On the Cauchy problem for noneffectively hyperbolic operators: The Gevrey 4 well-posedness

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Abstract For a hyperbolic second-order differential operator P, we study the relations between the maximal Gevrey index for the strong Gevrey well-posedness and some algebraic and geometric properties of the principal symbol p. If the Hamilton map F_p of p (the linearization of the Hamilton field H_p along double characteristics) has nonzero real eigenvalues at every double characteristic (the so-called effectively hyperbolic case), then it is well known that the Cauchy problem for P is well posed in any Gevrey class $1 \le s < +\infty$ for any lower-order term. In this paper we prove that if p is noneffectively hyperbolic and, moreover, such that $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \neq \{0\}$ on a C^∞ double characteristic manifold Σ of codimension 3, assuming that there is no null bicharacteristic landing Σ tangentially, then the Cauchy problem for P is well posed in the Gevrey class $1 \le s < 4$ for any lower-order term (strong Gevrey well-posedness with threshold 4), extending in particular via energy estimates a previous result of Hörmander in a model case.

1. Introduction

Let

$$P(x,D) = -D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x) D^\alpha = P_2 + P_1 + P_0 \tag{1.1}$$

be a second-order differential operator with real analytic or Gevrey s class $a_\alpha(x)$ (s is close to 1) defined in an open neighborhood of the origin of $\mathbb{R}^{n+1}$ with the principal symbol $p(x,\xi)$, hyperbolic with respect to the x_0-direction, where $x = (x_0, x_1, \dots, x_n) = (x_0, x')$, $\xi = (\xi_0, \xi_1, \dots, \xi_n) = (\xi_0, \xi')$. We are interested in the Cauchy problem for P in the Gevrey classes when p has double characteristics $\rho \in T^*\mathbb{R}^{n+1} \setminus \{0\}$, $p(\rho) = 0$, $dp(\rho) = 0$. We say that $f(x) \in \gamma^{(s)}(\mathbb{R}^n)$, the Gevrey class s (≥ 1), if for any compact set $K \subset \mathbb{R}^n$, there exist $C > 0$, $h > 0$ such that

$$|\partial_x^\alpha f(x)| \le C h^{|\alpha|} |\alpha|!^s, \quad x \in K, \quad \forall \alpha \in \mathbb{N}^n.$$

We set $\gamma_0^{(s)}(\mathbb{R}^n) = \gamma^{(s)}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$.

Let ρ be a double characteristic; then the Taylor expansion of p around ρ starts with a quadratic polynomial p_ρ called the localization of p at ρ, which is

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a hyperbolic polynomial (see [10]). The linearization of the Hamilton field H_p at ρ is called the Hamilton map $F_p(\rho)$ of p at ρ (see, e.g., [6], [10]). Note that $H_{p_\rho}(\theta) = F_p\theta$. If the Hamilton map has nonzero real eigenvalues at every double characteristic (effectively hyperbolic case), then the Cauchy problem for P is well posed in C^∞; in particular, in any Gevrey class s, $1 \le s < +\infty$ for any lower-order term. To check this fact it is enough to apply the energy method developed in [13] to $e^{-x_0\langle D'\rangle^{1/s}} P e^{x_0\langle D'\rangle^{1/s}}$ $(0 < s \ll 1)$. In this paper we are thus interested in the optimal (maximal) Gevrey index s^* such that the Cauchy problem for the noneffectively hyperbolic operator P is well posed in the Gevrey class s for $1 \le s < s^*$ for any lower-order term and how this index s^* relates to the geometry of the double characteristic manifold and null bicharacteristics.

As proved in [7], a hyperbolic quadratic form Q in $\mathbb{R}^{2(n+1)}$ such that the Hamilton map F_Q of Q has no nonzero real eigenvalues can be written in a suitable system of symplectic coordinates according to the spectral structure of F_Q; according to whether $\operatorname{Ker} F_Q^2 \cap \operatorname{Im} F_Q^2 = \{0\}$ or $\operatorname{Ker} F_Q^2 \cap \operatorname{Im} F_Q^2 \neq \{0\}$, we have

$$Q(x,\xi) = -\xi_0^2 + \sum_{j=1}^{k} \mu_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2, \tag{1.2}$$

$$Q(x,\xi) = (-\xi_0^2 + 2x_1\xi_0 + \xi_1^2)/\sqrt{2} + \sum_{j=2}^{k} \mu_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2, \tag{1.3}$$

respectively.

To symbols (1.2) or (1.3) there correspond (apart from harmonic oscillator contributions) the differential operators

$$P_{ne,1} = -D_0^2 + \sum_{j=1}^{r} D_j^2, \quad r < n, \tag{1.4}$$

$$P_{ne,2} = -D_0^2 + 2x_1 D_0 D_n + D_1^2 + \sum_{j=2}^{r} D_j^2, \quad r < n, \tag{1.5}$$

respectively. It is easy to see that the Cauchy problem for $P_{ne,1} + SD_n$ is not well posed in the Gevrey class $s > 2$ for $S \neq 0$, while it is well known that the Cauchy problem for P is well posed in the Gevrey class s, $1 \le s < 2$, for any lower-order term $P_i (i = 0, 1)$ which is a special case of a general result (see [4], [9]). On the other hand, in [7] an explicit formula of the forward fundamental solution of $P_{ne,2} + SD_n$ is obtained for every $S \in \mathbb{C}$ which is a distribution on the Gevrey class 4. Thus we see that the Cauchy problem for $P_{ne,2}$ is well posed in the Gevrey class $1 \le s < 4$ for any $S \in \mathbb{C}$ and not well posed in the Gevrey class $s > 4$ for $S \neq 0$ which follows from the explicit formula in [7] (for another proof of this fact, which is available for the more general case, see [16]). Therefore, in this paper, we consider the case (1.3) in a more general setting.

In what follows we assume that p vanishes exactly to order 2 on a C^∞-manifold Σ, which means that near every $\rho \in \Sigma$ one can write

$$p = -\xi_0^2 + \sum_{j=1}^{r} \phi_j(x,\xi')^2,$$

where $d\phi_j$ are linearly independent at ρ and Σ is given near ρ by

$$\Sigma = \{(x,\xi) \mid \xi_0 = 0, \phi_j(x,\xi') = 0, j = 1,\dots,r\}$$

on which $F_p(\rho)$ has no nonzero real eigenvalues and

$$\operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho) \neq \{0\}, \quad \rho \in \Sigma. \tag{1.6}$$

In this case the spectral properties of $F_p(\rho)$ are not enough by themselves to determine completely the behavior of null bicharacteristics near Σ (see [15]), while the behavior of null bicharacteristic,

(1.7) there is no null bicharacteristic falling on Σ tangentially,

is crucial to the C^∞ well posedness (see [3]). In this paper we prove the following.

THEOREM 1.1

Assume (1.6) and (1.7). We also assume that the codimension of Σ is 3. Then the Cauchy problem for P is well posed in the Gevrey class $1 \le s < 4$ for any lower-order term; for any $f(x) \in \gamma^{(s)}(\mathbb{R}^{n+1})$ vanishing for $x_0 \le 0$, there is $u(x)$ which is C^∞, vanishing for $x_0 \le 0$ verifying

$$Pu = f \tag{1.8}$$

near the origin.

The Gevrey index 4 is optimal in the following sense. Consider a model operator

$$P_{\text{mod}} = -D_0^2 + 2D_0D_1 + x_1^2 D_n^2 \quad (n \ge 2)$$

which verifies (1.6) and (1.7).

THEOREM 1.2 ([7, SECTION 9], [16, PROPOSITION 1.3])

The Cauchy problem for $P_{\text{mod}} + SD_n$ with $S \neq 0$ is not locally solvable in any Gevrey class $s > 4$.

This paper is organized as follows. In Section 2, we analyze our assumptions and we rewrite the principal symbol p in a suitable form microlocally. In Section 3, giving heuristic arguments, we explain the idea of the proof of Theorem 1.1. In Section 4 we prepare symbol classes which are used in this paper and introduce a weight $\exp(\tilde{\phi})$ for an energy estimate which plays a crucial role to derive a priori estimates. In Sections 5 and 6, we justify the heuristic arguments in Section 3. In Section 5, we prove required properties for the transformed operator $\exp(\tilde{\phi})P\exp(-\tilde{\phi})$. In Section 6 we derive a priori estimates for the transformed operator, and using these a priori estimates we prove the existence of a parametrix of P with finite propagation speed of wave front sets which proves Theorem 1.1. In Appendix A we collect several results about symbols without proofs which are used in this paper, and in Appendix B we present some formulas about $\exp(\tilde{\phi})P\exp(-\tilde{\phi})$.

2. Preliminaries

Let $\bar{s} < 4$; we prove Theorem 1.1 by proving the existence of a parametrix at any $\hat{\rho}' = (0, \hat{x}', \hat{\xi}')$ of the Cauchy problem

$$\begin{cases} Pu = f, & f \in C^0([-T,T]; \gamma^{(\bar{s})}(\mathbb{R}^n)), \\ u = 0, & x_0 \le 0, \end{cases}$$

with finite propagation speed of wave front sets, abbreviated as a parametrix with finite propagation speed in the following (see [11, Appendix]; here we define such parametrices by just requiring (A.3), (A.5) without (A.4)), where the support of f is contained in $[0,T] \times \{|x'| \le K\}$ with some $K = K(f) > 0$.

Let κ be a local homogeneous canonical transformation $(y, \eta) \mapsto (x, \xi)$ from a neighborhood of $(\hat{y}, \hat{\eta})$ to a neighborhood of $(\hat{x}, \hat{\xi})$ such that $y_0 = x_0$. Since κ preserves the y_0-coordinate, a generating function of this canonical transformation has the form $x_0\eta_0 + H(x, \eta')$. We assume that $H(x, \eta') = x'\eta' + \phi(x, \eta')$ with $|\nabla_{x,\eta'}\phi(x, \eta')| \ll 1$ which is actually in our case (after a quadratic change of coordinates x if necessary). Recall that

$$\xi' = \frac{\partial H(x, \eta')}{\partial x'}, \qquad y' = \frac{\partial H(x, \eta')}{\partial \eta'}.$$

Let us denote by $S_{(s)}(m, g_{\rho,\delta})$, $0 \le \delta < \rho \le 1$, the set of all smooth $a(x, \xi')$ verifying

$$|\partial_x^\beta \partial_{\xi'}^\alpha a(x, \xi')| \le CA^{|\alpha+\beta|}|\alpha + \beta|!^s m \langle \xi' \rangle^{m+\delta|\beta|-\rho|\alpha|}$$

for all α, β. Extending $\phi(x, \xi')$ so that it becomes homogeneous of degree 1 in ξ' and hence cutting off near $\xi' = 0$, which is in $S_{(s)}(\langle \xi' \rangle, g_{1,0})$, we consider the Fourier integral operators

$$\mathrm{Op}^0(e^{i\phi})u = (2\pi)^{-n} \int e^{i(x'-y')\eta' + i\phi(x,\eta')} u(y')\, dy'\, d\eta',$$

$$\mathrm{Op}^1(e^{-i\phi})u = (2\pi)^{-n} \int e^{i(x'-y')\eta' - i\phi(x_0,y',\eta')} u(y')\, dy'\, d\eta',$$

where x_0 is regarded as a parameter. Let $0 < \rho \le 1$; then we have the following.

PROPOSITION 2.1

Let $p \in S_{(s)}(\langle \xi' \rangle^m, g_{1,0})$ and $\phi \in S_{(s)}(\langle \xi' \rangle, g_{1,0})$ be real valued. Then we have

$$\mathrm{Op}^0(e^{i\phi})\,\mathrm{Op}^0(p)\,\mathrm{Op}^1(e^{-i\phi}) = \mathrm{Op}^0(\tilde{p}) + \mathrm{Op}^0(r),$$

where $\tilde{p} \in S_{(s)}(\langle \xi' \rangle^m, g_{1,0})$ and $r \in S_{((1+2\rho)s+\epsilon)}(e^{-c\langle \xi' \rangle^{1/s}}, g_{\rho,0})$ for any $\epsilon > 0$ with some $c > 0$. Here

$$\tilde{p}(x, \xi') = J(x, \xi')p\big(x_0, \nabla_{\eta'}H(x, \Xi'), \Xi'\big) + S_{(s)}(\langle \xi' \rangle^{m-1}, g_{1,0}),$$

where $\Xi'(x,\xi')$ *verifies* $\nabla_{x'} H(x,\Xi') = \xi'$, $H(x,\eta') = x'\eta' + \phi(x,\eta')$, *and*

$$J(x,\xi') = \det\Big[\frac{\partial \Xi'(x,\xi')}{\partial \xi'}\Big].$$

Proof

The proof is standard (see e.g., [5], [12]) except for the Gevrey estimates for $\tilde{p}$ and r. We only sketch how to get the Gevrey estimates for r. Let us write $\mathrm{Op}^0(e^{i\phi})\,\mathrm{Op}^0(p) = \mathrm{Op}^0(b_1) + \mathrm{Op}^0(b_2)$, where

$$b_i = (2\pi)^{-2}\int e^{-iy'\eta' + i\phi(x,\xi'+\eta')}\chi_i(\xi',\eta')p(x'+y',\xi')\,dy'\,d\eta'$$

with $\chi_1 = \chi(\eta'\langle\xi'\rangle^{-1})$, $\chi_2 = 1-\chi_1$. Here $\chi(x') \in \gamma_0^{(s)}(\mathbb{R}^n)$ is such that $\chi(x') = 1$ for $|x'| \le 1/4$ and $\chi(x') = 0$ for $|x'| \ge 1/2$. We see easily that $b_1 = e^{i\phi}q$ with $q \in S_{(s)}(\langle\xi'\rangle^m, g_{1,0})$. We examine b_2. Let $s_1 = s+\epsilon > s$. From Corollary A.3 we have

$$\begin{aligned}
&\big|\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha} e^{i\phi(x,\xi'+\eta')}\langle D_{y'}\rangle^N\langle\eta'\rangle^{-N}\chi_2 p(x'+y',\xi')\big| \\
&\quad\le CA^{|\alpha+\beta|+N}|\alpha+\beta|!^{s_1}e^{c\langle\eta'\rangle^{1/s_1}}N!^s\langle\eta'\rangle^{-N} \qquad (2.1)\\
&\quad\le CA^{|\alpha+\beta|+N}|\alpha+\beta|!^{s_1}e^{c\langle\eta'\rangle^{1/s_1}}\langle\xi'\rangle^{-\rho|\alpha|}N!^s\langle\eta'\rangle^{-N+\rho|\alpha|}.
\end{aligned}$$

We choose $N = [\rho|\alpha| + \ell]$ such that the right-hand side of (2.1) is bounded by

$$CA^{|\alpha+\beta|}(|\alpha+\beta|!)^{s_1+\rho s}\langle\xi'\rangle^{-\rho|\alpha|}e^{c\langle\eta'\rangle^{1/s_1}}(A\ell^s/\langle\eta'\rangle)^{\ell}.$$

Choosing ℓ such that $\ell = [(A^{-1}e^{-1}\langle\eta'\rangle)^{1/s}]$ and noting $s_1 > s$, we conclude that

$$b_2 \in S_{(s_1+\rho s)}(e^{-c\langle\xi'\rangle^{1/s}}, g_{\rho,0}).$$

By standard arguments and the same type estimates as (2.1), we see

$$\mathrm{Op}^0(e^{i\phi}q)\,\mathrm{Op}^0(e^{-i\phi}) = \mathrm{Op}^0(\tilde{p}) + \mathrm{Op}^0(\tilde{r}),$$

where $\tilde{p} \in S_{(s)}(\langle\xi'\rangle^m, g_{1,0})$ verifies the assertion of Proposition 2.1 and $\tilde{r} \in S_{((1+\rho)s+\epsilon)}(e^{-c\langle\xi'\rangle^{1/s}}, g_{\rho,0})$. Repeating arguments similar to (2.1), we get

$$\mathrm{Op}^0(b_2)\,\mathrm{Op}^0(e^{-i\phi}) = \mathrm{Op}^0(c)$$

with $c \in S_{(s_1+2\rho s)}(e^{-c\langle\xi'\rangle^{1/s}}, g_{\rho,0})$, which proves the assertion. □

For any $k \in \mathbb{N}$ there is $j \in S_{(s)}(1, g_{1,0})$ such that $\mathrm{Op}^0(J)\,\mathrm{Op}^0(j) - 1 \in \mathrm{Op}^0(S_{(s)}(\langle\xi'\rangle^{-k}, g_{1,0}))$, and hence we conclude that

$$\begin{aligned}
&\mathrm{Op}^0(e^{i\phi})P\,\mathrm{Op}^1(e^{-i\phi})\,\mathrm{Op}^0(j) \\
&\quad= \mathrm{Op}^0\big(p(\kappa^{-1}(x,\xi))\big) \qquad (2.2)\\
&\qquad+ \mathrm{Op}^0\Big(S_{(s)}(\langle\xi'\rangle^{-k}, g_{1,0})\xi_0^2 + \sum_{j=0}^{1} S_{(s)}(\langle\xi'\rangle^{1-j}, g_{1,0})\xi_0^j\Big) \\
&\qquad+ \mathrm{Op}^0\Big(\sum_{j=0}^{2} S_{((1+3\rho)s+\epsilon)}(e^{-c\langle\xi'\rangle^{1/s}}, g_{\rho,0})\xi_0^j\Big),
\end{aligned}$$

where $\kappa : (x_0, x' + \nabla_{\eta'}\phi(x,\eta'), \eta_0, \eta') \mapsto (x_0, x', \eta_0 + \partial_{x_0}\phi, \eta' + \nabla_{x'}\phi(x,\eta'))$.

Let us fix any $\rho = (0, x', 0, \xi') \in \Sigma$; we work in a conic neighborhood of $\rho' = (0, x', \xi')$. Since the codimension of Σ is 3, one can write

$$p = -\xi_0^2 + \phi_1^2 + \phi_2^2.$$

We set $\phi_0 = \xi_0$. Consider $\{\phi_0, \phi_j\}(\rho)$, $j = 1, 2$, and suppose $\{\phi_0, \phi_j\}(\rho) = 0$, $j = 1, 2$. With $q = \phi_1^2 + \phi_2^2$ we would have $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \operatorname{Ker} F_q^2 \cap \operatorname{Im} F_q^2$ which contradicts (1.6) since $\operatorname{Ker} F_q^2 \cap \operatorname{Im} F_q^2 = \{0\}$ for q is nonnegative. Thus considering $(\tilde{\phi}_j)_{j=1,2} = O(\phi_j)_{j=1,2}$ with a suitable smooth orthogonal O, we may assume that $\{\phi_0, \phi_1\}(\rho) \neq 0$ and $\{\phi_0, \phi_2\}$ is a linear combination of ϕ_1 and ϕ_2. We next examine $\{\phi_1, \phi_2\}(\rho) \neq 0$. If $\{\phi_1, \phi_2\}(\rho) = 0$, then p would be effectively hyperbolic at ρ. Indeed, $H_{\phi_1} \in (T\Sigma)^\sigma$ and

$$p_\rho(H_{\phi_1}) = -\sigma(H_{\phi_0}, H_{\phi_1})^2 + \sigma(H_{\phi_2}, H_{\phi_1})^2 = -\sigma(H_{\phi_0}, H_{\phi_1})^2 < 0;$$

hence it follows from [6, Corollary 1.4.7] that p is effectively hyperbolic at ρ.

LEMMA 2.1

One can write p as

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \phi_2^2,$$
$$\{\xi_0 + \phi_1, \phi_j\} = 0,\ j = 1, 2, \{\phi_1, \phi_2\} \neq 0 \quad \textit{on } \Sigma.$$

Proof

Recall that

$$p = -\xi_0^2 + \phi_1^2 + \phi_2^2.$$

Let $0 \neq X = aH_{\phi_0} + bH_{\phi_1} + cH_{\phi_2} \in \operatorname{Im} F_p^2 \cap \operatorname{Ker} F_p^2$, which exists by our assumption. Since $X = F_p(\alpha H_{\phi_0} + \beta H_{\phi_1} + \gamma H_{\phi_2})$, it follows that

$$a = -2\beta\{\phi_0, \phi_1\}, \qquad c = 2\beta\{\phi_2, \phi_1\}.$$

From $F_p^2(X) = 0$ we see that

$$b[\{\phi_0, \phi_1\}^2 - \{\phi_1, \phi_2\}^2] = 0, \qquad \beta[\{\phi_0, \phi_1\}^2 - \{\phi_1, \phi_2\}^2] = 0.$$

If $\{\phi_0, \phi_1\}^2 - \{\phi_1, \phi_2\}^2 \neq 0$, then we would have $X = 0$, which is a contradiction. Thus we have proved

$$\{\xi_0, \phi_1\}^2 = \{\phi_1, \phi_2\}^2 \quad \text{on } \Sigma.$$

We may assume that $\{\xi_0, \phi_1\} = \{\phi_1, \phi_2\}$ so that $\{\xi_0 + \phi_2, \phi_1\} = 0$ on Σ. Writing

$$p = -(\xi_0 + \phi_2)(\xi_0 - \phi_2) + \phi_1^2$$

and exchanging ϕ_1 and ϕ_2, we get the desired assertion. □

Since $\{\xi_0 + \phi_1, \phi_2\} = 0$ and $\{\phi_1, \phi_2\} \neq 0$, it follows that $\{\xi_0, \phi_2\} \neq 0$. Hence one can write $\phi_2 = a(x, \xi')(x_0 - \psi(x, \xi'))$, where $a(x, \xi')$ is nonvanishing and ψ is independent of x_0. Since $\{\phi_1, \phi_2\} \neq 0$ and hence $d\psi \neq 0$, one can take a new system of homogeneous symplectic coordinates (y,η) so that

$$y_0 = x_0, \qquad \eta_0 = \xi_0, \qquad y_1 = \psi(x,\xi').$$

Hence we may assume, for instance, that $\phi_2 = a(x,\xi')(x_0 - x_1)$. Make a linear change of coordinates

$$y_0 = x_0, \qquad y_1 = x_1 - x_0.$$

We get

$$(2.3) \qquad \begin{cases} p = -(\xi_0 + \xi_1 + \phi_1)(\xi_0 + \xi_1 - \phi_1) + \phi_2^2, \\ \phi_2 = a(x,\xi')x_1, \qquad a(x,\xi') \neq 0, \\ \{\xi_0 + \xi_1 + \phi_1, \phi_j\} = 0, \quad j = 1,2, \{\phi_1,\phi_2\} \neq 0 \text{ on } \Sigma. \end{cases}$$

Here we recall a result that characterizes when (1.7) occurs. Choose a smooth vector field $z(\rho)$ on Σ such that $z(\rho) \in \operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho)$ and $F_p(\rho)z(\rho) \neq 0$. Then we have the following.

