatorname{Spec}(\bar{R})$, there exists at least one $\bar{p}$ such that $\bar{p} \in \bar{\mathfrak{p}}$.

We fix a surjective mapping $\rho^* : \mathbb{N} \to \mathcal{P}^*$, which we call a *numbering* on $\mathcal{P}^*$, and set $\rho^*(i) = p_i$. By the remark above, we may assume that $p_1 = z_1 + \cdots + z_n$

and that ρ^* satisfies the following:

$$p_k \in S_{k-2} \quad \text{for every } k \geq 2. \tag{3.0.3}$$

Remark that if ρ^* is the numbering above, the induced mapping $\bar{\rho}\colon \mathbb{N} \to \bar{\mathcal{P}}$, which applies i to $\bar{p}_i$, is a *numbering* on $\bar{\mathcal{P}}$ such that $\bar{p}_1 = \bar{z}_1 + \cdots + \bar{z}_n$ and that

$$\bar{p}_k \in \bar{S}_{k-2} = S_{k-2}/xS_{k-2} \quad \text{for every } k \geq 2. \tag{3.0.4}$$

As in Section 1, for a sequence of strictly increasing natural numbers $\varepsilon_1, \ldots, \varepsilon_k, \ldots$, we define

$$z_{i0} = z_i, \tag{3.0.5}$$

$$q_k = p_1 \cdots p_k, \tag{3.0.6}$$

$$z_{ik} = z_i + a_{i1}q_1^{\varepsilon_1} + \cdots + a_{ik}q_k^{\varepsilon_k} \quad \text{for } k \geq 1. \tag{3.0.7}$$

Similarly, we define

$$\bar{z}_{i0} = \bar{z}_i, \tag{3.0.8}$$

$$\bar{q}_k = \bar{p}_1 \cdots \bar{p}_k, \tag{3.0.9}$$

$$\bar{z}_{ik} = \bar{z}_i + \bar{a}_{i1}\bar{q}_1^{\varepsilon_1} + \cdots + \bar{a}_{ik}\bar{q}_k^{\varepsilon_k} \quad \text{for } k \geq 1. \tag{3.0.10}$$

Then $Q_k = (x, z_{1k}, \ldots, z_{mk})R$ becomes a prime ideal of height $m+1$ for $k \geq 0$. The same reasoning as in Heitmann's lemma shows the following.

LEMMA 3.1 (SPECIFIED HEITMANN'S NUMBERING; CF. [12, PROPOSITION 1])
With the notation and assumptions above, suppose $m < n$. Let ρ^* be a numbering on $\mathcal{P}^*$ that satisfies (3.0.3). Then $(x, z_{1k}, \ldots, z_{\ell k})S_k$ is a prime ideal, generated by an S_k-regular sequence x, $z_{1k}, \ldots, z_{\ell k}$ for every $\ell = 1, \ldots, m$, and

$$p_h \notin Q_k \quad \textit{whenever } h \leq k+1. \tag{3.1.1}$$

3.2. Relations

Let $n, r, m \in \mathbb{N}$ with $m < n$. For each j with $1 \leq j \leq r$, let $G_j := G_j(X, Z_1, \ldots, Z_m)$ be a polynomial in $m+1$ variables with coefficients in K_0 such that

$$G_j \in (X, Z_1, \ldots, Z_m)K_0[X, Z_1, \ldots, Z_m].$$

Identifying $K_0[X, Z_1, \ldots, Z_m]/XK_0[X, Z_1, \ldots, Z_m]$ with $K_0[Z_1, \ldots, Z_m]$, let

$$F_j(Z_1, \ldots, Z_m) := G_j(0, Z_1, \ldots, Z_m) \in K_0[Z_1, \ldots, Z_m].$$

Take a sequence of strictly increasing natural numbers $\nu_1, \ldots, \nu_k, \ldots$ such that $\nu_k \leq \varepsilon_k$ for every k, and set

$$\alpha_{jk} := \frac{1}{q_k^{\nu_k}} G_j(x, z_{1k}, \ldots, z_{mk}) \in Q(R) \quad \text{for } j = 1, \ldots, r \tag{3.2.1}$$

$$\bar{\alpha}_{jk} := \frac{1}{\bar{q}_k^{\nu_k}} G_j(0, \bar{z}_{1k}, \ldots, \bar{z}_{mk}) = \frac{1}{\bar{q}_k^{\nu_k}} F_j(\bar{z}_{1k}, \ldots, \bar{z}_{mk}) \in Q(\bar{R}). \tag{3.2.2}$$

Here $Q(R) = K(x, z_1, \ldots, z_n)$ and $Q(\bar{R}) = K(\bar{z}_1, \ldots, \bar{z}_n)$ are the fields of fractions of R and of $\bar{R} = R/xR$, respectively (cf. (3.0.0)). Then

$$\begin{aligned}\alpha_{j(k+1)} &= \frac{1}{q_{k+1}^{\nu_{k+1}}} G_j(x, z_{1(k+1)}, \ldots, z_{m(k+1)}) \\ &= \frac{1}{q_{k+1}^{\nu_{k+1}}} G_j(x, z_{1k} + a_{1(k+1)} q_{k+1}^{\varepsilon_{k+1}}, \ldots, z_{mk} + a_{m(k+1)} q_{k+1}^{\varepsilon_{k+1}}).\end{aligned}$$

Thus we have the following relation between α_{jk} and $\alpha_{j(k+1)}$:

(3.2.3) $$\alpha_{jk} = \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{j(k+1)} + \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} s_{jk} \quad \text{with } s_{jk} \in S_{k+1}.$$

Let

(3.2.4) $$B := \bigcup_{k \in \mathbb{N}} R[\alpha_{1k}, \ldots, \alpha_{rk}] \subset Q(R).$$

Then the same proof as in Lemma 1.3 shows the following lemma.

LEMMA 3.3

With the notation above, let $M = (x, z_1, \ldots, z_n)B$. Then M is a maximal ideal of B.

We define

(3.3.1) $$A := B_M \subset Q(R) = Q(K[x, z_1, \ldots, z_n]).$$

Then A is a quasi-local domain with its maximal ideal $\mathfrak{m} = MA$. In addition, we put

(3.3.2) $$\zeta_i := z_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ik} q_k^{\varepsilon_k} + \cdots = z_i + \sum_{k=1}^{\infty} a_{ik} q_k^{\varepsilon_k},$$

(3.3.3) $$\begin{aligned} g_j &:= G_j(x, \underline{\zeta}) \\ &= G_j(x, \zeta_1, \ldots, \zeta_m) \in K_0[[x, \zeta_1, \ldots, \zeta_m]] \subset K[[x, \zeta_1, \ldots, \zeta_n]] = \hat{R},\end{aligned}$$

(3.3.4) $$\bar{\zeta}_i := \bar{z}_i + \bar{a}_{i1} \bar{q}_1^{\varepsilon_1} + \cdots + \bar{a}_{ik} \bar{q}_k^{\varepsilon_k} + \cdots = \bar{z}_i + \sum_{k=1}^{\infty} \bar{a}_{ik} \bar{q}_k^{\varepsilon_k},$$

(3.3.5) $$f_j := F_j(\underline{\bar{\zeta}}) = F_j(\bar{\zeta}_1, \ldots, \bar{\zeta}_m) \in K_0[[\bar{\zeta}_1, \ldots, \bar{\zeta}_m]] \subset K[[\bar{\zeta}_1, \ldots, \bar{\zeta}_n]] = \bar{R}^\wedge$$

for $i = 1, \ldots, n$ and for $j = 1, \ldots, r$.

