p-Adic period domains and toroidal partial compactifications, I

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Abstract We construct toroidal partial compactifications of p-adic period domains.

Contents

1. Introduction

1.1

For classifying spaces of Hodge structures, toroidal partial compactifications have been constructed in [20] and [14, Part III]. (Summaries are given in [18], [15]–[17].)

The aim of this paper is to show that a similar theory exists over p-adic local fields, replacing Hodge structures by p-adic Hodge structures.

We describe what we construct in this paper, comparing the theory over $\mathbb{C}$ and the theory over a p-adic local field. To make the descriptions simple, we will be sometimes rather rough in this introduction, but we will try to present the ideas clearly. We compare Hodge structures and p-adic Hodge structures in Sections 1.2–1.4; we compare the Hodge conjecture and Fontaine's p-adic Hodge conjecture in Section 1.5; we compare classifying spaces of Hodge structures (so-called period domains) and classifying spaces of p-adic Hodge structures (p-adic period domains) in Sections 1.6–1.7; and we compare the toroidal partial compactifications of the period domains and those of p-adic period domains in Sections 1.8–1.11.

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1.2. Hodge structures

Roughly speaking, a Hodge structure is a pair (H,F) of a finitely generated $\mathbb{Z}$-module H and a decreasing filtration F on $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H$, satisfying certain conditions. For a proper smooth scheme X over $\mathbb{C}$ and for $m \in \mathbb{Z}$, we have a Hodge structure (H,F), where $H = H^m(X(\mathbb{C}),\mathbb{Z})$ and F is the Hodge filtration on $H_{\mathbb{C}}$.

1.3. p-Adic Hodge structures

Let K be a finite extension of $\mathbb{Q}_p$. We consider p-adic Hodge structures over K.

Let k be the residue field of K. Let K_0 be the largest unramified extension of $\mathbb{Q}_p$ contained in K, which is identified with the field of fractions of the ring $W(k)$ of Witt vectors. The pth power map $k \to k, x \mapsto x^p$ induces an automorphism $\varphi : W(k) \to W(k)$ of the ring $W(k)$ and induces an automorphism $\varphi : K_0 \to K_0$ of the field K_0.

Roughly speaking, a p-adic Hodge structure over K is a triple (H,N,F) consisting of a finite-dimensional K_0-vector space H endowed with a bijective Frobenius-linear operator $\varphi : H \to H$ (Frobenius-linear means that $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(ax) = \varphi(a)\varphi(x)$ for $x, y \in H$ and $a \in K_0$), a K_0-linear map $N : H \to H$ such that $N\varphi = p\varphi N$ (from this condition, we have that N is nilpotent), and a decreasing filtration F on $H_K := K \otimes_{K_0} H$ satisfying certain admissibility conditions formulated by Fontaine [8] (see Section 3.2 of this paper for a review).

For a proper smooth scheme X over K with semistable reduction and for $m \in \mathbb{Z}$, we have a p-adic Hodge structure (H,N,F), similar to Hodge theory. If X is of good reduction and Y denotes the reduction of X over k, then H is the crystalline cohomology $H^m_{\mathrm{crys}}(Y)$, $\varphi : H \to H$ is induced from the pth power morphism $Y \to Y$, $N = 0$, and F is the Hodge filtration on the de Rham cohomology $H^m_{\mathrm{dR}}(X/K) = K \otimes_{K_0} H^m_{\mathrm{crys}}(Y)$, where the last $=$ is by the Berthelot-Ogus isomorphism [3]. In the general, semistable reduction case, the crystalline cohomology should be replaced by log crystalline cohomology (see [12]), and N need not be zero.

1.4. Examples: Elliptic curves

As an example, we compare Hodge structures and p-adic Hodge structures of elliptic curves.

We first consider elliptic curves over $\mathbb{C}$. An elliptic curve over $\mathbb{C}$ is isomorphic to the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ over $\mathbb{C}$ for some element τ of the upper half-plane (i.e., the imaginary part $\mathrm{Im}(\tau)$ of τ is > 0). Assume that $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. Consider the case $X = E$ and $m = 1$ in Section 1.2. Then $H = H^1(E(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}^2 = \mathbb{Z}e_1 + \mathbb{Z}e_2$, where (e_1, e_2) is the dual base of the base of $(\tau, 1)$ of the $\mathbb{Z}$-dual $H_1(E(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}\tau + \mathbb{Z}$ of $H^1(E(\mathbb{C}),\mathbb{Z})$. The Hodge filtration F on $H_{\mathbb{C}}$ is given by

$$0 = F^2 \subset \mathbb{C}(\tau e_1 + e_2) = F^1 \subset H_{\mathbb{C}} = F^0.$$

Next, let K be a finite extension of $\mathbb{Q}_p$, and let E be the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ over K, where q is an element of K with p-adic absolute value $|q|_p < 1$ and where $q^{\mathbb{Z}}$ denotes the discrete subgroup of $K^\times$ generated by q. Note that E is of semistable reduction but of bad reduction. This presentation of the Tate elliptic curve as a quotient of $K^\times$ is a p-adic analogue of the presentation of the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ over $\mathbb{C}$ as a quotient of $\mathbb{C}^\times$ via the isomorphism $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}, z \mapsto e^{2\pi i z}$ with $q = e^{2\pi i \tau}$. Consider the case $X = E$ and $m = 1$ in Section 1.3. Then

$$H = K_0^2 = K_0 e_1 + K_0 e_2,$$

$$\varphi(e_1) = e_1, \qquad \varphi(e_2) = pe_2, \qquad N(e_1) = 0, \qquad N(e_2) = \mathrm{ord}_p(q)e_1,$$

$$0 = F^2 \subset K\bigl(\log(q)e_1 + e_2\bigr) = F^1 \subset H_K = F^0.$$

Here

$$\mathrm{ord}_p : K^\times \to \mathbb{Q}, \qquad \log : K^\times \to K$$

are the homomorphisms characterized by the following properties (see Section 3.1.4): ord_p kills the unit group $(O_K)^\times$ of the valuation ring O_K of K, $\mathrm{ord}_p(p) = 1$, the restriction of log to $\mathrm{Ker}((O_K)^\times \to k^\times)$ coincides with the usual p-adic logarithm $x \mapsto \sum_{n=1}^{\infty}(-1)^{n-1}(x-1)^n/n$, and $\log(p) = 0$.

1.5. Hodge conjecture and p-adic Hodge conjecture

In this paper (see Section 3.4), we introduce the p-adic Hodge conjecture of Fontaine, which is a p-adic analogue of the Hodge conjecture, to present the philosophy of this paper clearly by using it. Here in this introduction, we introduce this conjecture shortly to observe the nature of the analogy between the theory over $\mathbb{C}$ and the theory over K and also to write about this conjecture in Section 1.14.

Let X be a proper smooth scheme over $\mathbb{C}$, and let $r \in \mathbb{Z}$. Then the Hodge conjecture predicts that the cycle map

$$\mathrm{CH}^r(X)_{\mathbb{Q}} \to H^{2r}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr) \cap F^r$$

is surjective, where F^r is the Hodge filtration.

Let K be a finite extension of $\mathbb{Q}_p$, and let X be a proper smooth scheme over K with good reduction. Let Y be the reduction. Then the p-adic Hodge conjecture of Fontaine predicts that the cycle map

$$\mathrm{CH}^r(X)_{\mathbb{Q}} \to \bigl(\text{Image of } \mathrm{CH}^r(Y)_{\mathbb{Q}} \to H^{2r}_{\mathrm{crys}}(Y)\bigr) \cap F^r$$

is surjective, where F^r is the Hodge filtration on $H^{2r}_{\mathrm{dR}}(X/K) = K \otimes_{K_0} H^{2r}_{\mathrm{crys}}(Y)$.

This is an old conjecture of Fontaine, but is not yet written in the literature to the knowledge of the author. In Section 3.4, we introduce the above p-adic Hodge conjecture again (Conjecture 3.4.2) with its variant (Conjecture 3.4.9).

1.6. Griffiths period domains D

Roughly speaking, the period domain of Griffiths (the classifying space of Hodge structures) is defined in the following way. Fix a finitely generated free $\mathbb{Z}$-module H. We also fix Hodge numbers. We also have to fix a bilinear form on $H_{\mathbb{Q}}$ to treat polarizations, but this is not explained in detail in this introduction, for simplicity. The Griffiths period domain D is the set of all decreasing filtrations F on $H_{\mathbb{C}}$ such that (H, F) is a Hodge structure with the fixed Hodge numbers polarized by the given bilinear form (see Section 2.2 for details).

In a special case, $H = \mathbb{Z}^2 = \mathbb{Z}e_1 + \mathbb{Z}e_2$ and we have $D = \mathfrak{h}$, where $\mathfrak{h}$ is the upper half-plane and $\tau \in \mathfrak{h}$ corresponds to $F = F(\tau) \in D$ defined by $0 = F^2 \subset \mathbb{C}(\tau e_1 + e_2) = F^1 \subset H_{\mathbb{C}} = F^0$ (see Section 2.2, Example b). This $F = F(\tau)$ is identified with the Hodge filtration of the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$.

1.7. p-Adic period domains D

The p-adic period domain (the classifying space of p-adic Hodge structures) is defined in the following way. Fix a finite-dimensional K_0-vector space H endowed with a bijective Frobenius-linear map $\varphi : H \to H$. Fix also Hodge numbers. Then, roughly speaking, the p-adic period domain D is the space of pairs (N, F), where N is a K_0-linear map $H \to H$ such that $N\varphi = p\varphi N$ and where F is a decreasing filtration on H_K such that (H, N, F) is a p-adic Hodge structure with fixed Hodge numbers (see Section 5.2 for details).

In a special case related to Tate elliptic curves, $H = K_0^2 = K_0 e_1 + K_0 e_2$ with $\varphi(e_1) = e_1$ and $\varphi(e_2) = pe_2$, and the p-adic Hodge structure of the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ in Section 1.4 determines a point (N, F) of D, where N (resp., F) is described as in Section 1.4 by using $\mathrm{ord}_p(q)$ (resp., $\log(q)$; see Section 5.5, Example b).

We study degeneration in this paper. The elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ over $\mathbb{C}$ and the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ over K degenerate when q tends to zero. Over $\mathbb{C}$, $q = e^{2\pi i\tau} \to 0$ means that $\mathrm{Im}(\tau)$ tends to ∞. Over K, $q \to 0$ means that $\mathrm{ord}_p(q)$ tends to ∞. Thus in this analogy, $\mathrm{Im}(\tau)$ in the theory over $\mathbb{C}$ is similar to $\mathrm{ord}_p(q)$ in the theory over K. It seems that the real part $\mathrm{Re}(\tau)$ in the theory over $\mathbb{C}$ is similar to $\log(q)$ in the theory over K.

The p-adic period domains were studied by Rapoport and Zink [23]–[25]. They considered the case $N = 0$. We need to consider N because nontrivial N always appears in degeneration.

1.8. Toroidal partial compactifications $\Gamma \setminus D \subset \Gamma \setminus D_\Sigma$ over $\mathbb{C}$

In [20] and [14, Part III], a toroidal partial compactification $\Gamma \setminus D_\Sigma$ of the quotient space $\Gamma \setminus D$ of the period domain D is constructed. (Precisely speaking, the period domain D in [14, Part III] is the mixed Hodge theoretic version (see [28]) of the Griffiths period domain.) Here Γ is a discrete group acting on D, and Σ is a collection of monodromy cones, which satisfy certain conditions. These works are attempts to generalize the toroidal compactification in [1] to general Hodge type.

In a special case related to elliptic curves (see Section 2.2, Example b), the diagram

$$D \to \Gamma \backslash D \subset \Gamma \backslash D_\Sigma \tag{1}$$

becomes

$$\mathfrak{h} \to \Delta^* \subset \Delta.$$

Here

$$\Delta = \{q \in \mathbb{C} \mid |q| < 1\}, \qquad \Delta^* = \Delta - \{0\}, \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix},$$

and we identify Δ^* with $\Gamma \backslash \mathfrak{h}$ via $\mathfrak{h} \to \Delta^*; \tau \mapsto e^{2\pi i \tau}$. The Hodge structure of the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ gives the point $q = e^{2\pi i\tau}$ of Δ^*. When q tends to zero, the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ and its Hodge structure degenerate, and the corresponding point $q \in \Delta^*$ converges to $0 \in \Delta$ in the toroidal partial compactification.

1.9. Toroidal partial compactifications ${}_\Gamma D \subset {}_\Gamma D_\Sigma$ over K

The aim of this paper is to show that for the p-adic period domain D over the p-adic local field K, we have a diagram

$$D \leftarrow {}_\Gamma D \subset {}_\Gamma D_\Sigma, \tag{2}$$

which is similar to diagram (1) in Section 1.8. Note that the first arrow in (2) has the converse direction when compared with (1).

In a special case related to Tate elliptic curves (see Section 5.5, Example b), this diagram becomes

$$D \leftarrow \Delta^* \subset \Delta,$$

where

$$\Delta = \{q \in K \mid |q|_p < 1\}, \qquad \Delta^* = \Delta - \{0\}.$$

The first arrow sends $q \in \Delta^*$ to $(N, F) \in D$, where (N, F) is the p-adic Hodge structure associated to the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ described in Section 1.4. This first arrow $\leftarrow$ is essentially $q \mapsto (\mathrm{ord}_p(q), \log(q))$, while the analogous arrow over $\mathbb{C}$ was $\tau \mapsto e^{2\pi i\tau}$ in the converse direction. When $q \to 0$, the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ and its p-adic Hodge structure degenerate, and the corresponding point $q \in \Delta^*$ converges to $0 \in \Delta$ in the toroidal partial compactification.

In the special case related to elliptic curves over $\mathbb{C}$ (resp., Tate elliptic curves over a p-adic local field K), the spaces D, $\Gamma \backslash D$, and $\Gamma \backslash D_\Sigma$ (resp., D, ${}_\Gamma D$, and ${}_\Gamma D_\Sigma$) are described in detail in Example b in Sections 2.2.4 and 2.3.13 (resp., Example b in Sections 5.5.2 and 6.6.2).

1.10. More about ${}_\Gamma D$

We give more explanations about ${}_\Gamma D$ in p-adic theory (see Section 5 for details; Γ is a compact adelic group, but we do not explain it here).

In p-adic Hodge theory, a very important theorem is that the category of p-adic Hodge structures is equivalent to the category of semistable p-adic representations of $\mathrm{Gal}(\bar{K}/K)$, where $\bar{K}$ denotes the algebraic closure of K (see [5]; see Section 3.2.6 of this paper for a review). In this equivalence, for a proper smooth scheme X over K of semistable reduction, the mth p-adic Hodge structure of X (see Section 1.3) corresponds to the étale cohomology $H^m_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Q}_p)$ regarded as a p-adic representation of $\mathrm{Gal}(\bar{K}/K)$. Note that Galois representations have nice integral structures (e.g., the integral structure $H^m_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Z}_p)$ of $H^m_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Q}_p)$) and often some level structures. (A level structure is a finer version of an integral structure; see Section 4.3.)

Roughly speaking, the space ${}_\Gamma D$ is defined to be the set of triples (N, F, μ), where $(N, F) \in D$ and μ is a level structure on a related Galois representation. The canonical arrow $D \leftarrow {}_\Gamma D$ is given by $(N, F, \mu) \mapsto (N, F)$ forgetting the level structure μ.

In the special case related to Tate elliptic curves $E = K^\times/q^{\mathbb{Z}}$, $q \in \Delta^*$ corresponds to $(N, F, \mu) \in {}_\Gamma D$ with (N, F) associated to E as in Section 1.4 and with μ associated to the integral structures $H^1_{\text{ét}}(E \otimes_K \bar{K}, \mathbb{Z}_\ell)$ of $H^1_{\text{ét}}(E \otimes_K \bar{K}, \mathbb{Q}_\ell)$, where ℓ ranges over all prime numbers.

Level structures were considered by Rapoport and Zink [25] in their study of p-adic period domains related to p-divisible groups. In this case, involved p-adic Hodge structures satisfy $F^r = H_K$ and $F^{r+2} = 0$ for some $r \in \mathbb{Z}$. In this paper, we can treat any Hodge type.

We define a structure of an analytic manifold over K on ${}_\Gamma D$. (We do not define a rigid analytic structure but define only a naive analytic structure. The attempt to define something like a rigid analytic structure will be given in the later part of this series of papers.)

1.11. More about ${}_\Gamma D_\Sigma$

We give more explanations about ${}_\Gamma D_\Sigma$ in p-adic theory (see Section 6 for more details).

In the theory (see [20], [14, Part III]) for Hodge structures, the toroidal partial compactifications $\Gamma \backslash D_\Sigma$ are obtained by adding nilpotent orbits to $\Gamma \backslash D$ as points at infinity. This is a natural idea because the infinity of the period domain should consist of limit points of degenerating families of Hodge structures, and nilpotent orbits are associated to such families (see [27]). In the p-adic theory in this paper, we have the notion of a p-adic nilpotent orbit (see Sections 6.1, 6.2), and to obtain toroidal partial compactifications ${}_\Gamma D_\Sigma$, we add p-adic nilpotents to ${}_\Gamma D$ as points at infinity.

In the theory for Hodge structures, it is proved in [20] and [14, Part III] that $\Gamma \backslash D_\Sigma$ is not necessarily a complex analytic space but is a log manifold which is like a complex analytic manifold with slits. Similarly, in this paper, we prove that ${}_\Gamma D_\Sigma$ in p-adic theory is a p-adic log manifold (see Theorem 6.5.6), which is like a p-adic analytic manifold with slits.

1.12

The organization of this paper is as follows. In Section 2, we review toroidal partial compactifications of period domains over $\mathbb{C}$ constructed in [20] and [14]. In Section 3, we review the theory of p-adic Hodge structures, we introduce the p-adic Hodge conjecture of Fontaine, and we give (in Section 3.5) some results on p-adic Hodge structures which we use in Section 6. In Sections 4 and 5, we consider p-adic period domains D and ${}_{\Gamma}D$. In Section 6, we construct toroidal partial compactifications ${}_{\Gamma}D_{\Sigma}$ of ${}_{\Gamma}D$.

1.13

This paper is part I of series of papers. In [19], [20], and [14], in the theory over $\mathbb{C}$, various kinds of enlargements of D were constructed with maps between them as in the following diagram.

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \xrightarrow{\subset} & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \downarrow & & \downarrow \\ D_{\Sigma,\mathrm{val}} & \longleftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \longrightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}}. \\ \downarrow & & \downarrow & & & & \\ D_{\Sigma} & \longleftarrow & D^{\sharp}_{\Sigma} & & & & \end{array}$$

In the later part of this series, we study a p-adic analogue of a part of this diagram.

1.14

In a later part of the series, we review the theory of p-adic intermediate Jacobians of [26] from our point of view and study degenerations of p-adic intermediate Jacobians. This study of degeneration is a p-adic analogue of the study [15] over $\mathbb{C}$. Degenerations of intermediate Jacobians over $\mathbb{C}$ are now studied intensively (see, e.g., [22]). We hope that the complex analytic study and the p-adic study stimulate each other. Degeneration of the intermediate Jacobian over $\mathbb{C}$ is related to the Hodge conjecture (see [22]). We hope that degeneration of the p-adic intermediate Jacobian is related to the p-adic Hodge conjecture.

2. Review of the theory over $\mathbb{C}$

In Section 2.2, we review classifying spaces D of mixed Hodge structures and the quotient spaces $\Gamma \setminus D$. In Section 2.3, we review toroidal partial compactifications $\Gamma \setminus D_{\Sigma}$ of $\Gamma \setminus D$. The p-adic version of Section 2.2 is given in Sections 4 and 5, and the p-adic version of Section 2.3 is given in Section 6.

We explain the theory over $\mathbb{C}$ using Examples a–d (Sections 2.2.3–2.2.6). The corresponding Examples a–d in p-adic theory are given in Sections 5.5 and 6.6.

2.1. Notation in Section 2

2.1.1

In Section 2, fix a quadruple

$$\big(H, W, (\langle\ ,\ \rangle_w)_{w\in\mathbb{Z}}, (h_{w,i})_{w,i\in\mathbb{Z}}\big),$$

where

- H is a finitely generated free $\mathbb{Z}$-module;
- W is an increasing filtration on the $\mathbb{Q}$-vector space $H_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} H$ such that $W_w = H_{\mathbb{Q}}$ for $w \gg 0$ and $W_w = 0$ for $w \ll 0$;
- $\langle\ ,\ \rangle_w$ for each $w \in \mathbb{Z}$ is a nondegenerate $\mathbb{Q}$-bilinear form $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbb{Q}$ which is symmetric if w is even and antisymmetric if w is odd;
- $h_{w,i}$ are nonnegative integers which are zero for almost all (w,i), such that $\sum_i h_{w,i} = \dim(\mathrm{gr}^W_w)$ for all w and $h_{w,i} = h_{w,w-i}$ for all (w,i).

2.2. Period domains D and $\Gamma \setminus D$

We review the definition of the classifying space D of mixed Hodge structures with polarized graded quotients, defined by Usui [28]. This is the mixed Hodge theoretic version of the classifying space in the pure case defined by Griffiths [11].

2.2.1

As in Usui [28], let D be the set of all decreasing filtrations F on $H_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Z}} H$ satisfying the following conditions (i)–(iii).

(i) (H, W, F) is a mixed Hodge structure.

(ii) For any $w \in \mathbb{Z}$, the Hodge structure $((H \cap W_w)/(H \cap W_{w-1}), \mathrm{gr}^W_w(F))$ of weight w is polarized by $\langle\ ,\ \rangle_w$ for any w.

Here $\mathrm{gr}^W_w(F)$ denotes the filtration on $\mathrm{gr}^W_{w,\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Q}} \mathrm{gr}^W_w$ induced by F.

(iii) We have $h_{w,i} = \dim \mathrm{gr}^i_F \mathrm{gr}^W_{w,\mathbb{C}}$ for any $w, i \in \mathbb{Z}$.

On the other hand, let $\check{D}$ be the set of all decreasing filtrations F on $H_{\mathbb{C}}$ satisfying the above condition (iii) and the following condition (ii′).

(ii′) For any $w, i \in \mathbb{Z}$, the annihilator of $\mathrm{gr}^W_w(F^i)$ in $\mathrm{gr}^W_{w,\mathbb{C}}$ for the pairing $\langle\ ,\ \rangle_w$ coincides with $\mathrm{gr}^W_w(F^{w+1-i})$.

Then $\check{D}$ has a natural structure of a complex analytic manifold, and D is an open set of $\check{D}$. By this, D has a structure of a complex analytic manifold.

2.2.2

For $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let

$$G_R = \big\{ g \in \mathrm{Aut}_R(H_R) \;\big|\; g \text{ respects } W \text{ and } \langle\ ,\ \rangle_w \text{ for all } w \in \mathbb{Z} \big\}.$$

Then $G_{\mathbb{C}}$ acts naturally on $\check{D}$, and $G_{\mathbb{R}}$ acts naturally on D.

For a torsion-free subgroup Γ of $G_{\mathbb{Z}}$, the quotient space $\Gamma \backslash D$ is a complex analytic manifold, and the projection $D \to \Gamma \backslash D$ is locally an isomorphism of complex analytic manifolds.