PROPOSITION 2.2 ([2, THEOREM 2.1], [14, THEOREM 4.1])

Let $S(x,\xi)$ be a smooth real-valued function vanishing on Σ such that $H_S(\rho)$ is proportional to $z(\rho)$ modulo $\operatorname{Ker} F_p(\rho)$. Then there is no null bicharacteristic falling on Σ if and only if $H_S^3 p = 0$ on Σ.

Let us write with $\Lambda = \xi_0 + \xi_1 + \phi_1$,

$$p = -\Lambda^2 + 2\phi_1\Lambda + \phi_2^2, \quad \{\Lambda, \phi_2\} = \alpha\phi_1 + \beta\phi_2.$$

LEMMA 2.2

Assume (1.7). Then we have $\alpha = 0$ on Σ. In particular, we have

$$(2.4) \qquad \{\Lambda, \phi_2\} = a\phi_1^2 + b\phi_2.$$

Proof

Note that $F_p H_{\phi_1} = 2\{\phi_2,\phi_1\}H_{\phi_2}$ and hence $H_{\phi_2} \in \operatorname{Im} F_p^2$. On the other hand, since $F_p H_{\phi_2} = \{\phi_1,\phi_2\}H_\Lambda$, hence $H_{\phi_2} \in \operatorname{Ker} F_p^2$ because $F_p H_\Lambda = 0$. Thus we may take $S = \phi_2$ in Proposition 2.2. Then $H_S^3 p = 0$ implies that

$$\big\{\phi_2, \{\phi_2, \Lambda\}\big\} = 0,$$

which proves $\alpha = 0$ on Σ and hence the result. □

Since the existence of a parametrix with finite propagation speed (i.e., a parametrix verifying (A.3), (A.5) in [11]) is invariant under conjugation with Fourier integral operator associated to a local homogeneous canonical transformation preserving the x_0-coordinate (see the proof of [11, Proposition A.5]), we can assume that the operator we are studying has the form in the right-hand side in (2.2), and in particular, we can assume that (2.3) and (2.4) hold near the reference double point ρ.

We extend p globally outside the reference double point.

LEMMA 2.3

Assume that (2.3) and (2.4) are satisfied in a conic neighborhood of $\rho' = (0, x', \xi')$. *Then we can extend* $\phi_1(x, \xi')$ *such that* $\phi_1(x, \xi') \in S(\langle \xi' \rangle, g_{1,0})$,

$$\phi_2(x, \xi') = a(x, \xi')\psi(x_1),$$

where $a \in S(1, g_{1,0})$ *with* $C^{-1} \le a(x, \xi') \le C$ *and* $\psi(x_1)$ *is nondecreasing,* $|\psi(x_1)| \le 1$ *and* $\psi(x_1) = x_1$ *near* $x_1 = 0$, *and*

$$\{\phi_1, \psi\} \ge c > 0 \tag{2.5}$$

provided $|\psi(x_1)| + \langle \xi' \rangle^{-2} |\phi_1(x, \xi')|^2$ *is small. Moreover, there exists* $c_i(x, \xi') \in S(\langle \xi' \rangle^i, g_{1,0})$ *which vanishes in a conic neighborhood of* ρ' *such that*

$$\begin{cases} \{\Lambda + c_1\psi + c_0\phi_1 + c_{-1}\phi_1^2, \phi_1\} \\ \qquad = d_0\phi_1 + d_0'\phi_2 + d_1\sqrt{\psi^2 + \langle \xi' \rangle^{-4}\phi_1^4}, \\ \{\Lambda + c_1\psi + c_0\phi_1 + c_{-1}\phi_1^2, \psi\} \\ \qquad = d_{-2}\phi_1^2 + d_{-1}\phi_2 + d_0''\sqrt{\psi^2 + \langle \xi' \rangle^{-4}\phi_1^4} \end{cases} \tag{2.6}$$

with some d_i, d_i', $d_i'' \in S(\langle \xi' \rangle^i, g_{1,0})$, *where* $\Lambda = \xi_0 + \xi_1 + \phi_1$.

Proof

We may assume that $\{\phi_1, \phi_2\}(\rho') > 0$ and hence $\partial_{\xi_1}\phi_1(\rho') > 0$. Thus one can write $\phi_1 = b(x, \xi')(\xi_1 - \psi_1(x, \xi'))$ near ρ', where ψ_1 is independent of ξ_1. Extending b and ψ_1 so that $b \in S(1, g_{1,0})$, $C^{-1} \le b \le C$ and $\psi_1 \in S(\langle \xi' \rangle, g_{1,0})$, the assertion (2.5) follows immediately because $x_1 - \psi = 0$ near $x_1 = 0$. We turn to (2.6). From (2.4) it follows that $\{\Lambda, \psi\} = d_{-2}\phi_1^2 + d_{-1}\phi_2 + R$ with $R \in S(1, g_{1,0})$ which vanishes in a neighborhood of ρ'. Note that

$$\{\phi_1, \psi\} + K\sqrt{\psi^2 + \langle \xi' \rangle^{-4}\phi_1^4} \ge c > 0$$

with a large $K > 0$ thanks to (2.5). Hence we can write

$$R = a(\{\phi_1, \psi\} + r), \quad r = K\sqrt{\psi^2 + \langle \xi' \rangle^{-4}\phi_1^4},$$

with $a \in S(1, g_{1,0})$ vanishing in a neighborhood of ρ'. Thus we have

$$\begin{aligned} \{\Lambda + c_0\phi_1 + c_{-1}\phi_1^2, \psi\} &= (d_{-2} + \{c_{-1}, \psi\})\phi_1^2 + d_{-1}\phi_2 \\ &+ (a + c_0)\{\phi_1, \psi\} + ar + (\{c_0, \psi\} + 2c_{-1}\{\phi_1, \psi\})\phi_1. \end{aligned}$$

Choose $c_0 = -a$ and $2c_{-1} = \{a, \psi\}(\{\phi_1, \psi\} + r)^{-1}$ so that $c_i \in S(\langle \xi' \rangle^i, g_{1,0})$ vanishes in a neighborhood of ρ'; we get

$$\{\Lambda + c_0\phi_1 + c_{-1}\phi_1^2, \psi\} = d_{-2}\phi_1^2 + d_{-1}\phi_2 + d_0 r.$$

A similar argument proves that there is c_1-vanishing in a neighborhood of ρ' such that $\{\Lambda + c_1\psi, \phi_1\} = d_0\phi_1 + d_0'\phi_2 + d_1 r$. These prove the assertion (2.6). □

Replacing Λ by $\Lambda + c_1\psi + c_0\phi + c_{-1}\phi_1^2$, the resulting symbol $-\Lambda^2 + 2\phi_1\Lambda + \phi_2^2$ differs from the original one by

$$C_0\xi_0 + C_1,$$

where $C_i \in S(\langle \xi' \rangle^i, g_{1,0})$ vanishes in a neighborhood of ρ'. Thus it sufffices to show the existence of a parametrix with finite propagation speed for the operator where Λ is replaced by $\Lambda + c_1\psi + c_0\phi + c_{-1}\phi_1^2$ (see the proof of [11, Lemma A.1].

3. Idea of the proof of Theorem 1.1

We prove Theorem 1.1 by deriving a priori estimates for the transformed operator. In this section we give a heuristic argument on how to do it, in which symbols and operators are not strictly distinguished. Moreover, we write $p \sim q$ if the main parts of p and q coincide, and write $p \preceq q$ if the main part of $q - p$ is nonnegative.

The symbol of the operator that we are studying looks like (replacing $\xi_0 + \xi_1 + \phi_1$ by ξ_0)

$$P = -\xi_0^2 + 2\phi_1\langle\xi\rangle\xi_0 + \phi_2^2\langle\xi\rangle^2,$$

where ϕ_j are homogeneous of degree 0, and verifies

$$(3.1) \quad \{\xi_0, \phi_2\} = a\phi_1^2 + b\phi_2, \qquad \{\xi_0, \phi_1\} = a'\phi_1 + b'\phi_2, \quad \langle\xi\rangle\{\phi_1, \phi_2\} \geq c(>0).$$

With $w = \sqrt{\phi_1^4 + \langle\xi\rangle^{-1}}$ we introduce a weight function

$$\Phi = i\langle\xi\rangle^{1/4}\big\{\log(\phi_2 + iw) - \log(\phi_2 - iw)\big\} = -2\langle\xi\rangle^{1/4}\arg(\phi_2 + iw).$$

Conjugate $e^{\gamma\langle D\rangle^{1/4}x_0}$ to P; then $e^{-\gamma\langle D\rangle^{1/4}x_0}Pe^{\gamma\langle D\rangle^{1/4}x_0}$ yields

$$P \sim -(\xi_0 - i\gamma\langle\xi\rangle^{1/4})^2 + 2\phi_1\langle\xi\rangle(\xi_0 - i\gamma\langle\xi\rangle^{1/4}) + \phi_2^2\langle\xi\rangle^2.$$

We rewrite this in the form

$$\begin{aligned} P &\sim -(\xi_0 - i\gamma\langle\xi\rangle^{1/4} + k\phi_1^3\langle\xi\rangle)(\xi_0 - i\gamma\langle\xi\rangle^{1/4} - k\phi_1^3\langle\xi\rangle) \\ &\quad + 2\phi_1\langle\xi\rangle(\xi_0 - i\gamma\langle\xi\rangle^{1/4} - k\phi_1^3\langle\xi\rangle) \\ &\quad + 2k\phi_1^4\langle\xi\rangle^2 + \phi_2^2\langle\xi\rangle^2 - k^2\phi_1^6\langle\xi\rangle^2 \\ &= -M\Lambda + 2\phi_1\langle\xi\rangle\Lambda + Q \end{aligned}$$

with a positive constant $k > 0$. We next conjugate e^{Φ} with P; that is, we study

$$e^{\Phi}Pe^{-\Phi} \sim -e^{\Phi}\Lambda e^{-\Phi} \cdot e^{\Phi}Me^{-\Phi} + 2e^{\Phi}\phi_1\langle\xi\rangle e^{-\Phi} \cdot e^{\Phi}\Lambda e^{-\Phi} + e^{\Phi}Qe^{-\Phi}.$$

Let $e^{\Phi}\Lambda e^{-\Phi} = \Lambda + \Lambda'$; then the main part of Λ' consists of $e^{\Phi}\{\xi_0, e^{-\Phi}\}/i = i\{\xi_0, \Phi\}$ and $ie^{\Phi}\{\phi_1^3\langle\xi\rangle, e^{-\Phi}\} = -\{\phi_1^3\langle\xi\rangle, \Phi\}$. Note that

$$(3.2) \qquad i\{\xi_0, \Phi\} = -2\langle\xi\rangle^{1/4}\{\xi_0, \phi_2\}\frac{w}{\phi_2^2 + w^2} + 2\langle\xi\rangle^{1/4}\{\xi_0, w\}\frac{\phi_2}{\phi_2^2 + w^2}$$

and $\{\xi_0, w\} = 2w^{-1}\phi_1^3\{\xi_0, \phi_1\}$. By the assumption (3.1), we have

$$|e^{\Phi}\{\xi_0, e^{-\Phi}\}| \leq C\langle\xi\rangle^{1/4}.$$

We also note

$$\big\{\phi_1^3\langle\xi\rangle, \Phi\big\} \sim 6\langle\xi\rangle^{1/4}\phi_1^2\langle\xi\rangle\{\phi_1, \phi_2\}\frac{w}{\phi_2^2 + w^2}$$

which shows that

$$\big|e^{\Phi}\big\{k\phi_1^3\langle\xi\rangle, e^{-\Phi}\big\}\big| \leq Ck\langle\xi\rangle^{1/4}.$$

We now study $e^{\Phi}\phi_1\langle D\rangle e^{-\Phi}=\phi_1\langle D\rangle+\tilde{\phi}_1$. The main part of $\tilde{\phi}_1$ is

$$\frac{1}{i}e^{\Phi}\{\phi_1\langle\xi\rangle,e^{-\Phi}\}\sim 2i\langle\xi\rangle^{5/4}\{\phi_1,\phi_2\}\frac{w}{\phi_2^2+w^2},$$

and hence

$$\mathsf{Im}\Big(\frac{1}{i}e^{\Phi}\{\phi_1\langle\xi\rangle,e^{-\Phi}\}\Big)\geq c\langle\xi\rangle^{1/4}\frac{w}{\phi_2^2+w^2},\quad c>0$$

by assumption (3.1), which gives a positive contribution, crucial to the control of not only lower-order terms but also of other terms caused by conjugation of e^{Φ}.

We consider $e^{\Phi}Qe^{-\Phi}=Q+iQ'$. The main part of Q' is

$$\begin{aligned}-e^{\Phi}\{\phi_2^2\langle\xi\rangle^2+k\phi_1^4\langle\xi\rangle^2,e^{-\Phi}\}&=\{\phi_2^2\langle\xi\rangle^2+k\phi_1^4\langle\xi\rangle^2,\Phi\}\\&\sim -8\langle\xi\rangle^{2+1/4}\{\phi_2,\phi_1\}\Big[\frac{\phi_2^2\phi_1^3}{(\phi_2^2+w^2)w}+k\frac{\phi_1^3 w}{\phi_2^2+w^2}\Big]\end{aligned}$$

which gives

$$|Q'|\preceq\langle\xi\rangle^{5/4}w^{1/2}$$

because $|\phi_1|\leq w^{1/2}$. Now $e^{\Phi}Pe^{-\Phi}$ looks as follows:

$$\tilde{P}=-M\Lambda+2B\Lambda+(Q+\langle D\rangle)-\langle D\rangle,$$

where we regard $Q+\langle D\rangle$ as a new Q and $-\langle D\rangle$ as a lower-order term. Note that

$$\begin{aligned}\Lambda&\sim\xi_0-i\gamma\langle\xi\rangle^{1/4}-k\phi_1^3\langle\xi\rangle+i\tilde{\lambda}=\xi_0-i\gamma\langle\xi\rangle^{1/4}+\lambda,\quad|\mathsf{Im}\,\lambda|\leq C\langle\xi\rangle^{1/4},\\M&\sim\xi_0-i\gamma\langle\xi\rangle^{1/4}+k\phi_1^3\langle\xi\rangle+i\tilde{m}=\xi_0-i\gamma\langle\xi\rangle^{1/4}+m,\quad|\mathsf{Im}\,m|\leq C\langle\xi\rangle^{1/4},\\B&\sim\phi_1\langle\xi\rangle+2i\langle\xi\rangle^{5/4}\{\phi_1,\phi_2\}\frac{w}{\phi_2^2+w^2},\quad\mathsf{Im}\,B\geq c\langle\xi\rangle^{1/4}\frac{w}{\phi_2^2+w^2},\\Q+\langle D\rangle&\sim\phi_2^2\langle\xi\rangle^2+k\phi_1^4\langle\xi\rangle^2+\langle\xi\rangle+iQ',\quad|Q'|\preceq\langle\xi\rangle^{5/4}w^{1/2}.\end{aligned}$$

We recall an energy identity (see Proposition 6.1)

$$\begin{aligned}2\,\mathsf{Im}(\tilde{P}u,\Lambda u)=&\frac{d}{dx_0}\big(\|\Lambda u\|^2+((\mathsf{Re}\,Q)u,u)\big)\\&+2\gamma\|\langle D\rangle^{1/8}\Lambda u\|^2+2\gamma\mathsf{Re}\big(\langle D\rangle^{1/4}(\mathsf{Re}\,Q)u,u\big)\\&+2\big((\mathsf{Im}\,B)\Lambda u,\Lambda u\big)+2\big((\mathsf{Im}\,m)\Lambda u,\Lambda u\big)+2\mathsf{Re}\big(\Lambda u,(\mathsf{Im}\,Q)u\big)\\&+\mathsf{Im}([D_0-\mathsf{Re}\lambda,\mathsf{Re}\,Q]u,u)+2\mathsf{Re}\big((\mathsf{Re}\,Q)u,(\mathsf{Im}\,\lambda)u\big).\end{aligned}$$

The terms $2((\mathsf{Im}\,m)\Lambda u,\Lambda u)$ and $2\mathsf{Re}((\mathsf{Re}\,Q)u,(\mathsf{Im}\,\lambda)u)$ are easily estimated because $|\mathsf{Im}\,\lambda|$, $|\mathsf{Im}\,m|\leq C\langle\xi\rangle^{1/4}$. In what follows we note that the terms

$$\epsilon\Big|\Big(\frac{\langle\xi\rangle^{1/4}w}{\phi_2^2+w^2}\Lambda u,\Lambda u\Big)\Big|,\qquad K\big|\big(\langle\xi\rangle^{1/4}(\phi_2^2\langle\xi\rangle^2+\phi_1^4\langle\xi\rangle^2+\langle\xi\rangle)u,u\big)\big|$$

with small $\epsilon>0$ and any $K>0$ can be controlled by $2((\mathsf{Im}B)\Lambda u,\Lambda u)$ and $2\gamma\mathsf{Re}(\langle D\rangle^{1/4}(\mathsf{Re}\,Q+\langle D\rangle)u,u)$, taking γ large if necessary.

To estimate $(\mathsf{Im}\, Qu, \Lambda u)$ it suffices to note that

$$\begin{aligned}\big|(\langle\xi\rangle^{5/4}w^{1/2}u, \Lambda u)\big| &\le \epsilon\Big|\Big(\frac{\langle\xi\rangle^{1/4}w}{\phi_2^2+w^2}\Lambda u, \Lambda u\Big)\Big| \\ &\quad + \epsilon^{-1}\big|\big(\langle\xi\rangle^{2+1/4}(\phi_2^2+w^2)u, u\big)\big| \\ &\le \epsilon\Big|\Big(\frac{\langle\xi\rangle^{1/4}w}{\phi_2^2+w^2}\Lambda u, \Lambda u\Big)\Big| \\ &\quad + \epsilon^{-1}\big|\big(\langle\xi\rangle^{1/4}(\phi_2^2\langle\xi\rangle^2+\phi_1^4\langle\xi\rangle^2+\langle\xi\rangle)u, u\big)\big|.\end{aligned}$$

To see how any lower-order term can be controlled, it is enough to note that

$$\begin{aligned}&2K|(\langle\xi\rangle u, \Lambda u)| \\ &\le \epsilon^{-1}K^2\Big|\Big(\frac{(\phi_2^2+w^2)\langle\xi\rangle^2}{\langle\xi\rangle^{1/4}w}u, u\Big)\Big| + \epsilon\Big|\Big(\frac{\langle\xi\rangle^{1/4}w}{\phi_2^2+w^2}\Lambda u, \Lambda u\Big)\Big| \\ &\le \epsilon^{-1}K^2\big|\big(\langle\xi\rangle^{2+1/4}(\phi_2^2+w^2)u, u\big)\big| + \epsilon\Big|\Big(\frac{\langle\xi\rangle^{1/4}w}{\phi_2^2+w^2}\Lambda u, \Lambda u\Big)\Big| \\ &\le \epsilon^{-1}K^2\big|\big(\langle\xi\rangle^{1/4}(\phi_2^2\langle\xi\rangle^2+\phi_1^4\langle\xi\rangle^2+\langle\xi\rangle)u, u\big)\big| + \epsilon\Big|\Big(\frac{\langle\xi\rangle^{1/4}w}{\phi_2^2+w^2}\Lambda u, \Lambda u\Big)\Big|\end{aligned}$$

because $(\langle\xi\rangle^{1/4}w)^{-1} \le \langle\xi\rangle^{1/4}$. We finally check the commutator term, $[D_0 - \mathsf{Re}\lambda, \mathsf{Re}\, Q]$. Note that

$$\begin{aligned}[D_0 - \mathsf{Re}\lambda, \mathsf{Re}\, Q] &\sim -i\big\{\xi_0 - k\phi_1^3\langle\xi\rangle, \phi_2^2\langle\xi\rangle^2 + k\phi_1^4\langle\xi\rangle^2\big\} \\ &\sim -2i\phi_2\langle\xi\rangle^2\{\xi_0, \phi_2\} - 4ik\langle\xi\rangle^2\phi_1^3\{\xi_0, \phi_1\} + 6ik\langle\xi\rangle^3\phi_1^2\phi_2\{\phi_1, \phi_2\}\end{aligned}$$

and hence

$$|\mathsf{Im}[D_0 - \mathsf{Re}\lambda, \mathsf{Re}\, Q]| \preceq C_M(\phi_2^2\langle\xi\rangle^2 + \phi_1^4\langle\xi\rangle^2)$$

because of (3.1). Thus we conclude

$$\begin{aligned}2\,\mathsf{Im}\big((\tilde{P} + K\langle D\rangle)u, \Lambda u\big) &\succeq \frac{d}{dx_0}\big(\|\Lambda u\|^2 + ((\mathsf{Re}\, Q)u, u)\big) \\ &\quad + c\gamma\|\langle D\rangle^{1/8}\Lambda u\|^2 + c\gamma\mathsf{Re}\big(\langle D\rangle^{1/4}(\mathsf{Re}\, Q)u, u\big),\end{aligned}$$

and hence an a priori estimate is obtained. We justify these heuristic arguments in Sections 4–6.

4. Symbols

In this section we precisely define our weight function and the symbols with which we work. As observed in the end of Section 2, we can assume that $P(x,\xi)$ is globally defined and the principal symbol $p(x,\xi)$,

$$p(x,\xi) = -\Lambda^2 + 2\phi_2\Lambda + \phi_2^2, \quad \Lambda = \xi_0 + \lambda_1,$$

verifies the conditions (2.5) and

$$
\tag{4.1}
\begin{cases}
\{\Lambda, \phi_1\} = d_0\phi_1 + d_0'\phi_2 + d_1\sqrt{\psi^2 + \langle\xi'\rangle^{-4}\phi_1^4}, \\
\{\Lambda, \psi\} = d_{-2}\phi_1^2 + d_{-1}\phi_2 + d_0''\sqrt{\psi^2 + \langle\xi'\rangle^{-4}\phi_1^4}.
\end{cases}
$$

We dilate the variable: $x_0 \to \mu x_0$ (small $\mu > 0$) so that we have

$$
\begin{aligned}
P(x, \xi, \mu) &= \mu^2 P(\mu x_0, x', \mu^{-1}\xi_0, \xi') \\
&= p(\mu x_0, x', \xi_0, \mu\xi') + \mu P_1(\mu x_0, x', \xi_0, \mu\xi') + \mu^2 P_0(\mu x_0, x', \xi_0, \mu\xi') \\
&= p(x, \xi, \mu) + P_1(x, \xi, \mu) + P_0(x, \xi, \mu).
\end{aligned}
$$

In what follows we often write $p(x, \xi)$, $\phi_j(x, \xi')$ for $p(x, \xi, \mu)$, $\phi_j(x, \xi', \mu)$, dropping μ.