THEOREM 3.4

Let K be a purely transcendental extension field of countably infinite degree over a countable field K_0. Take polynomials $G_j := G_j(X, Z_1, \ldots, Z_m)$ with $1 \le j \le r$, in $m+1$ variables over K_0 without constant term.

By identifying $K_0[X, \underline{Z}]/XK_0[X, \underline{Z}]$ with $K_0[\underline{Z}]$, let

$$F_j(Z_1, \ldots, Z_m) := G_j(0, Z_1, \ldots, Z_m) \in K_0[Z_1, \ldots, Z_m] \quad \textit{for } j = 1, \ldots, r.$$

Taking $r+1$ indeterminates Q, $T_1,\dots,T_r$, let $\tilde{\phi}$ and ϕ be ring homomorphisms:

$$\tilde{\varphi}:\ K_0[X,\underline{Z},Q][T_1,\dots,T_r]\to K_0[X,\underline{Z},Q][G_1/Q,\dots,G_r/Q] \quad \text{with } T_j\mapsto G_j/Q$$

$$\varphi:\ K_0[\underline{Z},Q][T_1,\dots,T_r]\to K_0[\underline{Z},Q][F_1/Q,\dots,F_r/Q] \quad \text{with } T_j\mapsto F_j/Q.$$

Suppose that, by regarding $K_0[\underline{Z},Q]$ as $K_0[X,\underline{Z},Q]/XK_0[X,\underline{Z},Q]$, we have

$$\operatorname{Ker}\varphi = K_0[\underline{Z},Q]\otimes_{K_0[X,\underline{Z},Q]}\operatorname{Ker}\tilde{\varphi}. \tag{3.4.0}$$

Then, for every $n>m$, the quasi-local domain $(A,\mathfrak{m})$ defined in (3.3.1) is Noetherian with a prime element $x\in\mathfrak{m}$ that satisfies the following:

$$\tilde{\iota}\colon K[[x,\zeta_1,\dots,\zeta_n]]/\big(G_1(x,\underline{\zeta}),\dots,G_r(x,\underline{\zeta})\big)=\hat{R}/(g_1,\dots,g_r)\overset{\cong}{\hookrightarrow}\hat{A}, \tag{3.4.1}$$

$$\tilde{\bar{\iota}}\colon K[[\bar{\zeta}_1,\dots,\bar{\zeta}_n]]/\big(F_1(\underline{\bar{\zeta}}),\dots,F_r(\underline{\bar{\zeta}})\big)=\widehat{R/xR}/(f_1,\dots,f_r)\overset{\cong}{\hookrightarrow}\hat{A}/x\hat{A}, \tag{3.4.2}$$

$$\hat{\mathfrak{q}}:=\big(\tilde{\iota}(x),\tilde{\iota}(\zeta_1),\dots,\tilde{\iota}(\zeta_m)\big)\hat{A} \text{ is a prime ideal of } \hat{A} \quad \text{and} \quad \hat{\mathfrak{q}}\cap A = xA, \tag{3.4.3}$$

$$A/\mathfrak{p} \text{ is essentially of finite type over } K \text{ for every } \mathfrak{p}\in\operatorname{Spec}(A)\setminus\{xA,(0)\}. \tag{3.4.4}$$

Proof
We follow the proof of Theorem 1.4. We first show that A is Noetherian. Namely, we check that every nonzero prime ideal $\mathfrak{p}$ of A is finitely generated. Note that $\mathfrak{p}\cap R\neq(0)$, and we consider two cases.
First case. There exists $\ell\in\mathbb{N}$ such that $p_\ell\in\mathfrak{p}\cap\mathcal{P}^*$. Then $\alpha_{jk}\in R+p_\ell A$ for every $j=1,\dots,r$ and for every $k=1,2,\dots$ by (3.2.3). Hence, we have a canonical surjection $\iota_\ell\colon R\to A/p_\ell A$, and $A/p_\ell A$ is essentially of finite type over K. Consequently $\mathfrak{p}$ is finitely generated and satisfies (3.4.4).
Second case. Suppose that $xR=\mathfrak{p}\cap R$. Then, our assumption (3.4.0) implies that

$$B/xB\cong\bigcup_{k\in\mathbb{N}}\bar{R}[\bar{\alpha}_{1k},\dots,\bar{\alpha}_{rk}]\subset Q(\bar{R}). \tag{3.4.5}$$

Thus $\mathfrak{p}=xA$, and therefore A is Noetherian. Moreover, (3.4.5) shows that A/xA has the same structure as local domains in Theorem 1.4.

Next we consider canonical surjections

$$\hat{\iota}\colon \hat{R}\to\hat{A} \qquad\text{and}\qquad \hat{\bar{\iota}}\colon \hat{R}/x\hat{R}\to\hat{A}/x\hat{A}.$$

It is clear that the same reasoning as in the proof of Theorem 1.4 guarantees

$$g_j\in\operatorname{Ker}\hat{\iota},\qquad \operatorname{Ker}\hat{\bar{\iota}}=(f_1,\dots,f_r),\qquad\text{and}\qquad \hat{Q}\cap R=xR,$$

where $\hat{Q}=(x,\zeta_1,\dots,\zeta_m)\hat{R}$ (cf. (1.4.5)). Then

$$(g_1,\dots,g_r)\subset\operatorname{Ker}\hat{\iota}\subset(x,g_1,\dots,g_r).$$

Therefore $\operatorname{Ker}\hat{\imath} = (g_1, \dots, g_r)$, because x is a nonzero divisor in $\hat{A}$. Finally $\hat{\mathfrak{q}} = \hat{Q}/(g)$ is a prime ideal of $\hat{A}$ and $\hat{\mathfrak{q}} \cap A = xA$, because A/xA and R/xR have a common field of fractions. Thus (3.4.3) holds. □

We end this section with the following result, which is a corollary to the proof of Theorem 3.4. The additional hypotheses enable us to bypass some parts of the proof and thus obtain a slight generalization of the theorem, so that $n = m$.

COROLLARY 3.5

We use the notation above, except that $n = m$. Let $G_1(X, \underline{Z}), \dots, G_r(X, \underline{Z})$ be polynomials in the variables X and $\underline{Z} := (Z_1, \dots, Z_n)$ over K_0 with zero constant term.

By identifying $K_0[X, \underline{Z}]/XK_0[X, \underline{Z}]$ with $K_0[\underline{Z}]$, let

$$F_j(\underline{Z}) = G_j(0, \underline{Z}) \in K_0[Z_1, \dots, Z_n].$$

Let $g_{jk} = G_j(x, z_{1k}, \dots, z_{nk})$, $J_k = (g_{1k}, \dots, g_{rk})R$, and $f_{jk} = F_j(\bar{z}_{1k}, \dots, \bar{z}_{nk})$. Suppose that

$$p_h \notin \sqrt{J_k} \quad \text{whenever } h \le k \text{ for every sufficiently large } k,$$

(3.5.0) $\hat{R}/(x, G_1(x, \underline{\zeta}), \dots, G_r(x, \underline{\zeta}))\hat{R}$ *is R/xR-torsion-free,*

$$R\Big[\frac{g_{1k}}{q_k^{\nu_k}}, \dots, \frac{g_{rk}}{q_k^{\nu_k}}\Big]/xR\Big[\frac{g_{1k}}{q_k^{\nu_k}}, \dots, \frac{g_{rk}}{q_k^{\nu_k}}\Big] \cong \bar{R}\Big[\frac{f_{1k}}{\bar{q}_k^{\nu_k}}, \dots, \frac{f_{rk}}{\bar{q}_k^{\nu_k}}\Big] \quad \text{for every } k,$$

where $\underline{\zeta}$ abbreviates $\zeta_1, \dots, \zeta_n$.