2.2.3. Example a (The multiplicative group $\mathbb{C}^\times$ appears here.)
Define the quadruple $(H, W, (\langle\, ,\, \rangle_w)_w, (h_{w,i}))$ as follows:

- H is a free $\mathbb{Z}$-module of rank 2 with basis e_1, e_2;
- W is the increasing filtration on $H_{\mathbb{Q}}$ defined by

$$0 = W_{-3} \subset \mathbb{Q}e_1 = W_{-2} = W_{-1} \subset H_{\mathbb{Q}} = W_0;$$

- $\langle\, ,\, \rangle_w$ and $h_{w,i}$ are defined as

$$\langle e_1, e_1 \rangle_{-2} = 1, \qquad \langle e_2, e_2 \rangle_0 = 1,$$

where we denote the element e_2 mod W_{-1} of gr_0^W simply by e_2,

$$h_{0,0} = h_{-2,-1} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

Then we have an isomorphism

$$D \cong \mathbb{C},$$

where $z \in \mathbb{C}$ corresponds to the following $F = F(z) \in D$:

$$0 = F^1 \subset \mathbb{C}(ze_1 + e_2) = F^0 \subset H_{\mathbb{C}} = F^{-1}.$$

For

$$\Gamma = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset G_{\mathbb{Z}},$$

we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \cong & D \\ \downarrow & & \downarrow \\ \mathbb{C}^\times & \cong & \Gamma \backslash D \end{array}$$

Here the left vertical arrow is $z \mapsto \exp(2\pi i z)$.

2.2.4. Example b (The upper half-plane $\mathfrak{h}$ and the unit disc Δ^* without the origin appear here.)
Define the quadruple $(H, W, (\langle\, ,\, \rangle_w)_w, (h_{w,i}))$ as follows:

- H is a free $\mathbb{Z}$-module of rank 2 with basis e_1, e_2;
- W is the increasing filtration on $H_{\mathbb{Q}}$ defined by

$$0 = W_{-2} \subset H_{\mathbb{Q}} = W_{-1};$$

- $\langle\, ,\, \rangle_w$ and $h_{w,i}$ are defined by

$$\langle e_2, e_1 \rangle_{-1} = 1,$$

$$h_{-1,0} = h_{-1,-1} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

For $\tau \in \mathbb{C}$, define $F = F(\tau) \in \check{D}$ by

$$0 = F^1 \subset \mathbb{C}(\tau e_1 + e_2) = F^0 \subset H_{\mathbb{C}} = F^{-1}.$$

Then we have an isomorphism

$$D \cong \mathfrak{h} := \{x + iy \mid x, y \in \mathbb{R}, y > 0\},$$

where $\tau = x + iy \in \mathfrak{h}$ corresponds to $F(\tau) \in D$.

For

$$\Gamma = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix},$$

we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{h} & \cong & D \\ \downarrow & & \downarrow \\ \Delta^* & \cong & \Gamma \backslash D \end{array}$$

Here

$$\Delta^* = \Delta - \{0\} \quad \text{with } \Delta = \big\{q \in \mathbb{C} \,\big|\, |q| < 1\big\}.$$

The left vertical arrow is $\tau \mapsto \exp(2\pi i \tau)$.

The example of D in the introduction related to elliptic curves over $\mathbb{C}$ is essentially this Example b. Precisely, it is the (-1)-Tate twist of this Example b. In general, Tate twists D, $\Gamma \backslash D$, and also $\Gamma \backslash D_\Sigma$ in Section 2.3 are canonically isomorphic to the original D, $\Gamma \backslash D$, and $\Gamma \backslash D_\Sigma$, respectively.

2.2.5. Example c (The universal elliptic curve appears here.)
Define the quadruple $(H, W, (\langle\, , \,\rangle_w)_w, (h_{w,i}))$ as follows:

- H is a free $\mathbb{Z}$-module of rank 3 with basis e_1, e_2, e_3;
- W is the increasing filtration on $H_{\mathbb{Q}}$ defined by

$$0 = W_{-2} \subset \mathbb{Q}e_1 + \mathbb{Q}e_2 = W_{-1} \subset H_{\mathbb{Q}} = W_0;$$

- $\langle\, , \,\rangle_w$ and $h_{w,i}$ are defined by

$$\langle e_2, e_1 \rangle_{-1} = 1, \qquad \langle e_3, e_3 \rangle_0 = 1,$$

where we denote the element e_3 mod W_{-1} of gr^W_0 simply by e_3,

$$h_{0,0} = h_{-1,0} = h_{-1,-1} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

For $\tau, z \in \mathbb{C}$, define $F = F(\tau, z) \in \check{D}$ by

$$0 = F^1 \subset \mathbb{C}(\tau e_1 + e_2) + \mathbb{C}(z e_1 + e_3) = F^0 \subset H_{\mathbb{C}} = F^{-1}.$$

Then we have an isomorphism

$$D \cong \mathfrak{h} \times \mathbb{C},$$

where $(\tau, z) \in \mathfrak{h} \times \mathbb{C}$ corresponds to $F(\tau, z) \in D$.

Let

$$\Gamma_1 = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

With the identification $D = \mathfrak{h} \times \mathbb{C}$, the fiber of $\Gamma_1 \backslash D \to \mathfrak{h}$ on $\tau \in \mathfrak{h}$ is the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{h} \times \mathbb{C} & \cong & D \\ \downarrow & & \downarrow \\ \Delta^* \times \mathbb{C}^\times & \cong & \Gamma_2 \backslash D \\ \downarrow & & \downarrow \\ \bigcup_{q \in \Delta^*} \mathbb{C}^\times / q^{\mathbb{Z}} & \cong & \Gamma_3 \backslash D \end{array}$$

The upper left vertical arrow is $(\tau, z) \mapsto (\exp(2\pi i \tau), \exp(2\pi i z))$.

The fibrations $\Gamma_1 \backslash D \to \mathfrak{h}$ and $\Gamma_3 \backslash D \to \Delta^*$ are the so-called universal elliptic curves.

2.2.6. Example d (The dilog function appears in this example; see Section 2.3.15.) Define the quadruple $(H, W, (\langle\, ,\, \rangle_w)_w, (h_{w,i}))$ as follows:

- H is a free $\mathbb{Z}$-module of rank 3 with basis e_1, e_2, e_3;
- W is the increasing filtration on $H_{\mathbb{Q}}$ defined by

$$0 = W_{-5} \subset \mathbb{Q}e_1 = W_{-4} = W_{-3} \subset W_{-3} + \mathbb{Q}e_2 = W_{-2} = W_{-1} \subset H_{\mathbb{Q}} = W_0;$$

- $\langle\, ,\, \rangle_w$ and $h_{w,i}$ are defined by

$$\langle e_1, e_1 \rangle_{-4} = 1, \qquad \langle e_2, e_2 \rangle_{-2} = 1, \qquad \langle e_3, e_3 \rangle_0 = 1,$$

where we denote the element e_2 mod W_{-3} of gr^W_{-2} and the element e_3 mod W_{-1} of gr^W_0 simply by e_2 and e_3, respectively,

$$h_{0,0} = h_{-2,-1} = h_{-4,-2} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

Then we have an isomorphism

$$D \cong \mathbb{C}^3,$$

where $(z_1, z_2, z_3) \in \mathbb{C}^3$ corresponds to the following $F = F(z_1, z_2, z_3) \in D$:

$$0 = F^{-1} \subset \mathbb{C}(z_1 e_1 + z_2 e_2 + e_3) = F^0 \subset F^0 + \mathbb{C}(z_3 e_1 + e_2) = F^{-1} \subset H_{\mathbb{C}} = F^{-2}.$$

Let

$$\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\Gamma \setminus D \cong \Gamma \setminus \begin{pmatrix} 1 & \mathbb{C} & \mathbb{C} \\ 0 & 1 & \mathbb{C} \\ 0 & 0 & 1 \end{pmatrix}, \qquad F(z_1, z_2, z_3) \leftrightarrow \begin{pmatrix} 1 & z_3 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The fibration $\Gamma \setminus D \to \mathbb{C}^\times \times \mathbb{C}^\times, F(z_1, z_2, z_3) \mapsto (\exp(2\pi i z_3), \exp(2\pi i z_2))$ is a $\mathbb{C}/\mathbb{Z}$-torsor.

2.3. Toroidal partial compactifications

We review the toroidal partial compactification $\Gamma \setminus D_\Sigma$ of $\Gamma \setminus D$ constructed in [20] and [14] ([20] treats the pure case, i.e., the case $W_w = H_\mathbb{Q}$ and $W_{w-1} = 0$ for some $w \in \mathbb{Z}$).

2.3.1

For $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let $\mathfrak{g}_R$ be the set of all R-linear maps $N : H_R \to H_R$ such that $N(W_{w,R}) \subset W_{w,R}$ for all $w \in \mathbb{Z}$ and such that

$$\langle \mathrm{gr}_w^W(N)x, y\rangle_w + \langle x, \mathrm{gr}_w^W(N)y\rangle_w = 0 \quad \text{for all } w \in \mathbb{Z}, x, y \in \mathrm{gr}_{w,R}^W.$$

2.3.2

We review the notion of *relative monodromy filtration* (see [6, Section 1.6.13]). For an abelian group A with an increasing filtration W and for a nilpotent homomorphism $N : A \to A$ such that $N(W_w) \subset W_w$ for all $w \in \mathbb{Z}$, an increasing filtration M on A is called a *relative monodromy filtration of N with respect to W* if $N(M_w) \subset M_{w-2}$ for all $w \in \mathbb{Z}$ and

$$N^m : \mathrm{gr}_{w+m}^M \mathrm{gr}_w^W \xrightarrow{\cong} \mathrm{gr}_{w-m}^M \mathrm{gr}_w^W$$

for all $w \in \mathbb{Z}$ and all integers $m \geq 0$. A relative monodromy filtration of N with respect to W is unique if it exists. If it exists, it is denoted by $M(N, W)$.

2.3.3

We call a subset σ of $\mathfrak{g}_\mathbb{Q}$ a *nilpotent cone* in $\mathfrak{g}_\mathbb{Q}$ if the following conditions (i)–(iii) are satisfied.

(i) The set σ is a finitely generated $\mathbb{Q}_{\geq 0}$-cone. That is,

$$\sigma = \mathbb{Q}_{\geq 0}N_1 + \cdots + \mathbb{Q}_{\geq 0}N_n$$

for some $n \geq 0$ and $N_1, \ldots, N_n \in \mathfrak{g}_\mathbb{Q}$.

(ii) Any element of σ is nilpotent as a linear map $H_\mathbb{Q} \to H_\mathbb{Q}$.

(iii) We have $NN' = N'N$ for any $N, N' \in \sigma$.

For a nilpotent cone σ in $\mathfrak{g}_\mathbb{Q}$ and for $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let σ_R be the R-linear span of σ in $\mathfrak{g}_R$.

2.3.4. Nilpotent orbit

Let σ be a nilpotent cone in $\mathfrak{g}_\mathbb{Q}$, and let Z be a subset of $\check{D}$. We say that Z is a σ-nilpotent orbit if the following conditions (i)–(iv) are satisfied.

(i) Z is an orbit under the action of $\exp(\sigma_{\mathbb{C}})$ on $\check{D}$. That is, there is $F \in \check{D}$ such that $Z = \exp(\sigma_{\mathbb{C}})F$.

(ii) We have $NF^r \subset F^{r-1}$ for any $N \in \sigma$, $F \in Z$, and $r \in \mathbb{Z}$ (Griffiths transversality).

(iii) Write $\sigma = \mathbb{Q}_{\geq 0}N_1 + \cdots + \mathbb{Q}_{\geq 0}N_n$. Let $F \in Z$. Then if $z_1, \ldots, z_n \in \mathbb{C}$ and if $\mathrm{Im}(z_j) \gg 0$, we have $\exp\bigl(\sum_{j=1}^n z_j N_j\bigr)F \in D$.

(iv) For any $N \in \sigma$, the relative monodromy filtration $M(N, W)$ exists. (This is a condition on σ.)

2.3.5

By a *fan* in $\mathfrak{g}_{\mathbb{Q}}$, we mean a nonempty set Σ of nilpotent cones in $\mathfrak{g}_{\mathbb{Q}}$ satisfying the following conditions (i)–(iii).

(i) All elements of Σ are sharp. That is, $\sigma \cap (-\sigma) = \{0\}$ for any $\sigma \in \Sigma$.
(ii) If $\sigma \in \Sigma$, then all faces of σ belong to Σ.
(iii) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of σ.

By a *weak fan* in $\mathfrak{g}_{\mathbb{Q}}$, we mean a nonempty set Σ of nilpotent cones in $\mathfrak{g}_{\mathbb{Q}}$ satisfying the above conditions (i) and (ii) and the following condition (iii′).

(iii′) Let $\sigma, \sigma' \in \Sigma$, and assume that σ and σ' have a common interior point. Assume further that there are a σ-nilpotent orbit Z and a σ'-nilpotent orbit Z' such that $Z \cap Z' \neq \emptyset$. Then $\sigma = \sigma'$.

As is seen easily, a fan in $\mathfrak{g}_{\mathbb{Q}}$ is a weak fan in $\mathfrak{g}_{\mathbb{Q}}$.

2.3.6

For a weak fan Σ in $\mathfrak{g}_{\mathbb{Q}}$, we define

$$D_\Sigma = \bigl\{(\sigma, Z) \bigm| \sigma \in \Sigma,\ Z \text{ is a } \sigma\text{-nilpotent orbit}\bigr\}.$$

2.3.7

We have an embedding

$$D \hookrightarrow D_\Sigma; F \mapsto (\{0\}, \{F\}).$$

In fact, D is identified with the subset of D_Σ consisting of all elements (σ, Z) such that $\sigma = \{0\}$.

2.3.8

Let Σ be a weak fan in $\mathfrak{g}_{\mathbb{Q}}$, and let Γ be a subgroup of $G_{\mathbb{Z}}$.

We say that Σ and Γ are *compatible* if the following condition (i) is satisfied.

(i) If $\sigma \in \Sigma$ and $\gamma \in \Gamma$, then $\gamma\sigma\gamma^{-1} \in \Sigma$.

If Σ and Γ are compatible, we have an action of Γ on D_Σ for which $\gamma \in \Gamma$ sends $(\sigma, Z) \in D_\Sigma$ to $(\gamma\sigma\gamma^{-1}, \gamma Z)$.

We say that Σ and Γ are *strongly compatible* if the above condition (i) and the following condition (ii) are satisfied.

(ii) As a $\mathbb{Q}_{\geq 0}$-cone, σ is generated by $\log(\gamma)$ when γ ranges over all unipotent elements of Γ such that $\log(\gamma) \in \sigma$.

2.3.9

Let Σ be a weak fan in $\mathfrak{g}_{\mathbb{Q}}$, let Γ be a subgroup of $G_{\mathbb{Z}}$, and assume that they are strongly compatible. Then as in [20] and [14, Part III], $\Gamma \setminus D_\Sigma$ is endowed with a topology for which $\Gamma \setminus D$ is a dense open subset and with a sheaf of rings of complex analytic functions whose restriction to $\Gamma \setminus D$ coincides with the usual sheaf of complex analytic functions on $\Gamma \setminus D$. This space $\Gamma \setminus D_\Sigma$ is also endowed with a canonical log structure whose restriction to $\Gamma \setminus D$ is trivial.

2.3.10

A basic fact about the topology of $\Gamma \setminus D_\Sigma$ is the following.

Let $(\sigma, Z) \in D_\Sigma$, and write $\sigma = \mathbb{Q}_{\geq 0}N_1 + \cdots + \mathbb{Q}_{\geq 0}N_n$. Let $F \in Z$. Then the class of (σ, Z) in $\Gamma \setminus D_\Sigma$ is the limit of the classes of $\exp\bigl(\sum_{i=1}^n z_i N_i\bigr)F$ in $\Gamma \setminus D$ for $z_i \in \mathbb{C}$, $\mathrm{Im}(z_i) \to \infty$.

2.3.11

Let Σ and Γ be as above. In general, $\Gamma \setminus D_\Sigma$ is not necessarily a complex analytic space since it may have slits caused by Griffiths transversality. Such a phenomenon was first observed in [29].

If Γ is neat, then $\Gamma \setminus D_\Sigma$ is a log manifold in the following sense. (Neat means that for any $\gamma \in \Gamma$, the subgroup of $\mathbb{C}^\times$ generated by all eigenvalues of the action of γ on $H_{\mathbb{C}}$ is torsion free. If Γ is neat, then Γ is torsion free. It is known that there is a neat subgroup Γ of $G_{\mathbb{Z}}$ of finite index.)

By a *log manifold* we mean a local ringed space over $\mathbb{C}$ endowed with an fs log structure which has an open covering $(U_\lambda)_\lambda$ with the following property: for each λ, there exist an affine toric variety Z_λ endowed with the canonical log structure, a finite subset I_λ of $\Gamma(Z_\lambda, \Omega^1_{Z_\lambda}(\log))$ ($\Omega^1(\log)$ denotes the sheaf of differential forms with log poles), and an isomorphism of local ringed spaces over $\mathbb{C}$ with log structures between U_λ and an open subset of

$$S_\lambda = \bigl\{ z \in Z_\lambda \bigm| \text{the image of } I_\lambda \text{ in } \Omega^1_z(\log) \text{ is zero} \bigr\},$$

where S_λ is endowed with the strong topology in Z_λ, with the inverse image $\mathcal{O}_{Z_\lambda}$ of the sheaf of complex analytic functions $\mathcal{O}_{Z_\lambda}$ on Z_λ, and with the inverse image M_{S_λ} of the log structure M_{Z_λ} of Z_λ.

Here $\Omega^1_z(\log)$ is the space of differential forms with log poles on the log point z. (It is zero if the log structure of the point z is trivial.) *Affine toric variety* and *strong topology* are as follows.

An affine toric variety is an analytic space over $\mathbb{C}$ of the form $\mathrm{Hom}(\mathcal{S}, \mathbb{C})$, where $\mathcal{S}$ is the intersection of a finitely generated $\mathbb{Q}_{\geq 0}$-cone τ and a finitely generated $\mathbb{Z}$-submodule of $\tau_{\mathbb{Q}}$, and Hom is the set of homomorphisms of monoids where $\mathbb{C}$ is regarded as a multiplicative monoid. $\mathrm{Hom}(\mathcal{S}, \mathbb{C})$ is regarded as an analytic space over $\mathbb{C}$ in the natural way. An affine toric variety is endowed with a canonical log structure.

For a subset S of a complex analytic space Z, the *strong topology* of S in Z is defined as follows. A subset U of S is open in the strong topology of S in Z if and only if, for any analytic space Y over $\mathbb{C}$ and any morphism $\lambda : Y \to Z$ such that $\lambda(Y) \subset S$, $\lambda^{-1}(U)$ is open on Y.

For basic facts about log manifolds (resp., strong topology), see [20, Section 3.5] (resp., [20, Section 3.1]).

2.3.12. Example a (The compactification $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{C}^\times$ appears here.)
Let $\Sigma = \{\{0\}, \sigma, -\sigma\}$, where σ (resp., $-\sigma$) denotes the cone of all $\mathbb{Q}$-linear maps $H_\mathbb{Q} \to H_\mathbb{Q}$ which send e_1 to zero and e_2 into $\mathbb{Q}_{\geq 0} \cdot e_1$ (resp., $\mathbb{Q}_{\leq 0} \cdot e_1$). Then Σ is a fan in $\mathfrak{g}_\mathbb{Q}$, and Σ and Γ are strongly compatible. The isomorphism $\mathbb{C}^\times \cong \Gamma \setminus D$ in Section 2.2.3 extends uniquely to an isomorphism

$$\mathbb{P}^1(\mathbb{C}) \cong \Gamma \setminus D_\Sigma$$

of complex analytic manifolds, in which $0 \in \mathbb{P}^1(\mathbb{C})$ (resp., $\infty \in \mathbb{P}^1(\mathbb{C})$) corresponds to the class of $(\sigma, Z) \in D_\Sigma$ (resp., $(-\sigma, Z) \in D_\Sigma$), where $Z = D = \{F(z) \mid z \in \mathbb{C}\}$.

2.3.13. Example b (The unit disc Δ (with the origin) appears here.)
Let $\Sigma = \{\{0\}, \sigma\}$, where σ denotes the cone of all $\mathbb{Q}$-linear maps $H_\mathbb{Q} \to H_\mathbb{Q}$ which send e_1 to zero and e_2 into $\mathbb{Q}_{\geq 0} \cdot e_1$. Then Σ is a fan in $\mathfrak{g}_\mathbb{Q}$, and Σ and Γ are strongly compatible. The isomorphism $\Delta^* \cong \Gamma \setminus D$ in Section 2.2.4 extends uniquely to an isomorphism

$$\Delta \cong \Gamma \setminus D_\Sigma$$

of complex analytic manifolds, in which $0 \in \Delta$ corresponds to the class of $(\sigma, Z) \in D_\Sigma$, where $Z = \{F(\tau) \mid \tau \in \mathbb{C}\} \subset \check{D}$.

2.3.14. Example c (A model of the universal elliptic curve with degenerate fiber on $0 \in \Delta$ appears here.)
For $n \in \mathbb{Z}$, let $N_n \in \mathfrak{g}_\mathbb{Q}$ be the $\mathbb{Q}$-linear map which sends e_1 to zero, e_2 to e_1, and e_3 to ne_1. Let $\Sigma = \{\{0\}, \sigma_n (n \in \mathbb{Z}), \sigma_{n,n+1} (n \in \mathbb{Z})\}$, where $\sigma_n = \mathbb{Q}_{\geq 0} N_n$, $\sigma_{n,n+1} = \mathbb{Q}_{\geq 0} N_n + \mathbb{Q}_{\geq 0} N_{n+1}$. Then Σ is a fan in $\mathfrak{g}_\mathbb{Q}$, Σ and Γ_2 are strongly compatible, and Σ and Γ_3 are also strongly compatible.

The space $\Gamma_3 \setminus D_\Sigma$ is the proper model over Δ of the universal elliptic curve $\Gamma_3 \setminus D$ over Δ^*. The fiber on $q \in \Delta^*$ in $\Gamma_3 \setminus D_\Sigma$ is the elliptic curve $\mathbb{C}^\times / q^\mathbb{Z}$ (see Section 2.2.5). The fiber on $0 \in \Delta$ in $\Gamma_3 \setminus D_\Sigma$ is $\mathbb{P}^1(\mathbb{C})/(0 \sim \infty)$, the quotient of $\mathbb{P}^1(\mathbb{C})$ obtained by identifying zero and ∞.