Let us denote by $S_{(s)}(m, g)$ with

$$
g = \sum_{j=0}^{n} \delta_j^2\, dx_j^2 + \sum_{j=1}^{n} \rho_j^{-2}\, d\xi_j^2
$$

the set of all smooth $a(x, \xi'; \mu)$ satisfying

$$
\tag{4.2}
|\partial_x^\beta \partial_{\xi'}^\alpha a(x, \xi'; \mu)| \le C A^{|\alpha+\beta|} |\alpha + \beta|!^s m(x, \xi'; \mu) \delta^\beta \rho^{-\alpha}
$$

with some $C > 0$, $A > 0$ independent of μ, where $\delta = (\delta_1(x, \xi', \mu), \ldots, \delta_n(x, \xi', \mu))$, $\rho = (\rho_1(x, \xi', \mu), \ldots, \rho_n(x, \xi', \mu))$, and δ_j, δ_j are assumed to be in $S_{(s)}(\delta_j, g)$, $S_{(s)}(\rho_j, g)$, respectively.

We also denote by $S(m, g)$ the symbol class consisting of all smooth $a(x, \xi', \mu)$ verifying (4.2) with $C_{\alpha,\beta}$, independent of μ, instead of $CA^{|\alpha+\beta|}|\alpha+\beta|!^s$. Note the following.

LEMMA 4.1

Let $a(x, \xi') \in S_{(s)}(\langle\xi'\rangle^k, g_{1,0})$. Then we have with, $g_0 = \mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2$, $\langle\xi'\rangle_\mu^2 = \mu^{-2} + |\xi'|^2 = \mu^{-2}\langle\mu\xi'\rangle^2$,

$$
a(\mu x_0, x', \mu\xi') \in S_{(s)}(\langle\mu\xi'\rangle^k, g_0).
$$

We rewrite $p(x, \xi)$ as

$$
\begin{aligned}
p = &-(\xi_0 + \lambda_1 + k\langle\mu\xi'\rangle^{-2}\phi_1^3)(\xi_0 + \lambda_1 - k\langle\mu\xi'\rangle^{-2}\phi_1^3) \\
&+ 2\phi_1(\xi_0 + \lambda_1 - k\langle\mu\xi'\rangle^{-2}\phi_1^3) + \phi_2^2 + 2k\langle\mu\xi'\rangle^{-2}\phi_1^4\{1 - k/2\langle\mu\xi'\rangle^{-2}\phi_1^2\}.
\end{aligned}
$$

Taking a positive constant k to be sufficiently small, we set

$$
Q = \phi_2^2 + \theta^2, \quad \theta^2 = 2k\langle\mu\xi'\rangle^{-2}\phi_1^4\{1 - k/2\langle\mu\xi'\rangle^{-2}\phi_1^2\},
$$

and note that $\theta(x, \xi') \in S(\langle\mu\xi'\rangle, g_0)$ verifies $C^{-1}\langle\mu\xi'\rangle^{-1}\phi_1^2 \le \theta \le C\langle\mu\xi'\rangle^{-1}\phi_1^2$ with some $C > 0$. Thus one can write

$$
\tag{4.3}
p = -M(x, \xi)\Lambda(x, \xi) + 2\phi_1(x, \xi')\Lambda(x, \xi) + Q(x, \xi'),
$$

where $M = \xi_0 + \lambda_1 + k\langle\mu\xi'\rangle^{-2}\phi_1^3$, $\Lambda = \xi_0 + \lambda_1 - k\langle\mu\xi'\rangle^{-2}\phi_1^3$. Let us set

$$w(x,\xi') = \sqrt{\langle\mu\xi'\rangle^{-4}\phi_1(x,\xi')^4 + \langle\xi'\rangle_\mu^{-1}};$$

then it follows from Corollary A.4 that

$$\begin{aligned} w &\in S_{(s)}\big(w, w^{-1}(\mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2)\big), \\ &\subset S_{(s)}\big(w, \langle\xi'\rangle_\mu^{1/2}(\mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2)\big). \end{aligned}$$

Let

$$0 < \kappa < \frac{1}{4}$$

be fixed hereafter. Eventually we take κ very close to $1/4$. We introduce the symbol

$$\begin{aligned} \phi &= i\langle\mu\xi'\rangle^{\kappa}\big\{\log\big(\psi(x_1) + iw(x,\xi')\big) - \log\big(\psi(x_1) - iw(x,\xi')\big)\big\} \\ &= \langle\mu\xi'\rangle^{\kappa}\arg\frac{\psi(x_1) - iw(x,\xi')}{\psi(x_1) + iw(x,\xi')} = -2\langle\mu\xi'\rangle^{\kappa}\arg\big(\psi(x_1) + iw\big) \end{aligned} \tag{4.4}$$

and set

$$r(x,\xi') = \sqrt{\psi(x_1)^2 + w(x,\xi')^2} = \sqrt{\psi^2 + \langle\mu\xi'\rangle^{-4}\phi_1^4 + \langle\xi'\rangle_\mu^{-1}}.$$

Then from Lemma A.6 it follows that

$$\phi(x,\xi') \in S_{(s)}(\langle\mu\xi'\rangle^{\kappa}, g), \qquad r(x,\xi') \in S_{(s)}(r, g),$$

where

$$g = \big(r(x,\xi')^{-1} + w^{-1/2}\big)^2 dx_1^2 + w^{-1}(\mu^2 dx_0^2 + |dx''|^2) + w^{-1}\langle\xi'\rangle_\mu^{-2}|d\xi'|^2 \tag{4.5}$$

with $x'' = (x_2, \ldots, x_n)$. We also use

$$\begin{aligned} g \le \hat{g} &= \langle\xi'\rangle_\mu\, dx_1^2 + \langle\xi'\rangle_\mu^{1/2}(\mu^2 dx_0^2 + |dx''|^2) + \langle\xi'\rangle_\mu^{-3/2}|d\xi'|^2 \\ &\le \langle\xi'\rangle_\mu |dx|^2 + \langle\xi'\rangle_\mu^{-3/2}|d\xi'|^2 = \bar{g}. \end{aligned}$$

We now recall conditions (2.5) and (2.6) in terms of symbol classes.

LEMMA 4.2

We have

$$\{\phi_1, \psi\} \ge c\mu \tag{4.6}$$

provided $r(x,\xi')$ is small. Moreover, we have

$$\begin{cases} \{\Lambda,\psi\} \in \mu S(r,g), \quad \{\Lambda,\phi_1\} \in \mu S((r + w^{1/2})\langle\mu\xi'\rangle, g), \\ \partial_{\xi_1}\Lambda \in \mu S(r,g), \quad \partial_{\xi_1}^2\Lambda \in \mu S((r + w^{1/2})\langle\xi'\rangle_\mu^{-1}, g). \end{cases} \tag{4.7}$$

Proof

The first two assertions follow from (2.5) and (4.1) immediately. Note that

$$\{\Lambda,\psi\} = C_{-2}\phi_1^2 + C_{-1}\phi_2 + C_0 r \tag{4.8}$$

with some $C_j \in \mu S_{(s)}(\langle \mu\xi' \rangle^j, g_0)$. Noting that $\{\Lambda, x_1\} = \{\Lambda, x_1 - \psi\} + \{\Lambda, \psi\}$ and $\{\Lambda, x_1 - \psi\}$ vanishes if x_1 is small, one can then write

$$\partial_{\xi_1}\Lambda = \{\Lambda, \psi\} + C\psi(x_1)$$

with $C \in \mu S(1, g)$, which shows $\partial_{\xi_1}\Lambda \in \mu S_{(s)}(r, g)$. From this expression it is clear that $\partial_{\xi_1}^2 \Lambda \in \mu S_{(s)}((r + w^{1/2})\langle \xi' \rangle_\mu^{-1}, g)$. □

5. Transformed symbols

Take κ' so that

$$\kappa' + \kappa = \frac{1}{2},$$

and assume that $s > 1$ verifies, with $\rho = 3/4$, $\delta = 1/2$,

$$(5.1) \qquad (s-1)\kappa', (s-1)(1 - \rho + \kappa) < \rho - \delta - \kappa, \quad s\kappa' < 1 - \delta.$$

Let us set

$$\tilde{\phi}(x, \xi') = -x_0 \langle \mu\xi' \rangle^{\kappa'} + \phi(x, \xi').$$

We study in detail the operator $\mathrm{Op}^0(e^{\tilde{\phi}}) \mathrm{Op}^0(p) \mathrm{Op}^1(e^{-\tilde{\phi}})$, where $\mathrm{Op}^t(p)$ is the t-quantization of p (see Appendix B). In what follows it is assumed that $|x_0| \le T$ with some $T > 0$. Our goal in this section is Proposition 5.4.

In this section we apply the results in Appendices A and B with $a_1 = 1/2$, $a_j = 1/4$, $j \ge 2$, $b_j = 3/4$, $\delta = 1/2$, $\rho = 3/4$, and

$$h = \langle \xi' \rangle_\mu^{-1/4}, \qquad k = \langle \mu\xi' \rangle^{\kappa} \langle \xi' \rangle_\mu^{\epsilon},$$

where $0 < \epsilon < 1/4 - \kappa$.

Recalling that p is a polynomial in ξ_0,

$$p(x, \xi) = -\xi_0^2 + p_1(x, \xi')\xi_0 + p_2(x, \xi'),$$

we apply Proposition B.1 ($\rho = 3/4$, $\delta = 1/2$) to get

$$\mathrm{Op}^0(e^{\tilde{\phi}}) \mathrm{Op}^0(p) = \mathrm{Op}^0(e^{\tilde{\phi}} q) + \mathrm{Op}^0(r_0\xi_0 + r_1),$$

where $r_i(x, \xi') \in S_{(sd)}(e^{-c\langle \mu\xi' \rangle^{1/2s}}, \bar{g})$ with $d = 5/2$ and

$$q(x, \xi) = \sum_{|\beta| < 5} \frac{1}{\beta!} \partial_{\eta'}^{\beta} p_{(\beta)}\big(x_0, x' - i\tilde{\Phi}(x, \xi', \eta'), \xi\big)_{\eta' = 0} + R_1(x, \xi') + R_0(x, \xi')\xi_0$$

with $R_1(x, \xi') \in \mu^{5/4} S_{(s)}(\langle \mu\xi' \rangle, \bar{g})$, $R_0(x, \xi') \in \mu^{5/4} S_{(s)}(1, \bar{g})$, where

$$\tilde{\Phi}(x, \xi', \eta') = \int_0^1 \nabla_{\xi'} \tilde{\phi}(x, \xi' + \theta\eta')\, d\theta.$$

We now conjugate $\mathrm{Op}^1(e^{-\tilde{\phi}})$ on the right:

$$\mathrm{Op}^0(e^{\tilde{\phi}} q) \mathrm{Op}^1(e^{-\tilde{\phi}}) + \mathrm{Op}^0(r_0\xi_0 + r_1) \mathrm{Op}^1(e^{-\tilde{\phi}}).$$

If an operator T is given by $T = \mathrm{Op}^0(p)$ with some $p \in S_{(s)}(m, g)$, then we abbreviate as $T = \mathrm{Op}^0(S_{(s)}(m, g))$. Since $1/2s > \kappa'$, it follows from Proposition B.3

that

$$\mathrm{Op}^0(r_0\xi_0+r_1)\,\mathrm{Op}^1(e^{-\tilde{\phi}})=\mu^k\,\mathrm{Op}^0\big(S_{(sd^2)}(\langle\mu\xi'\rangle^{-k},\bar{g})\xi_0+S_{(sd^2)}(\langle\mu\xi'\rangle^{-k},\bar{g})\big)$$

for any $k\in\mathbb{N}$. Since $p_i(x,\xi')\in S_{(s)}(\langle\mu\xi'\rangle^i,g_0)$, we see by Lemma A.2 that

$$p_{i(\beta)}\big(x_0,x'-i\tilde{\Phi}(x,\xi',\eta'),\xi'\big)\in S_{(s)}(\langle\mu\xi'\rangle^i,g|E)$$

for any β because $\tilde{\Phi}(x,\xi',\eta')\in\mu^{3/4}S_{(s)}(\langle\mu\xi'\rangle^{\kappa-3/4},g|E)$, where $E=\{(x,\xi',\eta')\mid |\eta'|<|\xi'|/2\}$ (for the definition $S_{(s)}(m,g|E)$, see Appendix A). This proves that

$$\partial_{\eta'}^{\beta}p_{i(\beta)}\big(x_0,x'-\tilde{\Phi}(x,\xi',\eta'),\xi'\big)\in\mu^{3(|\beta|+1)/4}S_{(s)}(\langle\mu\xi'\rangle^{i+\kappa-3/4}\langle\mu\xi'\rangle^{-3|\beta|/4},g|E)$$

for $|\beta|\geq 1$ by Lemma A.2 and hence

$$q(x,\xi)=p\big(x_0,x'-i\nabla_{\xi'}\tilde{\phi}(x,\xi'),\xi\big)+R_0(x,\xi')\xi_0+R_1(x,\xi')$$

with $R_0(x,\xi')\in\mu^{5/4}S_{(s)}(1,\bar{g})$ and $R_1(x,\xi')\in\mu^{5/4}S_{(s)}(\langle\mu\xi'\rangle,\bar{g})$. From Proposition B.2 we see that

$$\mathrm{Op}^0\big(e^{\tilde{\phi}}(R_1+R_0\xi_0)\big)\,\mathrm{Op}^1(e^{-\tilde{\phi}})=\mu^{5/4}\,\mathrm{Op}^0\big(S_{(sd^2)}(\langle\mu\xi'\rangle,\bar{g})+S_{(sd^2)}(1,\bar{g})\xi_0\big).$$

Thus we conclude that

$$\begin{aligned}\mathrm{Op}^0(e^{\tilde{\phi}})\,\mathrm{Op}^0(p)\,\mathrm{Op}^1(e^{-\tilde{\phi}})&=\mathrm{Op}^0\big(e^{\tilde{\phi}}p(x_0,x'-i\nabla_{\xi'}\tilde{\phi}(x,\xi'),\xi)\big)\,\mathrm{Op}^1(e^{-\tilde{\phi}})\\&\quad+\mu^{5/4}\,\mathrm{Op}^0\big(S(\langle\mu\xi'\rangle,\bar{g})+S(1,\bar{g})\xi_0\big).\end{aligned}$$

Let us set $q(x,\xi)=p(x_0,x'-i\nabla_{\xi'}\tilde{\phi}(x,\xi'),\xi)$ and study $\mathrm{Op}^0(e^{\tilde{\phi}}q)\,\mathrm{Op}^1(e^{-\tilde{\phi}})$. Since q is a polynomial in ξ_0 of order 2, we have $\mathrm{Op}^0(e^{\tilde{\phi}}q)\,\mathrm{Op}^1(e^{-\tilde{\phi}})=\mathrm{Op}^0(b)+\mu^{3/2}\,\mathrm{Op}^0(S_{(sd^2)}(\langle\mu\xi'\rangle^{\kappa+1/2},\bar{g}))$, where

$$\begin{aligned}b(x,\xi)=\lim_{\epsilon\to 0}(2\pi)^{-n}\int e^{-i(x'-y')(\xi'-\eta')+\tilde{\phi}(x,\eta')-\tilde{\phi}(x_0,y',\eta')}\chi_\epsilon(y',\xi',\eta')\\\times\,q\big(x,\xi_0+i\partial_{x_0}\tilde{\phi}(x_0,y',\eta'),\eta'\big)\,dy'\,d\eta'\end{aligned}$$

because

$$\mathrm{Op}^0(e^{\tilde{\phi}})\,\mathrm{Op}^1(e^{-\tilde{\phi}}\partial_{x_0}^2\tilde{\phi})\in\mu^{3/2}\,\mathrm{Op}^0\big(S_{(sd^2)}(\langle\mu\xi'\rangle^{\kappa+1/2},\bar{g})\big)$$

which follows from an assertion similar to Proposition B.2 because we have $\partial_{x_0}^2\tilde{\phi}\in\mu^{3/2}S_{(s)}(\langle\mu\xi'\rangle^{\kappa+1/2},\bar{g})$. Here we have set $\chi_\epsilon(y',\xi',\eta')=\chi(\epsilon y')\chi(\epsilon\langle\xi'\rangle_\mu^{-1}\eta')$ with $\chi(t)\in\gamma_0^{(s)}(\mathbb{R}^n)$ such that $\chi(t)=1$ near $t=0$. Let $\Xi'(x,y',\xi')=\xi'+G'(x,y',\xi')$ be the solution to

$$\Xi'-i\int_0^1\nabla_{x'}\phi\big(x_0,x'+\theta(y'-x'),\Xi'\big)\,d\theta=\xi'$$

given by Proposition A.3 and

$$J(x,y',\xi')=\det\Big[\frac{\partial\Xi(x,y',\xi')}{\partial\xi'}\Big].$$

Applying Proposition B.2 we get

$$b(x,\xi)=\sum_{|\alpha|<5}\frac{1}{\alpha!}\partial_{\xi'}^{\alpha}D_{y'}^{\alpha}\big[J(x,y',\xi')\\ \times q\big(x,\xi_0+i(\partial_{x_0}\tilde{\phi})(x_0,y',\Xi'(x,y',\xi')),\Xi'(x,y',\xi')\big)\big]_{y'=x'}\\ +R_1(x,\xi')+R_0(x,\xi')\xi_0+R_{-1}(x,\xi')\xi_0^2,$$

where $R_i(x,\xi')\in\mu^{5/4}S(\langle\mu\xi'\rangle^i,\bar{g})$. We summarize by the following.

PROPOSITION 5.1

We have

$$\operatorname{Op}^0(e^{\tilde{\phi}})\operatorname{Op}^0(p)\operatorname{Op}^1(e^{-\tilde{\phi}})\\ =\operatorname{Op}^0\big(b(x,\xi)\big)+\mu^{5/4}\operatorname{Op}^0\big(S(\langle\mu\xi'\rangle,\bar{g})+S(1,\bar{g})\xi_0+S(\langle\mu\xi'\rangle^{-1},\bar{g})\xi_0^2\big),$$

where

$$b(x,\xi)=\sum_{|\alpha|<5}\frac{1}{\alpha!}\partial_{\xi'}^{\alpha}D_{y'}^{\alpha}\big[J(x,y',\xi')p\big(x_0,x'-i(\nabla_{\xi'}\tilde{\phi})(x,\Xi'(x,y',\xi')),\xi_0\\ +i(\partial_{x_0}\tilde{\phi})(x_0,y',\Xi'(x,y',\xi')),\Xi'(x,y',\xi')\big)\big]_{y'=x'}.$$

To simplify notation we denote by $S^{\#}_{(s)}(m,g)$ the set of $a(x',y',\xi')$ verifying

$$\big[\partial_{x',y'}^{\beta}\partial_{\xi'}^{\alpha}a(x',y',\xi')\big]_{y'=x'}\in S_{(s)}\big(m(r^{-1}+\langle\xi'\rangle_\mu^{1/4})^{\beta_1}\langle\xi'\rangle_\mu^{|\beta'|/4-3|\alpha|/4},g\big),\quad\forall\alpha,\beta.$$

From Proposition A.3 with $\bar{k}(\xi')=\langle\mu\xi'\rangle^{\kappa}$, $\Delta_1=r^{-2}w+r^{-1}w^{1/2}$, $\Delta_j=r^{-1}w^{1/2}$, $j\neq 1$, it follows that

$$\begin{aligned}&G_j(x,y',\xi')\in S_{(s)}(\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_\mu^{1/4},\hat{g}),\quad j\neq 1,\\ &G_1(x,y',\xi')\in S_{(s)}(\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_\mu^{1/2},\hat{g}),\\ &G_j(x,y',\xi')\in S^{\#}_{(s)}(\Delta_j\langle\mu\xi'\rangle^{\kappa},g),\end{aligned}\tag{5.2}$$

where $G'(x,y',\xi')=(G_1,\ldots,G_n)$.

LEMMA 5.1

We have

$$w\big(x,\Xi'(x,x',\xi')\big)\in S_{(s)}\big(w(x,\xi'),g\big),\quad r\big(x,\Xi'(x,x',\xi')\big)\in S_{(s)}\big(r(x,\xi'),g\big),\\ w\big(x,\Xi'(x,x',\xi')\big)=w(x,\xi')\big(1+O(\mu^{1/4})\big),\\ (1+C\mu^{1/4})r(x,\xi')^2\geq\big|r\big(x,\Xi'(x,x',\xi')\big)\big|^2\geq(1-C\mu^{1/4})r(x,\xi')^2.$$

Proof

Note that $w(x,\xi')\in S_{(s)}(w(x,\xi'),g)$ and $r(x,\xi')\in S_{(s)}(r(x,\xi'),g)$ by Lemma A.6. Since $\Xi'(x,x',\xi')=\xi'+G'(x,x',\xi')$ taking into account (5.2), the first assertion follows from Lemma A.3. The second assertion follows from Corollary A.1. To check the third assertion it is enough to remark that

$$\begin{aligned}\big|r\big(x,\Xi'(x,x',\xi')\big)\big|^2 &= \big|\psi(x_1)^2 + w\big(x,\Xi'(x,x',\xi')\big)^2\big| \\ &= \big|\psi(x_1)^2 + w(x,\xi')^2\big(1+O(\mu^{1/4})\big)\big|. \qquad \square\end{aligned}$$

The next lemma is an immediate consequence of Corollary A.1, but we give a proof here.