Then $(A, \mathfrak{m})$, the quasi-local domain defined in (3.3.1), is Noetherian with a prime element x that satisfies the following:

(3.5.1) $$\tilde{\imath}\colon K[[x, \zeta_1, \dots, \zeta_n]]/\big(G_1(x, \underline{\zeta}), \dots, G_r(x, \underline{\zeta})\big) \overset{\cong}{\hookrightarrow} \hat{A},$$

(3.5.2) $$\tilde{\tilde{\imath}}\colon K[[\bar{\zeta}_1, \dots, \bar{\zeta}_n]]/\big(F_1(\underline{\bar{\zeta}}), \dots, F_r(\underline{\bar{\zeta}})\big) \overset{\cong}{\hookrightarrow} (A/xA)^\wedge = \hat{A}/x\hat{A},$$

(3.5.3) $A/\mathfrak{p}$ *is essentially of finite type over K for $\mathfrak{p} \in \operatorname{Spec}(A) \setminus \{xA, (0)\}$.*

4. Examples with a specified prime element

We start with Example 4.1, which can be obtained by Corollary 1.5. Then, using Theorem 3.4 and/or Corollary 3.5, we present examples of local domains with a specified prime element, whose residue rings have the structure of those constructed in Section 1.

EXAMPLE 4.1 ([21, EXAMPLE 3, P. 205])

A discrete valuation ring of positive characteristic, which is *not* a Nagata ring.

CONSTRUCTION

With notation as in Corollary 1.5, let K_0 be a countable field of characteristic

$p > 0$, and let $n = 2$. If we take

$$F(Z_1) = z_2 - Z_1^p \in K_0[z_2][Z_1],$$

this $F(Z_1)$ satisfies the conditions (1.5.0) (cf. the proof of Lemma 1.3). We get a discrete valuation ring $(A, \mathfrak{m})$ whose completion is

$$\hat{A} \cong K[[\zeta_1, \zeta_2]]/(z_2 - \zeta_1^p) = K[[\zeta_1, z_2]]/(z_2 - \zeta_1^p).$$

EXAMPLE 4.2 ([21, EXAMPLE 7, P. 209])

A two-dimensional normal local domain whose generic formal fiber is *not* connected.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 2$. Let

$$G(X, Z_1) = XZ_1 + Z_1^2 \in K_0[X, Z_1].$$

By Theorem 3.4, we obtain a two-dimensional local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[x, \zeta_1, \zeta_2]]/(x\zeta_1 + \zeta_1^2) = K[[x, \zeta_1, \zeta_2]]/(\zeta_1) \cap (x + \zeta_1).$$

Then A is normal, because $\mathrm{Sing}(\hat{A}) = V((x, \zeta_1))$ and because x is a prime element.

Remark. It might be interesting to study if the following example exists: a normal Nagata local domain with *non*connected generic formal fiber.

EXAMPLE 4.3 ([21, EXAMPLE 7, P. 209])

A two-dimensional regular local ring of *arbitrary* characteristic, which is *not* a Nagata ring.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 2$. Let

$$G(X, Z_1) = X + Z_1^c \quad \text{where } c \geq 2.$$

Then, we get a two-dimensional regular local ring $(A, \mathfrak{m})$ with a prime element x such that

$$\hat{A} \cong K[[x, \zeta_1, \zeta_2]]/(x + \zeta_1^c) \qquad \text{and} \qquad \hat{A}/x\hat{A} \cong K[[\zeta_1, \zeta_2]]/(\zeta_1^c).$$

Remark ([34]). Let p be a prime number. If we take $\mathbb{Z}_{p\mathbb{Z}}(a_{ik})$ and p for K and X in our construction above, we get similar examples of regular local rings of *mixed* characteristic.

EXAMPLE 4.4 ([27, SECTION 1])

A two-dimensional Nagata regular local ring of characteristic $p > 0$, which is *not* excellent.

CONSTRUCTION

Let K_0 be a countable field of characteristic p $(p>2)$ with $n=2$. Let

$$G(X,Z_1,Z_2)=Z_1^2+X+Z_2^p\in K_0[X,Z_1,Z_2].$$

Then by Corollary 3.5, we get a two-dimensional regular local ring $(A,\mathfrak{m})$ with a prime element x such that

$$\hat{A}\cong K[[x,\zeta_1,\zeta_2]]/(\zeta_1^2+x+\zeta_2^p).$$

We show that A is a Nagata ring whenever $\varepsilon_k\equiv 0 \pmod p$. Firstly, we show a special case of a theorem of André.

LEMMA (CF. [3])

Let R be an excellent local domain with a prime element x. Let $\hat{A}=\hat{R}/\hat{P}$ be a local domain, which is a homomorphic image of $\hat{R}$. Suppose that x is a nonzero prime element of $\hat{A}$ and that $Q(\hat{A}/x\hat{A})$ is a separable extension field of $Q(R/xR)$. Then $Q(\hat{A})$ is separable over $Q(R)$.

Proof

Let $D=R_{xR}$ and $E=\hat{A}_{x\hat{A}}$. Then we have a canonical exact sequence

$$H_2(D,E,E/xE)\longrightarrow H_1(D,E,E)\xrightarrow{\ x\ }H_1(D,E,E)\longrightarrow H_1(D,E,E/xE).$$

By our assumption, E/xE is separable over D/xD. Hence

$$H_i(D,E,E/xE)\cong H_i(D/xD,E/xE,E/xE)=0\quad(i=1,2)$$

(cf. [2, Propositions 4.54, 7.22, 7.23]). Thus

$$H_1\big(Q(R),Q(\hat{A}),Q(\hat{A})\big)=H_1(D,E,E)=\bigcap_{\nu=1}^{\infty}x^{\nu}H_1(D,E,E).$$

Therefore, to get the assertion, it suffices to show that $H_1(D,E,E)$ is x-adically separated. In fact, we claim that $H_1(D,E,E)$ is a finite E-module.

Indeed, let $\hat{Q}=\hat{P}+x\hat{R}$ be the prime ideal of $\hat{R}$. Then, because $\hat{Q}\cap R=xR$ by assumption, we have the canonical local homomorphisms

$$D=R_{xR}\xrightarrow{\ \varphi\ }\hat{R}_{\hat{Q}}\xrightarrow{\ \psi\ }\hat{A}_{x\hat{A}}=E$$

where φ is regular, because R is assumed to be excellent, and ψ is surjective. The following canonical exact sequence,

$$0=H_1(D,\hat{R}_{\hat{Q}},E)\longrightarrow H_1(D,E,E)\longrightarrow H_1(\hat{R}_{\hat{Q}},E,E),$$

implies that $H_1(D,E,E)$ is a finite E-module (cf. [2, Théorème 5.1]). □

By applying the lemma above to $A_{xA}=R_{xR}$ and to $\hat{A}_{x\hat{A}}$, we are only able to check that $Q(\hat{A}/x\hat{A})$ is separable over $Q(A/xA)$ (cf. (3.5.3)). Namely,

(4.4.1) $\quad Q\big(K[[\bar{\zeta}_1,\bar{\zeta}_2]]/(\bar{\zeta}_1^2+\bar{\zeta}_2^p)\big)$ is separable over $K(\bar{z}_1,\bar{z}_2)$.