Concerning $\Gamma_2 \setminus D_\Sigma \to \Delta$, the fiber on $q \in \Delta^*$ is $\mathbb{C}^\times$ (see Section 2.2.5). We describe the fiber S on $0 \in \Delta$ in $\Gamma_2 \setminus D_\Sigma$. It is an infinite chain of $\mathbb{P}^1(\mathbb{C})$. More precisely, for each $n \in \mathbb{Z}$, we have an open immersion

$$u_n : \mathbb{C}^\times \to S, a \mapsto \text{class}(\sigma_n, Z),$$
$$\text{where } Z = \big\{F(\tau, b + n\tau) \bigm| \tau \in \mathbb{C}\big\} \text{ with } a = \exp(2\pi i b).$$

We have $u_m(\mathbb{C}^\times) \cap u_n(\mathbb{C}^\times) = \emptyset$ if $m \neq n$. This u_n extends uniquely to a closed immersion $\bar{u}_n : \mathbb{P}^1(\mathbb{C}) \to S$, and $S = \bigcup_{n\in\mathbb{Z}} \bar{u}_n(\mathbb{P}^1(\mathbb{C}))$. If $m, n \in \mathbb{Z}$ and $n \notin \{m-1, m, m+1\}$, then $\bar{u}_m(\mathbb{P}^1(\mathbb{C})) \cap \bar{u}_n(\mathbb{P}^1(\mathbb{C})) = \emptyset$. We have $\bar{u}_n(\mathbb{P}^1(\mathbb{C})) \cap \bar{u}_{n+1}(\mathbb{P}^1(\mathbb{C})) = \{\bar{u}_n(0)\} = \{\bar{u}_{n+1}(\infty)\}$, and this point $\bar{u}_n(0)$ is the class of the nilpotent orbit $(\sigma_{n,n+1}, Z)$ with $Z = \{F(\tau, z) \mid \tau, z \in \mathbb{C}\}$. The action of a standard generator of Γ_3/Γ_2 sends $\bar{u}_n(a)$ $(a \in \mathbb{P}^1(\mathbb{C}))$ to $\bar{u}_{n+1}(a)$. The fiber $\mathbb{P}^1(\mathbb{C})/(0 \sim \infty)$ on $0 \in \Delta$ of $\Gamma_3 \setminus D_\Sigma \to \Delta$ is $(\Gamma_3/\Gamma_2) \setminus S$, that is, the quotient of S by this action.

2.3.15. Example d

Let Σ be the set of the cones $\mathbb{Q}_{\geq 0}N$, where N ranges over all $\mathbb{Q}$-linear maps $H_\mathbb{Q} \to H_\mathbb{Q}$ such that $N(W_w) \subset W_{w-1}$ for any $w \in \mathbb{Z}$. Then Σ is a fan in $\mathfrak{g}_\mathbb{Q}$, and Σ and Γ are strongly compatible. A remarkable fact is that $\Gamma \setminus D_\Sigma$ has a slit and is not a complex analytic space. We describe an open neighborhood of points of class(σ, Z_z) of $\Gamma \setminus D_\Sigma$, which has a slit. Here $\sigma = \mathbb{Q}_{\geq 0}N$ with $N(e_1) = N(e_2) = 0$, $N(e_3) = e_2$, and $Z_z = \{F(z, \tau, 0) \mid \tau \in \mathbb{C}\}$. We have an open immersion of log manifolds from a sufficiently small open neighborhood U of $(0,0,0)$ in the log manifold

$$\left\{(z_1, u, z_3) \in \mathbb{C}^3 \;\middle|\; \text{if } u = 0, \text{ then } z_3 = 0\right\}$$

to the log manifold $\Gamma \setminus D_\Sigma$, which sends $(z_1, e^{2\pi i w}, z_3) \in U$ to the class of $F(z_1, w, z_3)$ and sends $(z, 0, 0) \in U$ to the class of (σ, Z_z). Here the log structure of $U \subset \mathbb{C}^3$ is defined by the divisor $u = 0$ of $\mathbb{C}^3$. Note that U has a slit and is not a complex analytic space. This slit appears by Griffiths transversality; $(N, F(z_1, z_2, z_3))$ satisfies Griffiths transversality if and only if $z_3 = 0$.

The theory of dilog sheaves (a special case of polylog sheaves; see [2]) shows that there is a unique morphism $\mathbb{P}^1(\mathbb{C}) \to \Gamma \setminus D_\Sigma$ of log manifolds (here $\mathbb{P}^1(\mathbb{C})$ is endowed with the log structure defined by $\{0, 1, \infty\}$) which sends $u \in \mathbb{C}^\times \subset \mathbb{P}^1(\mathbb{C})$ with $|u| < 1$ to the class of $F(z_1, z_2, z_3)$ with

$$z_1 = -(2\pi i)^{-2} \sum_{n=1}^{\infty} \frac{u^n}{n^2}, \qquad z_2 = (2\pi i)^{-1} \log(u), \qquad z_3 = (2\pi i)^{-1} \log(1-u).$$

The image of $0 \in \mathbb{P}^1(\mathbb{C})$ under this morphism $\mathbb{P}^1(\mathbb{C}) \to \Gamma \setminus D_\Sigma$ is class(σ, Z_0).

3. p-Adic Hodge structures and Fontaine's p-adic Hodge conjecture

In this section, we review the theory of Fontaine on p-adic Hodge structures, and introduce the p-adic Hodge conjecture (See Conjectures 3.4.2, 3.4.9) of Fontaine. In Section 3.5, we give some results on p-adic Hodge structures which we use in Section 6.

3.1. Notation in Section 3

3.1.1

Let K be a complete discrete valuation field of characteristic zero with perfect residue field k of characteristic $p > 0$. Let O_K be the valuation ring of K, and let m_K be the maximal ideal of O_K.

Let $\bar{K}$ be an algebraic closure of K, and let $\mathbb{C}_p$ be the completion of $\bar{K}$. Let $O_{\mathbb{C}_p}$ and $m_{\mathbb{C}_p}$ be the valuation ring of $\mathbb{C}_p$ and its maximal ideal, respectively, and let $\bar{k}$ be the residue field of $\mathbb{C}_p$, which is an algebraic closure of k.

3.1.2

Let $W(k)$ be the ring of Witt vectors with entries in k, and let K_0 be the field of fractions of $W(k)$. We regard K as a finite extension of K_0 in the natural way.

Let $\varphi : W(k) \to W(k)$ be the canonical lifting of the pth power map $k \to k$, and denote the induced automorphism $K_0 \to K_0$ by the same letter φ.

3.1.3

We fix a primitive nth root ζ_n of 1 in $\bar{K}$ for each $n \geq 1$ satisfying $(\zeta_{mn})^n = \zeta_m$ for any $m, n \geq 1$.

3.1.4

We fix an element $\xi \in m_K - \{0\}$. We fix an nth root $\xi^{1/n}$ of ξ in $\bar{K}$ for each $n \geq 1$ satisfying $(\xi^{1/mn})^n = \xi^{1/m}$ for any $m, n \geq 1$.

Let

$$\mathrm{ord}_\xi : \mathbb{C}_p^\times \to \mathbb{Q}$$

be the unique homomorphism which sends $(O_{\mathbb{C}_p})^\times$ to zero and ξ to 1. Let

$$\log : \mathbb{C}_p^\times \to \mathbb{C}_p$$

be the unique homomorphism which coincides with the usual p-adic logarithm $x \mapsto \sum_{n=1}^{\infty} (-1)^{n-1}(x-1)^n/n$ on $1 + m_{\mathbb{C}_p}$, which is zero on the multiplicative representative of $k^\times$ in $(O_{\mathbb{C}_p})^\times$ and which sends ξ to zero. We have $\log(K^\times) \subset K$. (It may be better to denote this map log by $\log_{(\xi)}$, e.g., to indicate the dependence on ξ, but we just choose simple notation.)

A standard choice of ξ is p. The introduction of this paper is written by taking $\xi = p$.

3.2. p-Adic Hodge structures

We briefly review the formulation of p-adic Hodge structures by Fontaine (for details, see, e.g., [7], [8]).

3.2.1

A filtered module over K is a finite-dimensional K_0-vector space D endowed with a Frobenius-linear bijection $\varphi : D \to D$, with a K_0-linear map $N : D \to D$ such that $N\varphi = p\varphi N$, and with a decreasing filtration F on $D_K = K \otimes_{K_0} D$. We denote the category of filtered modules over K by MF_K.

The map N of a filtered module is automatically nilpotent.

The filtration $(F^i D_K)_i$ of a filtered module D is often denoted by $(D_K^i)_i$.

3.2.2

There is an important full subcategory $\mathrm{MF}_K^{\mathrm{adm}}$ of MF_K consisting of all admissible filtered modules over K. This is defined as follows.

First, assume that k is algebraically closed.

For a K_0-subspace S of D which is stable under φ and N, define integers $t_N(S)$ and $t_H(S)$ as follows. Let $r = \dim_{K_0} S$, and consider the rth exterior power $\bigwedge^r S$ of S, which is a one-dimensional K_0-vector space. Let $\varphi : \bigwedge^r S \to \bigwedge^r S$ be the Frobenius-linear bijection induced by $\varphi : D \to D$. Then if we write $\varphi(e) = ae$ for a basis e of $\bigwedge^r S$ and for $a \in K_0^\times$, a depends on the choice of e but $\mathrm{ord}_p(a)$ does not depend on the choice. Here ord_p is the additive valuation of K_0 which sends p to 1. Define $t_N(S) = \mathrm{ord}_p(a) \in \mathbb{Z}$. On the other hand, for the filtration on $\bigwedge_K^r S_K$ induced by the filtration F on S_K, there is a unique integer m such that $F^m(\bigwedge_K^r S_K) = \bigwedge_K^r S_K$ and $F^{m+1}(\bigwedge_K^r S_K) = 0$. Define $t_H(S) = m$.

An object D of MF_K is said to be *admissible* if $t_N(D) = t_H(D)$ and $t_N(S) \geq t_H(S)$ for any K_0-subspace S of D which is stable under φ and N.

The definition of admissibility without assuming that k is algebraically closed is as follows. Let K_0' be the field of fractions of $W(\bar{k})$, and let $K' = K \otimes_{K_0} K_0'$. Then K' is a complete discrete valuation field with algebraically closed residue field $\bar{k}$. For an object D of MF_K, we have an object $D' = K_0' \otimes_{K_0} D$ of $\mathrm{MF}_{K'}$ with $\varphi' = \varphi \otimes \varphi$, with $N' = 1 \otimes N$, and with the filtration $F' = K' \otimes_K F$. The admissibility of D is defined as the admissibility of D'.

In fact, this admissibility was called *weak admissibility* before (e.g., in [8]). But by [5], it is now known that weak admissibility is equivalent to the condition *admissibility*. To make the terminology simple, we use the word admissibility for weak admissibility (as experts do these days).

3.2.3

The following fact (see [8, Section 4.4.4(iii)]) is used often in this paper. Assume that we have an exact sequence $0 \to D' \to D \to D'' \to 0$ in MF_K. (Exactness here means that the sequences $0 \to D' \to D \to D'' \to 0$ and $0 \to F^r D'_K \to F^r D_K \to F^r D''_K \to 0$ for all $r \in \mathbb{Z}$ are exact.) Assume that two D, D', D'' are admissible. Then all D, D', D'' are admissible.

3.2.4

Fontaine defined important rings

$$A_{\mathrm{crys}} \subset B_{\mathrm{crys}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}}$$

such that

$$A_{\mathrm{crys}} \supset W(k), \qquad B_{\mathrm{crys}} \supset K_0, \qquad B_{\mathrm{dR}} \supset \bar{K}.$$

We review basic properties of these rings (see [7]).

B_{dR} is a complete discrete valuation field. For $r \in \mathbb{Z}$, let B_{dR}^r be the part of B_{dR} consisting of all elements whose normalized additive valuations are at least r. In particular, B_{dR}^0 is the valuation ring of B_{dR}, and B_{dR}^1 is the maximal ideal. We have

$$\bar{K} \subset B_{\mathrm{dR}}^0, \qquad A_{\mathrm{crys}} \subset B_{\mathrm{dR}}^0.$$

The residue field $B_{\mathrm{dR}}^0/B_{\mathrm{dR}}^1$ of B_{dR}, which is an extension of $\bar{K}$, is identified with $\mathbb{C}_p$.

The action of $\mathrm{Gal}(\bar{K}/K)$ on $\bar{K}$ extends to a canonical action of $\mathrm{Gal}(\bar{K}/K)$ on the field B_{dR}, which preserves the valuation of B_{dR} and induces the usual action on the residue field $\mathbb{C}_p$. The subrings A_{crys}, B_{crys}, and B_{st} of B_{dR} are stable under this Galois action.

The Frobenius map $\varphi : K_0 \to K_0$ extends to a special ring homomorphism

$$\varphi : B_{\mathrm{st}} \to B_{\mathrm{st}}$$

such that $\varphi(A_{\mathrm{crys}}) \subset A_{\mathrm{crys}}$ and $\varphi(B_{\mathrm{crys}}) \subset B_{\mathrm{crys}}$.

The ring B_{crys} is a localization of A_{crys}, and B_{st} is a polynomial ring in one variable over B_{crys}, as follows. There is a unique multiplicative map

$$[\ ] : \big\{(a_n)_{n\ge 0} \,\big|\, a_n \in O_{\mathbb{C}_p}(n \ge 0), a_{n+1}^p = a_n (n \ge 0)\big\} \to A_{\mathrm{crys}}$$

such that the composition with $A_{\mathrm{crys}} \to B_{\mathrm{dR}}^0/B_{\mathrm{dR}}^1 = \mathbb{C}_p$ is $(a_n)_n \mapsto a_0$. For example, we have an element $\epsilon := [(\zeta_{p^n})_n] \in A_{\mathrm{crys}}$, and $\epsilon \equiv 1 \bmod B_{\mathrm{dR}}^1$. Since B_{dR}^0 is a complete discrete valuation ring containing $\mathbb{Q}$,

$$t := \log(\epsilon) \in B_{\mathrm{dR}}^1$$

is defined. This element t is a prime element of the discrete valuation field B_{dR}. We have $t \in A_{\mathrm{crys}}$, $\varphi(t) = pt$, and

$$B_{\mathrm{crys}} = A_{\mathrm{crys}}\left[\frac{1}{t}\right] \quad \text{(see [7, Section 2.3.4])}.$$

We have $[(\xi^{1/p^n})_n]/\xi \equiv 1 \bmod B_{\mathrm{dR}}^1$, and hence

$$l_\xi := \log\big([(\xi^{1/p^n})_n]/\xi\big) \in B_{\mathrm{dR}}^1$$

is defined. This is also a prime element of B_{dR}. We have

$$B_{\mathrm{st}} = B_{\mathrm{crys}}[l_\xi],$$

and l_ξ is transcendental over B_{crys}; that is, B_{st} is isomorphic to a polynomial ring in one variable over B_{crys}. This element l_ξ is denoted by u in [7]. We have $\varphi(l_\xi) = pl_\xi$.

The canonical map $\bar{K} \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ is injective.

Let

$$N : B_{\mathrm{st}} \to B_{\mathrm{st}}$$

be the unique derivation over B_{crys} such that $N(l_\xi) = -1$. In fact, in [7], $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ was defined as the unique derivation over B_{crys} such that $N(l_\xi) = 1$ (see Section 3.2.7 for the advantage of our modification of the definition of N).

3.2.5

For a finite-dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$, define

$$D_{\mathrm{st}}(V) = H^0\big(\mathrm{Gal}(\bar{K}/K), B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\big).$$

Here the Galois group acts on $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ diagonally. Then $\dim_{K_0} D_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V$. The Galois representation V is called a *semistable representation* if $\dim_{K_0} D_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V$.

Let $\mathcal{C}_{K,p}$ be the category of semistable representations of $\mathrm{Gal}(\bar{K}/K)$.

3.2.6

By [5], we have an equivalence of categories

$$\mathcal{C}_{K,p} \simeq \mathrm{MF}_K^{\mathrm{adm}}.$$

The functor $V \mapsto D$ from the left to the right is defined as follows. Let V be an object of $\mathcal{C}_{K,p}$. Then $D = D_{\mathrm{st}}(V)$, $\varphi : D \to D$ is induced from $\varphi \otimes 1 : B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \to B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$, and $N : D \to D$ is induced from $N \otimes 1 : B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \to B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$. The filtration F on $K \otimes_{K_0} D$ is defined by

$$F^r = (K \otimes_{K_0} D) \cap (B_{\mathrm{dR}}^r \otimes_{\mathbb{Q}_p} V).$$

Here the intersection is taken by $K \otimes_{K_0} D \subset K \otimes_{K_0} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \subset B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$.

The inclusion map $D \overset{\subset}{\to} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ induces an isomorphism $B_{\mathrm{st}} \otimes_{K_0} D \overset{\cong}{\to} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$. We often identify $B_{\mathrm{st}} \otimes_{K_0} D$ with $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ via this isomorphism. V is recovered from D as

$$V = \big\{x \in B_{\mathrm{st}} \otimes_{K_0} D \,\big|\, \varphi(x) = x, N(x) = 0, x \in F^0(B_{\mathrm{dR}} \otimes_K D_K)\big\}.$$

Here $\varphi : B_{\mathrm{st}} \otimes_{K_0} D \to B_{\mathrm{st}} \otimes_{K_0} D$ is $\varphi \otimes \varphi$, $N : B_{\mathrm{st}} \otimes_{K_0} D \to B_{\mathrm{st}} \otimes_{K_0} D$ is $N \otimes 1 + 1 \otimes N$, and $F^r(B_{\mathrm{dR}} \otimes_K D_K) = \sum_{i+j=r} B_{\mathrm{dR}}^i \otimes_K F^j$.

3.2.7

As is stated in Section 3.2.4, we changed the sign of $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ in [7]. This changes the N of $D_{\mathrm{st}}(V)$ by sign for a semistable p-adic representation V of $\mathrm{Gal}(\bar{K}/K)$. This has the following advantage.

Let E be the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$, where $q \in m_K - \{0\}$. For a prime number ℓ, let

$$E(\bar{K})[\ell^n] = \mathrm{Ker}\big(\ell^n : E(\bar{K}) \to E(\bar{K})\big),$$
$$T_\ell(E) = \varprojlim_n E(\bar{K})[\ell^n], \qquad V_\ell(E) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(E).$$

Then

$$H^1_{\text{ét}}(E \otimes_K \bar{K}, \mathbb{Z}_\ell) \cong T_\ell(E)(-1), \qquad H^1_{\text{ét}}(E \otimes_K \bar{K}, \mathbb{Q}_\ell) \cong V_\ell(E)(-1),$$

where (-1) is the Tate twist. Since $E(\bar{K}) = (\bar{K})^\times/q^{\mathbb{Z}}$, we have an exact sequence

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z}(1) \to E(\bar{K})[\ell^n] \to \mathbb{Z}/\ell^n\mathbb{Z} \to 0$$

for any $n \geq 0$, where the map $E(\bar{K})[\ell^n] \to \mathbb{Z}/\ell^n\mathbb{Z}$ sends $q^{1/\ell^n} \bmod q^{\mathbb{Z}}$ to $1 \in \mathbb{Z}/\ell^n\mathbb{Z}$, and we obtain exact sequences

$$0 \to \mathbb{Z}_\ell(1) \to T_\ell(E) \to \mathbb{Z}_\ell \to 0, \qquad 0 \to \mathbb{Q}_\ell(1) \to V_\ell(E) \to \mathbb{Q}_\ell \to 0.$$

For $\ell = p$, D_{st} of the last exact sequence gives an exact sequence

$$0 \to K_0 \to D_{\mathrm{st}}\big(V_p(E)\big) \to K_0 \to 0.$$

With the definition of $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ in [7] (resp., in this paper), the N of $D_{\mathrm{st}}(V_p(E))$ is given as the multiplication by $-\mathrm{ord}_\xi(q)$ (resp., $\mathrm{ord}_\xi(q)$) from the right K_0 to the left K_0 of this exact sequence. For $\ell \neq p$, the N of $V_\ell(E)$ defined in Section 3.3 is given as the map from the right $\mathbb{Q}_\ell$ to the left $\mathbb{Q}_\ell(1)$ which is the composition of the multiplication $\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ by $\mathrm{ord}_\xi(q)$ and the map $\mathbb{Q}_\ell$ to $\mathbb{Q}_\ell(1)$ defined by $(\zeta_{\ell^n})_n$ (see Section 5.5, Example b). Thus our choice of the sign of $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ gives a nice choice of the sign of N on $D_{\mathrm{st}}(V_p(E))$ and also a nice compatibility with the ℓ-adic theory for $\ell \neq p$.

3.3. ℓ-Adic structures ($\ell \neq p$)

Let ℓ be a prime number that is different from p. We review basic facts about ℓ-adic representations of $\mathrm{Gal}(\bar{K}/K)$.

3.3.1

Let $\mathcal{C}_{K,\ell}$ be the following category.

An object is a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$ such that the action of the inertia subgroup $\mathrm{Ker}(\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k))$ of $\mathrm{Gal}(\bar{K}/K)$ is unipotent.

3.3.2

Let $\mathcal{C}_{k,\ell}$ be the following category.

Let $\kappa : \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}_\ell^\times$ be the ℓ-adic cyclotomic character.

An object is a pair (V, N), where V is a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous action of $\mathrm{Gal}(\bar{k}/k)$ and with a nilpotent linear map $N : V \to V$ such that $N \circ s = \kappa(s)s \circ N$ for any $s \in \mathrm{Gal}(\bar{k}/k)$.

3.3.3

We have an equivalence of categories

$$\mathcal{C}_{K,\ell} \simeq \mathcal{C}_{k,\ell}$$

defined as follows.

For $s \in \mathrm{Gal}(\bar{K}/K)$, define $a(s) \in \mathbb{Z}_\ell$ by $\sigma(\xi^{1/\ell^n}) = \xi^{1/\ell^n}\zeta_{\ell^n}^{a(s)}$.

For an object $\tilde{V}$ of $\mathcal{C}_{K,\ell}$ with the action $\tilde{\rho}$ of $\mathrm{Gal}(\bar{K}/K)$, the corresponding object (V, N) of $\mathcal{C}_{k,\ell}$ with the action ρ of $\mathrm{Gal}(\bar{k}/k)$ on V is the following. We have $V = \tilde{V}$ as a $\mathbb{Q}_\ell$-vector space. Let K^{tame} be the largest tame extension of K in $\bar{K}$. Then $\tilde{\rho} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}(\tilde{V})$ factors through the canonical surjection $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(K^{\mathrm{tame}}/K)$. Let $L = \bigcup_{(n,p)=1} K(\xi^{1/n})$. Then the canonical map $\mathrm{Gal}(K^{\mathrm{tame}}/L) \to \mathrm{Gal}(\bar{k}/k)$ is an isomorphism. We define ρ by $\rho(s) = \tilde{\rho}(\tilde{s})$, where

$\tilde{s}$ is the unique element of $\mathrm{Gal}(K^{\mathrm{tame}}/L)$ whose image in $\mathrm{Gal}(\bar{k}/k)$ coincides with s. We define $N = \log(\tilde{\rho}(s))/a(s)$ for an element s of the inertia subgroup of $\mathrm{Gal}(\bar{K}/K)$ such that $a(s) \neq 0$. Then N is independent of the choice of such s.

The converse functor $\mathcal{C}_{k,\ell} \to \mathcal{C}_{K,\ell}$ is as follows. Let (V, N) be an object of $\mathcal{C}_{k,\ell}$ with an action ρ of $\mathrm{Gal}(\bar{k}/k)$ on V. Then the corresponding object of $\mathcal{C}_{K,\ell}$ is the $\mathbb{Q}_\ell$-vector space V with the action $\tilde{\rho}$ of $\mathrm{Gal}(\bar{K}/K)$ defined by

$$\tilde{\rho}(s) = \exp\bigl(a(s)N\bigr)\rho(\bar{s}) \quad \text{for } s \in \mathrm{Gal}(\bar{K}/K),$$

where $\bar{s}$ is the image of s in $\mathrm{Gal}(\bar{k}/k)$.