LEMMA 5.2

Let $a(x',\xi') \in S_{(s)}(\langle\mu\xi'\rangle^m, g)$ and $a(x'+iy',\xi'+i\eta')$ be the almost-analytic extension given by Proposition A.1 with $k(\xi') = \langle\mu\xi'\rangle^\kappa\langle\xi'\rangle_\mu^\epsilon$, $0<\epsilon<1/4-\kappa$. Let $z(x',y',\xi',\eta') \in S_{(s)}(\langle\mu\xi'\rangle^\kappa\langle\xi'\rangle_\mu^{-3/4}, \hat g|E(k))$, $\zeta(x',y',\xi',\eta') \in S_{(s)}(\langle\mu\xi'\rangle^\kappa\langle\xi'\rangle_\mu^{1/2}, \hat g \mid E(k))$; then we have

$$\begin{aligned}a(x'+z,\xi'+\zeta) - \sum_{|\alpha+\beta|<\ell}\frac{1}{\alpha!\beta!}\partial_{x'}^\beta\partial_{\xi'}^\alpha a(x',\xi')z^\beta\zeta^\alpha \\ \in \mu^{\ell/4}S\big(\langle\mu\xi'\rangle^{m-(1/4-\kappa-\epsilon)\ell}, \hat g \mid E(k)\big).\end{aligned}$$

Proof

Let us denote $z = \hat x' + i\hat y'$ and $\zeta = \hat\xi' + i\hat\eta'$. With $\tilde a(x',y',\xi',\eta') = a(x'+iy',\xi'+i\eta')$, we have from Taylor formula

$$\begin{aligned}&a(x'+z,\xi'+\zeta)\\ &\quad= \sum_{|\alpha+\beta+\mu+\nu|<\ell}\frac{1}{\alpha!\beta!\mu!\nu!}\partial_{x'}^\beta\partial_{y'}^\nu\partial_{\xi'}^\alpha\partial_{\eta'}^\mu\tilde a(x',0,\xi',0)\hat x'^\beta\hat y'^\nu\hat\xi'^\alpha\hat\eta'^\mu\\ &\qquad+\sum_{|\alpha+\beta+\mu+\nu|=\ell}\frac{\ell}{\alpha!\beta!\mu!\nu!}\int_0^1(1-\theta)^{\ell-1}\partial_{x'}^\beta\partial_{y'}^\nu\partial_{\xi'}^\alpha\partial_{\eta'}^\mu\\ &\qquad\times\tilde a(x'+\theta\hat x',\theta\hat y',\xi'+\theta\hat\xi',\theta\hat\eta')\,d\theta\hat x'^\beta\hat y'^\nu\hat\xi'^\alpha\hat\eta'^\mu.\end{aligned}$$

Since $(\partial_{x_j}+i\partial_{y_j})\tilde a(x',0,\xi',0)$ and $(\partial_{\xi_j}+i\partial_{\eta_j})\tilde a(x',0,\xi',0)$ belong to the class $S_{(s)}(e^{-c\langle\mu\xi'\rangle^{(1/4-\kappa-\epsilon)/(s-1)}}, g)$ by Proposition A.1, one can replace ∂_{y_j}, ∂_{η_j} by $i\partial_{x_j}$, $i\partial_{\xi_j}$ with errors

$$S_{(s)}(e^{-c\langle\mu\xi'\rangle^{(1/4-\kappa-\epsilon)/(s-1)}}, g).$$

This shows that the first term in the right-hand side is

$$\sum_{|\alpha+\beta|<\ell}\frac{1}{\alpha!\beta!}\partial_{x'}^\beta\partial_{\xi'}^\alpha a(x',\xi')z^\beta\zeta^\alpha + S_{(s)}\big(e^{-c\langle\mu\xi'\rangle^{(1/4-\kappa-\epsilon)/(s-1)}}, \hat g \mid E(k)\big)$$

because $\partial_{x'}^\beta\partial_{\xi'}^\alpha\tilde a(x',0,\xi',0) = \partial_{x'}^\beta\partial_{\xi'}^\alpha a(x',\xi')$. From Lemma A.3 it follows that

$$\int_0^1(\cdots)\,d\theta \in S_{(s)}\big(\langle\mu\xi'\rangle^m\langle\xi'\rangle_\mu^{|\beta+\nu|/2-3|\alpha+\mu|/4}, \hat g \mid E(k)\big).$$

On the other hand, since

$$\hat{x'}^{\beta}\hat{y'}^{\nu}\hat{\xi'}^{\alpha}\hat{\eta'}^{\mu} \in S_{(s)}\big(\langle\mu\xi'\rangle^{\kappa\ell}\langle\xi'\rangle_\mu^{-3|\beta+\nu|/4+|\alpha+\mu|/2}, \hat{g} \mid E(k)\big)$$

for $|\alpha+\beta+\mu+\nu| = \ell$, we have

$$\int_0^1 (\cdots)\, d\theta \hat{x'}^{\beta}\hat{y'}^{\nu}\hat{\xi'}^{\alpha}\hat{\eta'}^{\mu} \in \mu^{\ell/4} S_{(s)}\big(\langle\mu\xi'\rangle^{m-(1/4-\kappa-\epsilon)\ell}, \hat{g} \mid E(k)\big),$$

which proves the desired assertion. □

LEMMA 5.3

Let us denote $\Xi' = \Xi'(x, y', \xi')$*. Then we have*

$$\partial_{\xi_j}\tilde{\phi}(x,\Xi')_{y'=x'} = \partial_{\xi_j}\tilde{\phi}(x,\xi') + \mu^{3/2}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa-5/4}, g) + \mu^{3/2}S(\langle\mu\xi'\rangle^{-1}, g),$$

$$\partial_{x_j}\phi(x,\Xi')_{y'=x'} = \partial_{x_j}\phi(x,\xi') + \mu^{1/2}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa-1/4}, g) + \mu S(1, g),$$

$$\partial_{x_1}\phi(x,\Xi')_{y'=x'} = \partial_{x_1}\phi(x,\xi') + \mu^{3/4}S(r^{-3}w\langle\mu\xi'\rangle^{\kappa-1/2}, g) + \mu S(1, g),$$

and

$$\partial_{x_0}\tilde{\phi}(x_0, y', \Xi') = \langle\mu\xi'\rangle^{\kappa'} + \mu S^{\#}\big((r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa} + r^{-1}\langle\mu\xi'\rangle^{-1/2}), g\big).$$

Proof

Recall that $\partial_{\xi_j}\tilde{\phi}(x, \xi' + i\eta')$ is the almost-analytic extension of $\partial_{\xi_j}\tilde{\phi}(x,\xi')$ with $k(\xi') = \langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_\mu^{\epsilon}$. Since $\Xi'(x,y',\xi') = \xi' + G'(x,y',\xi')$ it follows from Lemma 5.2 that

$$\partial_{\xi_j}\tilde{\phi}(x,\Xi') = \partial_{\xi_j}\tilde{\phi}(x,\xi') + \sum_{1\le|\alpha|<\ell} \frac{1}{\alpha!}\partial_{\xi'}^{\alpha}\partial_{\xi_j}\tilde{\phi}(x,\xi')G'(x,y',\xi')^{\alpha}$$
$$+ \mu^{3/2}S\big(\langle\mu\xi'\rangle^{-1}, \hat{g} \mid E^0(k)\times\mathbb{R}^n\big).$$

Since

$$\partial_{\xi'}^{\alpha}\partial_{\xi_j}\tilde{\phi}(x,\xi') \in S(\langle\mu\xi'\rangle^{\kappa'}\langle\xi'\rangle_\mu^{-|\alpha|-1}, g) + S(r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_\mu^{-1-3|\alpha|/4}, g)$$

and $G'(x,x',\xi') \in S((r^{-2}w + r^{-1}w^{1/2})\langle\mu\xi'\rangle^{\kappa}, g)$, we have the desired assertion. To prove the last two assertions of the first group it suffices to note that

$$\partial_{\xi'}^{\alpha}\partial_{x_j}\phi(x,\xi') \in S(\Delta_j\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_\mu^{-3|\alpha|/4}, g).$$

To prove the last assertion we note that

$$\partial_{x_0}\phi(x_0,y',\xi') \in \mu S^{\#}(r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa}, g)$$

and $\langle\mu\Xi'\rangle^{\kappa'} = \langle\mu\xi'\rangle^{\kappa'} + \mu S^{\#}(r^{-1}\langle\mu\xi'\rangle^{-1/2}, g)$, which follows from Lemma 5.2. □

LEMMA 5.4

We have

$$\{\Lambda, \phi\}(x,\xi') \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g),$$
$$\{\phi_1, \phi\}(x,\xi') = 2r^{-2}w\langle\mu\xi'\rangle^{\kappa}\{\phi_1, \psi\} + \mu S(\langle\mu\xi'\rangle^{\kappa}, g),$$

$$\{\phi_2, \phi\}(x, \xi') = f(x, \xi') + \mu S(\langle \mu\xi' \rangle^{\kappa}, g),$$

where $f(x, \xi') \in \mu S(r^{-1} w^{1/2} \langle \mu\xi' \rangle^{\kappa}, g)$.

Proof
We first note that

$$\{F, \phi\} = \langle \mu\xi' \rangle^{\kappa} \Big[2w \frac{\{F, \psi\}}{r^2} - 2\psi \frac{\{F, w\}}{r^2} \Big] + \big\{F, \langle \mu\xi' \rangle^{\kappa}\big\} S_{(s)}(1, g),$$
$$\{F, w\} = 2w^{-1} \phi_1^3 \langle \mu\xi' \rangle^{-4} \{F, \phi_1\}$$
$$+ 2w^{-1} \phi_1^4 \langle \mu\xi' \rangle^{-3} \big\{F, \langle \mu\xi' \rangle^{-1}\big\} + 2^{-1} w^{-1} \big\{F, \langle \xi' \rangle_{\mu}^{-1}\big\}.$$

The first assertion follows from (4.7) immediately. For the second assertion it suffices to note that $\{\phi_1, w\} \in \mu S((w + w^{-1}\langle \xi' \rangle_{\mu}^{-1}), g) \subset \mu S(w, g)$. To show the third assertion we note that $\{\phi_2, \psi\} = \{a, \psi\}\psi \in \mu S(r, g)$ and

$$\{\phi_2, w\} = 2w^{-1} \phi_1^3 \langle \mu\xi' \rangle^{-4} \{\phi_2, \phi_1\} + \mu S(w, g) \in \mu S(w^{1/2}, g). \qquad \square$$

To simplify notation we set

$$\tilde{\Lambda}(x, y', \xi) = \Lambda\big(x_0, x' - i\nabla_{\xi'} \tilde{\phi}(x, \Xi'), \xi_0 + i\partial_{x_0} \tilde{\phi}(x_0, y', \Xi'), \Xi'\big),$$
$$\tilde{\phi}_1(x, y', \xi') = \phi_1\big(x_0, x' - i\nabla_{\xi'} \tilde{\phi}(x, \Xi'), \Xi'\big)$$

with $\Xi' = \Xi'(x, y', \xi')$. We define $\tilde{M}(x, y', \xi)$, $\tilde{\phi}_2(x, y', \xi')$, $\tilde{\theta}(x, y', \xi')$ similarly.

PROPOSITION 5.2
Let $\tilde{\Lambda}(x, y', \xi)$, $\tilde{\phi}_1(x, y', \xi')$, $\tilde{\phi}_2(x, y', \xi')$, *and* $\tilde{\theta}(x, y', \xi')$ *be as above. Then we have*

$$\tilde{\Lambda}(x, y', \xi) = \Lambda(x, \xi) - i\langle \mu\xi' \rangle^{\kappa'} + \mu S^{\#}(r^{-1} w^{1/2} \langle \mu\xi' \rangle^{\kappa}, g)$$
$$+ \mu S^{\#}\big((r^{-1}\langle \mu\xi' \rangle^{-1/2} + \langle \mu\xi' \rangle^{\kappa'}), g\big),$$
$$\tilde{\Lambda}(x, x', \xi) = \Lambda(x, \xi) - i\langle \mu\xi' \rangle^{\kappa'} + \mu S(\langle \mu\xi' \rangle^{\kappa'}, g) + \sqrt{\mu} S(1, g),$$
$$\tilde{\phi}_1(x, y', \xi') = \phi_1(x, \xi') + \mu S^{\#}(r^{-1}\langle \mu\xi' \rangle^{\kappa}, g) + \mu S^{\#}(\langle \mu\xi' \rangle^{\kappa'}, g),$$
$$\tilde{\phi}_1(x, x', \xi') = \phi_1(x, \xi') + i\{\phi_1, \phi\}(x, \xi') + \mu^{5/4} S(r^{-2} w \langle \mu\xi' \rangle^{\kappa}, g)$$
$$+ \mu S(\langle \mu\xi' \rangle^{\kappa'}, g),$$
$$\tilde{\phi}_2(x, y', \xi') = \phi_2(x, \xi') + \mu S^{\#}(r^{-1} w^{1/2} \langle \mu\xi' \rangle^{\kappa}, g) + \mu S^{\#}(\langle \mu\xi' \rangle^{\kappa'}, g),$$
$$\tilde{\phi}_2(x, x', \xi') = \phi_2(x, \xi') + i\{\phi_2, \phi\}(x, \xi') + \mu^{5/4} S(r^{-2} w \langle \mu\xi' \rangle^{\kappa - 1/4}, g)$$
$$+ \mu S(\langle \mu\xi' \rangle^{\kappa'}, g),$$
$$\tilde{\theta}(x, y', \xi') = \theta(x, \xi') + \mu S^{\#}(r^{-1} w^{1/2} \langle \mu\xi' \rangle^{\kappa}, g) + \mu S^{\#}(\langle \mu\xi' \rangle^{\kappa'}, g),$$
$$\tilde{\theta}(x, x', \xi') = \theta(x, \xi') + i f(x, \xi') + \mu^{5/4} S(r^{-1} w^{1/2} \langle \mu\xi' \rangle^{2\kappa - 1/4}, g)$$
$$+ \mu S(\langle \mu\xi' \rangle^{\kappa'}, g),$$

where $f \in \mu S(r^{-1} w^{1/2} \langle \mu\xi' \rangle^{\kappa}, g)$ *is real.*

Proof

Recall that $\Xi' = \xi' + G(x, y', \xi')$. From Lemma 5.2 (or rather from its proof) and (5.2), we have

$$\begin{aligned}\tilde{\Lambda}(x, y', \xi) &= \Lambda\big(x_0, x' - i\nabla_{\xi'}\tilde{\phi}(x, \Xi'), \xi_0 + i\partial_{x_0}\tilde{\phi}(x_0, y', \Xi'), \Xi'\big)\\ &= \Lambda(x, \xi) + i\partial_{x_0}\tilde{\phi}(x_0, y', \Xi')\\ &\quad + \sum_{|\alpha+\beta|=1} \partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x, \xi)\big(-i\nabla_{\xi'}\tilde{\phi}(x, \Xi')\big)^{\beta} G'(x, y', \xi')^{\alpha}\\ &\quad + \sum_{2\le|\alpha+\beta|<\ell} \frac{1}{\alpha!\beta!}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x, \xi)\big(-i\nabla_{\xi'}\tilde{\phi}(x, \Xi')\big)^{\beta} G'(x, y', \xi')^{\alpha}\\ &\quad + \mu^{\ell/4} S^{\#}(\langle\mu\xi'\rangle^{\kappa}, g)\end{aligned} \tag{5.3}$$

taking ℓ large if necessary. Noting (4.7), $\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x, \xi) \in S(\langle\mu\xi'\rangle\langle\xi'\rangle_{\mu}^{-|\alpha|}, g)$, and $\nabla_{\xi'}\phi(x, \Xi') \in S^{\#}((1 + r^{-1}w^{1/2})\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_{\mu}^{-1}, g)$, it follows from (5.2) that

$$\begin{aligned}&i\partial_{x_0}\tilde{\phi}(x_0, y', \Xi') + \sum_{|\alpha+\beta|=1} \partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x, \xi)\big(-i\nabla_{\xi'}\tilde{\phi}(x, \Xi')\big)^{\beta} G'(x, y', \xi')^{\alpha}\\ &\quad \in \mu S^{\#}\big((r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa} + \langle\mu\xi'\rangle^{\kappa'}), g\big).\end{aligned}$$

Thanks to (4.7) similar arguments show

$$\sum_{2\le|\alpha+\beta|<\ell} \cdots \in \mu^{5/4} S^{\#}(\langle\mu\xi'\rangle^{\kappa}, g).$$

Thus we have the assertion about $\tilde{\Lambda}(x, y', \xi)$. We turn to the assertion for $\tilde{\phi}_1(x, y', \xi')$. The same arguments as above show that

$$\begin{aligned}&\sum_{|\alpha+\beta|=1} \phi_{1(\beta)}^{(\alpha)}(x, \xi')\big(-i\nabla_{\xi'}\tilde{\phi}(x, \Xi')\big)^{\beta} G'(x, y', \xi')^{\alpha}\\ &\quad \in \mu S^{\#}\big((r^{-1}\langle\mu\xi'\rangle^{\kappa} + \langle\mu\xi'\rangle^{\kappa'}), g\big).\end{aligned}$$

We turn to the term $|\alpha + \beta| \ge 2$. It is easy to see that

$$\begin{aligned}&\sum_{2\le|\alpha+\beta|<\ell} \frac{1}{\alpha!\beta!}\phi_{1(\beta)}^{(\alpha)}(x, \xi')\big(-i\nabla_{\xi'}\tilde{\phi}(x, \Xi')\big)^{\beta} G'(x, y', \xi')^{\alpha}\\ &\quad \in \mu S^{\#}\big((\langle\mu\xi'\rangle^{\kappa} + r^{-1}\langle\mu\xi'\rangle^{\kappa-1/4}), g\big)\end{aligned}$$

and hence the result. We show the assertion for $\tilde{\phi}_2(x, y', \xi')$. Let $|\alpha + \beta| = 1$. Noting that $\phi_2^{(\alpha)}(x, \xi') \in S(r\langle\mu\xi'\rangle\langle\xi'\rangle_{\mu}^{-|\alpha|}, g)$, the same arguments as above show that

$$\begin{aligned}&\sum_{|\alpha+\beta|=1} \phi_{2(\beta)}^{(\alpha)}(x, \xi')\big(-i\nabla_{\xi'}\tilde{\phi}(x, \Xi')\big)^{\beta} G'(x, y', \xi')^{\alpha}\\ &\quad \in \mu S^{\#}\big((r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa} + \langle\mu\xi'\rangle^{\kappa'}), g\big).\end{aligned}$$

We check the term $\sum_{|\alpha+\beta|\geq 2}\cdots$. It is easy to see that

$$\sum_{2\leq|\alpha+\beta|<\ell}\frac{1}{\alpha!\beta!}\phi_{2(\beta)}^{(\alpha)}(x,\xi')\big(-i\nabla_{\xi'}\tilde{\phi}(x,\Xi')\big)^{\beta}G'(x,y',\xi')^{\alpha}\in\mu S^{\#}(\langle\mu\xi'\rangle^{\kappa},g)$$

and hence the result. To check the assertion about $\tilde{\theta}(x,y',\xi')$ it suffices to note $\theta(x,\xi')\in S(\langle\mu\xi'\rangle,g_0)\cap S(w\langle\mu\xi'\rangle,g)$ and Lemma 5.2.

We prove the assertion for $\tilde{\Lambda}(x,x',\xi)$. Since $\Xi'(x,x',\xi')=\xi'+i\nabla_{x'}\phi(x,\Xi')$, we see with $\Xi'=\Xi'(x,x',\xi')$,

$$\begin{aligned}&\tilde{\Lambda}(x,x',\xi)\\&\quad=\Lambda\big(x_0,x'-i\nabla_{\xi'}\tilde{\phi}(x,\Xi'),\xi_0+i\partial_{x_0}\tilde{\phi}(x,\Xi'),\xi'+i\nabla_{x'}\phi(x,\Xi')\big)\\&\quad=\Lambda(x,\xi)+i\partial_{x_0}\tilde{\phi}(x,\Xi')+\sum_{1\leq|\alpha+\beta|<\ell}\frac{1}{\alpha!\beta!}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x,\xi)(-i\nabla_{\xi'}\tilde{\phi})^{\beta}(i\nabla_{x'}\phi)^{\alpha}\\&\qquad+\mu^{5/4}S(\langle\mu\xi'\rangle^{\kappa},g)\end{aligned}$$

by Lemma 5.2 taking ℓ large. From Lemma 5.3 and (4.7) it follows that

$$\sum_{2\leq|\alpha+\beta|<\ell}\frac{1}{\alpha!\beta!}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x,\xi)(-i\nabla_{\xi'}\tilde{\phi})^{\beta}(i\nabla_{x'}\phi)^{\alpha}\in\mu^{5/4}S(\langle\mu\xi'\rangle^{\kappa'},g).$$

Since $\xi'=\Xi'-i\nabla_{x'}\phi(x,\Xi')$, we see that

$$\begin{aligned}&i\partial_{x_0}\tilde{\phi}+\sum_{|\alpha+\beta|=1}\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\Lambda(x,\xi)(-i\nabla_{\xi'}\tilde{\phi})^{\beta}(i\nabla_{x'}\phi)^{\alpha}\\&\quad=-i\langle\mu\Xi'\rangle^{\kappa'}+\sum_{|\alpha+\beta|=1}\partial_{x'}^{\beta}\partial_{\xi}^{\alpha}\Lambda(x,\xi)(-i\nabla_{\xi'}\phi)^{\beta}(i\nabla_{x}\phi)^{\alpha}+\mu S(\langle\mu\xi'\rangle^{\kappa'},g)\\&\quad=-i\langle\mu\Xi'\rangle^{\kappa'}+i\{\Lambda,\phi\}(x,\Xi')\\&\qquad+\sum_{1\leq|\gamma|<\ell,|\alpha+\beta|=1}\frac{1}{\gamma!}\partial_{\xi'}^{\gamma}\partial_{x'}^{\beta}\partial_{\xi}^{\alpha}\Lambda(x,\xi_0,\Xi')(-i\nabla_{x'}\phi)^{\gamma}(-i\nabla_{\xi'}\phi)^{\beta}(i\nabla_{x}\phi)^{\alpha}\\&\qquad+\mu S(\langle\mu\xi'\rangle^{\kappa'},g)\end{aligned}$$

by Lemma 5.2 again. From Lemma 5.3 and (4.7) it is easy to check that the third term in the right-hand side is in $\mu^{5/4}S(\langle\mu\xi'\rangle^{\kappa},g)$. We now consider $\{\Lambda,\phi\}(x,\Xi')$. Note that

$$\begin{aligned}\{\Lambda,\phi\}(x,\Xi')=\{\Lambda,\phi\}(x,\xi')+\sum_{1\leq|\gamma|<\ell}\frac{1}{\gamma!}(\partial_{\xi'}^{\gamma}\{\Lambda,\phi\})(x,\xi')(i\nabla_{x'}\phi)^{\gamma}\\+\mu^{5/4}S(\langle\mu\xi'\rangle^{\kappa},g).\end{aligned}$$

Thanks to $\{\Lambda,\phi\}(x,\xi')\in\mu S(\langle\mu\xi'\rangle^{\kappa},g)$ by Lemma 5.4, one sees easily that the second term in the right-hand side is in $\mu^{5/4}S(\langle\mu\xi'\rangle^{\kappa},g)$, and hence we have $\{\Lambda,\phi\}(x,\Xi')\in\mu S(\langle\mu\xi'\rangle^{\kappa},g)$. These prove the assertion.