In fact, this is equivalent to the following:

(4.4.2) $$K[[\bar{\zeta}_1,\bar{\zeta}_2]]/(\bar{\zeta}_1^2+\bar{\zeta}_2^p)\otimes_{K[\bar{z}_1,\bar{z}_2]} K^{1/p}[\bar{z}_1^{1/p},\bar{z}_2^{1/p}] \text{ is reduced.}$$

Because $K[[\bar{\zeta}_1,\bar{\zeta}_2]]=K[[\bar{z}_1,\bar{z}_2]]$ and because

$$\begin{aligned}
&K[[\bar{z}_1,\bar{z}_2]]\otimes_{K[\bar{z}_1,\bar{z}_2]} K^{1/p}[\bar{z}_1^{1/p},\bar{z}_2^{1/p}]\\
&\quad\cong K[[\bar{z}_1^{1/p},\bar{z}_2^{1/p}]][K^{1/p}]\\
&\quad=\bigcup_k K[[\bar{z}_1^{1/p},\bar{z}_2^{1/p}]][a_{11}^{1/p},a_{21}^{1/p},\dots,a_{1k}^{1/p},a_{2k}^{1/p}]\\
&\quad=\bigcup_k K(a_{11}^{1/p},a_{21}^{1/p},\dots,a_{1k}^{1/p},a_{2k}^{1/p})[[\bar{z}_1^{1/p},\bar{z}_2^{1/p}]]
\end{aligned}$$

is a regular local ring that is a direct limit (cf. [21, (E3.1), p. 206]), to get (4.4.2), it suffices to show that, for every k

(4.4.3) $$K(a_{11}^{1/p},a_{21}^{1/p},\dots,a_{1k}^{1/p},a_{2k}^{1/p})[[\bar{z}_1^{1/p},\bar{z}_2^{1/p}]]/(\bar{\zeta}_1^2+\bar{\zeta}_2^p)\quad\text{is reduced.}$$

Indeed, let $R_k^*=K(a_{11}^{1/p},a_{21}^{1/p},\dots,a_{1k}^{1/p},a_{2k}^{1/p})[[\bar{z}_1^{1/p},\bar{z}_2^{1/p}]]$, and take

$$\frac{\partial}{\partial a_{1\ell}}\in \operatorname{Der}(R_k^*,R_k^*)\quad\text{for some } \ell>k.$$

Because $\varepsilon_h\equiv 0 \pmod p$ for every h by assumption, the assertion (4.4.3) follows from

$$q_\ell^{\varepsilon_\ell}\bar{\zeta}_1\notin\sqrt{(\bar{\zeta}_1^2+\bar{\zeta}_2^p)R_k^*}\quad\text{(cf. (3.3.4)).}$$

EXAMPLE 4.5 ([32])

A three-dimensional Nagata regular local ring of arbitrary characteristic, which is *not* excellent.

CONSTRUCTION

Let K_0 be a countable field of characteristic zero or $p>2$, and let $n=3$. Let

$$G(X,Z_1,Z_2)=Z_1^2+X+Z_2^p\in K_0[X,Z_1,Z_2,Z_3].$$

Here, in the case $\operatorname{char}K_0=0$, we may take as p every natural number greater than one. Then by Theorem 3.4, we get a three-dimensional regular local ring $(A,\mathfrak{m})$ that has a prime element x such that (cf. Example 2.5)

$$\hat{A}\cong K[[x,\zeta_1,\zeta_2,\zeta_3]]/(\zeta_1^2+x+\zeta_2^p)\qquad\text{and}\qquad \hat{A}/x\hat{A}\cong K[[\zeta_1,\zeta_2,\zeta_3]]/(\zeta_1^2+\zeta_2^p).$$

The same argument as in Example 4.4 shows that, when $\operatorname{char}K_0=p>2$, A is a Nagata ring if $\varepsilon_k\equiv 0 \pmod p$ for every k.

EXAMPLE 4.6 ([27])

A three-dimensional analytically irreducible Nagata normal local domain A that has $\mathfrak{p}\in\operatorname{Spec}(A)$ such that $A_{\mathfrak{p}}$ is analytically *reducible*.

CONSTRUCTION

Let K_0 be a countable field of characteristic zero or $p>2$, and let $n=3$. Taking as p every odd number greater than two in the case $\operatorname{char} K_0 = 0$, let

$$G(X, Z_1, Z_2) = X^2 Z_2^2 + Z_2^p - Z_1^2.$$

Then by Theorem 3.4, we get a three-dimensional local domain $(A, \mathfrak{m})$ with a prime element x such that

$$\hat{A} \cong K[[x, \zeta_1, \zeta_2, \zeta_3]]/(x^2\zeta_2^2 + \zeta_2^p - \zeta_1^2).$$

Further, $\hat{A}$ is a domain, because $\hat{A}/x\hat{A} \cong K[[\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3]]/(\bar{\zeta}_2^p - \bar{\zeta}_1^2)$.

Take $\mathfrak{p} = (z_1, z_2, z_3)A \in \operatorname{Spec}(A)$, and take $K_0(x)$ in place of K_0. Then

$$A_{\mathfrak{p}}{}^{\wedge} \cong K(x)[[\zeta_1, \zeta_2, \zeta_3]]/(x^2\zeta_2^2 + \zeta_2^p - \zeta_1^2).$$

Thus $A_{\mathfrak{p}}{}^{\wedge}$ is *reducible*, because

$$\sqrt{1 + \frac{1}{x^2}\zeta_2^{p-2}} \in K(x)[[\zeta_2]].$$

The same argument as in Example 4.4 shows that, when $\operatorname{char} K_0 = p > 2$, A is a Nagata ring whenever $\varepsilon_k \equiv 0 \pmod p$ for every k.

EXAMPLE 4.7 ([6], CF. [23])

A three-dimensional unmixed local domain A that has $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $A/\mathfrak{p}$ is *not* unmixed.