3.4. p-Adic Hodge conjecture of Fontaine

Here we introduce the p-adic version of the Hodge conjecture (see Conjectures 3.4.2, 3.4.9) formulated by Fontaine. The author learned this conjecture from Fontaine [10]. Any mistakes and insufficient points in the descriptions of the conjecture in the following are due to the author.

3.4.1

First, we introduce the p-adic Hodge conjecture in the form which does not use motives.

Recall that the Hodge conjecture is stated as follows.

HODGE CONJECTURE

Let X be a proper smooth algebraic variety X over $\mathbb{C}$, and let $r \in \mathbb{Z}$. Then the image of the cycle map $\mathbb{Q} \otimes \mathrm{CH}^r(X) \to H^{2r}(X(\mathbb{C}), \mathbb{Q})$ coincides with the intersection of $H^{2r}(X(\mathbb{C}), \mathbb{Q})$ and $\mathrm{fil}^r H^{2r}_{\mathrm{dR}}(X/\mathbb{C})$.

Here $\mathrm{CH}^r(X)$ is the Chow group of algebraic cycles on X of codimension r, fil is the Hodge filtration, and the intersection is taken in $\mathbb{C} \otimes_{\mathbb{Q}} H^{2r}(X(\mathbb{C}), \mathbb{Q}) = H^{2r}_{\mathrm{dR}}(X/\mathbb{C})$.

The following is a one-form of the p-adic Hodge conjecture of Fontaine.

CONJECTURE 3.4.2 (p-ADIC HODGE CONJECTURE)

Let $\mathfrak{X}$ be a proper, smooth scheme over O_K, let $X = \mathfrak{X} \otimes_{O_K} K$, and let $Y = \mathfrak{X} \otimes_{O_K} k$. Let $r \in \mathbb{Z}$. Then the image of the cycle map $\mathbb{Q} \otimes \mathrm{CH}^r(X) \to H^{2r}_{\mathrm{dR}}(X/K)$ coincides with the intersection of the image of the cycle map $\mathbb{Q} \otimes \mathrm{CH}^r(Y) \to H^{2r}_{\mathrm{crys}}(Y)$ and $\mathrm{fil}^r H^{2r}_{\mathrm{dR}}(X/K)$.

Here $H^m_{\mathrm{crys}}(Y) = K_0 \otimes_{W(k)} H^m_{\mathrm{crys}}(Y/W(k))$, fil is the Hodge filtration, and the intersection is taken in $K \otimes_{K_0} H^{2r}_{\mathrm{crys}}(Y) = H^{2r}_{\mathrm{dR}}(X/K)$. Here the last $=$ is the identification by the Berthelot-Ogus isomorphism (see [3]).

3.4.3

The Hodge conjecture has the following formulation in terms of motives.

In this paper, mixed motives are with $\mathbb{Q}$-coefficients. Morphisms of motives are considered modulo homological equivalence.

In the following, a $\mathbb{Q}$-Hodge structure means a Hodge structure with $\mathbb{Q}$-coefficients.

HODGE CONJECTURE

The realization functor from the category of mixed motives over $\mathbb{C}$ to the category of mixed $\mathbb{Q}$-Hodge structures is fully faithful.

This is related to the usual form of Hodge conjecture in Section 3.4.1 as follows. The intersection $H^{2r}(X(\mathbb{C}),\mathbb{Q}) \cap \mathrm{fil}^r H^{2r}_{\mathrm{dR}}(X/\mathbb{C})$ is identified with the space of homomorphisms $\mathbb{Q} \to H^{2r}(X(\mathbb{C}),\mathbb{Q})(r)$ of $\mathbb{Q}$-Hodge structures, and $\mathbb{Q} \otimes \mathrm{CH}^r(X)$ modulo homological equivalence is identified with the space of homomorphisms $\mathbb{Q} \to H^{2r}(X)(r)$ of motives over $\mathbb{C}$.

The p-adic Hodge conjecture (Conjecture 3.4.2) of Fontaine also has a formulation below in terms of motives (see Conjecture 3.4.9).

3.4.4

We consider motives in this paper only to make clearer the motivations and the ideas of various constructions, and we do not need motives in the results in this paper. So we do not discuss how to define the notion of mixed motive.

For a mixed motive M over K (resp., k) and a prime number ℓ (resp., prime number $\ell \neq p$), let M_ℓ be the ℓ-adic realization of M. It is a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous linear action of $\mathrm{Gal}(\bar{K}/K)$ (resp., $\mathrm{Gal}(\bar{k}/k)$). If M is the mixed motive $H^m(X)(r)$ for a scheme X of finite type over K (resp., k) and for $m, r \in \mathbb{Z}$, then $M_\ell = H^m_{\text{ét}}(\bar{X},\mathbb{Q}_\ell)(r)$, where $\bar{X} = X \otimes_K \bar{K}$ (resp., $X \otimes_k \bar{k}$).

For a mixed motive M over k, let M_{crys} be the crystalline realization of M which is a finite-dimensional K_0-vector space endowed with a Frobenius-linear bijection $\varphi : M_{\mathrm{crys}} \to M_{\mathrm{crys}}$. If $M = H^m(Y)(r)$ for a proper smooth scheme Y over k and for $m, r \in \mathbb{Z}$, then $M_{\mathrm{crys}} = H^m_{\mathrm{crys}}(Y) := K_0 \otimes_{W(k)} H^m_{\mathrm{crys}}(Y/W(k))$.

The following conjecture is well known. Let the categories $\mathcal{C}_{K,\ell}$ for prime numbers ℓ be as in Sections 3.2 and 3.3.

CONJECTURE 3.4.5

Let M be a mixed motive over K. Then for any prime numbers ℓ and ℓ' (they can be p), the ℓ-adic realization M_ℓ belongs to $\mathcal{C}_{K,\ell}$ if and only if $M_{\ell'}$ belongs to $\mathcal{C}_{K,\ell'}$.

3.4.6

For a mixed motive over K, we say that M is of *semistable reduction* if $M_\ell \in \mathcal{C}_{K,\ell}$ for any prime number ℓ.

3.4.7

We consider the categories $\mathcal{C}_K$, $\mathcal{C}_k$, $\mathcal{C}_{k,K}$.

Let $\mathcal{C}_K$ be the category of mixed motives over K of semistable reduction.

Let $\mathcal{C}_k$ be the category of triples (M, W, N), where

- M is a mixed motive over k,
- W is an increasing filtration on M (which need not coincide with the weight filtration of M),
- N is a homomorphism $M \to M(-1)$ of mixed motives which sends $W_w M$ into $(W_w M)(-1)$ for any $w \in \mathbb{Z}$ such that for any $w \in \mathbb{Z}$ and $m \geq 0$, the homomorphism $N^m : \mathrm{gr}^{\mathcal{W}}_{w+m} \mathrm{gr}^W_w(M) \to (\mathrm{gr}^{\mathcal{W}}_{w-m} \mathrm{gr}^W_w(M))(-m)$ is an isomorphism, where $\mathcal{W}$ is the weight filtration of the mixed motive M over k.

Let $\mathcal{C}_{k,K}$ be the category of quadruples (M, W, N, F), where

- (M, W, N) is an object of $\mathcal{C}_k$,
- F is a decreasing filtration on the K-module $K \otimes_{K_0} M_{\mathrm{crys}}$ such that $\mathrm{gr}^W_w(M_{\mathrm{crys}}, \varphi, N, F)$ is admissible (3.2.2) for any $w \in \mathbb{Z}$.

The p-adic Hodge conjecture formulated as Conjecture 3.4.9 says that there is a fully faithful functor $\mathcal{C}_K \to \mathcal{C}_{k,K}$ which is compatible with realization functors.

3.4.8

For a prime number ℓ (resp., prime number $\ell \neq p$), let $\tilde{\mathcal{C}}_{K,\ell}$ (resp., $\tilde{\mathcal{C}}_{k,\ell}$) be the category of pairs (V, W), where V is an object of $\mathcal{C}_{K,\ell}$ (resp., $\mathcal{C}_{k,\ell}$) and W is an increasing filtration on V by subobjects of V. Let $\tilde{\mathcal{C}}_{k,K,p}$ be the category of pairs (D, W), where D is an object of $\mathrm{MF}^{\mathrm{adm}}_K$ and W is an increasing filtration on D by subobjects of D in $\mathrm{MF}^{\mathrm{adm}}_K$.

We have the ℓ-adic realization functors

$$\mathcal{C}_K \to \tilde{\mathcal{C}}_{K,\ell} \quad (\text{resp., } \mathcal{C}_k \to \tilde{\mathcal{C}}_{k,\ell})$$

for any prime number ℓ (resp., prime number $\ell \neq p$), where the weight filtration W of the mixed motive over K (resp., the filtration W of an object of $\mathcal{C}_k$) induces the filtration W on the ℓ-adic realization. We have similarly the p-adic realization functor

$$\mathcal{C}_{k,K} \to \tilde{\mathcal{C}}_{k,K,p}.$$

CONJECTURE 3.4.9 (p-ADIC HODGE CONJECTURE)

There is a fully faithful functor $\mathcal{C}_K \to \mathcal{C}_{k,K}$ having the following properties (1) and (2).

(1) *The following diagrams of categories are commutative. Here ℓ is a prime number $\neq p$:*

$$\begin{array}{ccc} \mathcal{C}_K & \longrightarrow & \mathcal{C}_{k,K} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{C}}_{K,p} & \simeq & \tilde{\mathcal{C}}_{k,K,p} \end{array} \qquad \begin{array}{ccc} \mathcal{C}_K & \longrightarrow & \mathcal{C}_{k,K} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{C}}_{K,\ell} & \simeq & \tilde{\mathcal{C}}_{k,\ell} \end{array}$$

Here $\mathcal{C}_{k,K} \to \tilde{\mathcal{C}}_{k,\ell}$ is the composition $\mathcal{C}_{k,K} \to \mathcal{C}_k \to \tilde{\mathcal{C}}_{k,\ell}$, and the lower equivalences $\tilde{\mathcal{C}}_{K,p} \simeq \tilde{\mathcal{C}}_{k,K,p}$ and $\tilde{\mathcal{C}}_{K,\ell} \simeq \tilde{\mathcal{C}}_{k,\ell}$ are induced from the equivalences $\mathcal{C}_{K,p} \simeq \mathrm{MF}_K^{\mathrm{add}}$ (see Section 3.2.6) and $\mathcal{C}_{K,\ell} \simeq \mathcal{C}_{k,\ell}$ (see Section 3.3.3), respectively.

(2) *Let $\mathfrak{X}$ be a proper smooth scheme over O_K, let $X = \mathfrak{X} \otimes_{O_K} K$, and let $Y = \mathfrak{X} \otimes_{O_K} k$. Let $m, r \in \mathbb{Z}$. Then the functor $\mathcal{C}_K \to \mathcal{C}_{k,K}$ sends the motive $H^m(X)(r)$ to the following object (M, W, N, F): $M := H^m(Y)(r)$, $W_w := H^m(Y)(r)$ for $w \geq m - 2r$, $W_w = 0$ for $w < m - 2r$, $N := 0$, and $F^i := \mathrm{fil}^{i+r} H^m_{\mathrm{dR}}(X/K)$, where H_{dR} is the de Rham cohomology and* fil *is the Hodge filtration. Here we identify $H^m_{\mathrm{dR}}(X/K)$ and $K \otimes_{K_0} M_{\mathrm{crys}}$ by the Berthelot-Ogus isomorphism $H^m_{\mathrm{dR}}(X/K) \cong K \otimes_{K_0} H^m_{\mathrm{crys}}(Y)$.*

3.4.10

The p-adic Hodge Conjecture 3.4.9 is related to the p-adic Hodge Conjecture 3.4.2 as follows. Let X, Y and (M, W, N, F) be as in Conjecture 3.4.9(2) with $m = 2r$. Then $\mathbb{Q} \otimes \mathrm{CH}^r(X)$ modulo homological equivalence is identified with the space of homomorphisms $\mathbb{Q} \to H^{2r}(X)(r)$ in $\mathcal{C}_K$, and the intersection $\mathrm{Image}(\mathbb{Q} \otimes \mathrm{CH}^r(Y) \to H^{2r}_{\mathrm{crys}}(Y)) \cap \mathrm{fil}^r H^{2r}_{\mathrm{dR}}(X/K)$ is identified with the space of homomorphisms $\mathbb{Q} \to (M, W, N, F)$ in $\mathcal{C}_{k,K}$.

3.4.11

Evidence for Conjecture 3.4.9 is the Serre-Tate theory (see [21]). Let A and B be abelian varieties over K with good reduction, and let A_k and B_k be their reductions over k, respectively. Then by Serre-Tate theory, we have a bijection

$$\mathbb{Q} \otimes \mathrm{Hom}(A, B) \xrightarrow{\cong} F^0\big(\mathbb{Q} \otimes \mathrm{Hom}(A_k, B_k)\big),$$

where the right-hand side is the subset of $\mathbb{Q} \otimes \mathrm{Hom}(A_k, B_k)$ consisting of all homomorphisms such that the induced map $H^1_{\mathrm{dR}}(B/K) = K \otimes_{K_0} H^1_{\mathrm{crys}}(B_k) \to K \otimes_{K_0} H^1_{\mathrm{crys}}(A_k) = H^1_{\mathrm{dR}}(A/K)$ respects the Hodge filtrations. Note that $\mathbb{Q} \otimes \mathrm{Hom}(A, B)$ (resp., $\mathbb{Q} \otimes \mathrm{Hom}(A_k, B_k)$) is identified with the set of all homomorphisms of motives $H^1(B) \to H^1(A)$ (resp., $H^1(B_k) \to H^1(A_k)$) over K (resp., k).

3.4.12

Conjecture 3.4.9 contains the so-called weight-monodromy conjecture. Let X be a proper smooth scheme over K of semistable reduction. Then the weight-monodromy conjecture states that for any prime number $\ell \neq p$, the filtration on $H^m_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ given by the monodromy $N : H^m_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell) \to H^m_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ (it is $M(N, W)$, where W is the filtration defined by $W_m = H^m(\bar{X}, \mathbb{Q}_\ell)$ and $W_{m-1} = 0$) coincides with the filtration defined by Frobenius weights. If (M, N, W, F) is the object of $\mathcal{C}_{k,K}$ corresponding to the object $H^m(X)$ of $\mathcal{C}_K$, then the property $N^i : \mathrm{gr}^{\mathcal{W}}_{w+i} \mathrm{gr}^W_w(M) \xrightarrow{\cong} (\mathrm{gr}^{\mathcal{W}}_{w-i} \mathrm{gr}^W_w(M))(-i)$ for $w \in \mathbb{Z}$ and $i \geq 0$ in Section 3.4.7 implies that the weight-monodromy conjecture for $H^m_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ holds.

3.4.13

The functor $\mathcal{C}_K \to \mathcal{C}_{k,K}$ and also the composition $\mathcal{C}_K \to \mathcal{C}_{k,K} \to \mathcal{C}_k$ should depend on our choice of ξ. For example, we have the following.

(a) For $a \in K^\times$, we have a mixed motive M_a over K which is an extension of $\mathbb{Q}$ by $\mathbb{Q}(1)$. Write $a = \xi^c u$ $(c = \mathrm{ord}_\xi(a) \in \mathbb{Q}$ and $u \in \mathbb{Q} \otimes O_K^\times)$ in $\mathbb{Q} \otimes K^\times$. Then this motive M_a should be sent to the object (M, W, cN), where M is the extension of $\mathbb{Q}$ by $\mathbb{Q}(1)$ corresponding to the image of u in $\mathbb{Q} \otimes k^\times$, W is the increasing filtration on M given by

$$0 = W_{-3} \subset \mathbb{Q}(1) = W_{-2} = W_{-1} \subset M = W_0,$$

and N is the composition of the canonical morphisms $M \to \mathbb{Q} \to M(-1)$.

(b) For a nonzero element q of m_K, if we denote by E_q the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ and we write $q = \xi^c u$ $(c = \mathrm{ord}_\xi(q) \in \mathbb{Q}$, $u \in \mathbb{Q} \otimes O_K^\times)$ in $\mathbb{Q} \otimes K^\times$, the motive $H^1(E_q)(1)$ over K should be sent to the object (M, W, cN) of $\mathcal{C}_k$, where M and N are the same as above and W is given by

$$0 = W_{-2} \subset M = W_{-1}.$$

In Sections 5 and 6, these two examples (a) and (b) are treated in Examples a and b, respectively. In Sections 5 and 6, since we assume that k is finite there, $\mathbb{Q} \otimes k^\times = 0$ and $M = \mathbb{Q}(1) \oplus \mathbb{Q}$.

3.4.14

We consider here only mixed motives over K of semistable reduction. Concerning general mixed motives, it is believed that the category of mixed motives over K is equivalent to the category $\varinjlim_L \mathcal{C}_{L/K}$, where L ranges over all finite Galois extensions of K in $\bar{K}$ and $\mathcal{C}_{L/K}$ denotes the category of pairs (M, c), where M is an object of $\mathcal{C}_L$ and c is $\mathrm{Gal}(L/K)$-descent data of M (the family of isomorphisms $c_s : M^{(s)} \cong M$ given for $s \in \mathrm{Gal}(L/K)$ satisfying $c_{ss'} = c_s s(c_{s'})$ for $s, s' \in \mathrm{Gal}(L/K)$). Here $M^{(s)}$ denotes the mixed motive over L obtained from M by applying s. Admitting this, the case of semistable reduction can cover all mixed motives.

3.5. Toroidal modifications of p-adic Hodge structures

3.5.1

Let D be an admissible filtered module over K (see Section 3.2.2) with φ, N, and F, and let V be the corresponding semistable p-adic representation of $\mathrm{Gal}(\bar{K}/K)$ (see Section 3.2.6).

Let $\mathfrak{M}$ be the set of all homomorphisms $D \to D(-1)$ of filtered modules over K, where $D(-1)$ is the (-1)-Tate twist of D. That is, $\mathfrak{M}$ is the set of all K_0-linear maps $N' : D \to D$ such that $N'\varphi = p\varphi N'$, $NN' = N'N$ and such that the induced K-linear map $N' : D_K \to D_K$ satisfies $N'F^r \subset F^{r-1}$ for any $r \in \mathbb{Z}$. Via the equivalence of categories in Section 3.2.6, we can understand $\mathfrak{M}$ also as the set of all homomorphisms $V \to V(-1)$ of representations of $\mathrm{Gal}(\bar{K}/K)$.

The aim of this subsection is to show that we can modify the admissible filtered module D by using elements of $\mathfrak{M}$ or of $K \otimes_{\mathbb{Q}_p} \mathfrak{M}$ to obtain a new admissible filtered module. In this modification, we do not change the K_0-vector space D and the operator φ, but we modify N and F.

The results of Section 3.5 are used in Section 6.

PROPOSITION 3.5.2

Let $N' \in \mathfrak{M}$. Let $D' = (D, N + N', F)$ be the filtered module over K obtained from D by replacing N by $N + N'$. (The K_0-vector spaces D, φ, and F are unchanged.)

(1) *D' is admissible.*

(2) *The semistable p-adic representation of $\mathrm{Gal}(\bar{K}/K)$ corresponding to D' coincides with*

$$\exp(l_\xi \otimes N')V \subset B_{\mathrm{st}} \otimes_{K_0} D = B_{\mathrm{st}} \otimes_{K_0} D',$$

where l_ξ is as in Section 3.2.4.

Proof

Let $V' = \exp(l_\xi \otimes N')V \subset B_{\mathrm{st}} \otimes_{K_0} D$.

CLAIM 1

V' is stable under the action of $\mathrm{Gal}(\bar{K}/K)$ in $B_{\mathrm{st}} \otimes_{K_0} D$.

Proof

Let t be the generator of $\mathbb{Z}_p(1) \subset B_{\mathrm{crys}}$ given in Section 3.2. For any $s \in \mathrm{Gal}(\bar{K}/K)$, $s(l_\xi \otimes N') - l_\xi \otimes N' \in \mathbb{Q}_p tN'$. Since an element of $\mathfrak{M}$ sends $V \subset B_{\mathrm{st}} \otimes_{K_0} D$ into $V(-1) \subset B_{\mathrm{st}} \otimes_{K_0} D$ (3.5.1), tN' sends V into V. Hence $s(V') \subset \exp(l_\xi \otimes N') \exp(\mathbb{Q}_p tN')V = V'$. This proves Claim 1. $\square$

CLAIM 2

For any $v' \in V' \subset B_{\mathrm{st}} \otimes_{K_0} D$, we have $(\varphi \otimes \varphi)(v') = v'$, and $(N \otimes 1 + 1 \otimes (N + N'))(v') = 0$.

Proof

Write $v' = \exp(l_\xi \otimes N')v$ with $v \in V$.

Since $\varphi(l_\xi) = pl_\xi$ and $\varphi N' = p^{-1}N'\varphi$, and since $(\varphi \otimes \varphi)(v) = v$, we have

$$(\varphi \otimes \varphi)(v') = \exp(pl_\xi \otimes p^{-1}N')(\varphi \otimes \varphi)v = \exp(l_\xi \otimes N')(v) = v'.$$

Since $N(l_\xi) = -1$ and $(N \otimes 1 + 1 \otimes N)(v) = 0$,

$$(N \otimes 1 + 1 \otimes N)(v') = (-1 \otimes N') \exp(l_\xi \otimes N')(v) = -(1 \otimes N')(v').$$

This completes the proof of Claim 2. $\square$

CLAIM 3

Identify $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ with $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ via the isomorphism $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V' \xrightarrow{\simeq} B_{\mathrm{st}} \otimes_{K_0} D$ induced from the inclusion map $V' \to B_{\mathrm{st}} \otimes_{K_0} D$. Then $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V' = B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ in $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ for any $i \in \mathbb{Z}$.

Proof
This follows from the fact that $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V = \sum_{j\in\mathbb{Z}} B^j_{\mathrm{dR}} \otimes_K F^{i-j}$, $l_\xi \in B^1_{\mathrm{dR}}$ and $N'F^r \subset F^{r-1}$ for any $r \in \mathbb{Z}$. □

The $\mathrm{Gal}(\bar{K}/K)$-fixed part of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ is that of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$, and hence it is D. Hence V' is a semistable representation of $\mathrm{Gal}(\bar{K}/K)$, and $D_{\mathrm{st}}(V)$ is D as a K_0-vector space.

CLAIM 4
The φ of $D_{\mathrm{st}}(V')$ is the φ of D, and N of $D_{\mathrm{st}}(V')$ is $N+N'$ of D.

Proof
By definition, φ of $D_{\mathrm{st}}(V')$ is induced by $\varphi\otimes 1$ of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$. By Claim 2, $\varphi\otimes 1$ of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ is $\varphi\otimes\varphi$ of $B_{\mathrm{st}} \otimes_{K_0} D$. The last map induces φ of D.