Noting the fact

$$(5.4)\quad \begin{cases} \nabla_{\xi'}\phi(x,\Xi'(x,x',\xi'))=\nabla_{\xi'}\phi(x,\xi')+\mu^{3/2}S(r^{-2}w\langle\mu\xi'\rangle^{2\kappa-3/2},g),\\ G_1(x,x',\xi')=i\partial_{x_1}\phi(x,\xi')+\mu^{3/4}S(r^{-3}w\langle\mu\xi'\rangle^{2\kappa-3/4},g),\\ G_j(x,x',\xi')=i\partial_{x_j}\phi(x,\xi')+\mu^{1/2}S(r^{-2}w\langle\mu\xi'\rangle^{2\kappa-1/2},g),\end{cases}$$

which follows from Lemma 5.2 or rather its proof because we have $G'(x,x',\xi')=i\nabla_{x'}\phi(x_0,x',\xi'+G'(x,x',\xi'))$, the assertions on $\tilde\phi_1(x,x',\xi')$, $\tilde\phi_2(x,x',\xi')$ and $\tilde\theta_1(x,x',\xi')$ are checked by similar easier arguments. □

Let us denote

$$j(x,\xi')=\sum_{|\alpha|<5}\frac{1}{\alpha!}D_{y'}^{\alpha}\partial_{\xi'}^{\alpha}J(x,y',\xi')_{y'=x'}.$$

Let $a_i(x',\xi')\in S(\langle\mu\xi'\rangle^{m_i},g)$; then there exists $b\in S(\langle\mu\xi'\rangle^{m_1+m_2},g)$ such that $\mathrm{Op}^t(a_1)\,\mathrm{Op}^t(a_2)=\mathrm{Op}^t(b)$. We denote b as

$$b=a_1\# a_2.$$

PROPOSITION 5.3

We have

$$\sum_{|\alpha|<5}\frac{1}{\alpha!}D_{y'}^{\alpha}\partial_{\xi'}^{\alpha}[\tilde M(x,y',\xi)\tilde\Lambda(x,y',\xi)J(x,y',\xi')]_{y'=x'}$$
$$=[\tilde M(x,x',\xi)\tilde\Lambda(x,x',\xi)]\# j+C_1(x,\xi')\tilde\Lambda(x,x',\xi)+\mu S(\langle\mu\xi'\rangle,g)$$

with $C_1(x,\xi')\in\mu S(\langle\mu\xi'\rangle^{\kappa},g)$,

$$\sum_{|\alpha|<5}\frac{1}{\alpha!}D_{y'}^{\alpha}\partial_{\xi'}^{\alpha}[\tilde\phi_1(x,y',\xi')\tilde\Lambda(x,y',\xi)J(x,y',\xi')]_{y'=x'}$$
$$=[\tilde\phi_1(x,x',\xi')\tilde\Lambda(x,x',\xi)]\# j+C_2(x,\xi')\tilde\Lambda(x,x',\xi)+\mu S(\langle\mu\xi'\rangle,g)$$

with $C_2\in\mu^{5/4}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa-1/4},g)+\mu S(\langle\mu\xi'\rangle^{\kappa'},g)$, *and*

$$\sum_{|\alpha|<5}\frac{1}{\alpha!}D_{y'}^{\alpha}\partial_{\xi'}^{\alpha}[\tilde\phi_2(x,y',\xi')^2J(x,y',\xi')]_{y'=x'}=\tilde\phi_2(x,x',\xi')^2\# j(x,\xi')$$
$$+\mu S(\langle\mu\xi'\rangle,g),$$

$$\sum_{|\alpha|<5}\frac{1}{\alpha!}D_{y'}^{\alpha}\partial_{\xi'}^{\alpha}[\tilde\theta(x,y',\xi')^2J(x,y',\xi')]_{y'=x'}=\tilde\theta(x,x',\xi')^2\# j(x,\xi')$$
$$+\mu S(\langle\mu\xi'\rangle,g).$$

In what follows, in this section, to simplify notation we denote by C_1, C_2, R symbols belonging to (or the class itself)

$$\mu^{3/4}S(\langle\mu\xi'\rangle^{\kappa'},g),\qquad \mu^{5/4}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa-1/4},g),\qquad \mu S(\langle\mu\xi'\rangle,g),$$

respectively. To show the proposition we first prove the following.

LEMMA 5.5

We have

$$\begin{aligned}
&\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}\big(\tilde M(x,y',\xi)\tilde\Lambda(x,y',\xi)J(x,y',\xi')\big)_{y'=x'}\\
&\quad=\tilde M(x,x',\xi)\tilde\Lambda(x,x',\xi)\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}J(x,y',\xi')_{y'=x'}+C_1(x,\xi')\tilde\Lambda(x,x',\xi)+R,\\
&\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}\big(\tilde\phi_1(x,y',\xi')\tilde\Lambda(x,y',\xi)J(x,y',\xi')\big)_{y'=x'}\\
&\quad=\tilde\phi_1(x,x',\xi')\tilde\Lambda(x,x',\xi)\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}J(x,y',\xi')_{y'=x'}\\
&\qquad+C_1(x,\xi')\tilde\Lambda(x,x',\xi')+C_2(x,\xi')\tilde\Lambda(x,x',\xi)+R,\\
&\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}\big(\tilde\phi_2(x,y',\xi')^2J(x,y',\xi')\big)_{y'=x'}\\
&\quad=\tilde\phi_2(x,x',\xi')^2\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}J(x,y',\xi')_{y'=x'}+R,\\
&\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}\big(\tilde\theta(x,y',\xi')^2J(x,y',\xi')\big)_{y'=x'}\\
&\quad=\tilde\theta(x,x',\xi')^2\partial_{y'}^{\alpha}\partial_{\xi'}^{\alpha}J(x,y',\xi')_{y'=x'}+R.
\end{aligned}$$

Proof

From Proposition 5.2 and Lemma 5.4 one can write

$$\tilde\Lambda(x,y',\xi)=\Lambda(x,\xi)-i\langle\mu\xi'\rangle^{\kappa'}+\lambda(x,y',\xi'),$$
$$\tilde M(x,y',\xi)=M(x,\xi)-i\langle\mu\xi'\rangle^{\kappa'}+m(x,y',\xi')$$

with

$$\lambda(x,y',\xi'),\ m(x,y',\xi')\in\mu S^{\#}\big((r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa}+r^{-1}\langle\mu\xi'\rangle^{-1/2}+\langle\mu\xi'\rangle^{\kappa'}),g\big).$$

Recall that $J(x,y',\xi')\in S^{\#}(1,g)$ and, moreover,

$$\begin{aligned}
\partial_{\xi'}^{\beta}\partial_{y'}^{\alpha}J(x,y',\xi')_{y'=x'}&\in S\big((r^{-2}w+r^{-1}w^{1/2})\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_{\mu}^{-3/4}\\
&\quad\times(r^{-1}+\langle\xi'\rangle_{\mu}^{1/4})^{\alpha_1}\langle\xi'\rangle_{\mu}^{-3|\beta|/4+(|\alpha|-\alpha_1)/4},g\big)\\
&\subset S\big((r^{-2}w+r^{-1}w^{1/2})\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_{\mu}^{-3(1+|\beta|)/4+(|\alpha|+\alpha_1)/4},g\big)\\
&\subset S\big((r^{-1}+\langle\xi'\rangle_{\mu}^{1/4})\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_{\mu}^{|\alpha|/2-3(1+|\beta|)/4},g\big)\\
&\subset\mu^{1/4}S(\langle\mu\xi'\rangle^{\kappa-1/4}\langle\xi'\rangle_{\mu}^{|\alpha|/2-3|\beta|/4},g)
\end{aligned}\tag{5.5}$$

with $\alpha=(\alpha_1,\alpha')$ for $|\beta+\alpha|\ge 1$. Let us set $\bar\Lambda=\Lambda(x,\xi)-i\langle\mu\xi'\rangle^{\kappa'}$ and $\bar M=M(x,\xi)-i\langle\mu\xi'\rangle^{\kappa'}$ and consider

$$\tilde M\tilde\Lambda J=\bar M\bar\Lambda J+m\bar\Lambda J+\lambda\bar MJ+m\lambda J.$$

It is clear that $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(m\lambda J)_{y'=x'}\in R$. Here we note that $\bar M(x,\xi)=\bar\Lambda(x,\xi)+2k\phi_1(x,\xi')^3\langle\mu\xi'\rangle^{-2}$ and hence

$$\lambda\bar MJ=\lambda\bar\Lambda J+\phi_1^3\langle\mu\xi'\rangle^{-2}\lambda J.$$

Noting that $\phi_1^3\langle\mu\xi'\rangle^{-2}\in S(w^{3/2}\langle\mu\xi'\rangle,g)$, we see easily that

$$\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\phi_1^3\langle\mu\xi'\rangle^{-2}\lambda J)_{y'=x'}\in R,\quad|\alpha|\ge 1.$$

We now study $m\bar{\Lambda}J$ (or $\lambda\bar{\Lambda}J$). Since

$$\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}(mJ)_{y'=x'}\in\mu S\big((r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa}+r^{-1}\langle\mu\xi'\rangle^{-1/2}+\langle\mu\xi'\rangle^{\kappa'})\langle\xi'\rangle_{\mu}^{|\alpha|/2-3|\alpha''|/4},g\big),$$

hence $\partial_{\xi'}^{\alpha'}\bar{\Lambda}\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}(mJ)_{y'=x'}\in R$ if $|\alpha'|\geq 1$ where $\alpha'+\alpha''=\alpha$. This shows

$$\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\bar{\Lambda}mJ)_{y'=x'}=C_1\bar{\Lambda}+R,\quad |\alpha|\geq 1.$$

Thus we have

$$\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\tilde{M}\tilde{\Lambda}J)_{y'=x'}=\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\bar{M}\bar{\Lambda}J)_{y'=x'}+C_1\tilde{\Lambda}(x,x',\xi)+R$$

because $(C_1\lambda)_{y'=x'}\in R$. Finally, we consider $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\bar{M}\bar{\Lambda}J)$. Let $|\alpha'|\geq 1$. Then from (5.5) one sees that

$$\partial_{\xi'}^{\alpha'}(\bar{M}\bar{\Lambda})(\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}J)_{y'=x'}=C_1\bar{\Lambda}+R$$

because $\bar{M}=\bar{\Lambda}+S(w^{3/2}\langle\mu\xi'\rangle,w^{-1}g_0)$ and

$$\partial_{\xi_1}\bar{\Lambda},\quad \partial_{\xi_1}\bar{M}=\mu S(r,g)+S(\langle\mu\xi'\rangle^{\kappa'}\langle\xi'\rangle_{\mu}^{-1},g) \tag{5.6}$$

thanks to (4.7). Thus we conclude that

$$(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}\bar{M}\bar{\Lambda}J)_{y'=x'}=\bar{M}\bar{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'}+C_1\bar{\Lambda}+R.$$

Noting that

$$\bar{M}\bar{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'}=\tilde{M}\tilde{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'}+C_1\tilde{\Lambda}(x,x',\xi)+R,$$

we have the desired assertion.

We next consider $\tilde{\phi}_1\tilde{\Lambda}J$. From Proposition 5.2 we can write

$$\tilde{\phi}_1(x,y',\xi')=\tilde{\phi}_1(x,\xi')+\nu_1(x,y',\xi')$$

with $\nu_1(x,y',\xi)\in\mu S^{\#}((r^{-1}\langle\mu\xi'\rangle^{\kappa}+\langle\mu\xi'\rangle^{\kappa'}),g)$. Let $\tilde{\phi}_1\tilde{\Lambda}J=\phi_1\bar{\Lambda}J+\bar{\Lambda}\nu_1J+\phi_1\lambda J+\nu_1\lambda J$. It is easy to check that $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\nu_1\lambda J)_{y'=x'}\in R$ for $|\alpha|\geq 1$. Noting that

$$\phi_1(x,\xi')\in S(w^{1/2}\langle\mu\xi'\rangle,g)\cap S(\langle\mu\xi'\rangle,g_0), \tag{5.7}$$

we have $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\phi_1\lambda J)_{y'=x'}\in R$, $|\alpha|\geq 1$. Let $|\alpha'|\geq 1$; then from (5.6) we have

$$\partial_{\xi'}^{\alpha'}\bar{\Lambda}\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}(\nu_1J)_{y'=x'}\in R,\quad |\alpha'|\geq 1.$$

Since $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\nu_1J)_{y'=x'}\in\mu^{7/4}S(r^{-2}\langle\mu\xi'\rangle^{\kappa-3/4},g)+\mu S(\langle\mu\xi'\rangle^{\kappa'},g)$ and noting that $S(r^{-2}\langle\mu\xi'\rangle^{\kappa-3/4},g)\subset\mu^{-1/2}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa-1/4},g)$, we conclude that

$$\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\bar{\Lambda}\nu_1J)_{y'=x'}=C_1(x,\xi')\bar{\Lambda}+C_2(x,\xi')\bar{\Lambda}+R,\quad |\alpha|\geq 1.$$

Thus we can write

$$[\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\tilde{\phi}_1\tilde{\Lambda}J)]_{y'=x'}=[\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\phi_1\bar{\Lambda}J)]_{y'=x'}+C_1\bar{\Lambda}+C_2\bar{\Lambda}+R.$$

We check $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\phi_1\bar{\Lambda}J)$. Let $|\alpha'|\geq 1$ and $|\alpha|\geq 2$. Then from (5.7) and (5.5) it follows that

$$\partial_{\xi'}^{\alpha'}(\phi_1\bar{\Lambda})(\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}J)_{y'=x'}=C_1\bar{\Lambda}+R.$$

Let $|\alpha'| = |\alpha| = 1$, and consider $\bar{\Lambda}\partial_{\xi'}^{\alpha}\phi_1(\partial_{y'}^{\alpha}J)_{y'=x'}$ and $\phi_1(\partial_{\xi'}^{\alpha}\bar{\Lambda})(\partial_{y'}^{\alpha}J)_{y'=x'}$. From (5.6) and (5.7), taking (5.5) into account again we see that

$$\phi_1(\partial_{\xi'}^{\alpha}\bar{\Lambda})(\partial_{y'}^{\alpha}J)_{y'=x'} \in R, \quad \partial_{\xi'}^{\alpha}\phi_1(\partial_{y'}^{\alpha}J)_{y'=x'} \in C_1 + C_2.$$

Hence we conclude that

$$\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\phi_1\bar{\Lambda}J)_{y'=x'} = \phi_1\bar{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} + C_1(x,\xi')\bar{\Lambda} + C_2(x,\xi')\bar{\Lambda} + R.$$

Noting that $(C_2\lambda)_{y'=x'} \in R$ and

$$\begin{aligned}\phi_1\bar{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} &= \phi_1\tilde{\Lambda}(x,x',\xi)(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} + R\\ &= [\tilde{\phi}_1\tilde{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)]_{y'=x'} - [\nu_1\tilde{\Lambda}(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)]_{y'=x'} + R\end{aligned}$$

we get the second assertion for $(\nu_1\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} \in C_1 + C_2$.

We turn to considering $\tilde{\phi}_2^2 J$ and $\tilde{\theta}^2 J$. From Proposition 5.2 one can write

$$\begin{aligned}\tilde{\phi}_2(x,y',\xi') &= \phi_2(x,\xi') + \nu_2(x,y',\xi'),\\ \tilde{\theta}(x,y',\xi') &= \theta(x,\xi') + \nu_3(x,y',\xi')\end{aligned}$$

with $\nu_i(x,y',\xi') \in \mu S^{\#}((r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa} + \langle\mu\xi'\rangle^{\kappa'}), g)$. Since $\phi_2(x,\xi') \in S(r\langle\mu\xi'\rangle, g)$, writing $\tilde{\phi}(x,y',\xi')^2 = \phi_2(x,\xi')^2 + r(x,y',\xi')$ it is clear that $\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(rJ)_{y'=x'} \in R$ for $|\alpha| \geq 1$. Recalling (5.5) and $\partial_{\xi'}^{\alpha'}\phi_2^2 \in S(r^2\langle\mu\xi\rangle^2\langle\xi'\rangle_{\mu}^{-|\alpha'|}, g)$, we have

$$\partial_{\xi'}^{\alpha'}\phi_2^2(\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}J)_{y'=x'} \in R$$

if $|\alpha'| \geq 1$. Thus we get

$$\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\tilde{\phi}_2^2 J)_{y'=x'} = \phi_2^2(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} + R.$$

Since $(r\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} \in R$, $|\alpha| \geq 1$, we have the third assertion. Since

$$\theta(x,\xi') \in S(w\langle\mu\xi'\rangle, g) \cap S(\langle\mu\xi'\rangle, g_0), \tag{5.8}$$

then writing $\tilde{\theta}(x,y',\xi')^2 = \theta(x,\xi')^2 + r(x,y',\xi')$, it is easy to see

$$(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}rJ)_{y'=x'},\ (r\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} \in R, \quad |\alpha| \geq 1.$$

Since $\partial_{\xi'}^{\alpha'}\theta(x,\xi')^2 \in S(w^{(4-|\alpha'|)/2}\langle\mu\xi'\rangle^2\langle\xi'\rangle_{\mu}^{-|\alpha'|}, g)$ for $|\alpha'| \leq 4$, it follows from (5.5) that $\partial_{\xi'}^{\alpha'}\theta^2(\partial_{\xi'}^{\alpha''}\partial_{y'}^{\alpha}J)_{y'=x'} \in R$ if $|\alpha'| \geq 1$, and hence we have, for $|\alpha| \geq 1$,

$$[\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}(\tilde{\theta}^2 J)]_{y'=x'} = \theta(x,\xi')^2(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} + R.$$

Since $[\tilde{\theta}^2(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)]_{y'=x'} = \theta(x,\xi')^2(\partial_{\xi'}^{\alpha}\partial_{y'}^{\alpha}J)_{y'=x'} + R$, we get the fourth assertion. □

Proof of Proposition 5.3

We note that

$$j(x,\xi') = 1 + S\big((r^{-2}w + r^{-1}w^{1/2})\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_{\mu}^{-3/4}, g\big),$$

and hence

$$\partial_{\xi'}^{\beta}\partial_{x'}^{\alpha}j \in S\big((r^{-1} + \langle\xi'\rangle_{\mu}^{1/4})^{1+\alpha_1}\langle\mu\xi'\rangle^{\kappa}\langle\xi'\rangle_{\mu}^{-3/4+(|\alpha|-\alpha_1)/4-3|\beta|/4}, g\big), \quad |\alpha+\beta| \geq 1.$$

Then using similar easier arguments as in the proof of Lemma 5.5, we can show that with $\tilde\Lambda=\tilde\Lambda(x,x',\xi)$, $\tilde M=\tilde M(x,x',\xi)$,

$$\tilde\Lambda\tilde M\sum_{|\alpha|<5}\frac{1}{\alpha!}(D_{y'}^{\alpha}\partial_{\xi'}^{\alpha}J)_{y'=x'}=\tilde\Lambda\tilde M\#j+C_1\tilde\Lambda+R.$$

The proof for the other cases is similar. □

Since we can write $\mathrm{Op}^0(\mu^{5/4}S(\langle\mu\xi'\rangle^{-1},\bar g)\xi_0^2)=\tilde\Lambda\tilde M\#j'+C_1\tilde\Lambda+R$ with $j'\in\mu^{5/4}S(\langle\mu\xi'\rangle^{-1},\bar g)$, combining Propositions 5.1 and 5.3 we get

$$\begin{aligned}&\mathrm{Op}^0(e^{\tilde\phi})\,\mathrm{Op}^0(p)\,\mathrm{Op}^1(e^{-\tilde\phi})\\&\quad=\mathrm{Op}^0\big(\tilde p(x,\xi)\#\tilde j+C_1\tilde\Lambda(x,x',\xi)+C_2\tilde\Lambda(x,x',\xi)+\bar R\big),\end{aligned}$$

where $\tilde j=j+j'$, and hence $\mathrm{Op}^0(e^{\tilde\phi})\,\mathrm{Op}^1(e^{-\tilde\phi})=\mathrm{Op}^0(\tilde j)$ and

$$\begin{aligned}\tilde p&=-\tilde M(x,x',\xi)\tilde\Lambda(x,x',\xi)+2\tilde\phi_1(x,x',\xi')\tilde\Lambda(x,x',\xi)+\tilde Q(x,x',\xi'),\\\tilde Q(x,x',\xi')&=\tilde\phi_2(x,x',\xi')^2+\tilde\theta(x,x',\xi')^2,\\C_1(x,\xi')&\in\mu S(\langle\mu\xi'\rangle^{\kappa'},g)+\mu S(1,\bar g),\\C_2(x,\xi')&\in\mu^{5/4}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa-1/4},g),\end{aligned}$$

where $\bar R$ denotes the symbol class

$$\mu S(r\langle\mu\xi'\rangle^{\kappa'+1},g)+\mu S(\langle\mu\xi'\rangle,\bar g)$$

or, rather, a symbol itself belonging to $\bar R$. Thanks to Lemma 5.4 and Proposition 5.2, one can write

$$\tilde\phi_2=\phi_2(x,\xi')+ih(x,\xi')+\mu S\big((r^{-1}\langle\mu\xi'\rangle^{\kappa-1/4}+\langle\mu\xi'\rangle^{\kappa'}),g\big)$$

with $h\in\mu S(r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa},g)$ which is real; then we see that

$$\tag{5.9}\begin{cases}\mathsf{Re}\,\tilde\phi_2(x,x',\xi')^2=\phi_2(x,\xi')^2+\bar R,\\\mathsf{Im}\,\tilde\phi_2(x,x',\xi')^2=\mu S(w^{1/2}\langle\mu\xi'\rangle^{1+\kappa},g)+\bar R.\end{cases}$$

From Proposition 5.2 we can write

$$\tilde\theta(x,x',\xi')=\theta(x,\xi')+if+\mu^{5/4}S(r^{-1}w^{1/2}\langle\mu\xi'\rangle^{2\kappa-1/4},g)+\mu S(\langle\mu\xi'\rangle^{\kappa'},g)$$

with $f\in\mu S(r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa},g)$ which is real. Noting

$$S(w^{1/2}\langle\mu\xi'\rangle^{2\kappa+3/4},g)\subset\mu^{-3/4}S(w^2\langle\mu\xi'\rangle^2,g)$$

and (5.8), we obtain (because $w\le r$)

$$\begin{aligned}\mathsf{Re}\,\tilde\theta(x,x',\xi')^2&=\theta(x,\xi')^2+\mu^{1/2}S(w^2\langle\mu\xi'\rangle^2,g)+\bar R,\\\mathsf{Im}\,\tilde\theta(x,x',\xi')^2&=\mu S(w^{1/2}\langle\mu\xi'\rangle^{\kappa+1},g)+\bar R.\end{aligned}$$

Since $w^2\langle\mu\xi'\rangle^2=\langle\mu\xi'\rangle^{-2}\phi_1^4+\mu\langle\mu\xi'\rangle$, with $\alpha(x,\xi')\in S(1,g)$ such that $C^{-2}\le\alpha\le C$ we can write

$$\tag{5.10}\begin{cases}\mathsf{Re}\,\tilde\theta^2=\alpha(x,\xi')\theta(x,\xi')^2+\bar R,\\\mathsf{Im}\,\tilde\theta^2=\mu S(w^{1/2}\langle\mu\xi'\rangle^{\kappa+1},g)+\bar R.\end{cases}$$

From (5.9) and (5.10) we have

$$\mathrm{Re}\,\tilde{Q}(x,x',\xi') = \phi_2^2 + \alpha\theta^2 + \bar{R},$$
$$\mathrm{Im}\,\tilde{Q}(x,x',\xi') = \mu S(w^{1/2}\langle\mu\xi'\rangle^{1+\kappa}, g) + \bar{R}.$$

Let us write $\tilde{j} = 1 + \mu^{1/4}S(1,\bar{g})$. For small μ there exists an inverse of $\mathrm{Op}^0(\tilde{j})$ in L^2 which is actually given by $\mathrm{Op}^0(\tilde{j}^{-1})$ with a $\tilde{j}^{-1} \in S(1,\bar{g})$ (see [1]) so that

$$\mathrm{Op}^0(\tilde{j})\,\mathrm{Op}^0(\tilde{j}^{-1}) = I.$$

Let $C_i(x,\xi')$ be as above. Then it is easy to check that

$$(C_1\tilde{\Lambda})\#\tilde{j}^{-1} = C_1(x,\xi')\tilde{\Lambda}(x,x',\xi) + \tilde{C}_1\tilde{\Lambda} + \mu S(\langle\mu\xi'\rangle, \bar{g}),$$
$$(C_2\tilde{\Lambda})\#\tilde{j}^{-1} = C_2(x,\xi')\tilde{\Lambda}(x,x',\xi') + \tilde{C}_1\tilde{\Lambda} + \mu S(\langle\mu\xi'\rangle, \bar{g}),$$

where $\tilde{C}_1(x,\xi') \in \mu S(\langle\mu\xi'\rangle^{\kappa'}, g)$. With a constant $K > 0$, let us put

$$\omega = 2r^{-2}w\langle\mu\xi'\rangle^{\kappa}\{\phi_1,\psi\} + \mu K\langle\mu\xi'\rangle^{\kappa}. \tag{5.11}$$

Since $\{\phi_1,\psi\} \geq c_1\mu$ if r is small, then taking K large it is clear that

$$\omega \geq c\mu r^{-2}w\langle\mu\xi'\rangle^{\kappa}$$

with some $c > 0$. Finally, we can conclude the following.