CONSTRUCTION

Let K_0 be a countable field of arbitrary characteristic, and let $n = 4$. Take

$$G_1(X, Z_1, Z_2, Z_3) = Z_2^3 - Z_3^2, \qquad G_2(X, Z_1, Z_2, Z_3) = Z_2X^2 - Z_1^2,$$
$$G_3(X, Z_1, Z_2, Z_3) = Z_2Z_1 - XZ_3, \qquad G_4(X, Z_1, Z_2, Z_3) = Z_2^2X - Z_3Z_1.$$

Then, using Macaulay [16], we get (cf. [14, p. 61])

$$\begin{aligned}
\operatorname{Ker}\tilde{\varphi} = (&QT_1 - G_1, QT_2 - G_2, QT_3 - G_3, QT_4 - G_4,\\
&XT_1 - Z_3T_3 - Z_2T_4, Z_1T_1 - Z_2^2T_3 - Z_3T_4, Z_2T_2 + Z_1T_3 - XT_4,\\
&Z_3T_2 + Z_2XT_3 - Z_1T_4, T_1T_2 + Z_2T_3^2 - T_4^2),\\
\operatorname{Ker}\varphi = (&QT_1 - F_1, QT_2 - F_2, QT_3 - F_3, QT_4 - F_4,\\
&- Z_3T_3 - Z_2T_4, Z_1T_1 - Z_2^2T_3 - Z_3T_4, Z_2T_2 + Z_1T_3,\\
&Z_3T_2 - Z_1T_4, T_1T_2 + Z_2T_3^2 - T_4^2).
\end{aligned}$$

Thus $\operatorname{Ker}\varphi = K_0[\underline{Z}, Q] \otimes_{K_0[X, \underline{Z}, Q]} \operatorname{Ker}\tilde{\varphi}$ (cf. (3.4.0)). Therefore, Theorem 3.4 gives us a local domain $(A, \mathfrak{m})$ with a *prime* element x that satisfies the following:

$$\hat{A} \cong K[[x, \zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_2^3 - \zeta_3^2, \zeta_2x^2 - \zeta_1^2, \zeta_2\zeta_1 - x\zeta_3, \zeta_2^2x - \zeta_3\zeta_1),$$

$$\begin{aligned}
\hat{A}/x\hat{A} &\cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_2^3 - \zeta_3^2, \zeta_1^2, \zeta_2\zeta_1, \zeta_3\zeta_1)\\
&\cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_2^3 - \zeta_3^2, \zeta_1) \cap (\zeta_1, \zeta_2, \zeta_3)^2.
\end{aligned}$$

Namely, A is analytically irreducible but A/xA is *not* unmixed.

REMARK ([29])
With the notation above, take

$$G_1(X, Z_1, Z_2, Z_3) = Z_1 Z_3, \qquad G_2(X, Z_1, Z_2, Z_3) = Z_1(X + Z_2),$$
$$G_3(X, Z_1, Z_2, Z_3) = Z_2 Z_3, \qquad G_4(X, Z_1, Z_2, Z_3) = Z_2(X + Z_2).$$

Then, Macaulay gives us (cf. [29, Proposition 1.3])

$$\begin{aligned}\operatorname{Ker}\tilde{\varphi} = \big(&QT_1 - G_1, QT_2 - G_2, QT_3 - G_3, QT_4 - G_4,\\ &(X + Z_2)T_1 - Z_3T_2, (X + Z_2)T_1 - Z_1T_3,\\ &(X + Z_2)T_2 - Z_1T_4, (X + Z_2)T_3 - Z_3T_4, T_1T_4 - T_2T_3\big),\\ \operatorname{Ker}\varphi = \big(&QT_1 - F_1, QT_2 - F_2, QT_3 - F_3, QT_4 - F_4,\\ &Z_2T_1 - Z_3T_2, Z_2T_1 - Z_1T_3,\\ &Z_2T_2 - Z_1T_4, Z_2T_3 - Z_3T_4, T_1T_4 - T_2T_3\big).\end{aligned}$$

Consequently, $\operatorname{Ker}\varphi = K_0[\underline{Z}, Q] \otimes_{K_0[X,\underline{Z},Q]} \operatorname{Ker}\tilde{\varphi}$. Therefore, we get a local domain $(A, \mathfrak{m})$ with a *prime* element x such that

$$\begin{aligned}\hat{A} &\cong K[[x, \zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1, \zeta_2) \cap (\zeta_3, x + \zeta_2),\\ \hat{A}/x\hat{A} &\cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1\zeta_3, \zeta_1\zeta_2, \zeta_2\zeta_3, \zeta_2^2)\\ &\cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1, \zeta_2) \cap (\zeta_3, \zeta_2) \cap (\zeta_1, \zeta_2, \zeta_3)^2.\end{aligned}$$

We get another analytically unramified unmixed local domain A such that A/xA is *not* unmixed.

5. Construction of bad factorial local domains

In this section, thanks to T. Ogoma, we first define the *decomposition* of prime elements of a regular local ring with respect to the equations, or relations, formed by a subregular system of parameters. Then, we observe how this decomposition changes when the original regular local ring is extended by a finite number of indeterminates and when the subregular system of parameters is appropriately modified according to the extension.

Making use of the observation above, we give a so-called *factorial* numbering on the *subset* of prime elements of the regular local ring R in (1.0.0). Finally we show that the standard construction in Section 1 combined with this factorial numbering gives a desired factorial local domain.

5.0. Notation and enumeration on $\mathcal{P}$

With notation as in Section 1.0, we first fix an enumeration on a set of prime elements that represents the set of *all* height-one prime ideals of R.

Take a set of prime elements $\mathcal{P}$ of $\mathfrak{N}$ that contains, for each height-one prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, a *unique* $p \in S_k$ with the *least* possible k such that $pR = \mathfrak{p}$:

$$\mathcal{P} \subset \mathfrak{N} \setminus \{0\}. \tag{5.0.1}$$

Then, as before, $\mathcal{P}$ is a *countable* set, and we may assume that

$$z_1 + \cdots + z_n \in \mathcal{P}$$

and that $\mathcal{P}$ contains an infinite number of elements of S_0.

Let $\rho\colon \mathbb{N} \to \mathcal{P}$ be a *bi*jective mapping, and write $\rho(i) = \rho_i$ instead of p_i. By the remark above, we may assume that $\rho_1 = z_1 + \cdots + z_n$ and that ρ satisfies the following:

$$\rho_k \in S_{k-2} \quad \text{for every } k \geq 2. \tag{5.0.2}$$

In the next subsection, we show that if relation polynomials $F(Z)$ satisfy the condition (5.1.0), one can give a so-called *factorial* numbering on a *subset* Π of $\mathcal{P}$, which guarantees the realization of desired factorial local domains.

In fact, thanks to Ogoma's decomposition lemma below, we show that one can pick up elements of $\mathcal{P}$,

$$p_1, p_2, \ldots, p_k, \ldots,$$

and, at the same time, determine a strictly increasing sequence of natural numbers

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots,$$

so that they fulfill our inductive conditions (5.1.2)–(5.1.5).

However, we end this subsection by fixing some more notation. Namely, first let

$$p_1 := \rho_1 = z_1 + \cdots + z_n \tag{5.0.3}$$

$$z_{i0} := z_i \quad \text{for } i = 1, \ldots, n. \tag{5.0.4}$$

Assuming that $p_1, \ldots, p_\ell$ and $\varepsilon_1, \ldots, \varepsilon_{\ell-1}$, which satisfy (5.1.2)–(5.1.5), have been chosen, we define:

$$q_k := p_1 \cdots p_k \quad \text{for } 1 \leq k \leq \ell, \tag{5.0.5}$$

$$z_{ih} := z_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ih} q_h^{\varepsilon_h} \quad \text{for } 1 \leq h < \ell. \tag{5.0.6}$$

5.1. Relations and prime elements

Take polynomials in m variables over K_0 with no constant term,

$$F_1(\underline{Z}), \ldots, F_r(\underline{Z}) \in (Z_1, Z_2, \ldots, Z_m) K_0[Z_1, \ldots, Z_m],$$

that satisfy the following *absolute* irreducibility condition:

$$\begin{array}{l} L[Z_1, \ldots, Z_m]/\big(F_1(\underline{Z}), \ldots, F_r(\underline{Z})\big) \text{ is a domain, which is } \textit{not} \text{ a field,} \\ \text{for every extension field } L \text{ of } K_0. \end{array} \tag{5.1.0}$$

With the notation and assumptions above, for $0 \le h < \ell$, let

(5.1.1) $$P_h = (f_{1h}, \dots, f_{rh})R_h \quad \text{with } f_{jh} = F_j(z_{1h}, \dots, z_{mh}).$$

Then P_h is a prime ideal of $R_h = K_h[z_1, \dots, z_n]_{(z_1,\dots,z_n)}$, because the $F(\underline{Z})$'s above are assumed to satisfy the condition (5.1.0).