By definition, N of $D_{\mathrm{st}}(V')$ is induced by $N\otimes 1$ of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$. By Claim 2, $N\otimes 1$ of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ is $N\otimes 1 + 1\otimes(N+N')$ of $B_{\mathrm{st}} \otimes_{K_0} D$. The last map induces $N+N'$ of D. This completes the proof of Claim 4. □

CLAIM 5
We have $F^i D_{\mathrm{st}}(V')_K = F^i D_K$ for any $i \in \mathbb{Z}$.

Proof
By definition, $F^i D_{\mathrm{st}}(V')$ is the $\mathrm{Gal}(\bar{K}/K)$-fixed part of $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V'$. By Claim 3, this coincides with the $\mathrm{Gal}(\bar{K}/K)$-fixed part of $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ which is $F^i D_K$. □

By Claims 4 and 5, $D' = D_{\mathrm{st}}(V')$, and hence it is admissible. It follows that V' is the semistable p-adic representation of $\mathrm{Gal}(\bar{K}/K)$ corresponding to D'. This completes the proof of Proposition 3.5.2. □

3.5.3
Note that we have an exact sequence

$$0 \to \mathbb{Q}_p(1) \to B^{\varphi=p}_{\mathrm{crys}} \cap B^0_{\mathrm{dR}} \to \mathbb{C}_p \to 0,$$

where $B^{\varphi=p}_{\mathrm{crys}} = \{x \in B_{\mathrm{crys}} \mid \varphi(x) = px\}$.

PROPOSITION 3.5.4
Let $b \in K \otimes_{\mathbb{Q}_p} \mathfrak{M}$. Let $D' = (D, N, \exp(b)F)$ be the filtered module over K obtained from D by replacing F by $\exp(b)F$. (The K_0-vector spaces D, φ, and N are unchanged.)

(1) *D' is admissible.*

(2) *Take an element $\tilde{b}$ of $(B^{\varphi=p}_{\mathrm{crys}} \cap B^0_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} \mathfrak{M}$ whose image in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{M}$ coincides with b. Then the semistable p-adic representation of* $\mathrm{Gal}(\bar{K}/K)$ *corre-*

sponding to D' coincides with

$$\exp(\tilde{b})V \subset B_{\mathrm{st}} \otimes_{K_0} D = B_{\mathrm{st}} \otimes_{K_0} D'.$$

Proof
Let $V' = \exp(\tilde{b})V \subset B_{\mathrm{st}} \otimes_{K_0} D$.

CLAIM 1
V' is stable under the action of $\mathrm{Gal}(\bar{K}/K)$ in $B_{\mathrm{st}} \otimes_{K_0} D$.

Proof
For any $s \in \mathrm{Gal}(\bar{K}/K)$, $s(\tilde{b}) - \tilde{b} \in \mathbb{Q}_p tN'$ by Section 3.5.3. Since tN' sends V into V in $B_{\mathrm{st}} \otimes_{K_0} D$ (see the proof of Claim 1 in the proof of Proposition 3.5.2), we have $s(V') \subset \exp(\tilde{b})\exp(\mathbb{Q}_p tN')V = V'$. This proves Claim 1. $\square$

CLAIM 2
For any $v' \in V' \subset B_{\mathrm{st}} \otimes_{K_0} D$, we have $(\varphi \otimes \varphi)(v') = v'$, and $(N \otimes 1 + 1 \otimes N)(v') = 0$.

Proof
The proof is similar to the proof of Claim 2 in the proof of Proposition 3.5.2. In the place where we used $\varphi(l_\xi) = pl_\xi$ (resp., $N(l_\xi) = -1$), we use $\varphi(\tilde{b}) = p\tilde{b}$ (resp., $N(\tilde{b}) = 0$). $\square$

CLAIM 3
Identify $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ with $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ via the isomorphism $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V' \xrightarrow{\simeq} B_{\mathrm{st}} \otimes_{K_0} D$ induced from the inclusion map $V' \to B_{\mathrm{st}} \otimes_{K_0} D$. Then $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V' = \exp(b) \times (B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ in $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ for any $i \in \mathbb{Z}$.

Proof
This follows from the fact that $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V' = \exp(\tilde{b})(B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$, $B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V = \sum_{j\in\mathbb{Z}} B^j_{\mathrm{dR}} \otimes_K F^{i-j}$, $\tilde{b} \equiv b \bmod B^1_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{M}$, and $N'F^r \subset F^{r-1}$ for any $N' \in \mathfrak{M}$ and any $r \in \mathbb{Z}$. $\square$

The $\mathrm{Gal}(\bar{K}/K)$-fixed part of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V'$ is that of $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$, and hence it is D. Hence V' is a semistable representation of $\mathrm{Gal}(\bar{K}/K)$, and $D_{\mathrm{st}}(V)$ is D as a K_0-vector space.

CLAIM 4
The φ of $D_{\mathrm{st}}(V')$ is the φ of D, and N of $D_{\mathrm{st}}(V')$ is N of D.

Proof
The proof is similar to that of Claim 4 in the proof of Proposition 3.5.2. $\square$

CLAIM 5
$F^i D_{\mathrm{st}}(V')_K = \exp(b)F^i D_K$ *for any* $i \in \mathbb{Z}$.

Proof
This follows from Claim 3. $\square$

By Claims 4 and 5, $D' = D_{\mathrm{st}}(V')$, and hence it is admissible. It follows that V' is the semistable p-adic representation of $\mathrm{Gal}(\bar{K}/K)$ corresponding to D'. This completes the proof of Proposition 3.5.4. $\square$

PROPOSITION 3.5.5
Let $b \in K \otimes_{\mathbb{Q}_p} \mathfrak{M}$.

(1) *Assume that* $bF^r \subset F^r$ *for any* $r \in \mathbb{Z}$. *Then* $b = 0$.
(2) *Assume that* $\exp(b)F = F$. *Then* $b = 0$.

Proof
We prove (1). Let $U = \mathrm{Hom}_{\mathbb{Q}_p}(V, V)$, and let $U' = \mathrm{Hom}_{\mathrm{Gal}(\bar{K}/K)}(V, V(-1)) \subset U(-1)$. Then, via the identification $B_{\mathrm{st}} \otimes_{K_0} D = B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$, $S := \{b \in K \otimes_{\mathbb{Q}_p} \mathfrak{M} \mid bF^r \subset F^r \text{ for all } r\}$ is identified with $(K \otimes_{\mathbb{Q}_p} U') \cap (B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} U)$, where the intersection is taken in $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} U$ and where $K \otimes_{\mathbb{Q}_p} U(-1)$ is embedded in $B^{-1}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} U$ in the natural way. Hence

$$S \subset \big((K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-1)) \cap B^0_{\mathrm{dR}}\big) \otimes_{\mathbb{Q}_p} U = 0,$$

where $\mathbb{Q}_p(-1)$ is embedded in B^{-1}_{dR} in the natural way. Here $(K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-1)) \cap B^0_{\mathrm{dR}} = 0$ because $K \otimes \mathbb{Q}_p(-1) \to B^{-1}_{\mathrm{dR}}/B^0_{\mathrm{dR}}$ is injective.

We prove (2). Assume that $\exp(b)F = F$. Then $\exp(nb)F = F$ for any integer $n \geq 0$. By continuity, $\exp(nb)F = F$ for any $n \in \mathbb{Z}_p$. From this, we obtain $bF^r \subset F^r$ for any $r \in \mathbb{Z}$, and hence we are reduced to (1). $\square$

4. p-Adic period domains ${}_N D$ and ${}_{N,\Gamma} D$

In Section 4, we consider the p-adic period domains ${}_N D$ and ${}_{N,\Gamma} D$, where the latter is a refinement of the former by taking the p-adic level structure into account.

4.1. Notation in Section 4

4.1.1
We use the notation explained in Section 3.1.

4.1.2
In Section 4, we fix a triple

$$\big(H, W, (h_{w,i})_{w,i \in \mathbb{Z}}\big),$$

where we have the following:

· H is a finite-dimensional vector space over K_0 endowed with a Frobenius-linear bijection $\varphi : H \to H$;

· W is an increasing filtration on H such that $\varphi(W_w) \subset W_w$ for all $w \in \mathbb{Z}$, $W_w = H$ if $w \gg 0$, and $W_w = 0$ if $w \ll 0$;

· $h_{w,i}$ are integers which are zero for almost all (w,i) such that $\sum_i h_{w,i} = \dim(\mathrm{gr}_w^W)$ for any $w \in \mathbb{Z}$.

4.1.3

Let $d = \dim_{K_0}(H)$. To discuss level structures, we fix a d-dimensional $\mathbb{Q}_p$-vector space L endowed with an increasing filtration $W_\bullet L$ on L such that the $\mathbb{Q}_p$-dimension of $W_w L$ is equal to the K_0-dimension of $W_w \subset H$ for any $w \in \mathbb{Z}$.

Let G be the automorphism group of $(L, W_\bullet L)$, which we regard as an algebraic group over $\mathbb{Q}_p$.

4.1.4

Let $\check{D}$ be the set of all decreasing filtrations on the K-module $H_K = K \otimes_{K_0} H$ such that $h_{w,i} = \dim_K \mathrm{gr}_F^i \mathrm{gr}_{w,K}^W$ for any w, i. Then $\check{D}$ has a natural structure of an analytic manifold over K.

4.2. p-Adic period domains ${}_N D$

4.2.1

Let $N : H \to H$ be a K_0-linear map such that $N\varphi = p\varphi N$ and $NW_w \subset W_w$ for any $w \in \mathbb{Z}$. (Such an N is automatically nilpotent.) Define

$$ {}_N D = \left\{ F \in \check{D} \,\middle|\, \mathrm{gr}_w^W(N, F) \text{ is admissible (by Section 3.2.2) for any } w \in \mathbb{Z} \right\} \subset \check{D}. $$

Here $\mathrm{gr}_w^W(N, F)$ denotes the K_0-vector space gr_w^W endowed with $\mathrm{gr}_w^W(\varphi) : \mathrm{gr}_w^W \to \mathrm{gr}_w^W$, $\mathrm{gr}_w^W(N) : \mathrm{gr}_w^W \to \mathrm{gr}_w^W$, and the filtration $\mathrm{gr}_w^W(F)$ on $\mathrm{gr}_{w,K}^W := K \otimes_{K_0} \mathrm{gr}_w^W$, regarded as an object of MF_K.

PROPOSITION 4.2.2

${}_N D$ is open in $\check{D}$.

This is proved in [24] in the case $N = 0$ (W is not considered in [25]), and the proof in [24] works in the general case. Here we write down a proof which is essentially the same as that in [24].

Proof of Proposition 4.2.2

We may and do assume that $k = \bar{k}$ and that there is $w \in \mathbb{Z}$ such that $W_w = H$ and $W_{w-1} = 0$.

Let $t_N(H) \in \mathbb{Z}$ be as in Section 3.2.2. For any $F \in {}_N D$, $t_H(H, F) \in \mathbb{Z}$ in Section 3.2.2 coincides with $\sum_i i h_{w,i}$. Hence ${}_N D$ is empty if $t_N(H) \neq \sum_i i h_{w,i}$.

We assume that $t_N(H) = \sum_i i h_{w,i}$. For integers $r \geq 0$ and m, let

$$ \tilde{P}_{r,m} = \left\{ x \in \bigwedge^r H \,\middle|\, \varphi(x) = p^m x, N(x) = 0 \right\}, \qquad P_{r,m} = (\tilde{P}_{r,m} - \{0\})/\mathbb{Q}_p^\times. $$

Here φ and N denote the maps $\bigwedge^r H \to \bigwedge^r H$ which are induced from φ and $N : H \to H$, respectively. That is, $\varphi(v_1 \wedge \cdots \wedge v_r) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_r)$ and $N(v_1 \wedge \cdots \wedge v_r) = \sum_{i=1}^r v_1 \wedge \cdots \wedge v_{i-1} \wedge N(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_r$ for $v_1, \ldots, v_r \in H$. Then $P_{r,m}$ is compact. Note that $P_{r,m}$ are empty for almost all (r,m). Let

$$S_{r,m} = \Big\{(x,F) \in P_{r,m} \times \check{D} \;\Big|\; \tilde{x} \in F^{m+1}\Big(\bigwedge_K^r H_K\Big)\Big\}.$$

Here $\tilde{x}$ denotes a lifting of x to $\tilde{P}_{r,m} - \{0\}$ and F^{m+1} on $\bigwedge_K^r H_K$ is induced by F. Since $P_{r,m}$ is compact and since $S_{r,m}$ is closed in $P_{r,m} \times \check{D}$, we see that the map $S_{r,m} \to \check{D}, (x,F) \mapsto F$ is proper. Hence the image of this map is closed in $\check{D}$. Hence for the proof of Proposition 4.2.2, it is sufficient to prove the following lemma.

LEMMA 4.2.3

Assume that $k = \bar{k}$, $W_w = H$, and $W_{w-1} = 0$ for some w, and assume that $t_N(H) = \sum_i i h_{w,i}$. Then in $\check{D}$, ${}_N D$ coincides with the complement of $\bigcup_{r,m} \mathrm{Image}(S_{r,m} \to \check{D})$.

Proof

Let $(x,F) \in S_{r,m}$. We prove that $F \notin {}_N D$. Let $\tilde{x}$ be a lifting of $x \in P_{r,m}$ in $\tilde{P}_{r,m} - \{0\}$, and let S be the one-dimensional K_0-subspace of $\bigwedge^r H$ generated by $\tilde{x}$. Since $t_N(S) = m < t_H(S)$, $(\bigwedge^r H, N, F)$ is not admissible. Since the exterior powers of admissible filtered modules are admissible (this follows from the fact that exterior powers of semistable Galois representations are semistable, by the equivalence of categories in Section 3.2.6), this shows that (H, N, F) is not admissible. Hence $F \notin {}_N D$.

Next, assume that $F \notin {}_N D$. We show that there are (r,m) and $x \in P_{r,m}$ such that $(x,F) \in S_{r,m}$. Let S be a K_0-subspace of H such that $t_N(S) < t_H(S)$. Let $r = \dim_{K_0} H$, and let $m = t_N(S)$. Then there is a nonzero element $\tilde{x}$ of $\bigwedge^r S$ such that $\varphi(\tilde{x}) = p^m \tilde{x}$. Since $t_N(S) < t_H(S)$, we have $\tilde{x} \in F^{m+1}$. Let x be the image of $\tilde{x} \in \tilde{P}_{r,m}$ in $P_{r,m}$. Then $(x,F) \in S_{r.m}$. □

4.2.4

By Proposition 4.2.2, ${}_N D$ is regarded as an open analytic subspace over K of $\check{D}$.

4.3. p-Adic level structures and p-adic period domains ${}_{N,\Gamma}D$

We discuss p-adic level structures.

4.3.1

Let $N : H \to H$ be a K_0-linear map such that $N\varphi = p\varphi N$ and $NW_w \subset W_w$ for all $w \in \mathbb{Z}$.

Let $F \in {}_N D$. By Section 3.2.3, (H, N, F) is admissible. Let $V_p(N, F)$ be the semistable representation of $\mathrm{Gal}(\bar{K}/K)$ over $\mathbb{Q}_p$ corresponding to (H, N, F) (see Section 3.2.6).

4.3.2

Let Γ be a compact open subgroup of $G(\mathbb{Q}_p) = \mathrm{Aut}_{\mathbb{Q}_p}(L, W_\bullet L)$.

We define the notion of a Γ-level structure on $V_p(N, F)$.

4.3.3. Level structure

Let V be a d-dimensional $\mathbb{Q}_p$-vector space endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$ and with a $\mathrm{Gal}(\bar{K}/K)$-stable filtration $W_\bullet V$ such that $\dim_{\mathbb{Q}_p} W_w V = \dim_{\mathbb{Q}_p} W_w L$ for all $w \in \mathbb{Z}$.

For isomorphisms $\mu_i : (L, W_\bullet L) \xrightarrow{\cong} (V, W_\bullet V)$ $(i = 1, 2)$ of $\mathbb{Q}_p$-vector spaces with filtrations, we say that μ_1 and μ_2 are Γ-*equivalent* if $\mu_1^{-1}\mu_2 \in \Gamma$.

By a Γ-*level structure on* V we mean a Γ-equivalence class of an isomorphism $\mu : (L, W_\bullet L) \cong (V, W_\bullet V)$ such that $\mu^{-1} s \mu \in \Gamma$ for any $s \in \mathrm{Gal}(\bar{K}/K)$.

For example, if $L_{\mathbb{Z}_p}$ is a finitely generated $\mathbb{Z}_p$-submodule of L such that $L = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{\mathbb{Z}_p}$ and $\Gamma = \mathrm{Aut}_{\mathbb{Z}_p}(L_{\mathbb{Z}_p}, W_\bullet L_{\mathbb{Z}_p})$, where $W_\bullet L_{\mathbb{Z}_p}$ denotes the restriction of $W_\bullet L$ to $L_{\mathbb{Z}_p}$, then a Γ-level structure on V corresponds in a one-to-one manner to a $\mathrm{Gal}(\bar{K}/K)$-stable finitely generated $\mathbb{Z}_p$-submodule T of V such that $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$. The correspondence is given in the following way. If μ is a Γ-level structure on V and it is the class of $\tilde{\mu} : L \to V$, then we have $T = \tilde{\mu}(L_{\mathbb{Z}_p})$. Conversely, from T, we obtain a Γ-level structure on V as the class of $L \to V$ induced by any isomorphism $L_{\mathbb{Z}_p} \to T$ of $\mathbb{Z}_p$-modules with filtrations.

4.3.4

Let $F \in {}_N D$. Since $\mathrm{gr}_w^W(N, F)$ are admissible for all $w \in \mathbb{Z}$, $W_w(N, F)$ are admissible for any $w \in \mathbb{Z}$ by Section 3.2.3. Let $V = V_p(N, F)$ (see Section 4.3.1). Let $W_w V$ be the p-adic representation of $\mathrm{Gal}(\bar{K}/K)$ corresponding to $W_w(N, F)$, which is a subrepresentation of V. We have $\dim_{\mathbb{Q}_p} W_w V = \dim_{K_0} W_w$ (the last W_w is that of H) $= \dim_{\mathbb{Q}_p} W_w L$. Thus we have the situation of Section 4.3.3.

4.3.5

Define

$$ {}_{N,\Gamma} D = \big\{ (F, \mu) \bigm| F \in {}_N D,\ \mu \text{ is a } \Gamma\text{-level structure on } V_p(N, F) \big\}. $$

4.4. The analytic structure over K of ${}_{N,\Gamma} D$

Let N and Γ be as in Section 4.3.

4.4.1

To define a structure of an analytic space over K on ${}_{N,\Gamma} D$, the key idea is to define a "partial section" (see Section 4.4.8) of the projection ${}_{N,\Gamma} D \to {}_N D$ and transfer the analytic structure of ${}_N D$ (4.2.4) to ${}_{N,\Gamma} D$.

Let $\alpha = (F, \mu) \in {}_{N,\Gamma}D$. Then, as is explained below, for a sufficiently small neighborhood U of F in ${}_N D$, we have a canonical map (see Section 4.4.8)

$$e_\alpha : U \to {}_{N,\Gamma}D$$

such that $e_\alpha(F) = \alpha$ and such that for any $F' \in U$, the image of $e_\alpha(F')$ under the projection ${}_{N,\Gamma}D \to {}_N D$ is F'. Using this map, we define the analytic structure over K of ${}_{N,\Gamma}D$ as in Section 4.4.11 in such a way that the map ${}_{N,\Gamma}D \to {}_N D$ is a local isomorphism of analytic spaces over K and e_α is a local section of it.

In some examples, ${}_{N,\Gamma}D \to {}_N D$ is interpreted as a p-adic logarithm map, and this map e_α is interpreted as a p-adic exponential map (see Section 5.5, Example a).

4.4.2

Let $\alpha = (F, \mu) \in {}_{N,\Gamma}D$, let $V = V_p(N, F)$ (see Section 4.3.1), and take a representative $\tilde{\mu} : (L, W_\bullet L) \xrightarrow{\cong} (V, W_\bullet V)$ of μ. Let

$$B_{\mathrm{crys}}^{\varphi=1} = \big\{ x \in B_{\mathrm{crys}} \bigm| \varphi(x) = x \big\}.$$

As in Section 4.1.3, let G be the automorphism group of $(L, W_\bullet L)$ regarded as an algebraic group over $\mathbb{Q}_p$.

Then we have an injective map

$$h_{\tilde{\mu}} : {}_N D \to G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$$

defined as follows. Let $F' \in {}_N D$, and let $V' = V_p(N, F')$. Then

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V' = B_{\mathrm{st}} \otimes_{K_0} H = B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V.$$

Taking the part $\{x \mid \varphi(x) = x, N(x) = 0\}$ of this, we have

$$B_{\mathrm{crys}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V' = B_{\mathrm{crys}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V \quad \text{in } B_{\mathrm{st}} \otimes_{K_0} H. \tag{3}$$

Take any isomorphism $\tilde{\mu}' : (L, W_\bullet L) \xrightarrow{\cong} (V', W_\bullet V')$. By (3), we have the composition

$$\tilde{\mu}^{-1}\tilde{\mu}' : B_{\mathrm{crys}}^{\varphi=1} \otimes_{\mathbb{Q}_p} L \xrightarrow{\tilde{\mu}'} B_{\mathrm{crys}}^{\varphi=1} \otimes_{\mathbb{Q}_p} V \xrightarrow{\tilde{\mu}^{-1}} B_{\mathrm{crys}}^{\varphi=1} \otimes_{\mathbb{Q}_p} L,$$

which is regarded as an element of $G(B_{\mathrm{crys}}^{\varphi=1})$. Any other choice of $\tilde{\mu}'$ is given by $\tilde{\mu}'\gamma$ with $\gamma \in G(\mathbb{Q}_p)$. We define $h_{\tilde{\mu}}(F') \in G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$ to be the class of $\tilde{\mu}^{-1}\tilde{\mu}'$, which is independent of the choice of $\tilde{\mu}'$.

If we replace the representative $\tilde{\mu}$ of μ by $\tilde{\mu}\gamma^{-1}$ with $\gamma \in \Gamma$, we have

$$h_{\tilde{\mu}\gamma^{-1}}(F') = \gamma h_{\tilde{\mu}}(F') \gamma^{-1}. \tag{4}$$

4.4.3

For an integer $c \geq 0$, define subrings B_c and $B_c^{\varphi=1}$ of B_{crys} by

$$B_c := A_{\mathrm{crys}}\left[\frac{p^c}{t}\right] \subset B_{\mathrm{crys}}, \qquad B_c^{\varphi=1} := B_c \cap B_{\mathrm{crys}}^{\varphi=1}.$$

Note that

$$B_{c+1} \subset B_c, \qquad \varphi(B_{c+1}) \subset B_c.$$

LEMMA 4.4.4

Assume that $c \geq 2$. Then

$$B_c \cap \mathbb{Q}_p = \mathbb{Z}_p.$$

This is proved in Section 4.5.8.