PROPOSITION 5.4

We have

$$\mathrm{Op}^0(e^{\tilde{\phi}})\,\mathrm{Op}^0(p)\,\mathrm{Op}^1(e^{-\tilde{\phi}})\,\mathrm{Op}^0(\tilde{j}^{-1}) = \mathrm{Op}^0\left(\hat{p}\right)$$

with

$$\hat{p} = -(M - i\langle\mu\xi'\rangle^{\kappa'} + \tilde{m})(\Lambda - i\langle\mu\xi'\rangle^{\kappa'} + \tilde{\lambda})$$
$$+ 2(\phi_1 + i\omega + C)(\Lambda - i\langle\mu\xi'\rangle^{\kappa'} + \tilde{\lambda})$$
$$+ Q + \mu S(r\langle\mu\xi'\rangle^{1+\kappa'}, g) + \mu S(\langle\mu\xi'\rangle, \bar{g}),$$

where $\tilde{m}(x,\xi')$, $\tilde{\lambda}(x,\xi') \in \mu S(\langle\mu\xi'\rangle^{\kappa'}, g) + \sqrt{\mu}S(1,g)$, $\omega \in \mu S(r^{-2}w\langle\mu\xi'\rangle^{\kappa}, g)$ *which is real,* $C(x,\xi') \in \mu^{5/4}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa}, g)$, *and*

$$Q = \phi_2^2 + \alpha\theta^2 + iQ_1, \quad Q_1 \in \mu S(w^{1/2}\langle\mu\xi'\rangle^{1+\kappa}, g), Q_1 \text{ is real,}$$
$$\Lambda = \xi_0 + \lambda_1 - k\langle\mu\xi'\rangle^{-2}\phi_1^3, \qquad M = \xi_0 + \lambda_1 + k\langle\mu\xi'\rangle^{-2}\phi_1^3.$$

Moreover, we have $\omega \geq c\mu r^{-2}w\langle\mu\xi'\rangle^{\kappa}$ *with some* $c > 0$ *and*

$$\{\Lambda,\phi_2\} \in \mu S(r\langle\mu\xi'\rangle, g), \qquad \{\Lambda,\phi_1\} \in \mu S\big((r + w^{1/2})\langle\mu\xi'\rangle, g\big). \tag{5.12}$$

Proof

From Proposition 5.2 we have

$$\tilde{M}(x,x',\xi) = M(x,\xi) - i\langle\mu\xi'\rangle^{\kappa'} + \tilde{m},$$
$$\tilde{\Lambda}(x,x',\xi) = \Lambda(x,\xi) - i\langle\mu\xi'\rangle^{\kappa'} + \tilde{\lambda}$$

where $\tilde{\lambda}, \tilde{m} \in \mu S(\langle\mu\xi'\rangle^{\kappa'}, g) + \sqrt{\mu} S(1, g)$. From Proposition 5.2 and (5.11) we can write

$$\tilde{\phi}_1(x, x', \xi')\tilde{\Lambda}(x, x', \xi) = \big(\phi_1 + i\omega + C + \mu S(\langle\mu\xi'\rangle^{\kappa'}, g)\big)\tilde{\Lambda}(x, x', \xi)$$

with $C \in \mu^{5/4}(r^{-2}w\langle\mu\xi'\rangle^{\kappa}, g)$, where we move the term $\mu S(\langle\mu\xi'\rangle^{\kappa'}, g)$ into $\tilde{m}$, which completes the proof. □

6. Energy estimate (proof of Theorem 1.1)

Let P be the operator with symbol given in Section 4, that is, the symbol which is equal, in a conic neighborhood of the reference point, to that given by the right-hand side of (2.2).

In this section we derive a priori estimates for the transformed operator $\tilde{P}$,

$$\tilde{P} = \mathrm{Op}^0(e^{\tilde{\phi}}) P \,\mathrm{Op}^1(e^{-\tilde{\phi}})\,\mathrm{Op}^0(\tilde{j}^{-1}),$$

where the principal symbol of $\tilde{P}$ is given in Proposition 5.4. Let us denote $\mathrm{Op}^{1/2}(a) = a^w$, the Weyl quantization of a. By the same M, Λ, we denote $M = D_0 - i\langle\mu D'\rangle^{\kappa'} - m$, $\Lambda = D_0 - i\langle\mu D'\rangle^{\kappa'} - \lambda$, where $\lambda = -(\lambda_1 - k\langle\mu\xi'\rangle^{-2}\phi_1^3 + \tilde{\lambda})^w$, $m = -(\lambda_1 + k\langle\mu\xi'\rangle^{-2}\phi_1^3 + \tilde{m})^w$. We also denote $(Q + 2\mu\langle\mu\xi'\rangle)^w$ by the same Q and $2(\phi_1 + i\omega + C)^w$ by B. Note that one can write

$$\tilde{P} = -M\Lambda + B\Lambda + Q + \tilde{P}_1$$

with $\tilde{P}_1 = a^w\Lambda + b^w$, $a \in S(1, \bar{g})$, $b \in \bar{R}$. Recall the following.

PROPOSITION 6.1 ([3, PROPOSITION 4.1])

We have

$$\begin{aligned} 2\,\mathsf{Im}\big((\tilde{P} - \tilde{P}_1)v, \Lambda v\big) &= \frac{d}{dx_0}\big(\|\Lambda v\|^2 + ((\mathsf{Re}\,Q)v, v)\big) \\ &\quad + 2\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2 + 2\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v, v\big) \\ &\quad + 2\big((\mathsf{Im}\,B)\Lambda v, \Lambda v\big) + 2\big((\mathsf{Im}\,m)\Lambda v, \Lambda v\big) + 2\mathsf{Re}\big(\Lambda v, (\mathsf{Im}\,Q)v\big) \\ &\quad + \mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}\,Q]v, v) + 2\mathsf{Re}\big((\mathsf{Re}\,Q)v, (\mathsf{Im}\,\lambda)v\big) \end{aligned}$$

for any $v \in C^2((-T, -T); \mathcal{S}(\mathbb{R}^n))$.

In this energy identity, the main term that controls (any) lower-order term is

$$\big((\mathsf{Im}\,B)\Lambda v, \Lambda v\big) + \mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v, v\big).$$

We first consider $((\mathsf{Im}\,B)\Lambda v, \Lambda v)$. Since $\mathsf{Im}\,C \in \mu^{5/4}S(r^{-2}w\langle\mu\xi'\rangle^{\kappa}, g)$, hence one can write

$$\omega + \mathsf{Im}\,C = \mu(\sqrt{w}r^{-1}\langle\mu\xi'\rangle^{\kappa/2}a)\#(\sqrt{w}r^{-1}\langle\mu\xi'\rangle^{\kappa/2}a) + \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$$

with $a = \sqrt{\mu^{-1}w^{-1}r^2\langle\mu\xi'\rangle^{-\kappa}(\omega + \mathsf{Im}\,C)} \in S(1, g)$, where $a \ge c > 0$. This shows that

$$\big((\mathsf{Im}\,B)\Lambda v, \Lambda v\big) \ge \mu\|(\sqrt{w}r^{-1}\langle\mu\xi'\rangle^{\kappa/2}a)^w\Lambda v\|^2 - C\mu\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2. \tag{6.1}$$

We next study the terms $((\mathsf{Re}\,Q)v,v)$ and $\mathsf{Re}(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v)$. Note that $(\langle\xi'\rangle_\mu^{-1}\langle\mu\xi'\rangle^2=\mu\langle\mu\xi'\rangle)$,

$$\mathsf{Re}\,Q=q\#q+\mu\langle\mu\xi'\rangle+\mu S(r\langle\mu\xi'\rangle,g)+\mu^{3/2}S(\langle\mu\xi'\rangle,g),$$
$$q=\sqrt{\phi_2^2+\alpha\theta^2+\mu\langle\mu\xi'\rangle}\in S(r\langle\mu\xi'\rangle,g).$$

Since $q\geq c\langle\mu\xi'\rangle r$ with some $c>0$,

$$\big((\mathsf{Re}\,Q)v,v\big)\geq(1-C\mu^{1/2})\|q^w v\|^2+\mu(1-C\mu^{1/2})\|\langle\mu D'\rangle^{1/2}v\|^2 \tag{6.2}$$

because one can write $\mu S(r\langle\mu\xi'\rangle,g)=C\#q+\mu^{3/2}S(\langle\mu\xi'\rangle^{1/2},g)$ with $C\in\mu S(1,g)$. Noting that

$$\langle\mu D'\rangle^{\kappa'}\,\mathsf{Re}\,Q=(\langle\mu\xi'\rangle^{\kappa'}q^2)^w+\mu\langle\mu D'\rangle^{1+\kappa'}$$
$$+\mu^{3/4}S(r\langle\mu\xi'\rangle^{\kappa'+5/4},g)+\mu^2S(\langle\mu\xi'\rangle^{\kappa'},g),$$

one can write $S(r\langle\mu\xi'\rangle^{\kappa'+5/4},g)=C\#(\langle\mu\xi'\rangle^{\kappa'/2}q)+\mu S(\langle\mu\xi'\rangle^{1/2+\kappa'},g)$ with $C\in S(\langle\mu\xi'\rangle^{1/4+\kappa'/2},g)$, and we have

$$\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v\big)\geq\big(((\langle\mu\xi'\rangle^{\kappa'}q^2)^w v,v\big)+\mu\|\langle\mu D'\rangle^{\kappa'/2+1/2}v\|^2$$
$$-C\mu^{1/4}\big\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^w v\big\|^2-C\mu^{5/4}\|\langle\mu D'\rangle^{1/4+\kappa'/2}v\|^2.$$

On the other hand, since

$$\langle\mu\xi'\rangle^{\kappa'}q^2=(\langle\mu\xi'\rangle^{\kappa'/2}q)\#(\langle\mu\xi'\rangle^{\kappa'/2}q)+\mu S(r\langle\mu\xi'\rangle^{1+\kappa'},g)+\mu^{3/2}S(\langle\mu\xi'\rangle^{\kappa'+1/2},g)$$

it follows that

$$\big((\langle\mu\xi'\rangle^{\kappa'}q^2)^w v,v\big)\geq(1-C\mu^{1/2})\big\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^w v\big\|^2-C\mu^{3/2}\|\langle\mu D'\rangle^{1/4+\kappa'/2}u\|^2$$

and hence

$$\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v\big)\geq(1-C\mu^{1/4})\big\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^w v\big\|^2 \tag{6.3}$$
$$+\mu(1-C\mu^{1/4})\|\langle\mu D'\rangle^{\kappa'/2+1/2}v\|^2.$$

We estimate the terms $((\mathsf{Im}\,m)\Lambda v,\Lambda v)$ and $\mathsf{Re}((\mathsf{Re}\,Q)v,(\mathsf{Im}\,\lambda)v)$. Noting that $\mathsf{Im}\,m\in\mu S(\langle\mu\xi'\rangle^{\kappa'},g)+\sqrt{\mu}S(1,g)$, it is clear that

$$\big|\big((\mathsf{Im}\,m)\Lambda v,\Lambda v\big)\big|\leq C\mu\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2+C\sqrt{\mu}\|\Lambda v\|^2. \tag{6.4}$$

With a large constant $K>0$, let us write

$$\big(K(\mu\langle\mu\xi'\rangle^{\kappa'}+\sqrt{\mu})-\mathsf{Im}\lambda\big)\#(\mathsf{Re}\,Q)$$
$$=\big((K(\mu\langle\mu\xi'\rangle^{\kappa'}+\sqrt{\mu})-\mathsf{Im}\,\lambda)^{1/2}q\big)\#\big(((K\mu\langle\mu\xi'\rangle^{\kappa'}+\sqrt{\mu})-\mathsf{Im}\lambda)^{1/2}q\big)$$
$$+\mu^{5/4}(\langle\mu\xi'\rangle^{\kappa'/2}q)\#C+\mu^2S(\langle\mu\xi'\rangle^{\kappa'+1/2},g)$$

with $C\in S(\langle\mu\xi'\rangle^{1/4+\kappa'/2},g)$ which shows that

$$K\mu\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v\big)+K\sqrt{\mu}\big((\mathsf{Re}\,Q)v,v\big)\geq\mathsf{Re}\big((\mathsf{Re}\,Q)v,(\mathsf{Im}\,\lambda)v\big)$$
$$-C\mu\big\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^w v\big\|^2-C\mu^{3/2}\|\langle\mu D'\rangle^{1/2+\kappa'/2}v\|^2$$

and hence

$$\mathsf{Re}\big((\mathsf{Re}\,Q)v,(\mathsf{Im}\,\lambda)v\big)\le K'\mu\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v\big)+K'\sqrt{\mu}\big((\mathsf{Re}\,Q)v,v\big) \tag{6.5}$$

with some $K'>0$.

Let us consider $\mathsf{Re}(\Lambda v,(\mathsf{Im}\,Q)v)$. Let $\mathsf{Im}\,Q=Q_1\in\mu S(w^{1/2}\langle\mu\xi'\rangle^{1+\kappa},g)$. Writing

$$Q_1=(\mu r^{-1}w^{1/2}\langle\mu\xi'\rangle^{\kappa/2}a)\#A+\mu^{3/2}S(\langle\mu\xi'\rangle^{\kappa+1/2},g)$$

with $A\in S(r\langle\mu\xi'\rangle^{1+\kappa/2},g)$, we get

$$\begin{aligned}|(\Lambda v,Q_1v)|\le\mu^{5/4}\big\|(w^{1/2}r^{-1}a\langle\mu\xi'\rangle^{\kappa/2})^w\Lambda v\big\|^2\\ +\mu^{3/4}\|Av\|^2+C\mu^{3/2}\|\langle\mu D'\rangle^{\kappa/2+1/4}v\|^2.\end{aligned}$$

Writing $A=C\#(\langle\mu\xi'\rangle^{\kappa'/2}q)+\mu^{1/2}S(\langle\mu\xi'\rangle^{1/2+\kappa/2},g)$ with $C\in S(1,g)$, we have

$$\mu^{3/4}\|Av\|^2\le C\mu^{3/4}\big\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^wv\big\|^2+C\mu^{7/4}\|\langle\mu D'\rangle^{1/2+\kappa'/2}v\|^2$$

and hence by (6.1), (6.3),

$$\begin{aligned}\big|\big(\Lambda v,(\mathsf{Im}\,Q)v\big)\big|\le C\mu^{3/4}\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v\big)\\ +C\mu^{1/4}\big((\mathsf{Im}\,B)\Lambda v,\Lambda v\big)+C\mu^{5/4}\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2.\end{aligned} \tag{6.6}$$

We turn to $([D_0-\mathsf{Re}\lambda,\mathsf{Re}\,Q]v,v)$. Note that

$$\begin{aligned}[D_0-\mathsf{Re}\lambda,\mathsf{Re}\,Q]=\frac{1}{i}\big(\big\{\xi_0+\lambda_1-k\langle\mu\xi'\rangle^{-2}\phi_1^3+\mathsf{Re}\tilde\lambda,\phi_2^2+\alpha\theta^2\big\}\big)^w\\ +\mu^2S(\langle\mu\xi'\rangle^{\kappa'+1},g).\end{aligned}$$

From (5.12) it follows that

$$\big\{\xi_0+\lambda_1-k\langle\mu\xi'\rangle^{-2}\phi_1^3,\phi_2^2\big\}\in\mu S(r^2\langle\mu\xi'\rangle^2,g).$$

Recalling that $\theta=\beta\phi_1^2$ with some $\beta\in S(\langle\mu\xi'\rangle^{-1},g_0)$, it is clear that

$$\big\{\xi_0+\lambda_1-k\langle\mu\xi'\rangle^{-2}\phi_1^3,\alpha\theta^2\big\}\in\mu S(w^{3/2}\langle\mu\xi'\rangle^2,g)\subset\mu S(r^{3/2}\langle\mu\xi'\rangle^2,g).$$

Let $A\in\mu S(r^{3/2}\langle\mu\xi'\rangle^2,g)$; then one can write

$$A=\mu^{1/4}(\langle\mu\xi'\rangle^{\kappa'/2}q)\#A_1+\mu^{7/4}S(\langle\mu\xi'\rangle^{1+\kappa'},g)$$

with $A_1\in\mu^{3/4}S(r^{1/2}\langle\mu\xi'\rangle^{1-\kappa'/2},g)$, and again we can write

$$A_1\#A_1=\mu^{1/2}(\langle\mu\xi'\rangle^{\kappa'/2}q)\#A_2+\mu^2S(\langle\mu\xi'\rangle^{1+\kappa'},g)$$

with $A_2\in\mu S(\langle\mu\xi'\rangle^{1-3\kappa'/2},g)\subset\mu S(\langle\mu\xi'\rangle^{1/2+\kappa'/2},g)$. These prove that

$$|(Av,v)|\le C\mu^{1/2}\big\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^wv\big\|^2+C\mu^{7/4}\|\langle\mu D'\rangle^{1/2+\kappa'/2}v\|^2. \tag{6.7}$$

On the other hand it is easy to see

$$\{\mathsf{Re}\tilde\lambda,\phi_2^2+\alpha\theta^2\}=\mu^{1/4}(\langle\mu\xi'\rangle^{\kappa'/2}q)\#A_3+\mu^2S(\langle\mu\xi'\rangle^{1+\kappa'},g)$$

with $A_3\in\mu S(\langle\mu\xi'\rangle^{1/2+\kappa'/2},g)$. Hence taking (6.3) and (6.7) into account, we have

$$C\sqrt{\mu}\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v,v\big)+\mathsf{Im}([D_0-\mathsf{Re}\lambda,\mathsf{Re}\,Q]v,v)\ge0 \tag{6.8}$$

with some $C>0$.

Finally, we consider $P_1 = S(1,\bar g)\Lambda + \mu S(r\langle\mu\xi'\rangle^{1+\kappa'}, g) + \mu S(\langle\mu\xi'\rangle, \bar g)$. We first study the term $A \in \mu S(\langle\mu\xi'\rangle, \bar g)$ and estimate $|(Av, \Lambda v)|$. Write

$$A = \mu^{5/8}(w^{1/2}r^{-1}a\langle\mu\xi'\rangle^{\kappa/2})\#(\mu^{-5/8}A') + \mu^{5/4}S(\langle\mu\xi'\rangle^{3/4}, \bar g)$$

with $A' \in \mu S(w^{-1/2}r\langle\mu\xi'\rangle^{1-\kappa/2}, \bar g)$. Thus we have

$$|(Av,\Lambda v)| \le C\mu^{5/4}\big\|(w^{1/2}r^{-1}a\langle\mu\xi'\rangle^{\kappa/2})^w\Lambda v\big\|^2 + C\mu^{-5/4}\|(A')^w v\|^2 + C\mu\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2 + C\mu^{3/2}\|\langle\mu D'\rangle^{1/2+\kappa'/2}v\|^2.$$

Since $r^2w^{-1}\langle\mu\xi'\rangle^{2-\kappa} = r^2\langle\mu\xi'\rangle^{2+\kappa'}w^{-1}\langle\mu\xi'\rangle^{-1/2} \le \mu^{-1/2}r^2\langle\mu\xi'\rangle^{2+\kappa'}$ so that

$$\mu^{-5/4}A'\#A' \in \mu^{1/4}S(r^2\langle\mu\xi'\rangle^{2+\kappa'}, \bar g),$$

one can write

$$K\mu^{1/4}\langle\mu\xi'\rangle^{\kappa'}q^2 - \mu^{-5/4}A'\#A' = \mu^{1/4}(b\langle\mu\xi'\rangle^{\kappa'/2}q)\#(b\langle\mu\xi'\rangle^{\kappa'/2}q) + \mu^{3/4}((\langle\mu\xi'\rangle^{\kappa'/2}q)\#C + \mu^{5/4}S(\langle\mu\xi'\rangle^{1+\kappa'}, \bar g)$$

with $b = \sqrt{K - \mu^{-3/2}\langle\mu\xi'\rangle^{-\kappa'}q^{-2}(A'\#A')} \in S(1,g)$ with a large $K > 0$ and $C \in S(\langle\mu\xi'\rangle^{1/2+\kappa'/2}, \bar g)$. This shows that

$$C\mu^{1/4}\big((\langle\mu\xi'\rangle^{\kappa'}q^2)^w v, v\big) \ge \mu^{-5/4}\|(A')^w v\|^2 - C\mu^{1/4}\|(\langle\mu\xi'\rangle^{\kappa'/2}q)^w v\|^2 - C\mu^{5/4}\|\langle\mu D'\rangle^{1/2+\kappa'/2}v\|^2.$$

Thus we get, from (6.1) and (6.3),

$$|(Av,\Lambda v)| \le C\mu^{1/4}\big((\mathsf{Im}\,B)\Lambda v, \Lambda v\big) + C\mu^{1/4}\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v, v\big) + C\mu\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2. \tag{6.9}$$

In particular, this shows that one can control any lower-order term by the right-hand side of (6.9). We next estimate $|(Av,\Lambda v)|$, $A \in \mu S(r\langle\mu\xi'\rangle^{1+\kappa'}, g)$. Write

$$A = \mu\langle\mu\xi'\rangle^{\kappa'/2}\#A' + \mu^{3/2}S(\langle\mu\xi'\rangle^{\kappa'+1/2}, \bar g), \quad A' \in S(r\langle\mu\xi'\rangle^{1+\kappa'/2}, \bar g),$$

and hence

$$|(Av,\Lambda v)| \le C\mu\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2 + C\mu\|(A')^w v\|^2 + C\mu^2\|\langle\mu D'\rangle^{\kappa'/2+1/2}v\|^2.$$

Writing $A' = \langle\mu\xi'\rangle^{\kappa'/2}q\#C + \mu^{1/2}S(\langle\mu\xi'\rangle^{\kappa'/2+1/2}, \bar g)$ with $C \in S(1,\bar g)$, we get

$$|(Av,\Lambda v)| \le C\mu\|\langle\mu D\rangle^{\kappa'/2}\Lambda v\|^2 + C\mu\mathsf{Re}\big(\langle\mu D'\rangle^{\kappa'}(\mathsf{Re}\,Q)v, v\big). \tag{6.10}$$

For $A \in S(1,\bar g)$ it is clear that

$$|(A\Lambda v, \Lambda v)| \le C\|\Lambda v\|^2. \tag{6.11}$$

From (6.1) to (6.11) we now have

$$\begin{aligned}\mathsf{Im}(\tilde P v, \Lambda v) &= \mathsf{Im}\big((\tilde P - \tilde P_1)v, \Lambda v\big) + \mathsf{Im}(\tilde P_1 v, \Lambda v)\\ &\ge \frac{d}{dx_0}\big(\|\Lambda v\|^2 + ((\mathsf{Re}\,Q)v, v)\big) + (1 - C\mu^{3/4})\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2\\ &\quad + \mu(1 - C\mu^{1/4})\|\langle\mu D'\rangle^{\kappa'/2+1/2}v\|^2 - C\mu\big((\mathsf{Re}\,Q)v, v\big) - C\|\Lambda v\|^2.\end{aligned} \tag{6.12}$$

Multiplying (6.12) by $e^{-\gamma x_0}$ we get

$$\begin{aligned} & e^{-\gamma x_0}\|\langle\mu D'\rangle^{-\kappa'/2}\tilde{P}v\|^2 \\ & \quad \geq \frac{d}{dx_0}\big[e^{-\gamma x_0}\big(\|\Lambda v\|^2+((\mathrm{Re}\,Q)v,v)\big)\big] \\ & \qquad +(1-C\mu^{3/4})e^{-\gamma x_0}\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2 \\ & \qquad +\mu(1-C\mu^{1/4})e^{-\gamma x_0}\|\langle\mu D'\rangle^{\kappa'/2+1/2}v\|^2 \\ & \qquad +(\gamma-C)e^{-\gamma x_0}\|\Lambda v\|^2+(\gamma-C\mu)e^{-\gamma x_0}\big((\mathrm{Re}\,Q)v,v\big). \end{aligned}$$

Thus for $0<\mu<\mu_0$ and $\gamma>\gamma_0(\mu_0)$, taking (6.2) into account, one has

$$\begin{aligned} & \int_{-T}^{t} e^{-\gamma x_0}\|\langle\mu D'\rangle^{-\kappa'/2}\tilde{P}v\|^2\,dx_0 \\ & \quad \geq ce^{-\gamma t}\big(\|\Lambda v(t)\|^2+\mu\|\langle\mu D'\rangle^{1/2}v(t)\|^2\big) \\ & \qquad + c\int_{-T}^{t} e^{-\gamma x_0}\big\{\|\langle\mu D'\rangle^{\kappa'/2}\Lambda v\|^2+\mu\|\langle\mu D'\rangle^{\kappa'/2+1/2}v\|^2\big\}\,dx_0 \\ & \qquad + c\gamma\int_{-T}^{t} e^{-\gamma x_0}\big\{\|\Lambda v\|^2+\mu\|\langle\mu D'\rangle^{1/2}v\|^2\big\}\,dx_0 \end{aligned} \tag{6.13}$$

for $v\in C^2((-T,T);\mathcal{S}(\mathbb{R}^n))$ vanishing in $x_0\leq 0$.