With the notation above, we state the inductive conditions:

(5.1.2) $p_1, \dots, p_k$ are nonzero divisors on R_{k-1}/P_{k-1} for $1 \le k \le \ell$;

(5.1.3) $p_1R_{k-1} + P_{k-1}, \dots, p_kR_{k-1} + P_{k-1}$ are mutually distinct prime ideals for $1 \le k \le \ell$;

(5.1.4) $\rho_h \equiv p_1^{e_{h1}} \cdots p_h^{e_{hh}} \cdot u_{h(k-1)} \pmod{P_{k-1}}$ with a unit $u_{h(k-1)} \in R_{k-1}$ and $e_{hg} \in \mathbb{N}_0$ for $1 \le g \le h \le k \le \ell$;

(5.1.5) $\varepsilon_h \ge \max\{\varepsilon_{h-1} + 1, e_{11}, e_{21}, \dots, e_{(h+1)h}, e_{(h+1)(h+1)}\}$ for $1 \le h < \ell$.

Note that p_1 fulfills the conditions above when $\ell = 1$. The following Ogoma's decomposition lemma [28, Proposition 2.3] makes us possible to climb our induction steps up.

LEMMA 5.2 (OGOMA'S DECOMPOSITION LEMMA)

With notation and inductive assumptions above, take an element $q \in R_{k-1}$. Let $y_{ik} := z_{i(k-1)} + a_{ik}q$ where the a_{ik}'s are indeterminates over R_{k-1} (cf. (1.0.6)). Let $g_{jk} := F_j(y) \in R_{k-1}[a]$ and $Q_k = (g_{1k}, \dots, g_{rk})R_k$. Then

(5.2.0) $$g_{jk} = f_{j(k-1)} + qH_j(a) \quad \text{with } H_j(a) \in (a)R_{k-1}[a].$$

Hence $p_h^\varepsilon R_k + P_{k-1}R_k = p_h^\varepsilon R_k + Q_k$ if $q \in p_h^\varepsilon R_{k-1}$ for some h. In particular, $p_hR_k + Q_k$ is a prime ideal when $q \in p_h^\varepsilon R_{k-1}$ and when $\varepsilon > 0$. Suppose that

(5.2.1) $$p_1, \dots, p_k \text{ are nonzero-divisors on } R_k/Q_k.$$

Take an element $r \in R_{k-1} \setminus P_{k-1}$. By (5.1.2) and (5.1.3), we have

(5.2.2) $$r \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot s \pmod{P_{k-1}} \quad \text{with } s \in R_{k-1}.$$

Under the circumstances, suppose that $q \in \bigcap_{h=1}^k p_h^{e_h} R_{k-1}$. Then

(5.2.3) $$r \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot t \pmod{Q_k} \quad \text{with } t \in R_k,$$

(5.2.4) *if s above is a unit in R_{k-1}, t is also a unit in R_k.*

Moreover, suppose that r is a prime element of R_{k-1} and that $q = p_1^{\varepsilon_1} \cdots p_k^{\varepsilon_k} \cdot u$ with u a unit in R_{k-1} and $\varepsilon_h \ge \max\{e_h, 1\}$ for every h. Then

(5.2.5) *t is either a prime element in R_k/Q_k or a unit if $t \notin \bigcup_{h=1}^{k} (p_hR_k + Q_k)$.*

Proof
Indeed, we get the assertion (5.2.3) by (5.2.0) and the following:

$$p_1^{e_1}\cdots p_k^{e_k}R_k + P_{k-1}R_k = p_1^{e_1}\cdots p_k^{e_k}R_k + Q_k.$$

Next express

$$r = p_1^{e_1}\cdots p_k^{e_k}\cdot s + p \quad \text{with } p \in P_{k-1}.$$

Then, because $q = p_1^{e_1}\cdots p_k^{e_k}\cdot u$ and $p = \sum_{j=1}^{r} r_j f_{j(k-1)}$ with $u,\ r_j \in R_{k-1}$, we have

$$r = p_1^{e_1}\cdots p_k^{e_k}\Big(s - u\sum_{j=1}^{r} r_j H_j(a)\Big) + \sum_{j=1}^{r} r_j g_{jk}$$

and

$$t \equiv s - u\sum_{j=1}^{r} r_j H_j(a) \pmod{Q_k} \quad (\text{cf. } (5.2.1)).$$

Consequently t is a unit in R_k, because s is a unit in R_{k-1} and because

$$u\sum_{j=1}^{r} r_j H_j(a) \in (a)R_{k-1}[a] \quad (\text{cf. } (5.2.0)).$$

Finally we have the canonical isomorphisms

$$\begin{aligned}\big(R_{k-1}(a)/(t,g)\big)[1/q] &= \big((R_{k-1}[y]/(r,g))[1/q]\big)_T\\ &= \big(((R_{k-1}/rR_{k-1})[y]/\big(F(y)\big))[1/q]\big)_T\\ &\cong \big((R_{k-1}/rR_{k-1})[1/q]\otimes_{K_0} K_0[y]/(F(y))\big)_T.\end{aligned}$$

Then $(R_{k-1}(a)/(t,g))[1/q]$ is either a domain or (0) by assumption (5.1.0). Thus we get the assertion, because t and q form an (R_k/Q_k)-sequence. □

5.3. Inductive step: Decomposition of $\rho_{\ell+1}$ and choice of $p_{\ell+1}$ and ε_ℓ

By the inductive hypotheses in Section 5.1, we may assume

(5.3.0) $$\rho_{\ell+1} \equiv p_1^{e_{(\ell+1)1}}\cdots p_\ell^{e_{(\ell+1)\ell}}\cdot v_\ell \pmod{P_{\ell-1}}$$

where $v_\ell \in R_{\ell-1}$ and $v_\ell \notin \bigcup_{h=1}^{\ell}(p_h R_{\ell-1} + P_{\ell-1})$, because $\rho_{\ell+1} \in R_{\ell-1}$ by (5.0.2). Then, take

(5.3.1) $$\varepsilon_\ell > \max\{\varepsilon_{\ell-1}, e_{11}, e_{21}, \ldots, e_{\ell\ell}, e_{(\ell+1)1}, \ldots, e_{(\ell+1)\ell}\}.$$

And we define

(5.3.2) $$z_{i\ell} = z_i + a_{i1}q_1^{\varepsilon_1} + \cdots + a_{i\ell}q_\ell^{\varepsilon_\ell} \quad \text{with } q_\ell = p_1\cdots p_\ell$$

(5.3.3) $$P_\ell = (f_{1\ell},\ldots,f_{r\ell})R_\ell \quad \text{with } f_{j\ell} = F_j(z_{1\ell},\ldots,z_{m\ell}).$$

Here we remark that

(5.3.4) every $P \in \mathrm{Ass}(R_\ell/P_\ell)$ is contained in $(z_{1\ell},\ldots,z_{m\ell})$.