4.4.5

Since Γ is compact, there is a finitely generated $\mathbb{Z}_p$-submodule $L_{\mathbb{Z}_p}$ of L which satisfies $L = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{\mathbb{Z}_p}$ and which is stable under the action of Γ. In the rest of Section 4.4, we fix such $L_{\mathbb{Z}_p}$. Let $W_\bullet L_{\mathbb{Z}_p}$ be the restriction of the filtration $W_\bullet L$ to $L_{\mathbb{Z}_p}$. By abuse of notation, we denote also by G the automorphism group of $(L_{\mathbb{Z}_p}, W_\bullet L_{\mathbb{Z}_p})$ regarded as a smooth group scheme over $\mathbb{Z}_p$. ($G(R)$ for a ring R over $\mathbb{Z}_p$ is defined in the evident way.) We have $\Gamma \subset G(\mathbb{Z}_p)$.

For $n \geq 0$, let

$$G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} = \mathrm{Ker}\big(G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^n\mathbb{Z})\big),$$
$$G(B_c)_{\equiv 1 \bmod p^n} := \mathrm{Ker}\big(G(B_c) \to G(B_c/p^n B_c)\big),$$
$$G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n} := \mathrm{Ker}\big(G(B_c^{\varphi=1}) \to G(B_c^{\varphi=1}/p^n B_c^{\varphi=1})\big).$$

4.4.6

Let $\alpha = (F, \mu) \in {}_{N,\Gamma}D$. Let $\tilde{\mu} : (L, W_\bullet L) \xrightarrow{\cong} (V, W_\bullet V)$ with $V = V_p(N, F)$ be a representative of μ. For integers $c \geq 2$ and $n \geq 0$, let

$$U_{\alpha,c,n} = \big\{F' \in {}_N D \,\big|\, h_{\tilde{\mu}}(F') \in \mathrm{Image}(G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n} \to G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p))\big\}.$$

Here $h_{\tilde{\mu}}$ is as in 4.4.2. Then $U_{\alpha,c,n}$ is independent of the choice of the representative $\tilde{\mu}$ of μ. This follows from (4) in Section 4.4.2 and from $\Gamma \subset G(\mathbb{Z}_p)$.

We have

$$U_{\alpha,c',n'} \subset U_{\alpha,c,n} \quad \text{if } c' \geq c \text{ and } n' \geq n.$$

PROPOSITION 4.4.7

For any $c \geq 2$ and $n \geq 0$, $U_{\alpha,c,n}$ is a neighborhood of F in ${}_N D$.

The proof of this proposition is given in Section 4.6.

4.4.8

We define the "partial section" e_α of the projection ${}_{N,\Gamma}D \to {}_N D$.

Let $\alpha = (F, \mu) \in {}_{N,\Gamma}D$, and let $\tilde{\mu}$ be a representative of μ. Since Γ is open in $G(\mathbb{Q}_p)$, there is an integer $n \geq 0$ such that

$$G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma.$$

Take such an $n \geq 0$, and take an integer $c \geq 2$. Let $U_{\alpha,c,n}$ be as above. We define the map

$$e_\alpha : U_{\alpha,c,n} \to {}_{N,\Gamma}D$$

as follows. Let $F' \in U_{\alpha,c,n}$, let $V' = V_p(N, F')$, and let h be an element of $G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n}$ whose image in $G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$ coincides with $h_{\tilde{\mu}}(F')$. Then we have an isomorphism $\tilde{\mu}' := \tilde{\mu}h : (L, W_\bullet L) \xrightarrow{\cong} (V', W_\bullet V')$.

We show that the Γ-equivalence class μ' of $\tilde{\mu}'$ is independent of the choices of $\tilde{\mu}$ and h and gives a Γ-level structure on V'. Note that $G(\mathbb{Q}_p) \cap G(B_c)_{\equiv 1 \bmod p^n} = G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n}$ by Lemma 4.4.4. For another choice h_1 of h, we have $h_1 = h\gamma$ with $\gamma \in G(\mathbb{Q}_p)$ and $\gamma = h^{-1}h_1 \in G(\mathbb{Q}_p) \cap G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n} = G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma$, and hence $\tilde{\mu}h_1 \in \tilde{\mu}h\Gamma$. For another choice $\tilde{\mu}\gamma$ ($\gamma \in \Gamma$) of $\tilde{\mu}$, $\gamma h \gamma^{-1} \in G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n}$ and the image of $\gamma h \gamma^{-1}$ in $G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$ coincides with $h_{\tilde{\mu}\gamma}(F')$ by (4) in Section 4.4.2, and $\tilde{\mu}\gamma h \in \tilde{\mu}\gamma h \gamma^{-1}\Gamma$. It remains to prove that for $s \in \mathrm{Gal}(\bar{K}/K)$, we have $s\tilde{\mu}' \in \tilde{\mu}'\Gamma$. Since V' is stable under the action of $\mathrm{Gal}(\bar{K}/K)$, we have $s\tilde{\mu}' = \tilde{\mu}'\gamma'$ for some $\gamma' \in G(\mathbb{Q}_p)$. We prove that $\gamma' \in \Gamma$. Since μ is a Γ-level structure, we have $s\mu = \mu\gamma$ for some $\gamma \in \Gamma$. Since $\mu' = \mu h$, we have

$$\gamma'\gamma^{-1} = h^{-1}(\gamma h \gamma^{-1}) \in G(\mathbb{Q}_p) \cap G(B_c)_{\equiv 1 \bmod p^n} = G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n},$$

and hence $\gamma' \in \Gamma$.

We define

$$e_\alpha(F') = (F', \mu').$$

Clearly we have $e_\alpha(F) = \alpha$, and for $F' \in U_{\alpha,c,n}$, the image of $e_\alpha(F')$ under ${}_{N,\Gamma}D \to {}_N D$ coincides with F'.

When c and n vary, e_α defined by various (c, n) are compatible.

PROPOSITION 4.4.9

Let $\alpha \in {}_{N,\Gamma}D$. Let $c \geq 2$, $n \geq 0$, and assume that $G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma$. Let $F' \in U_{\alpha,c,n}$, and let $\alpha' = e_\alpha(F')$. Then $U_{\alpha',c,n} = U_{\alpha,c,n}$, and we have

$$e_\alpha(F'') = e_{\alpha'}(F'') \quad \text{for any } F'' \in U_{\alpha,c,n}.$$

This is seen easily.

By Propositions 4.4.7 and 4.4.9, we have the following.

COROLLARY 4.4.10

The set $U_{\alpha,c,n}$ ($c \geq 2$, $n \geq 0$) is open in ${}_N D$.

4.4.11

Using the map e_α in Section 4.4.8, we define the analytic structure over K of ${}_{N,\Gamma}D$ as follows.

We first define the topology of ${}_{N,\Gamma}D$. A subset U of ${}_{N,\Gamma}D$ is open if and only if for each $\alpha = (F, \mu) \in U$, we have $e_\alpha(U') \subset U$ for sufficiently small neighborhoods U' of F in ${}_N D$.

By Proposition 4.4.9, for a sufficiently small open neighborhood U of F in ${}_N D$, the map $e_\alpha : U \to {}_{N,\Gamma}D$ is an injective open map and it is a local section of the projection ${}_{N,\Gamma}D \to {}_N D$. Hence the projection ${}_{N,\Gamma}D \to {}_N D$ is a local homeomorphism. By this, we transfer the analytic structure over K of ${}_N D$ to ${}_{N,\Gamma}D$.

The canonical map ${}_{N,\Gamma}D \to {}_N D$ becomes locally an isomorphism of analytic manifolds over K, and e_α gives local sections of it.

4.4.12

The analytic structure over K of ${}_{N,\Gamma}D$ is independent of the choice of $L_{\mathbb{Z}_p}$. This is seen by the following facts. Let $L'_{\mathbb{Z}_p}$ be another choice of $L_{\mathbb{Z}_p}$, and let G' be the automorphism group of $(L'_{\mathbb{Z}_p}, W_\bullet L'_{\mathbb{Z}_p})$ regarded as a group scheme over $\mathbb{Z}_p$. G and G' coincide over $\mathbb{Q}_p$. Take an integer $a \geq 0$ such that $p^a L_{\mathbb{Z}_p} \subset L'_{\mathbb{Z}_p}$ and $p^a L'_{\mathbb{Z}_p} \subset L_{\mathbb{Z}_p}$. Then, as is easily shown, for any flat ring R over $\mathbb{Z}_p$ and for $m = n + 2a$, we have

$$G(R)_{\equiv 1 \bmod p^m} \subset G'(R)_{\equiv 1 \bmod p^n}, \qquad G'(R)_{\equiv 1 \bmod p^m} \subset G(R)_{\equiv 1 \bmod p^n}.$$

4.5. Subrings A_c and B_c of B_{crys}

We prove Lemma 4.4.4, and we give preparations for the proof of Proposition 4.4.7.

The results given in Section 4.5 are taken from [13] by M. Kurihara, T. Tsuji, and the author.

4.5.1

For an integer $c \geq 0$, we define a subring A_c of B_c as follows.

As in [7], for $r \geq 0$, let

$$\mathrm{Fil}^r A_{\text{crys}} = A_{\text{crys}} \cap B^r_{\text{dR}},$$

$$\mathrm{Fil}^r_p A_{\text{crys}} = \{x \in \mathrm{Fil}^r A_{\text{crys}} \mid \varphi(x) \in p^r A_{\text{crys}}\}.$$

Define

$$A_c = \sum_{r \geq 0} \left(\frac{p^c}{t}\right)^r \cdot \mathrm{Fil}^r_p A_{\text{crys}} \subset B_c.$$

We have

$$A_{c+1} \subset A_c, \qquad \varphi(A_c) \subset B_c.$$

LEMMA 4.5.2

We have

(1) $B_c\left[\frac{1}{p}\right] = B_{\text{crys}}$;
(2) $A_c\left[\frac{1}{p}\right] = B_{\text{crys}} \cap B^0_{\text{dR}}$.

Proof

Assertion (1) follows from the fact that $A_{\text{crys}}[1/t] = B_{\text{crys}}$. The last fact shows that $B_{\text{crys}} \cap B^0_{\text{dR}} = \sum_{r \geq 0} t^{-r} \mathrm{Fil}^r A_{\text{crys}}$, and (2) follows from it. $\square$

PROPOSITION 4.5.3

Assume that $c \geq 1$. In the case $p = 2$, assume that $c \geq 2$. Then the sequence

$$0 \to \mathbb{Z}_p \to A_c \xrightarrow{1-\varphi} B_c \to 0$$

is exact.

Proof

We deduce this from the fundamental exact sequences of Fontaine and Messing (see [9], [7, Section 5.3.6]):

$$0 \to p^{-\lambda(r)}\mathbb{Z}_p(r) \to \mathrm{Fil}_p^r A_{\mathrm{crys}} \xrightarrow{1-p^{-r}\varphi} A_{\mathrm{crys}} \to 0 \tag{5}$$

for $r \geq 0$. Here

$$\lambda(r) := \sum_{i=0}^{\infty} \Big[\frac{r}{(p-1)p^i}\Big],$$

where $[x]$ for $x \in \mathbb{R}$ denotes the largest integer n such that $n \leq x$. In the exact sequence (5), $\mathbb{Z}_p(r)$ is identified with $\mathbb{Z}_p t^r \subset \mathrm{Fil}_p^r A_{\mathrm{crys}}$. Note that

$$\lambda(r) \leq \sum_{i=0}^{\infty} \frac{r}{(p-1)p^i} = \frac{pr}{(p-1)^2}. \tag{6}$$

The following exact sequence is a corollary of the case $r = 0$ of the exact sequence (5) (see [7, Section 5.3.7(iii)]):

$$0 \to \mathbb{Q}_p \to B_{\mathrm{crys}} \cap B_{\mathrm{dR}}^0 \xrightarrow{1-\varphi} B_{\mathrm{crys}} \to 0. \tag{7}$$

We prove the surjectivity of $1 - \varphi : A_c \to B_c$. For $x \in A_{\mathrm{crys}}$ and $r \geq 0$, by the exact sequence (5), there is $y \in \mathrm{Fil}_p^r A_{\mathrm{crys}}$ such that $x = (1 - p^{-r}\varphi)y$. We have

$$\Big(\frac{p^c}{t}\Big)^r x = (1-\varphi)\Big(\Big(\frac{p^c}{t}\Big)^r y\Big) \in (1-\varphi)A_c.$$

Next, we prove that the kernel of $1 - \varphi : A_c \to B_c$ is $\mathbb{Z}_p$. By the exact sequence (7), the kernel of $1 - \varphi : A_c \to B_c$ is contained in $\mathbb{Q}_p$ which contains $\mathbb{Z}_p$. Hence, to prove that the kernel coincides with $\mathbb{Z}_p$, since $\mathbb{Q}_p \cap O_{\mathbb{C}_p} = \mathbb{Z}_p$, it is sufficient to prove that the image of A_c in $B_{\mathrm{dR}}^0/B_{\mathrm{dR}}^1 = \mathbb{C}_p$ is contained in $O_{\mathbb{C}_p}$. By [7, Section 5.3.6(ii)], there exists an element x of $\mathrm{Fil}^1 A_{\mathrm{crys}}$ such that

$$\mathrm{Fil}_p^r A_{\mathrm{crys}} \subset \Big(\sum_{i,j\geq 0, i+j\geq r} A_{\mathrm{crys}} \cdot x^i p^{-\lambda(j)} t^j\Big) + B_{\mathrm{dR}}^m \quad \text{for any } m \geq 1. \tag{8}$$

The case $m = r + 1$ of (8) shows that

$$t^{-r}\mathrm{Fil}_p^r A_{\mathrm{crys}} \subset \Big(\sum_{i=0}^{r} A_{\mathrm{crys}} \cdot p^{-\lambda(r-i)}(x/t)^i\Big) + B_{\mathrm{dR}}^1. \tag{9}$$

Since the image of $t^{-1}\mathrm{Fil}^1 A_{\mathrm{crys}}$ in $B_{\mathrm{dR}}^0/B_{\mathrm{dR}}^1 = \mathbb{C}_p$ coincides with $p^{-1/(p-1)}O_{\mathbb{C}_p}$ (see [7]), (9) shows that the image of $t^{-r}\mathrm{Fil}_p^r A_{\mathrm{crys}}$ in $\mathbb{C}_p$ is contained in $\sum_{i=0}^{r} p^{-\lambda(r-i)-i/(p-1)}O_{\mathbb{C}_p}$. Hence it is sufficient to prove that $cr - \lambda(r-i) -$

$i/(p-1) \geq 0$ for any integers r, i such that $0 \leq i \leq r$. By (6),

$$cr - \lambda(r-i) - i/(p-1) \geq cr - p(r-i)/(p-1)^2 - i/(p-1)$$
$$\geq \Big(c - \frac{p}{(p-1)^2}\Big) r \geq 0. \qquad \square$$

PROPOSITION 4.5.4

Let $c \geq 2$, and let

$$\hat{B}_c = \varprojlim_m B_c/p^n B_c, \qquad \hat{A}_c = \varprojlim_n A_c/p^n A_c.$$

Then the canonical maps $B_c \to \hat{B}_c$ and $A_c \to \hat{A}_c$ are injective.

Proof

The injectivity of $B_c \to \hat{B}_c$ is reduced to that of $A_c \to \hat{A}_c$ by the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & A_c & \xrightarrow{1-\varphi} & B_c & \longrightarrow 0 \\ & & \| & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \hat{A}_c & \xrightarrow{1-\varphi} & \hat{B}_c & \longrightarrow 0 \end{array}$$

After preparations in Lemmas 4.5.5 and 4.5.6, we prove the injectivity of $A_c \to \hat{A}_c$ by defining a ring homomorphism $\hat{A}_c \to B^0_{\mathrm{dR}}$ which induces the inclusion map $A_c \to B^0_{\mathrm{dR}}$.

LEMMA 4.5.5

Let $m \geq 1$, and let M be a finitely generated ($A_{\mathrm{crys}}/\mathrm{Fil}^m A_{\mathrm{crys}}$)-module. Then $M \xrightarrow{\cong} \varprojlim_n M/p^n M$.

Proof

Consider the following property (P) of an abelian group A.

(P) We have $A \xrightarrow{\cong} \varprojlim_n A/p^n A$, and there is an integer $a \geq 1$ such that p^a kills the p-primary torsion part of A.

If $0 \to A' \to A \to A'' \to 0$ is an exact sequence of abelian groups and if A' and A'' have the property (P), then A also has the property (P).

First, we consider the case $m = 1$ of Lemma 4.5.5. In this case, $A_{\mathrm{crys}}/\mathrm{Fil}^1 A_{\mathrm{crys}} = O_{\mathbb{C}_p}$ is a valuation ring, and hence a finitely generated $O_{\mathbb{C}_p}$-module has the form $O_{\mathbb{C}_p}^{\oplus r} \oplus T$, where T is a finitely generated torsion $O_{\mathbb{C}_p}$-module. Hence M has the property (P) in this case.

We consider the general case. Since $\mathrm{Fil}^i A_{\mathrm{crys}}/\mathrm{Fil}^m A_{\mathrm{crys}}$ is a finitely generated A_{crys}-module, the submodules $(\mathrm{Fil}^i A_{\mathrm{crys}})M$ of M are finitely generated A_{crys}-modules for $0 \leq i \leq m-1$. From the case $m = 1$, we see that $(\mathrm{Fil}^i A_{\mathrm{crys}})M/$

$(\mathrm{Fil}^{i+1}A_{\mathrm{crys}})M$ $(0 \le i \le m-1)$ have the property (P). Hence M has the property (P). $\square$

LEMMA 4.5.6

Assume that $c \ge 2$. Then for any $m \ge 0$, the image of A_c in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ is contained in a finitely generated A_{crys}-submodule of $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$.

Proof

Let x be as in (8). Since the images of $(x/t)^{p-1}p$ and $(x/t)p$ in $\mathbb{C}_p$ are contained in $O_{\mathbb{C}_p}$ ([7]), we can write

$$\left(\frac{x}{t}\right)^{p-1}p = f + u, \qquad \left(\frac{x}{t}\right)p = g + v \quad \text{in } B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$$

with

$$f, g \in A_{\mathrm{crys}}, \qquad u, v \in B^1_{\mathrm{dR}}/B^m_{\mathrm{dR}}.$$

Let I be the A_{crys}-submodule of $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ generated by elements of the following forms (10) and (11):

$$x^e p^{-2m}(x/t)^{-s} \quad (0 \le e < m, 0 \le s < m), \tag{10}$$

$$x^e p^{-p} u^i v^j \quad (0 \le e < m, 0 \le i < m, 0 \le j < m). \tag{11}$$

We prove that the image of A_c in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ is contained in I.

By (8),

$$\mathrm{Fil}^r_p A_{\mathrm{crys}} \subset \Bigl(\sum_{i,j \ge 0, i+j \ge r} A_{\mathrm{crys}} \cdot x^i p^{-\lambda(j)} t^j\Bigr) + B^{r+m}_{\mathrm{dR}}$$

for all $r \ge 0$. From this, we have

$$A_c \subset \Biggl(\sum_{i,j,r \ge 0, 0 \le i+j-r < m} A_{\mathrm{crys}} \cdot \left(\frac{x}{t}\right)^{r-j} x^{i+j-r} p^{cr-\lambda(j)}\Biggr) + B^m_{\mathrm{dR}}.$$

That is,

$$A_c \subset \Biggl(\sum_{r,d,e} A_{\mathrm{crys}} \cdot x^e p^{cr-\lambda(r-d)} \left(\frac{x}{t}\right)^d\Biggr) + B^m_{\mathrm{dR}}, \tag{12}$$

where r, d, e ranges over all integers satisfying $r \ge d \ge -e$, $0 \le e < m$, $r \ge 0$. We prove that in the case $d \le 0$ (resp., $d \ge 0$), the class of $x^e p^{cr-\lambda(r-d)}(x/t)^d$ in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ is contained in the A_{crys}-submodule of $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ generated by the classes of elements of the form (10) (resp., (11)). For $d \le 0$, this follows from

$$\begin{aligned} cr - \lambda(r-d) &\ge cr - (r-d)p(p-1)^{-2} = \bigl(c - p(p-1)^{-2}\bigr)r + dp(p-1)^{-2} \\ &\ge -mp(p-1)^{-2} \ge -2m. \end{aligned}$$

We consider the case $d \ge 0$. Write

$$d = (p-1)d' + d'' \quad \text{with } d', d'' \in \mathbb{Z}, d' \ge 0, 0 \le d'' < p-1.$$

We have $(x/t)^d = p^{-d'-d''}(f+u)^{d'}(g+v)^{d''}$ in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$. Since $u^m = v^m = 0$ in $B^0_{\mathrm{dR}}/B^m_{dR}$, the case $d \geq 0$ is reduced to

$$\begin{aligned} & cr - \lambda\big(r-(p-1)d'-d''\big) - d' - d'' \\ & \quad \geq cr - \big(r-(p-1)d'-d''\big)p(p-1)^{-2} - d' - d'' \\ & \quad = \big(c - p(p-1)^{-2}\big)r + d'(p-1)^{-1} + d''\big(p(p-1)^{-2}-1\big) \geq d''\big(p(p-1)^{-2}-1\big) \\ & \quad \geq -p. \end{aligned}$$

□

4.5.7

We define a ring homomorphism $\hat{A}_c \to B^0_{\mathrm{dR}}$ which is compatible with the inclusion map $A_c \to B^0_{\mathrm{dR}}$. Let $m \geq 1$, and let I be a finitely generated A_{crys}-submodule of $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ which contains the image of A_c in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ (see Lemma 4.5.6). By Lemma 4.5.5, we have $I \xrightarrow{\cong} \varprojlim_n I/p^n I$. Hence the canonical map $A_c \to I$ extends to $\hat{A}_c \to I \subset B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$. By taking the inverse limit for m, we obtain the desired ring homomorphism $\hat{A}_c \to B^0_{\mathrm{dR}}$.

This proves the injectivity of $A_c \to \hat{A}_c$ and completes the proof of Proposition 4.5.4. □

4.5.8. Proof of Lemma 4.4.4

Since a subring of $\mathbb{Q}_p$ which contains $\mathbb{Z}_p$ is either $\mathbb{Z}_p$ or $\mathbb{Q}_p$, if $B_c \cap \mathbb{Q}_p$ is not $\mathbb{Z}_p$, it should be $\mathbb{Q}_p$. Then this contradicts the injectivity of $B_c \to \varprojlim_n B_c/p^n B_c$. Thus Lemma 4.4.4 is proved. □

4.6. Proof of Proposition 4.4.7

LEMMA 4.6.1

Let $m \geq 1$. Then the quotient $(B^0_{\mathrm{dR}}/B^m_{dR})/\operatorname{Image}(A_{\mathrm{crys}})$ is a p-primary torsion group.

Proof

The map $A_{\mathrm{crys}}[1/p] \to B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ is surjective because the map $A_{\mathrm{crys}}[1/p] \to B^0_{\mathrm{dR}}/B^1_{\mathrm{dR}} = \mathbb{C}_p$ is surjective and A_{crys} contains a prime element t of B^0_{dR}. Lemma 4.6.1 follows from this. □

LEMMA 4.6.2

For $m \geq 1$, $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ has a unique structure of a topological ring such that the images of $p^n A_{\mathrm{crys}}$ in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ for $n \geq 0$ form a fundamental system of neighborhoods of zero.