Let $h(x',\xi')\in S_{(s)}(1,g_0)$, and assume $\{\Lambda, x_0+h\}\geq c>0$ with some $c>0$. Then it is easy to see that the a priori estimates (6.13) holds with phase function

$$\tilde{\phi}_h=-(x_0+h)\langle\mu\xi'\rangle^{\kappa'}-2\langle\mu\xi'\rangle^{\kappa}\arg(\phi_2+iw)$$

since $\langle\mu\xi'\rangle^{\kappa'}h\in S_{(s)}(\langle\mu\xi'\rangle^{\kappa'},g_0)$, and hence $\{\Lambda,\langle\mu\xi'\rangle^{\kappa'}h\}\in\mu S_{(s)}(\langle\mu\xi'\rangle^{\kappa'},g_0)$. Starting with P^* instead of P, we get $\mathrm{Op}^0(e^{\tilde{\phi}_h})P^*\,\mathrm{Op}^1(e^{-\tilde{\phi}_h})\,\mathrm{Op}^0(j_h)$, where $\mathrm{Op}^0(e^{\tilde{\phi}_h})\,\mathrm{Op}^1(e^{-\tilde{\phi}_h})\,\mathrm{Op}^0(j_h)=I$. Taking its adjoint we have

$$T_hPS_h=\tilde{P}_h,$$

where

$$T_h=\mathrm{Op}^1(\bar{j}_h)\,\mathrm{Op}^0(e^{-\tilde{\phi}_h}),\qquad S_h=\mathrm{Op}^1(e^{\tilde{\phi}_h})$$

because $\mathrm{Op}^0(p)^*=\mathrm{Op}^1(\bar{p})$. Recall that $T_hS_h=S_hT_h=I$. We prove the existence of a parametrix of finite propagation speed of P by using the a priori estimates for $\tilde{P}_h$. Recall that P can be assumed to be of the form (2.2).

As for the fourth term in the right-hand side of (2.2), assuming that s verifies $(1+3\rho)s\kappa'<1$, we apply the following ($\rho=3/4,\delta=1/2$).

LEMMA 6.1

Let $s_1>s$ and $1>s_1\kappa'$, and assume $R\in S_{(s_1)}(e^{-c_1\langle\mu\xi'\rangle^{1/s}},g_\rho)$ with $g_\rho=|dx|^2+\langle\xi'\rangle_\mu^{-2\rho}|d\xi'|^2$. Then we have

$$\mathrm{Op}^0(e^{\tilde{\phi}_h})\,\mathrm{Op}^0(R)\,\mathrm{Op}^1(e^{-\tilde{\phi}_h})=\mathrm{Op}^0(c),$$

where

$$c \in S_{((1+\rho)s_1+\rho s/(1-\delta))}(e^{-c_2\langle\xi'\rangle^{s_2}}, \bar{g})$$

with $s_2 = \min\{1/s_1, (1-\delta)/s\}$. *In particular,* $c \in S(\langle\mu\xi'\rangle^{\ell}, \bar{g})$ *for any* $\ell \in \mathbb{Z}$.

Proof

We sketch the proof. To simplify notations we write $\tilde{\phi}$ for $\tilde{\phi}_h$. Let $\mathrm{Op}^0(e^{\tilde{\phi}}) \times \mathrm{Op}^0(R) = \mathrm{Op}^0(b)$. Then repeating arguments similar to those in the proof of Proposition 2.1, we see that

$$b \in S_{((1+\rho)s_1)}(e^{-c\langle\mu\xi'\rangle^{1/s_1}}, \bar{g}).$$

We next consider $\mathrm{Op}^0(b)\,\mathrm{Op}^1(e^{-\tilde{\phi}}) = \mathrm{Op}^0(c_1) + \mathrm{Op}^0(c_2)$, where

$$c_i = (2\pi)^{-n} \int e^{-iy'\eta' - \tilde{\phi}(x'+y',\xi'+\eta')} \chi_i b(x', \xi'+\eta')\, dy'\, d\eta'$$

with $\chi_1 = \chi((\eta'-\xi')\langle\xi'\rangle_{\mu}^{-1})$, $\chi_2 = 1 - \chi_1$. It is easy to check that

$$c_1 \in S_{(\hat{s})}(e^{-c\langle\mu\xi'\rangle^{1/s_1}}, \bar{g})$$

with $\hat{s} = (1+\rho)s_1$. We consider c_2. Note that

$$\begin{aligned} &|\partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}\langle D_{y'}\rangle^{N}\langle\eta'\rangle^{-N} e^{-\tilde{\phi}(x'+y',\xi'+\eta')}\chi_2 b(x',\xi'+\eta')| \\ &\quad \le CA^{|\alpha+\beta|}|\alpha+\beta|!^{\hat{s}}\langle\eta'\rangle^{\delta|\beta|+\rho|\alpha|}\langle\xi'\rangle_{\mu}^{-\rho|\alpha|} e^{c\langle\mu\eta'\rangle^{\kappa'}} A^N N!^s \langle\eta'\rangle^{-(1-\delta)N}. \end{aligned}$$

Choose N so that $N = [(\rho|\alpha| + \delta|\beta| + \ell)/(1-\delta)]$, $\ell \in \mathbb{N}$, and hence the right-hand side is bounded by

$$CA^{|\alpha+\beta|}|\alpha+\beta|!^{\hat{s}+\rho s/(1-\delta)}\langle\xi'\rangle_{\mu}^{-\rho|\alpha|} e^{c\langle\mu\eta'\rangle^{\kappa'}} \Big(\frac{A\ell^{s/(1-\delta)}}{\langle\eta'\rangle}\Big)^{\ell}.$$

Taking ℓ such that $\ell = [(A^{-1}e^{-1}\langle\eta'\rangle)^{(1-\delta)/s}]$ and noting $(1-\delta)/s > \kappa'$, we conclude that

$$c_2 \in S_{(\hat{s}+\rho s/(1-\delta))}(e^{-c\langle\xi'\rangle_{\mu}^{(1-\delta)/s}}, g_\rho).$$

These prove the desired assertion. □

Let us fix a small $T > 0$ and $h = \epsilon$. Since the energy estimates (6.13) hold for $\tilde{P}_\epsilon$, then from the standard arguments on functional analysis we conclude that for any given $F \in C^0([-T,T]; H^{\infty}(\mathbb{R}^n))$ vanishing in $x_0 \le 0$, there is a unique $U \in C^2([-T,T]; H^{\infty}(\mathbb{R}^n))$ vanishing in $x_0 < 0$ such that $\tilde{P}_\epsilon U = F$. Let $1 \le \bar{s} < 4$, and let $f \in C^0([-T,T]; \gamma_0^{(\bar{s})}(\mathbb{R}^n))$ be such that $f(x) = 0$ for $x_0 \le 0$. We choose κ' and κ such that

$$\bar{s} < \frac{1}{\kappa'} < 4, \quad \kappa' + \kappa = \frac{1}{2}.$$

Since $-\tilde{\phi}_\epsilon \le C\langle\mu\xi'\rangle^{\kappa'}$, it is easy to see that $\mathrm{Op}^0(e^{-\tilde{\phi}_\epsilon})f \in C^0([-T,T]; H^{\infty}(\mathbb{R}^n))$ because $1/\bar{s} > \kappa'$, and hence $F = T_\epsilon f \in C^0([-T,T]; H^{\infty}(\mathbb{R}^n))$. Then as remarked above, there exists a unique $U \in C^2([-T,T];H^\infty(\mathbb{R}^n))$ such that

$$T_\epsilon P S_\epsilon U = \tilde{P}_\epsilon U = F = T_\epsilon f, \tag{6.14}$$

where $U = 0$ in $x_0 \le 0$. This implies that $P(S_\epsilon U) = f$. Let us denote

$$u = S_\epsilon U = Gf$$

and prove that G is a parametrix of P with finite propagation speed.

We first examine $u = S_\epsilon U \in C^2([-T,T];H^\infty(\mathbb{R}^n))$. Since $\tilde{\phi}_\epsilon \le -\epsilon\langle\mu\xi'\rangle^{\kappa'}$ for $x_0 \ge 0$, it is easy to see from Corollary A.3 that $e^{\tilde{\phi}_\epsilon} \in S_{(s)}(1,\bar{g})$, and this proves the assertion. We now prove that G is a parametrix with finite propagation speed. Let $h_1(x',\xi') \in S_{(s)}(1,g_0)$ be such that

$$\operatorname{supp} h_1 \cap \{0 \le x_0 \le \tau\} \subset \{x_0 + h < 0\}. \tag{6.15}$$

LEMMA 6.2

Assume (6.15) and $\rho > s\kappa'$. Then we have

$$\mathrm{Op}^0(e^{-\tilde{\phi}_h})\,\mathrm{Op}^0(h_1) = \mathrm{Op}^0\big(S_{((1+\rho)s)}(\langle\mu\xi'\rangle^\ell,\bar{g})\big), \quad 0 \le x_0 \le \tau,$$

for any $\ell \in \mathbb{Z}$.

Proof

We sketch the proof. Write $\mathrm{Op}^0(e^{-\tilde{\phi}_h})\,\mathrm{Op}^0(h_1) = \mathrm{Op}^0(b_1) + \mathrm{Op}^0(b_2)$, where

$$\mathrm{Op}^0(b_i)u = (2\pi)^{-n}\int e^{-iy'\eta' + \tilde{\phi}_h(x,\xi'+\eta')}\chi_i(\xi',\eta')h_1(x'+y',\xi')u(y')\,dy'\,d\eta'$$

with $\chi_1 = \chi(M\eta'\langle\xi'\rangle_\mu^{-1})$, $\chi_2 = 1 - \chi_1$. We consider only $\mathrm{Op}^0(b_1)$. Let us write $\mathrm{Op}^0(b_1) = \mathrm{Op}^0(b_{11}) + \mathrm{Op}^0(b_{12})$, where

$$\mathrm{Op}^0(b_{1i})u = (2\pi)^{-n}\int e^{-iy'\eta' + \tilde{\phi}_h(x,\xi'+\eta')}\tilde{\chi}_i(y')\chi_1(\xi',\eta')h_1(x'+y',\xi')u(y')\,dy'\,d\eta'$$

with $\tilde{\chi}_1(y') = \chi(Ky')$ and $\tilde{\chi}_2 = 1 - \tilde{\chi}_1$. Choose $K > 0$ large so that

$$-\tilde{\phi}_h(x,\xi'+\eta') \le -c'\langle\mu\xi'\rangle^{\kappa'}, \quad 0 \le x_0 \le \tau,$$

on the support of $\chi_1(\xi',\eta')\tilde{\chi}_1(y')$ with some $c' > 0$. Then from integration by parts we see $b_{11} \in S_{(s)}(e^{-c\langle\mu\xi'\rangle^{\kappa'}},\bar{g})$. We turn to $\mathrm{Op}^0(b_{12})$. By Corollary A.3 we see

$$\begin{aligned}&|\partial_{x'}^\beta\partial_{\xi'}^\alpha\partial_{\eta'}^\gamma(y')^{-\gamma}e^{\tilde{\phi}_h(x,\xi'+\eta')}\chi_1(\xi',\eta')\tilde{\chi}_2(y')h_1(x'+y',\xi')|\\ &\quad\le CA^{|\alpha+\beta+\gamma|}|\alpha+\beta|!^s\langle\xi'\rangle_\mu^{-\rho|\alpha|+\delta|\beta|}|\gamma|!^s\langle\xi'\rangle_\mu^{-\rho|\gamma|}e^{c\langle\mu\xi'\rangle^{\kappa'}}.\end{aligned}$$

Take γ such that $|\gamma| = [(A^{-1}e^{-1}\langle\xi'\rangle_\mu^\rho)^{1/s}]$ and hence the right-hand side is bounded by

$$CA^{|\alpha+\beta|}|\alpha+\beta|!^s\langle\xi'\rangle_\mu^{-\rho|\alpha|+\delta|\beta|}e^{-c'\langle\mu\xi'\rangle^{\rho/s}}$$

because $\rho/s > \kappa'$. This proves $b_{12} \in S_{(s)}(e^{-c'\langle\mu\xi'\rangle^{\rho/s}},\bar{g})$. It is not difficult to see that $b_2 \in S_{((1+\rho)s)}(e^{-c'\langle\xi'\rangle_\mu^{1/s}},\bar{g})$. These prove the assertion. □

We now consider $Pu = h_1 f$. From $T_h P S_h T_h u = T_h h_1 f$ we have

$$\tilde{P}_h(T_h u) = T_h h_1 f.$$

From Lemma 6.2 it follows that for any p, q we have

$$\|T_h h_1 f\|_p \le C_{p,q}\|f\|_q,$$

where $\|u\|_p = \|u\|_{H^p(\mathbb{R}^n)}$. Thus one has, from (6.13),

$$\|T_h u\|_p \le C_p \int^t \|T_h h_1 f\|_p \, dx_0 \le C_{p,q} \int^t \|f\|_q \, dx_0.$$

Assume that $h_2 \in S_{(s)}(1, g_0)$ verifies

$$\operatorname{supp} h_2 \cap \{0 \le x_0 \le \tau\} \subset \{x_0 + h > 0\}. \tag{6.16}$$

A repetition of similar arguments proving Lemma 6.2 gives

$$h_2 S_h \in S_{((1+\rho)s)}(\langle \mu\xi'\rangle^{\ell}, \bar{g})$$

for $0 \le x_0 \le \tau$ and for any $\ell \in \mathbb{Z}$, and hence

$$\|h_2 G h_1 f\|_p = \|h_2 u\|_p = \|h_2 S_h T_h u\|_p \le C_{p,q}\|T_h u\|_p \le C_{p,q} \int^t \|f\|_q \, dx_0$$

for any p, q.

Assume $\operatorname{supp} h_1 \cap \operatorname{supp} h_2 = \emptyset$. Then it is clear that one can take $\tau > 0$ and h satisfying (6.15) and (6.16). Indeed it is enough to take h as a finite sum of such

$$\epsilon\sqrt{|\xi'\langle\xi'\rangle_\mu^{-1} - \tilde{\xi}'\langle\tilde{\xi}'\rangle_\mu^{-1}|^2 + |x' - \tilde{x}'| + \epsilon_1^2} - \epsilon_2,$$

where $0 < \epsilon_1 < \epsilon_2$, $0 < \epsilon < 1$ are small. Thus we have proved that G is a parametrix of P with finite propagation speed at $(0, \hat{x}', \hat{\xi}')$.

Since the existence of a parametrix with finite propagation speed is invariant under conjugation of Fourier integral operators, then the original operator (1.1) has a parametrix with finite propagation speed at every $(0, x', \xi')$, $|\xi'| \ne 0$. Then it follows (see the proof of Proposition A.6 in [11]) that there is $\tau_1 > 0$ such that for any $f \in C^0([-T,T]; \gamma_0^{(\bar{s})}(\mathbb{R}^n))$ vanishing in $x_0 \le 0$ there exists a unique $u \in C^2([-\tau_1, \tau_1]; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 \le 0$ verifying (1.8).

Appendices

A

Here and in Appendix B we collect several results about symbol classes used in this paper and Fourier integral operators with complex phase function (4.4) without proofs. We refer to [17] for the proofs.

A.1 Almost analytic extension of Gevrey functions

In this subsection we study an almost analytic extension of $a(x, \xi, \mu) \in S_{(s)}(m, g)$ with

$$g_{x,\xi}(y, \eta) = \sum_{j=1}^n \delta_j^2 y_j^2 + \rho_j^{-2}\eta_j^2,$$

where $\delta=(\delta_1(x,\xi,\mu),\dots,\delta_n(x,\xi,\mu))$, $\rho=(\rho_1(x,\xi,\mu),\dots,\rho_n(x,\xi,\mu))$ and $\delta_j(x,\xi,\mu)$, $\rho_j(x,\xi,\mu)$ are assumed to be in $S_{(s)}(\delta_j,g)$, $S_{(s)}(\rho_j,g)$, respectively. Let $\chi(t)\in\gamma_0^{(s)}(\mathbb{R})$ which is 1 in $|t|\le 1/4$ and 0 in $|t|\ge 1/2$, and set

$$\chi(x)=\chi(x_1)\chi(x_2)\cdots\chi(x_n).$$

In what follows, to simplify notation we often drop a small parameter $\mu>0$.

We assume that there exists a metric

$$g\le \hat g_\xi(y,\eta)=\sum_{j=1}^n\langle\xi\rangle_\mu^{2a_j}y_j^2+\langle\xi\rangle_\mu^{-2b_j}\eta_j^2,$$

where $0\le a_j<b_j\le 1$. Let us set

$$\hat g_\xi^\sigma(y,\eta)=\sum_{j=1}^n\langle\xi\rangle_\mu^{2b_j}y_j^2+\langle\xi\rangle_\mu^{-2a_j}\eta_j^2,\quad h(\xi,\mu)=\langle\xi\rangle_\mu^{-\min_{i,j}(b_i-a_j)}.$$

Let $k(\xi,\mu)\in S_{(s)}(k,\hat g)$, and set

$$E(k)=\big\{(x,y,\xi,\eta)\in\mathbb{R}^{4n}\;\big|\;\hat g_\xi^\sigma(y,\eta)<k(\xi,\mu)^2\big\},$$
$$E^0(k)=\big\{(x,\xi,\eta)\in\mathbb{R}^{3n}\;\big|\;\hat g_\xi^\sigma(0,\eta)<k(\xi,\mu)^2\big\}.$$

We denote by $S_{(s)}(m,g|E(k))$ the set of all $a(x,y,\xi,\eta)\in C^\infty(E(k))$ verifying

$$|\partial_{x,y}^\beta\partial_{\xi,\eta}^\alpha a(x,y,\xi,\eta)|\le CA^{|\alpha+\beta|}|\alpha+\beta|!^s m\delta^\beta\rho^{-\alpha},\quad\forall\alpha,\beta,$$

for $(x,y,\xi,\eta)\in E(k)$ with some $C>0,A>0$. Similarly we define $S_{(s)}(m,\hat g\mid E^0(k))$. Let

$$d_j=Bj^{s-1},\quad j\ge 1, d_0=1,$$

where B is some positive constant, take $k(\xi,\mu)\in S_{(s)}(k,\hat g)$ such that

$$k(\xi,\mu)h(\xi,\mu)\le C\langle\xi\rangle_\mu^{-\epsilon}\tag{A.1}$$

with some $\epsilon>0$, and define an almost analytic extension $\tilde a(z,\zeta)=a(x+iy,\xi+i\eta)$ of $a(x,\xi)\in S_{(s)}(m,g)$ by

$$a(x+iy,\xi+i\eta)=\sum_{\alpha,\beta}\frac{1}{\alpha!\beta!}\partial_x^\beta\partial_\xi^\alpha a(x,\xi)(iy)^\beta(i\eta)^\alpha\chi(kd_\beta\langle\xi\rangle_\mu^{a-b})\chi(kd_\alpha\langle\xi\rangle_\mu^{a-b}),$$

where $d_\beta\langle\xi\rangle_\mu^{a-b}=(d_{\beta_1}\langle\xi\rangle_\mu^{a_1-b_1},\dots,d_{\beta_n}\langle\xi\rangle_\mu^{a_n-b_n})$.