Then, by Lemma 1.1 plus (1.1.3) and by Lemma 5.2, we see that

(5.3.5) $p_1, \dots, p_\ell$ are nonzero divisors on R_ℓ/P_ℓ;

(5.3.6) $p_1R_\ell + P_\ell, \dots, p_\ell R_\ell + P_\ell$ are mutually distinct prime ideals;

(5.3.7) $\rho_k \equiv p_1^{e_{k1}} \cdots p_k^{e_{kk}} \cdot u_{k\ell} \pmod{P_\ell}$ with unit $u_{k\ell} \in R_\ell$ for $1 \le h \le k \le \ell$;

(5.3.8) $\rho_{\ell+1} \equiv p_1^{e_{(\ell+1)1}} \cdots p_\ell^{e_{(\ell+1)\ell}} \cdot v_{\ell+1} \pmod{P_\ell}$;

where $v_{\ell+1} \in R_\ell$ and $v_{\ell+1} \notin \bigcup_{k=1}^{\ell}(p_kR_\ell + P_\ell)$ (cf. proof of (5.2.4)).

Further, (5.2.5) implies that $v_{\ell+1}$ is either a nonzero *prime* element of R_ℓ/P_ℓ or a unit.

Firstly we have the case when $v_{\ell+1}$ is a prime element of R_ℓ/P_ℓ. By the choice of $\mathcal{P}$, we can find an element $p \in \mathcal{P} \cap S_\ell$ such that $v_{\ell+1} = p \cdot u_{(\ell+1)\ell}$ where $u_{(\ell+1)\ell}$ is a unit in R_ℓ. In fact, $p_1, \dots, p_\ell$ and this p with ε_ℓ above satisfy the conditions (5.1.2)–(5.1.5). Consequently, we can take this p as $p_{\ell+1}$.

Secondly we have the case when $v_{\ell+1}$ is a unit. We can find an element $p \in \mathcal{P} \cap S_{\ell-1}$ such that p and $p_1, \dots, p_\ell$ satisfy the conditions (5.1.2)–(5.1.4), because $\mathcal{P}$ is assumed to contain an infinite number of elements of S_0 (cf. proof of (5.2.5)). In letting $v_{\ell+1}$ be a unit $u_{(\ell+1)\ell}$ in R_ℓ, we can take this p as $p_{\ell+1}$.

These complete our inductive process.

5.4. Construction

Notation being as above, let $\Pi = \{p_k \mid k = 1, 2, \dots\}$ be the subset of $\mathcal{P}$ chosen in Sections 5.1 and 5.3. For $j = 1, \dots, r$, we define

(5.4.1)
$$\alpha_{jk} = \frac{1}{q_k^{\nu_k}} F_j(z_{1k}, \dots, z_{mk}) = \frac{f_{jk}}{q_k^{\nu_k}} \in Q(R)$$

where $f_{jk} = F_j(z_{1k}, \dots, z_{mk})$, $Q(R) = K(z_1, \dots, z_n)$ is the field of fractions of R, and ν_k $(\le \varepsilon_k)$, $k = 1, 2, \dots$, is a sequence of strictly increasing natural numbers such that

$$\nu_k > \max\{\varepsilon_{k-1}, e_{11}, e_{21}, \dots, e_{kk}, e_{(k+1)1}, \dots, e_{(k+1)k}\},$$

for example, $\nu_k := \varepsilon_k$ (cf. (5.3.1)). Then

(5.4.2)
$$\alpha_{jk} = \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{j(k+1)} + \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} s_{jk} \quad \text{with } s_{jk} \in S_{k+1}.$$

Let

(5.4.3)
$$B = \bigcup_{k \in \mathbb{N}} R[\alpha_{1k}, \dots, \alpha_{rk}] \subset Q(R).$$

Lemma 1.3 and the remark after Lemma 1.1 show that $M = (z_1, \dots, z_n)B$ is a maximal ideal of B such that $B/M \cong R/\mathfrak{n} \cong K$. Thus let

(5.4.4)
$$A := B_M \subset Q(R) = Q(K[z_1, \dots, z_n]).$$

Then A is a quasi-local domain with its maximal ideal $\mathfrak{m} = MA$. For $i = 1, \ldots, n$ and for $j = 1, \ldots, r$, we define

$$\zeta_i = z_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ik} q_k^{\varepsilon_k} + \cdots = z_i + \sum_{k=1}^{\infty} a_{ik} q_k^{\varepsilon_k}, \tag{5.4.5}$$

$$f_j = F_j(\zeta_1, \ldots, \zeta_m) \in K_0[[\zeta_1, \ldots, \zeta_m]] \subset K[[\zeta_1, \ldots, \zeta_n]] = \hat{R}. \tag{5.4.6}$$

THEOREM 5.5

Let K be a purely transcendental extension field of countably infinite degree over a countable field K_0, let $n, r, m \in \mathbb{N}$ with $m < n$, and let $z_1, \ldots, z_n$ be indeterminates over K. Let $R := K[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)}$, and let $\hat{R}$ denote the completion of R; that is, $\hat{R} = K[[z_1, \ldots, z_n]]$. For each j with $1 \le j \le r$, let $F_j := F_j(Z_1, \ldots, Z_m)$ be a polynomial in m variables over K_0 with no constant term. Suppose that $F_1(\underline{Z}), \ldots, F_r(\underline{Z})$ satisfy the absolute irreducibility condition (5.1.0):

$$L[Z_1, \ldots, Z_m]/\big(F_1(\underline{Z}), \ldots, F_r(\underline{Z})\big) \text{ is a domain, which is not a field,}$$

$$\text{for every extension field } L \text{ of } K_0.$$

Then there exist

(1) *elements $\zeta_1, \zeta_2, \ldots, \zeta_n \in \hat{R}$ that are analytically independent over K such that $K[[\zeta_1, \ldots, \zeta_n]] = K[[z_1, \ldots, z_n]]$,*

(2) *a factorial local domain $(A, \mathfrak{m})$ with $R \overset{\iota}{\subset} A \subset Q(R)$, where $Q(R)$ denotes the field of fractions of R, and*

(3) *a natural isomorphism $\tilde{\iota}$*

that satisfy the following:

$$\tilde{\iota}\colon K[[\zeta_1, \ldots, \zeta_n]]/\big(F_1(\underline{\zeta}), \ldots, F_r(\underline{\zeta})\big) = \hat{R}/(f_1, \ldots, f_r) \overset{\cong}{\hookrightarrow} \hat{A}, \tag{5.5.1}$$

$$\hat{\mathfrak{p}} := \big(\tilde{\iota}(\zeta_1), \ldots, \tilde{\iota}(\zeta_m)\big)\hat{A} \text{ is a prime ideal of } \hat{A} \text{ and } \hat{\mathfrak{p}} \cap A = (0), \tag{5.5.2}$$

$$A/\mathfrak{p} \text{ is essentially of finite type over } K \quad \text{for every } \mathfrak{p} \in \operatorname{Spec}(A) \setminus \{(0)\}. \tag{5.5.3}$$