Proof

As an additive group, $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ clearly has a unique structure of a topological abelian group with this fundamental system of neighborhoods of zero. The continuity of the multiplication follows from Lemma 4.6.1. □

Endow $B_{\mathrm{dR}} = \varprojlim_m B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ with the inverse limit of these topologies of $B^0_{\mathrm{dR}}/B^m_{dR}$.

LEMMA 4.6.3

The canonical map $K \to B^0_{\mathrm{dR}}$ is continuous.

Proof

It is sufficient to prove that $K \to B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$ is continuous for each $m \geq 1$. Let U be a neighborhood of zero in $B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$. Since O_K is a finitely generated $W(k)$-module and $W(k) \subset A_{\mathrm{crys}}$, Lemma 4.6.1 shows that there exists $n \geq 0$ such that $p^n O_K \subset U$. This proves the continuity of $K \to B^0_{\mathrm{dR}}/B^m_{\mathrm{dR}}$. $\square$

4.6.4

In general, if R is a topological ring and M is a finitely generated R-module, then we have a canonical topology of M as follows. Take a surjective R-homomorphism $h : R^n \to M$ for some $n \geq 0$, and endow M with the quotient topology of the product topology of R^n. Then this topology of M is independent of the choice of (n, h). With this topology, M is a topological R-module.

4.6.5

For $r \in \mathbb{Z}$, B^r_{dR} is regarded as a topological B^0_{dR}-module by Section 4.6.4. For $i \geq j$, the topology of B^i_{dR} coincides with the restriction of that of B^j_{dR}.

Fix a finitely generated $\mathbb{Z}_p$-submodule $L_{\mathbb{Z}_p}$ of L such that $L = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{\mathbb{Z}_p}$, and let G denote the automorphism group of $(L, W_\bullet L)$ regarded as a smooth group scheme over $\mathbb{Z}_p$.

LEMMA 4.6.6

Let $\delta \in G(B_{\mathrm{dR}})$. Let $c \geq 2$, $n \geq 0$. Then there exists a neighborhood U of 1 in $G(B^0_{\mathrm{dR}})$ such that for any $g \in U$, the class of $\delta g \delta^{-1}$ in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ belongs to the image of $G(B_c)_{\equiv 1 \bmod p^n}$.

Proof

Let $\mathfrak{g}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-vector space of all $\mathbb{Q}_p$-linear maps $L \to L$ which respect $W_\bullet L$, and let $\mathfrak{g}(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-module of all $\mathbb{Z}_p$-linear maps $L_{\mathbb{Z}_p} \to L_{\mathbb{Z}_p}$ which respect $W_\bullet L_{\mathbb{Z}_p}$. Take an integer $m \geq 1$ such that $\delta(B^m_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p))\delta^{-1} \subset B^1_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p)$, and take an integer $m' \geq 1$ such that $\delta^{-1}(B^{m'}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p))\delta \subset B^m_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p)$. Take an integer $r \geq 0$ such that $\delta(B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p))\delta^{-1} \subset B^{-r}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p)$. If $g \in G(B^0_{\mathrm{dR}})$ is sufficiently near to 1, $\delta g \delta^{-1}$ is sufficiently near to 1 in $B^{-r}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p)$, and we can write $\delta g \delta^{-1} = 1 + x + y$, where $x \in p^{nr} t^{-r} A_{\mathrm{crys}} \otimes \mathfrak{g}(\mathbb{Z}_p)$ and $y \in B^{m'}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p)$. Since $\delta^{-1}(1+x)\delta - g = -\delta^{-1} y \delta \in B^m_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p) \subset B^1_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}(\mathbb{Q}_p)$, we have $\delta^{-1}(1+x)\delta \in G(B^0_{\mathrm{dR}})$. Hence $\delta^{-1}(1+x)^{-1}\delta \in G(B^0_{\mathrm{dR}})$, and this proves that $\delta^{-1}(1+x)^{-1} y \delta = \delta^{-1}(1+x)^{-1}\delta \cdot \delta^{-1} y \delta \in B^m_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}_p$. Hence $(1+x)^{-1} y \in B^1_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathfrak{g}_p$. This shows that $1 + (1+x)^{-1} y \in G(B^0_{\mathrm{dR}})$ and hence shows that the class of $\delta g \delta^{-1} = (1+x)(1+(1+x)^{-1} y)$ in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ coincides with the class of $1 + x \in G(B_c)_{\equiv 1 \bmod p^n}$. $\square$

LEMMA 4.6.7

Fix $\alpha=(F,\mu)\in {}_{N,\Gamma}D$, and let $V=V_p(N,F)$. Fix a representative $\tilde{\mu}:(L,W_\bullet L)\xrightarrow{\cong}(V,W_\bullet V)$ of μ. Let $c\geq 2$, $n\geq 0$. Then there is a neighborhood U of F in ${}_N D$ such that for any $F'\in U$, the image of $h_{\tilde{\mu}}(F')$ under the canonical map

$$G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)\to G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^0)$$

belongs to the image of $G(B_c)_{\equiv 1 \bmod p^n}$.

Proof

Take an isomorphism $\nu:(H_K,W_K)\xrightarrow{\cong}K\otimes_{\mathbb{Q}_p}(L,W_\bullet L)$ of K-vector spaces with filtrations. Let $\delta\in G(B_{\mathrm{dR}})$ be the composite isomorphism

$$B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}L\xrightarrow{\tilde{\mu}}B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V\cong B_{\mathrm{dR}}\otimes_K H_K\xrightarrow{\nu}B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}L.$$

If $g\in G(K)$ and if $F':=(\nu^{-1}g\nu)F\in\check{D}$ belongs to ${}_N D$, then as is easily seen, the image of $h_{\tilde{\mu}}(F')$ under

$$G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)\to G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^0)$$

coincides with the class of $\delta g\delta^{-1}\in G(B_{\mathrm{dR}})$. If $F'\in {}_N D$ converges to F, we can write $F'=gF$ with $g\in G(K)$ which converges to 1. By Lemma 4.6.3, g converges to 1 in $G(B_{\mathrm{dR}}^0)$. Hence we are reduced to Lemma 4.6.6. □

LEMMA 4.6.8

Let $c\geq 2$, $n\geq 1$. Then the map

$$G(\hat{A}_c)_{\equiv 1 \bmod p^n}\to G(\hat{B}_c)_{\equiv 1 \bmod p^n};a\mapsto a^{-1}\varphi(a)$$

is surjective.

Proof

Let $b\in G(\hat{B}_c)_{\equiv 1 \bmod p^n}$. Since G is a smooth group scheme over $\mathbb{Z}_p$ and since $1-\varphi:A_c\to B_c$ is surjective (see Proposition 4.5.3), we can find $b_m\in G(\hat{B}_c)_{\equiv 1 \bmod p^m}$ and $a_m\in G(\hat{A}_c)_{\equiv 1 \bmod p^m}$ for integers $m\geq n$ such that $b_n=b$ and $b_m=a_m^{-1}\times b_{m+1}\varphi(a_m)$ for any $m\geq n$. Let $a\in G(\hat{A}_c)_{\equiv 1 \bmod n}$ be the limit of $a_{n+m}a_{n+m-1}\cdot\ldots\cdot a_{n+2}a_{n+1}a_n$ $(m\to\infty)$. Then $b=a^{-1}\varphi(a)$. □

LEMMA 4.6.9

Let $c\geq 2$, $n\geq 1$, $b\in G(B_{c+1})_{\equiv 1 \bmod p^n}$. Then the class of b in $G(\hat{B}_c)/G(\hat{A}_c)$ belongs to the image of $G(\hat{B}_c^{\varphi=1})_{\equiv 1 \bmod p^n}$.

Proof

By Lemma 4.6.8, we have $b^{-1}\varphi(b)=a^{-1}\varphi(a)$ in $G(\hat{B}_c)$ for some $a\in G(\hat{A}_c)$. Let $b'=ba^{-1}\in G(\hat{B}_c)$. Then $\varphi(b')=b'$, and hence b' belongs to $G(\hat{B}_c^{\varphi=1})$. □

LEMMA 4.6.10

Let $c\geq 2$, $n\geq 1$, and let h be an element of $G(B_{\mathrm{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$ whose image in

$G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ belongs to the image of $G(B_{c+1})_{\equiv 1 \bmod p^n}$. Then h belongs to the image of $G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n}$.

Proof

Assume that the image of h in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ coincides with the image of $b \in G(B_{c+1})_{\equiv 1 \bmod p^n}$. By Lemma 4.6.9, we can write $b = b'a$ with $b' \in G(\hat{B}_c^{\varphi=1})_{\equiv 1 \bmod p^n}$ and $a \in G(\hat{A}_c)_{\equiv 1 \bmod p^n}$. In $G(\hat{B}_c[1/p])/G(\hat{A}_c[1/p])$, the class of h and that of b' coincide. Since $((\hat{B}_c)^{\varphi=1})[1/p] \cap \hat{A}_c[1/p] = \mathbb{Q}_p$, the map $G((\hat{B}_c)^{\varphi=1}[1/p])/G(\mathbb{Q}_p) \to G(\hat{B}_c[1/p])/G(\hat{A}_c[1/p])$ is injective. Hence in $G((\hat{B}_c)^{\varphi=1}[1/p])/G(\mathbb{Q}_p)$, the class of h coincides with that of b'. Since $\hat{B}_c \cap B_c[1/p] = B_c$, b' belongs to $G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n}$. □

4.6.11

Now Proposition 4.4.7 follows from Lemmas 4.6.7 and 4.6.10. In fact, by Lemma 4.6.7, there is a neighborhood U of F in ${}_N D$ such that for any $F' \in U$, the image of $h_{\tilde{\mu}}(F')$ in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ belongs to the image of $G(B_{c+1})_{\equiv 1 \bmod p^n}$. By Lemma 4.6.10, $h_{\tilde{\mu}}(F')$ belongs to the image of $G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n}$. This completes the proof of Proposition 4.4.7. □

5. p-Adic period domains D and ${}_\Gamma D$ (finite residue field case)

In Section 5, assuming that the residue field k of K is finite and modifying the formulation in Section 4 slightly, we consider p-adic period domains D and ${}_\Gamma D$. The difference from Section 4 is that in Sections 5 and 6, we consider ℓ-adic Galois representations for all prime numbers ℓ. The space ${}_\Gamma D$ is a refinement of the space D by taking ℓ-adic level structures for all prime numbers ℓ into account.

5.1. Notation in Sections 5 and 6

5.1.1

We use the notation explained in Section 3.1. In Sections 5 and 6, we assume that k is a finite field, that is, K is a finite extension of $\mathbb{Q}_p$.

5.1.2

Let

$$\mathbb{Q}_A^f = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \quad \text{with } \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

be the nonarchimedean part of the adèle ring of $\mathbb{Q}$. Let $\mathbb{Q}^f_{A,\text{non-}p}$ be the non-p-part of $\mathbb{Q}_A^f$.

In Sections 5 and 6, assume that we are given a quadruple

$$\big(H, W, (h_{w,i})_{w,i\in\mathbb{Z}}, \mathfrak{N}\big).$$

- H is a free $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module of finite rank. Write $H = H_p \times H_{\text{non-}p}$, where H_p is the K_0-component of H and $H_{\text{non-}p}$ is the non-p component of H.

We assume that we are given a Frobenius-linear bijection $\varphi : H_p \to H_p$ and a continuous $\mathbb{Q}^f_{A,\text{non-}p}$-linear action of $\mathrm{Gal}(\bar{k}/k)$ on $H_{\text{non-}p}$, which satisfy condition (C) in Section 5.1.3. For each prime number $\ell \neq p$, let H_ℓ be the $\mathbb{Q}_\ell$-component of $H_{\text{non-}p}$.

• W is an increasing filtration on H by $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-submodules W_w ($w \in \mathbb{Z}$) such that $W_w = H$ for $w \gg 0$ and $W_w = 0$ for $w \ll 0$. We assume that for any $w \in \mathbb{Z}$, W_w is a free $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module and is a $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-direct summand of H, that $W_w H_p$ is stable under φ, and that $W_w H_{\text{non-}p}$ is stable under the action of $\mathrm{Gal}(\bar{k}/k)$.

• $h_{w,i}$ are integers which are zero for almost all (w, i) and which satisfy $\sum_i h_{w,i} = \operatorname{rank} \operatorname{gr}^W_w$ for all $w \in \mathbb{Z}$.

• $\mathfrak{N}$ is a finite-dimensional $\mathbb{Q}$-vector subspace of the space of all $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-linear homomorphisms $N : H \to H$ such that $N\varphi = p\varphi N$ on H_p, $sNs^{-1} = \kappa(s)N$ on $H_{\text{non-}p}$ for any $s \in \mathrm{Gal}(\bar{k}/k)$, where κ is the cyclotomic character, and $NW_w \subset W_w$ for all $w \in \mathbb{Z}$. We assume that the canonical map $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p}) \otimes_{\mathbb{Q}} \mathfrak{N} \to \mathrm{End}_{K_0 \times \mathbb{Q}^f_{A,\text{non-}p}}(H)$ is injective and that the image of this injection is a $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-direct summand.

Note that $N : H \to H$ is nilpotent for any $N \in \mathfrak{N}$.

See Section 5.4 for the motivation to consider this space $\mathfrak{N}$ from the point of view of motives.

5.1.3

Condition (C) is as follows.

Let $f = [k : \mathbb{F}_p]$. Define polynomials $P_\ell(T)$ for all prime numbers ℓ as follows. Let $P_p(T) \in K_0[T]$ be the eigenpolynomial of the K_0-linear operator φ^f on H_p. For $\ell \neq p$, let σ be the element of $\mathrm{Gal}(\bar{k}/k)$ defined by $\sigma(x) = x^{-1/p^f}$ for $x \in \bar{k}$, and let $P_\ell(T) \in \mathbb{Q}_\ell[T]$ be the eigenpolynomial of the $\mathbb{Q}_\ell$-linear action of σ on H_ℓ.

(C) $P_\ell(T) \in \mathbb{Q}[T]$ for any prime number ℓ, and $P_\ell(T)$ is independent of the prime number ℓ. Write $P_\ell(T)$ as $P(T)$. Then for any root α of $P(T)$ in $\mathbb{C}$, there is an integer $w \in \mathbb{Z}$ such that all conjugates of α over $\mathbb{Q}$ have complex absolute value $p^{fw/2}$.

5.1.4

We define an increasing filtration $\mathcal{W}$ on H, which we call the *filtration by Frobenius weights*.

Let the notation $P(T) \in \mathbb{Q}[T]$, $\sigma \in \mathrm{Gal}(\bar{k}/k)$, and $f = [k : \mathbb{F}_p]$ be as in Section 5.1.3.

Clearly we have a unique decomposition $P(T) = \prod_{w \in \mathbb{Z}} P^{(w)}(T)$ with $P^{(w)}(T) \in \mathbb{Q}[T]$ such that all roots of $P^{(w)}(T)$ in $\mathbb{C}$ have absolute value $p^{fw/2}$.

For each prime number ℓ, we have a direct decomposition $H_\ell = \bigoplus_{w \in \mathbb{Z}} H_\ell^{(w)}$ defined as follows. If $\ell = p$, $H_p^{(w)}$ is the kernel of the K_0-linear operator $P^{(w)}(\varphi^f)$ on H_p. If $\ell \neq p$, $H_\ell^{(w)}$ is the kernel of the $\mathbb{Q}_\ell$-linear operator $P^{(w)}(\sigma)$ on H_ℓ.

We show that there is a unique decomposition

$$H = \bigoplus_{w \in \mathbb{Z}} H^{(w)} \tag{13}$$

as a $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module such that the K_0-component of $H^{(w)}$ is $H_p^{(w)}$ and the $\mathbb{Q}_\ell$-component of $H^{(w)}$ for any prime number $\ell \neq p$ is $H_\ell^{(w)}$. Write $\hat{\mathbb{Z}} = \mathbb{Z}_p \times \hat{\mathbb{Z}}_{\text{non-}p}$, where $\hat{\mathbb{Z}}_{\text{non-}p} = \prod_{\ell \neq p} \mathbb{Z}_\ell$. Take a $\mathrm{Gal}(\bar{k}/k)$-stable free $\hat{\mathbb{Z}}_{\text{non-}p}$-submodule $T_{\text{non-}p}$ of $H_{\text{non-}p}$ of finite rank which generates $H_{\text{non-}p}$ over $\mathbb{Q}^f_{A,\text{non-}p}$. If $w, w' \in \mathbb{Z}$ and $w \neq w'$, then $P^{(w)}(T)$ and $P^{(w')}(T)$ are prime to each other in $\mathbb{Q}[T]$. Hence there is a finite set S of prime numbers such that $p \in S$ and such that if ℓ is a prime number which is not contained in S, then $P^{(w)}(T) \in \mathbb{Z}_\ell[T]$ for any $w \in \mathbb{Z}$ and $P^{(w)}(T)\mathbb{Z}_\ell[T] + P^{(w')}(T)\mathbb{Z}_\ell[T] = \mathbb{Z}_\ell[T]$ for any integers $w, w' \in \mathbb{Z}$ such that $w \neq w'$. For any prime number $\ell \neq p$, let T_ℓ be the $\mathbb{Z}_\ell$-component of $T_{\text{non-}p}$, and write $T_{\text{non-}S} = \prod_{\ell \notin S} T_\ell$. Then $T_{\text{non-}S} = \bigoplus_{w \in \mathbb{Z}} T^{(w)}_{\text{non-}S}$, where $T^{(w)}_{\text{non-}S}$ is the kernel of $P^{(w)}(\sigma)$ in $T_{\text{non-}S}$. This proves that we have the direct decomposition (13).

We define the increasing filtration $\mathcal{W}$ on H by

$$\mathcal{W}_w = \bigoplus_{w' \leq w} H^{(w')}.$$

Note that

$$N\mathcal{W}_w \subset \mathcal{W}_{w-2} \quad \text{for any } N \in \mathfrak{N} \text{ and any } w \in \mathbb{Z}. \tag{14}$$

5.1.5
Let d be the rank of the $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module H. To discuss level structures, we fix a free $\mathbb{Q}^f_A$-module L of rank d endowed with an increasing filtration $W_\bullet L$ having the following property. For each $w \in \mathbb{Z}$, $W_w L$ is free and is a direct summand of L as a $\mathbb{Q}^f_A$-module, and its rank is the same as the rank of $W_w \subset H$ as a $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module. For each prime number ℓ, let L_ℓ be the $\mathbb{Q}_\ell$-component of L.

Let G be the automorphism group of $(L, W_\bullet L)$, which we regard as a group scheme over $\mathbb{Q}^f_A$.

5.1.6
Let

$$H_K = K \otimes_{K_0} H_p.$$

As in Section 4.1.4, let $\check{D}$ be the set of all decreasing filtrations on the K-module H_K such that $h_{w,i} = \dim_K \mathrm{gr}^i_F \mathrm{gr}^W_{w,K}$ for any w, i.

5.1.7
The triple $(H_p, W_\bullet H_p, (h_{w,i}))$ becomes a triple in Section 4.1. The $\mathbb{Q}_p$-component $(L_p, W_\bullet L_p)$ of $(L, W_\bullet L)$ becomes $(L, W_\bullet L)$ of Section 4.1. When we refer to Section 4, we take the data in Section 4.1 in this way.

5.1.8

Thus we consider ℓ-adic structures for all prime numbers ℓ in Sections 5 and 6, though we considered only p-adic structures in Section 4. We intend to apply Sections 5 and 6 to motives over K (see Section 5.4) which have ℓ-adic realizations for all prime numbers ℓ and whose adelic level structures are considered. On the other hand, Section 4 can be applied also to objects like p-divisible groups which have p-adic realizations (but not ℓ-adic realizations for $\ell \neq p$) and whose p-adic level structures are considered.

5.2. Spaces C of monodromy operators and p-adic period domains D

5.2.1

Let $C \subset \mathfrak{N}$ be the set of all elements N of $\mathfrak{N}$ satisfying the following condition.

(i) $\mathcal{W}$ is the relative monodromy filtration (see Section 2.3.2) of $N : H \to H$ with respect to W.

5.2.2

We define

$$D = \{(N, F) \mid N \in C, F \in {}_N D\}.$$

5.2.3

We define the analytic structure over K of D by regarding it as the disjoint union over $N \in C$ of the analytic manifolds ${}_N D$ over K (see Section 4.2.4).

5.3. Level structures and the space ${}_\Gamma D$

5.3.1

For $(N, F) \in D$, we define a representation $V(N, F)$ over $\mathbb{Q}_A^f$ as follows.

First, for $N \in \mathfrak{N}$, let $V_{\text{non-}p}(N)$ be the following representation of $\mathrm{Gal}(\bar{K}/K)$ over $\mathbb{Q}^f_{A,\text{non-}p}$. As a $\mathbb{Q}^f_{A,\text{non-}p}$-module, $V_{\text{non-}p}(N) = H_{\text{non-}p}$. For each prime number $\ell \neq p$, let $V_\ell(N)$ be the space H_ℓ having the action of $\mathrm{Gal}(\bar{K}/K)$ associated to N defined as in Section 3.3.3. Then the subset $H_{\text{non-}p}$ of $\prod_{\ell \neq p} H_\ell$ is stable under this action. This is the action of $\mathrm{Gal}(\bar{K}/K)$ on $V_{\text{non-}p}(N)$.

For $(N, F) \in D$, let $V_p(N, F)$ be the representation of $\mathrm{Gal}(\bar{K}/K)$ over $\mathbb{Q}_p$ associated to the admissible filtered module (H_p, N, F) (see Section 3.2.6). We define

$$V(N, F) = V_p(N, F) \times V_{\text{non-}p}(N).$$

5.3.2

Let Γ be a compact open subgroup of $G(\mathbb{Q}_A^f)$.

Let V be a free $\mathbb{Q}_A^f$-module of finite rank endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$ and with a $\mathrm{Gal}(\bar{K}/K)$-stable filtration $W_\bullet V$ by $\mathbb{Q}_A^f$-direct summands such that $\operatorname{rank} V = \operatorname{rank} L$ and $\operatorname{rank} W_w V = \operatorname{rank} W_w L$ for all $w \in \mathbb{Z}$.

For isomorphisms $\mu_i : (L, W_\bullet L) \xrightarrow{\cong} (V, W_\bullet V)$ $(i = 1, 2)$ of $\mathbb{Q}_A^f$-modules with filtrations, we say that μ_1 and μ_2 are Γ-*equivalent* if $\mu_1^{-1}\mu_2 \in \Gamma$.

By a Γ-level structure on V we mean a Γ-equivalence class of an isomorphism $\mu : (L, W_\bullet L) \xrightarrow{\cong} (V, W_\bullet V)$ such that $\mu^{-1} s\mu \in \Gamma$ for any $s \in \mathrm{Gal}(\bar{K}/K)$.

For example, if $L_{\hat{\mathbb{Z}}}$ is a finitely generated $\hat{\mathbb{Z}}$-submodule of L such that $L = \mathbb{Q}_A^f \otimes_{\hat{\mathbb{Z}}} L_{\hat{\mathbb{Z}}}$ and $\Gamma = \mathrm{Aut}_{\hat{\mathbb{Z}}}(L_{\hat{\mathbb{Z}}}, W_\bullet L_{\hat{\mathbb{Z}}})$, where $W_\bullet L_{\hat{\mathbb{Z}}}$ denotes the restriction of $W_\bullet L$ to $L_{\hat{\mathbb{Z}}}$, then a Γ-level structure on V corresponds in a one-to-one manner to a $\mathrm{Gal}(\bar{K}/K)$-stable finitely generated $\hat{\mathbb{Z}}$-submodule T of V such that $V = \mathbb{Q}_A^f \otimes_{\hat{\mathbb{Z}}} T$ (see Section 4.3.3).