PROPOSITION A.1

Let $a(x,\xi)\in S_{(s)}(m,g)$. Then we have

$$\tilde a(z,\zeta)\in S_{(s)}\big(m,g\;\big|\;E(k)\big).$$

Moreover, with $\bar\partial_{z_j}=\partial_{x_j}+i\partial_{y_j}$ and $\bar\partial_{\zeta_j}=\partial_{\xi_j}+i\partial_{\eta_j}$ we have, with some $c>0$,

$$\bar\partial_{z_j}\tilde a(z,\zeta)\in S_{(s)}\big(m\delta^{e_j}e^{-c(hk)^{-1/(s-1)}},g\;\big|\;E(k)\big),$$
$$\bar\partial_{\zeta_j}\tilde a(z,\zeta)\in S_{(s)}\big(m\rho^{-e_j}e^{-c(hk)^{-1/(s-1)}},g\;\big|\;E(k)\big).$$

COROLLARY A.1

Let $Y_j \in S_{(s)}(k\langle\xi\rangle_\mu^{-b_j}, g \mid E(k))$, $H_j \in S_{(s)}(k\langle\xi\rangle_\mu^{a_j}, g|E(k))$, and let $Y = (Y_1, \ldots, Y_n)$, $H = (H_1, \ldots, H_n)$. Then we have

$$\tilde{a}(x, Y, \xi, H) - \sum_{|\alpha+\beta|<N} \frac{1}{\alpha!\beta!} \partial_x^\beta \partial_\xi^\alpha a(x,\xi)(iY)^\beta (iH)^\alpha \in S_{(s)}\big(m(kh)^N, g \bigm| E(k)\big)$$

for any $N \in \mathbb{N}$.

PROPOSITION A.2

Let $\theta(x,\xi,v)$ be a positive function. Assume that $a(x,\xi,v) \in C^\infty(\mathbb{R}^{2n} \times \Omega)$ verifies

$$|\partial_x^\beta \partial_\xi^\alpha \partial_v^\gamma a(x,\xi,v)| \le C A^{|\alpha+\beta+\gamma|} |\alpha+\beta+\gamma|!^s m \delta^\beta \rho^{-\alpha} \theta^\gamma, \qquad (x,\xi,v) \in \mathbb{R}^{2n} \times \Omega$$

for all α, β, γ. Then an almost analytic extension $\tilde{a}(z,\zeta,v)$ verifies

$$|\partial_{x,y}^\beta \partial_{\xi,\eta}^\alpha \partial_v^\gamma \tilde{a}(z,\zeta,v)| \le C A^{|\alpha+\beta+\gamma|} |\alpha+\beta+\gamma|!^s m \delta^\beta \rho^{-\alpha} \theta^\gamma,$$

$$|\partial_{x,y}^\beta \partial_{\xi,\eta}^\alpha \partial_v^\gamma \bar{\partial}_{z_j} \tilde{a}(z,\zeta,v)| \le C A^{|\alpha+\beta+\gamma|} |\alpha+\beta+\gamma|!^s \times m \delta^{e_j} e^{-c(hk)^{-1/(s-1)}} \delta^\beta \rho^{-\alpha} \theta^\gamma,$$

$$|\partial_{x,y}^\beta \partial_{\xi,\eta}^\alpha \partial_v^\gamma \bar{\partial}_{\zeta_j} \tilde{a}(z,\zeta,v)| \le C A^{|\alpha+\beta+\gamma|} |\alpha+\beta+\gamma|!^s \times m \rho^{-e_j} e^{-c(hk)^{-1/(s-1)}} \delta^\beta \rho^{-\alpha} \theta^\gamma$$

for $(x,y,\xi,\eta,v) \in E(k) \times \Omega$ and for any α, β, γ.

A.2 Estimates of composite functions

Let us put

$$\Gamma_s(k) = M \frac{k!^s}{k^{s+2}}.$$

Then we have the following.

LEMMA A.1

Choosing a positive constant M suitably, we have

(i)

$$\sum_{\alpha'+\alpha''=\alpha} \binom{\alpha}{\alpha'} \Gamma_s(|\alpha'|+p+1)\Gamma_s(|\alpha''|+q) \le \Gamma_s(|\alpha|+p+q)$$

for $p \ge 0$ and $q \ge 1$, $p, q \in \mathbb{N}$,

(ii)

$$\sum_{\alpha'+\alpha''=\alpha} \binom{\alpha}{\alpha'} \Gamma_s(|\alpha'|+p)\Gamma_s(|\alpha''|) \le \Gamma_s(|\alpha|+p)$$

for $p \ge 0$.

LEMMA A.2

Assume that

$$|\partial_x^\alpha \phi_j(x)| \le C_j A_1^\alpha \Gamma_s(|\alpha|-1), \quad |\alpha|\ge 1, j=1,2,\ldots,N,$$
$$|\partial_z^\alpha u(z)| \le C A_2^\alpha \Gamma_s(|\alpha|-1), \quad |\alpha|\ge 1,$$

where $C_j = C_j(x),\ A_1(x) = (A_{11}(x),\ldots,A_{1n}(x)),\ A_2(z) = (A_{21}(z),\ldots,A_{2N}(z))$, *and* $u(z) = u(z_1,\ldots,z_N)$. *Let*

$$d(x,z) = \sum_{j=1}^N C_j(x) A_{2j}(z).$$

Then we have, with $\phi(x) = (\phi_1(x),\ldots,\phi_n(x))$,

$$\big|\partial_x^\gamma u(\phi(x))\big| \le C d\big(x,\phi(x)\big)\big(1+d(x,\phi(x))\big)^{|\gamma|-1} A_1^\gamma \Gamma_s(|\gamma|-1), \quad |\gamma|\ge 1.$$

LEMMA A.3

Assume that

$$|\partial_x^\alpha \phi_j(x)| \le C_j A_1^\alpha \Gamma_s(|\alpha|-1), \quad |\alpha|\ge 1, j=1,\ldots,N,$$
$$\big|[\partial_x^\alpha \partial_z^\gamma F(x,z)]_{z=\phi(x)}\big| \le C A_2^\alpha B_2^\gamma \Gamma_s(|\alpha+\gamma|-1), \quad |\alpha+\gamma|\ge 1,$$

for $|\alpha+\gamma|\le k,\ |\alpha|\le k,$ *where* $C_j = C_j(x),\ A_1(x) = (A_{11}(x),\ldots,A_{1m}(x)),$ $A_{1j}(x) > 0,\ A_2(x) = (A_{21}(x),\ldots,A_{2m}(x)),$ *and* $B_2 = (B_{21}(x),\ldots,B_{2N}(x))$. *Assume that*

$$\sum_{j=1}^N C_j B_{2j} + \sum_{j=1}^m A_{2j} A_{1j}^{-1} \le 1;$$

then we have

$$\big|\partial_x^\mu F_{(\alpha)}^{(\gamma)}(x,\phi(x))\big| \le C A_2^\alpha B^\gamma A_1^\mu \Gamma_s(|\alpha+\mu+\gamma|-1)$$

for $|\alpha+\gamma+\mu|\le k$, *where* $F_{(\alpha)}^{(\gamma)}(x,z) = \partial_x^\alpha \partial_z^\gamma F(x,z)$.

COROLLARY A.2

Assume that

$$|\partial_x^\alpha \phi_j(x)| \le C_j A_1^\alpha \Gamma_s(|\alpha|-1), \quad |\alpha|\ge 1, j=1,\ldots,N,$$
$$\big|[\partial_x^\alpha \partial_y^\beta \partial_z^\gamma F(x,y,z)]_{z=\phi}\big| \le C A_2^\alpha B_2^\beta D_2^\gamma \Gamma_s(|\alpha+\beta+\gamma|-1), \quad |\alpha+\beta+\gamma|\ge 1,$$

for $|\alpha+\beta+\gamma|\le k,\ |\alpha|\le k,$ *where* $C_j = C_j(x),\ A_1(x) = (A_{11}(x),\ldots,A_{1m}(x)),$ $A_{1j}(x) > 0,\ A_2(x) = (A_{21}(x),\ldots,A_{2m}(x)),\ B_2(x,y) = (B_{21}(x,y),\ldots,B_{2\ell}(x,y)),$ *and* $D_2(x,y) = (D_{21}(x,y),\ldots,D_{2N}(x,y))$. *Assume that*

$$\sum_{j=1}^N C_j D_{2j} + \sum_{j=1}^m A_{2j} A_{1j}^{-1} \le 1;$$

then we have

$$\big|\partial_x^\mu \partial_y^\nu F\big(x,y,\phi(x)\big)\big| \le C A_1^\mu B_2^\nu \Gamma_s(|\mu+\nu|-1)$$

for $|\mu+\nu|\le k$.

A.3 Estimates of some special symbols

LEMMA A.4

Assume that $\phi \in S_{(s)}(k(\xi,\mu),g)$. *Let* $\omega_\beta^\alpha = e^{-\phi}\partial_x^\beta\partial_\xi^\alpha e^\phi$. *Then we have, with some constants* $A_i > 0$,

$$\left|\partial_x^\nu\partial_\xi^\mu\omega_\beta^\alpha\right| \le CA_1^{|\nu+\mu|}A_2^{|\alpha+\beta|}\delta^{\beta+\nu}\rho^{-\alpha-\mu} \times \sum_{j=0}^{|\alpha+\beta|} k(\xi)^{|\alpha+\beta|-j}(|\mu+\nu|+j)!.$$

COROLLARY A.3

We have

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha e^\phi| &\le Ce^\phi A^{|\alpha+\beta|}\delta^\beta\rho^{-\alpha}\sum_{j=0}^{|\alpha+\beta|}k(\xi)^{|\alpha+\beta|-j}j!^s \\ &\le CA_1^{|\alpha+\beta|}|\alpha+\beta|!^s\delta^\beta\rho^{-\alpha}e^\phi e^{k(\xi)^{1/s}}.\end{aligned}$$

LEMMA A.5

Assume that $\phi(x)$ *verifies*

$$|\partial_x^\alpha\phi(x)| \le C_1A_1(x)^\alpha|\alpha|!^s, \quad x\in\mathbb{R}^n.$$

For given $m,\ \ell\in\mathbb{N}$, *let us set*

$$w(x) = \big(\phi(x)^{2m}+B^{-2}\big)^{1/\ell}.$$

Then we have

$$|\partial_x^\alpha w(x)| \le CC_2^{|\alpha|}w(x)w(x)^{-\ell|\alpha|/2m}A_1(x)^\alpha|\alpha|!^s$$

for $x\in\mathbb{R}$.

COROLLARY A.4

Assume that

$$|\partial_x^\beta\partial_\xi^\alpha\phi(x,\xi)| \le CA^{|\alpha+\beta|}|\alpha+\beta|!^s\langle\xi\rangle_\mu^{-|\alpha|}.$$

Then we have, with $w(x,\xi) = (\phi(x,\xi)^{2m}+\langle\xi\rangle_\mu^{-p})^{1/\ell}$,

$$|\partial_x^\beta\partial_\xi^\alpha w(x,\xi)| \le C_1A_1^{|\alpha+\beta|}|\alpha+\beta|!^s w(x,\xi)w(x,\xi)^{-\ell|\alpha+\beta|/2m}\langle\xi\rangle_\mu^{-|\alpha|},$$

that is, $w\in S_{(s)}(w, w^{-\ell/m}(|dx|^2+\langle\xi\rangle_\mu^{-2}|d\xi|^2))$. *In particular,* $w\in S_{(s)}(w, \langle\xi\rangle_\mu^{p/m}(|dx|^2+\langle\xi\rangle_\mu^{-2}|d\xi|^2))$.

COROLLARY A.5

We have

$$w^{-1}\in S_{(s)}(w^{-1}, \langle\xi\rangle_\mu^{p/m}|dx|^2+\langle\xi\rangle_\mu^{-2+p/m}|d\xi|^2).$$

Proof
Since $|(d/dx)^k x^{-1}| = k!|x|^{-k-1}$ the assertion follows from Lemma A.2. $\square$

Let $\psi(s)$ satisfy

$$|\psi^{(k)}(s)| \le CA^k k!^s, \quad s \in \mathbb{R},$$

and consider

$$\phi = \log\big(\psi(x_1) + iw\big) - \log\big(\psi(x_1) - iw\big).$$

We set

$$r(x,\xi) = \sqrt{\psi(x_1)^2 + w(x,\xi)^2}.$$

LEMMA A.6
Let

$$g^* = \big(r(x,\xi)^{-1} + w^{-\ell/2m}\big)^2 dx_1^2 + \sum_{j=2}^{n} w^{-\ell/m} dx_j^2 + w^{-\ell/m}\langle\xi\rangle_\mu^{-2}|d\xi|^2.$$

Then we have $\phi(x,\xi) \in S_{(s)}(1,g^*)$ *and* $r(x,\xi) \in S_{(s)}(r,g^*)$.

Proof
For $|\alpha+\beta| = 1$ we have

$$\partial_x^\beta \partial_\xi^\alpha \phi = -2ir(x,\xi)^{-2}[w(x,\xi)\partial_x^\beta\partial_\xi^\alpha\psi(x_1) - \psi(x_1)\partial_x^\beta\partial_\xi^\alpha w].$$

Since it is clear that $\psi(x_1) \in S_{(s)}(r,g^*)$ and $w \in S_{(s)}(r,g^*)$, to prove the assertion it is enough to prove $r(x,\xi)^{-2} \in S_{(s)}(r^{-2},g^*)$. Noting that

$$|(d/dx)^k x^{1/2}| \le CA^k k! x^{1/2} x^{-k}, \quad x > 0,$$

and $r^2 = \psi(x_1)^2 + w^2(x,\xi) \in S_{(s)}(r^2,g^*)$, applying Lemma A.2 we get $r \in S_{(s)}(r,g^*)$. Again applying Lemma A.2 we conclude that $r^{-1} \in S_{(s)}(r^{-1},g^*)$, which proves the assertion. $\square$

A.4 Implicit functions $\Xi(x,y,\zeta)$

Let $\phi(x,\xi) \in S_{(s)}(\bar{k}(\xi,\mu),g)$ be real valued, and assume that $\partial_{x_j}\phi(x,\xi) \in S_{(s)}(\bar{k}\Delta_j,g)$, where $\bar{k}$ is assumed to satisfy (A.1) and $\Delta_j \in S_{(s)}(\Delta_j,g)$. Set

$$F(x,y,\zeta) = \int_0^1 \nabla_x\phi\big(y + \theta(x-y),\zeta\big)d\theta, \quad \zeta = \xi + i\eta,$$

where $\nabla_x\phi(x,\zeta)$ is an almost analytic extension of $\nabla_x\phi(x,\xi)$ with $k = \bar{k}\langle\xi\rangle_\mu^{\epsilon'}$ and small $0 < \epsilon' < \epsilon$.

PROPOSITION A.3
There is a C^∞*-function* $\Xi(x,y,\zeta) = \zeta + G(x,y,\zeta)$ *defined for* $(x,\xi,\eta;y) \in E^0(\bar{k}\langle\xi\rangle_\mu^{\epsilon''}) \times \mathbb{R}^n$ $(0 < \epsilon'' < \epsilon')$ *such that*

$$\Xi(x,y,\zeta) = iF\big(x,y,\Xi(x,y,\zeta)\big) + \zeta, \tag{A.2}$$

where $G_j(x,y,\zeta)$ *satisfies, for* $(x,\xi,\eta;y)\in E^0(\bar{k}\langle\xi\rangle_\mu^{\epsilon''})\times\mathbb{R}^n$,

$$|\partial_{x,y}^{\beta}\partial_{\xi,\eta}^{\alpha}G_j(x,y,\zeta)|\le CA^{|\alpha+\beta|}|\alpha+\beta|!^s\bar{k}\langle\xi\rangle_\mu^{a_j}\langle\xi\rangle_\mu^{a\beta}\langle\xi\rangle_\mu^{-b\alpha}.$$

Moreover, we have

$$[\partial_{x,y}^{\beta}\partial_{\xi,\eta}^{\alpha}G_j(x,y,\zeta)]_{y=x}\in S_{(s)}\big(\bar{k}\Delta_j\delta^{\beta}\rho^{-\alpha},g\bigm| E^0(\bar{k}\langle\xi\rangle_\mu^{\epsilon''})\big),\quad |\alpha+\beta|\ge 1.$$

We have also

$$|\partial_{x,y}^{\beta}\partial_{\xi,\eta}^{\alpha}\bar{\partial}_{\zeta_j}\Xi(x,y,\zeta)|\le CA^{|\alpha+\beta|}|\alpha+\beta|!^s he^{-c(h\bar{k}\langle\xi\rangle_\mu^{\epsilon''})^{-1/(s-1)}}\langle\xi\rangle_\mu^{a\beta}\langle\xi\rangle_\mu^{-b\alpha}$$

with some $c>0$ *for* $(x,\xi,\eta;y)\in E^0(\bar{k}\langle\xi\rangle_\mu^{\epsilon''})\times\mathbb{R}^n$.

B

In this section we use

$$\bar{g}(y,\eta)=\sum_{j=1}^{n}\langle\xi\rangle_\mu^{2\delta}y_j^2+\langle\xi\rangle_\mu^{-2\rho}\eta_j^2,\qquad g_\rho(y,\eta)=\sum_{j=1}^{n}y_j^2+\langle\xi\rangle_\mu^{-2\rho}\eta_j^2,$$

where $0<\delta<\rho\le 1$. Let $p(x,\xi)\in S_{(s)}(\langle\mu\xi\rangle^m,\bar{g})$, and let $\phi\in S_{(s)}(\langle\mu\xi\rangle^{\kappa'},\bar{g})$ which is real valued such that

$$\nabla_\xi\phi\in S_{(s)}(\langle\mu\xi\rangle^{\kappa}\langle\xi\rangle_\mu^{-\rho},\bar{g}),\qquad \nabla_x\phi(x,\xi)\in S_{(s)}(\langle\mu\xi\rangle^{\kappa}\langle\xi\rangle_\mu^{\delta},\bar{g}).$$

We assume that ($\kappa'\ge\kappa$, $s>1$)

(B.1) $\qquad (s-1)\kappa',\ (s-1)(1-\rho+\kappa)<\rho-\delta-\kappa,\quad s\kappa'<1-\delta.$

Recall

$$\mathrm{Op}^t(e^{\phi}p)u=(2\pi)^{-n}\int e^{i(x-y)\xi+\phi(ty+(1-t)x,\xi)}p(ty+(1-t)x,\xi)u(y)\,dy\,d\xi,$$

where $0\le t\le 1$. Let us put

$$\Phi(x,\xi,\eta)=\int_0^1\nabla_\xi\phi(x,\xi+\theta\eta)\,d\theta$$

so that $\phi(x,\xi+\eta)-\phi(x,\xi)=\eta\Phi(x,\xi,\eta)$. Then we have the following.

PROPOSITION B.1
Assume (B.1). Then we have

$$\mathrm{Op}^0(e^{\phi})\,\mathrm{Op}^0(p)=\mathrm{Op}^0(e^{\phi}q)+\mathrm{Op}^0(r),$$

where $q\in S_{(s)}(\langle\mu\xi'\rangle^m,\bar{g})$ *and* $r\in S_{(sd)}(e^{-c(\mu^{-1}\langle\mu\xi\rangle)^{(1-\delta)/s}},g_\rho)$ *with* $d=(1+\rho-\delta)/(1-\delta)$, *and one can write*

$$q(x,\xi)=\sum_{|\beta|<N}\frac{1}{\beta!}\partial_\eta^{\beta}p_{(\beta)}\big(x-i\Phi(x,\xi,\eta),\xi\big)|_{\eta=0}+q_N(x,\xi)$$

with $q_N\in\mu^{(\rho-\delta)N}S_{(s)}(\langle\mu\xi\rangle^{m-(\rho-\delta)N},\bar{g})$, *where* $p_{(\beta)}(x+iy,\xi)$ *is the almost analytic extension of* $(-i)^{|\beta|}\partial_x^{\beta}p(x,\xi)$ *given by Proposition A.1.*

Consider $\mathrm{Op}^0(e^{\phi}q)\,\mathrm{Op}^1(e^{-\phi})$, where $q \in S_{(s)}(\langle\mu\xi\rangle^m, \bar{g})$. Let $\Xi(x,y,\xi)$ be the solution to

$$\Xi - i\int_0^1 \nabla_x\phi\big(x+\theta(y-x),\Xi\big)d\theta = \xi$$

which is given by Proposition A.3. Then we have the following.

PROPOSITION B.2

Assume (B.1). Then one can write

$$\mathrm{Op}^0(e^{\phi}q)\,\mathrm{Op}^1(e^{-\phi}) = \mathrm{Op}^0(p) + \mathrm{Op}^0(r),$$

where $p(x,\xi) \in S_{(s)}(\langle\mu\xi\rangle^m, \bar{g})$ *and* $r(x,\xi) \in S_{(sd)}(e^{-c(\mu^{-1}\langle\mu\xi\rangle)^{(1-\delta)/s}}, g_\rho)$, *and we can write*

$$p(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha\big[J(x,y,\xi)q\big(x,\Xi(x,y,\xi)\big)\big]_{y=x} + R_N(x,\xi)$$

with $R_N(x,\xi) \in \mu^{(\rho-\delta)N}S_{(s)}(\langle\mu\xi\rangle^{m-(\rho-\delta)N}, \bar{g})$, *where*

$$J(x,y,\xi) = \det\Big[\frac{\partial\Xi(x,y,\xi)}{\partial\xi}\Big].$$

PROPOSITION B.3

Let $s>1$. *Assume that*

$$p(x,\xi) \in S_{(s)}(e^{c\mu^{-a}\langle\mu\xi\rangle^{\kappa}}\langle\mu\xi\rangle^{m_1}, \bar{g}), \qquad q(x,\xi) \in S_{(s)}(e^{c'\mu^{-a}\langle\mu\xi\rangle^{\kappa}}\langle\mu\xi\rangle^{m_2}, \bar{g})$$

with $c+c'<0$ *and some* $a>0$. *Then one can write*

$$\mathrm{Op}^0(p)\,\mathrm{Op}^1(q) = \mathrm{Op}^0(r_1) + \mathrm{Op}^0(r_2)$$

with $r_1(x,\xi) \in S_{(s)}(\langle\mu\xi\rangle^{m_1+m_2}e^{-c_1\mu^{-a}\langle\mu\xi\rangle^{\kappa}}, \bar{g})$, $r_2(x,\xi) \in S_{(sd)}(e^{-c_2(\mu^{-1}\langle\mu\xi\rangle)^{(1-\delta)/s}}, g_\rho)$ *with some* $c_i>0$. *In particular, we have* $\mathrm{Op}^0(p)\,\mathrm{Op}^1(q) \in \mu^k S_{(sd)}(\langle\mu\xi\rangle^{-k}, \bar{g})$ *for any* $k \in \mathbb{N}$.

COROLLARY B.1

Assume (B.1) and $p(x,\xi) \in S_{(s)}(\langle\mu\xi\rangle^m, \bar{g})$. *Then we have*

$$\mathrm{Op}^0(e^{-\phi})\,\mathrm{Op}^0(p)\,\mathrm{Op}^1(e^{\phi}) = \mathrm{Op}^0(\tilde{p}) + \mathrm{Op}^0(r),$$

where $\tilde{p}(x,\xi) \in S_{(s)}(\langle\mu\xi\rangle^m, \bar{g})$ *and* $r(x,\xi) \in \mu^k S_{(sd^2)}(\langle\mu\xi\rangle^{-k}, \bar{g})$ *for any* $k \in \mathbb{N}$.

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