Proof

We prove that A is Noetherian and factorial; that is,

$$\begin{array}{l}\text{if } \mathfrak{p} \text{ is a prime ideal of height one, } A/\mathfrak{p} \text{ is essentially of finite type} \\ \text{over } K \text{ and } \mathfrak{p} \text{ is principal (cf. [21, (13.1)]).}\end{array} \tag{5.5.4}$$

Indeed, take a nonzero prime ideal $\mathfrak{p}$ of A. Then $\mathfrak{p} \cap R \neq (0)$, because R and A have the same field of fractions. Thus there exists $\ell \in \mathbb{N}$ such that $\rho_\ell \in \mathfrak{p} \cap R$. Then

$$\rho_\ell = p_1^{e_{\ell 1}} \cdots p_\ell^{e_{\ell\ell}} \cdot u_{\ell k} + s_k \quad \text{with a unit } u_{\ell k} \in R \text{ and } s_k = \sum_{j=1}^{r} r_j f_{jk} \in P_k$$

for each $k > \ell$ (cf. (5.1.4)). Hence

$$\begin{aligned}\rho_\ell &= p_1^{e_{\ell 1}} \cdots p_\ell^{e_{\ell\ell}} \cdot u_{\ell k} + \sum_{j=1}^{r} r_j f_{jk} \\ &= p_1^{e_{\ell 1}} \cdots p_\ell^{e_{\ell\ell}} \cdot u_{\ell k} + q_k^{\nu_k} \sum_{j=1}^{r} r_j \alpha_{jk} = p_1^{e_{\ell 1}} \cdots p_\ell^{e_{\ell\ell}} \cdot u_\ell\end{aligned}$$

where u_ℓ is a unit of A. Thus there exists $p_h \in \Pi$ $(h \le \ell)$ such that $p_h \in \mathfrak{p}$. Then $\alpha_{jk} \in R + p_h A$ for every $j = 1, \dots, r$ and for every $k = 1, 2, \dots$ by (5.4.2). Hence we get a canonical surjection $\iota_h \colon R \to A/p_h A$, and $A/p_h A$ is essentially of finite type over K. Consequently $\mathfrak{p}$ is finitely generated, and therefore A is Noetherian.

Moreover, our proof of Corollary 1.5 shows that we have a canonical isomorphism

$$\bar{\iota}_h \colon R/(p_h R + P_{h-1} R) \cong A/p_h A.$$

This implies that $A/p_h A$ is an integral domain (cf. (5.1.3)). Hence, further if $\mathfrak{p}$ is a prime ideal of height one, $\mathfrak{p} = p_h A$ and this completes the proof of (5.5.3) and (5.5.4).

Finally (5.5.1) and (5.5.2) follow from the same proof as that of Theorem 1.4 (cf. (5.3.4), (5.1.2), and the remark after Lemma 1.1). □

As in the preceding sections, the additional hypotheses enable us to bypass some parts of the proof and thus obtain a slight generalization of Theorem 5.5.

COROLLARY 5.6

We use the notation above, except that $n = m$. Let $F_1(\underline{Z}), \dots, F_r(\underline{Z})$ be polynomials in the variables $\underline{Z} := (Z_1, \dots, Z_n)$ over K_0 with zero constant term. Suppose that

(5.6.0) $L[Z_1, \dots, Z_n]/\big(F_1(\underline{Z}), \dots, F_r(\underline{Z})\big)$ *is a domain whose dimension is not less than 2 for every extension field L of K_0.*

Let $P_0 = (F_1(z_1, \dots, z_n), \dots, F_r(z_1, \dots, z_n))R_0$. Taking as p_1 a linear combination of $z_1, \dots, z_n$ over K_0, assume that

(5.6.1) $p_1 R_0 + P_0$ *is a prime ideal of* R_0 *with* $p_1 \notin P_0$

$\hat{R}/\big(F_1(\underline{\zeta}), \dots, F_r(\underline{\zeta})\big)\hat{R}$ *is R-torsion-free (cf. (1.5.0)).*

Then there exists a factorial local domain $(A, \mathfrak{m})$ that satisfies the following:

(5.6.2) $$\tilde{\iota} \colon K[[\zeta_1, \dots, \zeta_n]]/\big(F_1(\underline{\zeta}), \dots, F_r(\underline{\zeta})\big) \overset{\cong}{\hookrightarrow} \hat{A},$$

(5.6.3) $A/\mathfrak{p}$ *is essentially of finite type over K for every $\mathfrak{p} \in \operatorname{Spec}(A) \setminus \{(0)\}$.*

6. Examples

As applications of Theorem 5.5 and/or Corollary 5.6, we obtain following examples of factorial local domains.

EXAMPLE 6.1 ([35], [28, SECTION 4])

A two-dimensional Cohen–Macaulay factorial excellent local domain with a Gorenstein module, which has *no* dualizing (i.e., canonical) module.

CONSTRUCTION

With notation as in Corollary 5.6, let K_0 be a countable field of characteristic zero, and let $n = 4$. Take

$$F_1(Z_1, Z_2, Z_3, Z_4) = Z_1Z_3 - Z_2^2, \qquad F_2(Z_1, Z_2, Z_3, Z_4) = Z_2Z_4 - Z_3^3,$$
$$F_3(Z_1, Z_2, Z_3, Z_4) = Z_1Z_4 - Z_2Z_3^2.$$

With $p_1 = z_1 - z_4$, we see that $F_1(Z)$, $F_2(Z)$, $F_3(Z)$ satisfy the conditions (5.6.0) and (5.6.1). Then by Corollary 5.6, we get a two-dimensional factorial local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4]]/(\zeta_1\zeta_3 - \zeta_2^2, \zeta_2\zeta_4 - \zeta_3^3, \zeta_1\zeta_4 - \zeta_2\zeta_3^2).$$

Thus $\hat{A}$ is a two-dimensional non-Gorenstein normal local domain with $\mathrm{Cl}(\hat{A}) \cong \mathbb{Z}/5\mathbb{Z}$. Therefore, A is a desired example (cf. [35, (1.7)]).

EXAMPLE 6.2 ([35])

A three-dimensional excellent factorial Cohen–Macaulay local domain that has *no* Gorenstein module.

CONSTRUCTION

Let K_0 be a countable field of characteristic zero, and let $n = 5$. Take

$$F_1(Z_1, \dots, Z_5) = Z_1Z_5 - Z_2Z_4, \qquad F_2(Z_1, \dots, Z_5) = Z_1Z_2 - Z_3Z_4,$$
$$F_3(Z_1, \dots, Z_5) = Z_2^2 - Z_3Z_5.$$

Letting $p_1 = z_1 - z_5$, we see that $F_1(Z)$, $F_2(Z)$, $F_3(Z)$ satisfy the conditions (5.6.0) and (5.6.1). Hence by Corollary 5.6, we get a three-dimensional factorial local domain $(A, \mathfrak{m})$ such that

$$\hat{A} \cong K[[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]]/(\zeta_1\zeta_5 - \zeta_2\zeta_4, \zeta_1\zeta_2 - \zeta_3\zeta_4, \zeta_2^2 - \zeta_3\zeta_5).$$

Thus $\hat{A}$ is a three-dimensional non-Gorenstein normal local domain such that $\mathrm{Cl}(\hat{A}) \cong \mathbb{Z}$ and $\mathrm{Sing}(\hat{A}) = V((\zeta_1, \dots, \zeta_5))$. Therefore, A is a desired example (cf. [35, (1.4)]).

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