5.3.3

For a compact open subgroup Γ of $G(\mathbb{Q}_A^f)$, define

$$_\Gamma D = \big\{(N, F, \mu) \bigm| (N, F) \in D, \mu \text{ is a } \Gamma\text{-level structure of } V(N, F)\big\}.$$

5.3.4

We define the analytic structure over K of $_\Gamma D$ as follows, by using the results in Section 4.4.

Let $\alpha = (N, F, \mu) \in {_\Gamma D}$. Then for a sufficiently small neighborhood U of F in $_N D$, we have a canonical map $e_\alpha : U \to {_\Gamma D}$ defined as follows. Let $V = V(N, F)$, let $\tilde{\mu} : (L, W_\bullet L) \xrightarrow{\cong} (V, W_\bullet V)$ be a representative of μ, and let $\tilde{\mu}_p$ (resp., $\tilde{\mu}_{\text{non-}p}$) be the $\mathbb{Q}_p$ (resp., $\mathbb{Q}_{A,\text{non-}p}^f$)-component of $\tilde{\mu}$. We have the map $h_{\tilde{\mu}_p} : {_N D} \to G(B_{\text{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$ (see Section 4.4.2). Take a finitely generated $\mathbb{Z}_p$-submodule $L_{\mathbb{Z}_p}$ of the $\mathbb{Q}_p$-component L_p of L which satisfies $L_p = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{\mathbb{Z}_p}$ and which is stable under the action of Γ through $\Gamma \to G(\mathbb{Q}_p)$. By abuse of notation, we use the letter G also for the automorphism group of $(L_{\mathbb{Z}_p} \times H_{\text{non-}p}, W_\bullet(L_{\mathbb{Z}_p} \times H_{\text{non-}p}))$ regarded as a smooth group scheme over $\mathbb{Z}_p \times \mathbb{Q}_{A,\text{non-}p}^f$. Take integers $c \geq 2$ and $n \geq 0$ such that $G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma \cap G(\mathbb{Q}_p)$. As in Section 4.4.6, let $U_{\alpha,c,n}$ be the set of all $F' \in {_N D}$ such that $h_{\tilde{\mu}_p}(F')$ belongs to the image of $G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n} \to G(B_{\text{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$. Then by Section 4.4.2(4), $U_{\alpha,c,n}$ is independent of the choice of the representative $\tilde{\mu}$ of μ and, by Corollary 4.4.10, $U_{c,\alpha,n}$ is an open neighborhood of F in $_N D$. We define the map

$$e_\alpha : U_{\alpha,c,n} \to {_\Gamma D}$$

as follows. Let $F' \in U_{\alpha,c,n}$, and let h be an element of $G(B_c^{\varphi=1})_{\equiv 1 \bmod p^n}$ whose image in $G(B_{\text{crys}}^{\varphi=1})/G(\mathbb{Q}_p)$ coincides with $h_{\tilde{\mu}_p}(F')$. Let μ' be the Γ-equivalence class of $(\tilde{\mu}_p h, \tilde{\mu}_{\text{non-}p}) : (L, W_\bullet L) \xrightarrow{\cong} (V(N, F'), W_\bullet V_p(N, F'))$. Then μ' is independent of the choices of the representative $\tilde{\mu}$ of μ and the element h. We define $e_\alpha(F') = (N, F', \mu')$.

We define the topology of $_\Gamma D$ as follows. A subset U of $_\Gamma D$ is open if and only if for any $\alpha = (N, F, \mu) \in {_\Gamma D}$, $e_\alpha(U') \subset U$ for a sufficiently small neighborhood U of F in $_N D$. Then $_\Gamma D \to D$ is a local homeomorphism. We transfer the analytic structure over K of D to $_\Gamma D$ via this local homeomorphism.

The projection ${}_{\Gamma}D \to D$ becomes locally an isomorphism of analytic spaces over K, and e_α is a local section of it.

This topology and the analytic structure are independent of the choice of $L_{\mathbb{Z}_p}$ (see Section 4.4).

5.4. Remarks on motives

By using the p-adic Hodge conjecture of Fontaine (see Conjecture 3.4.9), we explain the ideas of some definitions in Section 5 from the point of view of motives.

5.4.1

Fix a triple $(M, W, (h_{w,i})_{w,i\in\mathbb{Z}})$, where M is a mixed motive over k, W is an increasing filtration on M by mixed submotives such that $W_w = M$ for $w \gg 0$ and $W_w = 0$ for $w \ll 0$, and $h_{w,i}$ are nonnegative integers such that for each i, $\sum_i h_{w,i}$ is equal to the rank of the motive $\mathrm{gr}^W_w(M)$.

Then H_p in Section 5.1.2 is obtained as the crystalline realization of M and $H_{\text{non-}p}$ in Section 5.1.2 is obtained as the $\mathbb{Q}^f_{A,\text{non-}p}$-adic étale realization of M. $\mathfrak{N}$ in Section 5.1.2 is given as

$$\mathfrak{N} = \mathrm{Hom}\big(M, M(-1)\big).$$

The filtration $\mathcal{W}$ in Section 5.1.4 is given by the weight filtration of the mixed motive M over k.

For $N \in \mathfrak{N}$, (M, W, N) belongs to the category $\mathcal{C}_k$ in Section 3.4.7 if and only if N belongs to $C \subset \mathfrak{N}$.

5.4.2

Let D_{motive} be the set of all isomorphism classes of pairs $(\tilde{M}, \iota)$, where

- $\tilde{M}$ is a mixed motive over K whose de Rham realization $\tilde{M}_{\mathrm{dR}}$ satisfies $\dim_K \mathrm{gr}^i_F \mathrm{gr}^W_w \tilde{M}_{\mathrm{dR}} = h_{w,i}$ for any $w, i \in \mathbb{Z}$, where W is the weight filtration of the motive $\tilde{M}$ over K and F is the Hodge filtration on $\tilde{M}_{\mathrm{dR}}$;
- ι is an isomorphism $(M', W') \cong (M, W)$, where we denote by (M', W', N, F) the object of $\mathcal{C}_{k,K}$ corresponding to $\tilde{M}$ in Conjecture 3.4.9.

If we assume Conjecture 3.4.9, the realization functors give an injection

$$D_{\text{motive}} \overset{\subset}{\to} D$$

into the space D in Section 5.2.

It seems to be a very difficult problem to determine the image of this injection.

5.5. Examples

In Sections 5.5 and 6.6, we describe the p-adic versions of Examples a–d in Section 2. Most things written in Sections 5.5 and 6.6 are checked easily, but their complete proofs will be given in a later part of this series of papers after we develop some general theory.

5.5.1. Example a

This is a p-adic analogue of Section 2.2, Example a. The multiplicative group $\mathbb{C}^\times$ appeared in Section 2.2, Example a, and the multiplicative group $K^\times$ appears here.

In the formulation of Section 5.4, let $(M, W, (h_{w,i})_{w,i\in\mathbb{Z}})$ be as follows:

$$M = \mathbb{Q}(1) \oplus \mathbb{Q},$$

$$0 = W_{-3} \subset \mathbb{Q}(1) = W_{-2} = W_{-1} \subset M = W_0,$$

$$h_{0,0} = h_{-2,-1} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

In the formulation in Section 5.1, we take a quadruple $(H, W, (h_{w,i}), \mathfrak{N})$ as follows.

H is a free $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module of rank 2 with basis e_1, e_2, $\varphi : H_p \to H_p$ is defined by

$$\varphi(e_{1,p}) = p^{-1}e_{1,p}, \qquad \varphi(e_{2,p}) = e_{2,p},$$

and the action of $\mathrm{Gal}(\bar{k}/k)$ on $H_{\text{non-}p}$ is given by

$$se_{1,\text{non-}p} = \kappa(s)e_{1,\text{non-}p}, \qquad se_{2,\text{non-}p} = e_{2,\text{non-}p} \quad \big(s \in \mathrm{Gal}(\bar{k}/k)\big),$$

where κ is the cyclotomic character,

$$0 = W_{-3} \subset (K_0 \times \mathbb{Q}^f_{A,\text{non-}p}) \cdot e_1 = W_{-2} = W_{-1} \subset H = W_0,$$

$$h_{w,i} \text{ are as above,}$$

$$\mathfrak{N} = \mathbb{Q}N \quad \text{where } N(e_1) = 0,\ N(e_2) = e_1.$$

Then we have an isomorphism of analytic manifolds over K,

$$D \cong \mathbb{Q} \times K,$$

where $\mathbb{Q}$ is discrete. Here $(c, z) \in \mathbb{Q} \times K$ corresponds to $(cN, F(z)) \in D$, where $F = F(z) \in \check{D}$ is defined as

$$0 = F^1 \subset K(ze_{1,p} + e_{2,p}) = F^0 \subset H_K = F^{-1}.$$

We describe $V_p(cN, F(z))$. We have an exact sequence

$$0 \to \mathbb{Q}_p(1) \to B_{\text{crys}}^{\varphi=p} \cap B_{\text{dR}}^0 \to \mathbb{C}_p \to 0,$$

where $B_{\text{crys}}^{\varphi=p} := \{x \in B_{\text{crys}} \mid \varphi(x) = px\}$. Recall that we denote by t the generator of $\mathbb{Z}_p(1) \subset B_{\text{crys}}$ corresponding to $(\zeta_{p^n})_{n\geq 0}$ as in Section 3.2. As a subset of $B_{\text{st}} \otimes_{K_0} H_p$, $V_p(cN, F(z))$ is the two-dimensional $\mathbb{Q}_p$-subspace with basis $te_{1,p}$ and $(cl_\xi + b)e_{1,p} + e_{2,p}$, where l_ξ is as in Section 3.2.4 and where b is any element of $B_{\text{crys}}^{\varphi=p} \cap B_{\text{dR}}^0$ whose image in $\mathbb{C}_p$ coincides with z.

To consider level structures, let L be the free $\mathbb{Q}^f_A$-module of rank 2 with basis $e_{1,L}, e_{2,L}$ endowed with the weight filtration

$$0 = W_{-3}L \subset \mathbb{Q}^f_A \cdot e_{1,L} = W_{-2}L = W_{-1}L \subset L = W_0L.$$

Let

$$\Gamma = \begin{pmatrix} \hat{\mathbb{Z}}^\times & \hat{\mathbb{Z}} \\ 0 & \hat{\mathbb{Z}}^\times \end{pmatrix} \subset \mathrm{Aut}(L).$$

Let ${}_\Gamma D^\circ$ be the subset of ${}_\Gamma D$ consisting of all elements $(cN, F(z), \mu)$ such that the restrictions of μ to gr^W_w are the standard ones for all w. The last condition means that some representative $\tilde{\mu} : L \to V(cN, F(z))$ of μ has the following properties:

$$\tilde{\mu}_p(e_{1,L,p}) = te_{1,p}, \qquad \tilde{\mu}_p(e_{2,L,p}) \equiv e_{2,p} \mod B_{\mathrm{st}} \cdot e_{1,p},$$
$$\tilde{\mu}_{\text{non-}p}(e_{1,L,\text{non-}p}) = e_{1,\text{non-}p},$$
$$\tilde{\mu}_{\text{non-}p}(e_{2,L,\text{non-}p}) \equiv e_{2,\text{non-}p} \mod \mathbb{Q}^f_{A,\text{non-}p} \cdot e_{1,\text{non-}p}.$$

Then ${}_\Gamma D^\circ$ is open and closed in ${}_\Gamma D$. We have an isomorphism of analytic manifolds over K,

$${}_\Gamma D^\circ \cong K^\times,$$

via which the projection ${}_\Gamma D^\circ \to D$ corresponds to $(\mathrm{ord}_\xi, \log) : K^\times \to \mathbb{Q} \times K$. We describe this isomorphism.

Let

$$P(K) = \{(a_n)_{n\geq 1} \mid a_n \in \bar{K}^\times, a^m_{mn} = a_n (m, n \geq 1), a_1 \in K^\times\}.$$

Then $P(K)$ is a torsion-free locally compact abelian group. We have an exact sequence

$$0 \to \hat{\mathbb{Z}}(1) \to P(K) \to K^\times \to 0,$$

where the map $P(K) \to K^\times$ sends (a_n) to a_1. Let

$$\log : P(K) \to B^{\varphi=p}_{\mathrm{st}} := \{x \in B_{\mathrm{st}} \mid \varphi(x) = px\}$$

be the logarithm (see [7]). It is the unique homomorphism which sends $(a_n)_{n\geq 1} \in P(K)$ such that $a_1 \in O^\times_K$ to $\log([(a_{p^n})_{n\geq 1}])$ (where $[(a_{p^n})_{n\geq 1}]$ is as in Section 3.2.4 and $\log : A^\times_{\mathrm{crys}} \to A_{\mathrm{crys}}[1/p]$ is the logarithm of A_{crys} (see [7])) and sends $(\xi^{1/n})_{n\geq 1}$ to l_ξ. For $a \in K^\times$ and for an element $\tilde{a}$ of $P(K)$ whose image in $K^\times$ is a, we have $\log(\tilde{a}) = \mathrm{ord}_\xi(a) l_\xi + b$ for some $b \in B^{\varphi=p}_{\mathrm{crys}} \cap B^0_{\mathrm{dR}}$ such that the image of b in $\mathbb{C}_p$ is $\log(a)$. On the other hand, define the homomorphism

$$\log_{\text{non-}p} : P(K) \to \mathbb{Q}^f_{A,\text{non-}p}$$

as follows. Since $k^\times$ is finite and $\mathrm{Ker}(O^\times_K \to k^\times)$ is a pro-p group, there is a unique continuous homomorphism $P(K) \to \mathbb{Q}^f_{A,\text{non-}p}$ which kills $(\xi^{1/n})_n \in P(K)$ and whose restriction to $\hat{\mathbb{Z}}(1) \subset P(K)$ is $\hat{\mathbb{Z}}(1) \cong \hat{\mathbb{Z}} \to \mathbb{Q}^f_{A,\text{non-}p}$, where the isomorphism is defined by $(\zeta_n)_n$.

Then $a \in K^\times$ corresponds to $(N_a, F_a, \mu_a) \in {}_\Gamma D^\circ$, where $N_a = \mathrm{ord}_\xi(a) N$, $F_a = F(\log(a))$, and where μ_a is as follows. Take a lifting $\tilde{a}$ of a to $P(K)$. Note that $V_p(N_a, F_a)$ is generated by $te_{1,p}$ and $\log(\tilde{a})e_{1,p} + e_{2,p}$ over $\mathbb{Q}_p$. The level structure μ_a is the Γ-equivalence class of the following isomorphism $\tilde{\mu}_a : L \to V(N_a, F_a)$ over $\mathbb{Q}^f_A$:

$$\tilde{\mu}_{a,p}(e_{1,L,p}) = te_{1,p}, \qquad \tilde{\mu}_{a,p}(e_{2,L,p}) = \log(\tilde{a})e_{1,p} + e_{2,p},$$
$$\tilde{\mu}_{a,\text{non-}p}(e_{1,L,\text{non-}p}) = e_{1,\text{non-}p},$$
$$\tilde{\mu}_{a,p}(e_{2,L,\text{non-}p}) = \log_{\text{non-}p}(\tilde{a}) e_{1,\text{non-}p} + e_{2,\text{non-}p}.$$

We describe a relation with Kummer theory. Let $T(a)$ be the $\hat{\mathbb{Z}}$-submodule of $V(N_a, F_a)$ generated by $\tilde{\mu}_a(e_{j,L})$ $(j = 1, 2)$. That is, $T(a)$ is the $\mathrm{Gal}(\bar{K}/K)$-stable $\hat{\mathbb{Z}}$-lattice in $V(N_a, F_a)$ corresponding to μ (see Section 5.3.2). Consider the exact sequence of $\mathrm{Gal}(\bar{K}/K)$-modules

$$0 \to \hat{\mathbb{Z}}(1) \to T(a) \to \hat{\mathbb{Z}} \to 0$$

in which the generator of $\hat{\mathbb{Z}}(1)$ corresponding to $(\zeta_n)_n$ is sent to $\tilde{\mu}(e_{1,L}) \in T(a)$ and the map $T(a) \to \hat{\mathbb{Z}}$ sends $\tilde{\mu}(e_{2,L})$ to 1. This exact sequence determines an element of $\mathrm{Ext}^1_{\mathrm{Gal}(\bar{K}/K)}(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)) = H^1(K, \hat{\mathbb{Z}}(1))$, where $H^m(K, \ )$ $(m \in \mathbb{Z})$ denotes the continuous Galois cohomology $H^m(\mathrm{Gal}(\bar{K}/K),)$. This element of $H^1(K, \hat{\mathbb{Z}}(1))$ coincides with the image of a under the canonical homomorphism $K^\times \to H^1(K, \hat{\mathbb{Z}}(1))$ defined by Kummer theory. This is because the image of a in $H^1(K, \hat{\mathbb{Z}}(1))$ by Kummer theory is the class of the exact sequence $0 \to \hat{\mathbb{Z}}(1) \to ? \to \mathbb{Z} \to 0$ which is obtained from the exact sequence $0 \to \hat{\mathbb{Z}}(1) \to P(K) \to K^\times \to 0$ by $\mathbb{Z} \to K^\times, 1 \mapsto a$.

We describe the map e_α in Section 4.4.8. For $a \in K^\times$, if α denotes the element of ${}_\Gamma D^\circ$ corresponding to a, then for integers $c \geq 2$ and $n \geq 0$, $U_{\alpha,c,n}$ coincides with the set of $F(z)$ for $z \in K$ such that $z \equiv \log(a) \bmod p^{n+c}O_K$, and for such z, $e_\alpha(F(z))$ coincides with the element of ${}_\Gamma D^\circ$ corresponding to $a \exp(z - \log(a)) \in K^\times$.

5.5.2. Example b

This is a p-adic analogue of Section 2.2, Example b. The unit disc without the origin Δ^* over $\mathbb{C}$ appeared in Section 2.2, Example b. The p-adic unit disc without the origin Δ^* appears here.

In the formulation of Section 5.4, let $(M, W, (h_{w,i})_{w,i \in \mathbb{Z}})$ be as follows:

$$M = \mathbb{Q}(1) \oplus \mathbb{Q},$$

$$0 = W_{-2} \subset M = W_{-1},$$

$$h_{-1,0} = h_{-1,-1} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

In the formulation in Section 5.1, we take a quadruple $(H, W, (h_{w,i}), \mathfrak{N})$.

Define a quadruple $(H, W, (h_{w,i}), \mathfrak{N})$ as follows:

- H and $\mathfrak{N}$ are the same as those of Example a,
- $0 = W_{-2} \subset H = W_{-1}$,
- $h_{w,i}$ are as above.

Then we have an isomorphism of analytic manifolds over K,

$$D \cong (\mathbb{Q} - \{0\}) \times K,$$

where $\mathbb{Q} - \{0\}$ is discrete. Let $N \in \mathfrak{N}$ be as in Section 5.5.1, Example a. Then in this isomorphism, $(c, \tau) \in (\mathbb{Q} - \{0\}) \times K$ corresponds to $(cN, F(\tau))$, where $F(\tau)$ is defined in the same way as in Example a. The subset C of $\mathfrak{N}$ in Section 5.2 in this case is $\mathfrak{N} - \{0\}$.

We have the description of $V_p(cN, F(\tau))$ of the same form as in Example a.

Let L and Γ be the same as in Example a. Let ${}_\Gamma D^\circ$ be the subset of ${}_\Gamma D$ consisting of all elements $(cN, F(\tau), \mu)$ such that $c > 0$ and such that for some representative $\tilde{\mu} : L \to V(cN, F(\tau))$, $\tilde{\mu}(e_{1,L})$ and $\tilde{\mu}(e_{2,L})$ mod $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p}) \cdot e_1$ are given as in the definition of ${}_\Gamma D^\circ$ in Example a. Then ${}_\Gamma D^\circ$ is open and closed in ${}_\Gamma D$. We have an isomorphism of analytic manifolds over K,

$${}_\Gamma D^\circ \cong \Delta^*,$$

where $\Delta^* = \Delta - \{0\}$, $\Delta = \{q \in K \mid |q|_p < 1\}$. The projection ${}_\Gamma D^\circ \to D$ corresponds to $\Delta^* \to \mathbb{Q} \times K, q \mapsto (\text{ord}_\xi(q), \log(q))$. The level structure of the element of ${}_\Gamma D^\circ$ corresponding to $q \in \Delta^*$ is described in the same form as in Example a.

Let E be the Tate elliptic curve $K^\times/q^{\mathbb{Z}}$ with $q \in m_K - \{0\}$. Then for any prime number ℓ, the ℓ-adic Tate module $T_\ell(E)$ is the extension of $\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell(1)$ associated to q by Kummer theory. The relation with Kummer theory explained in Example a shows the following. Let $c = \text{ord}_\xi(q)$, and let $\tau = \log(q)$. Then $V(cN, F(\tau)) = \mathbb{Q}^f_A \otimes_{\hat{\mathbb{Z}}} \prod_\ell T_\ell(E)$, and the level structure defined by $q \in \Delta^* = {}_\Gamma D^\circ$ is identified with the integral structure $\prod_\ell T_\ell(E)$ of $\mathbb{Q}^f_A \otimes_{\hat{\mathbb{Z}}} \prod_\ell T_\ell(E)$.

The example of (D, Γ) in the introduction related to Tate elliptic curves is the (-1)-Tate twist of this Example b with Γ as above. In general, Tate twists D, ${}_\Gamma D$, and also ${}_\Gamma D_\Sigma$ in Section 6 are canonically isomorphic to the original D, $\Gamma \setminus D$, and $\Gamma \setminus D_\Sigma$, respectively.

5.5.3. Example c

This is a p-adic analogue of Section 2.2, Example c. The universal elliptic curve appeared in Section 2.2, Example c. The universal Tate elliptic curve appears here.

In the formulation of Section 5.4, let $(M, W, (h_{w,i})_{w,i\in\mathbb{Z}})$ be as follows:

$$M = \mathbb{Q}(1) \oplus \mathbb{Q} \oplus \mathbb{Q},$$

$$0 = W_{-2} \subset \mathbb{Q}(1) \oplus \mathbb{Q} \oplus 0 = W_{-1} \subset M = W_0,$$

$$h_{0,0} = h_{-1,0} = h_{-1,-1} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

In the formulation in Section 5.1, we take a quadruple $(H, W, (h_{w,i}), \mathfrak{N})$ as follows.

H is a free $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module of rank 3 with basis e_1, e_2, e_3; $\varphi : H_p \to H_p$ is defined by

$$\varphi(e_{1,p}) = p^{-1} e_{1,p}, \qquad \varphi(e_{2,p}) = e_{2,p}, \qquad \varphi(e_{3,p}) = e_{3,p};$$

the action of $\text{Gal}(\bar{k}/k)$ on $H_{\text{non-}p}$ is given by

$$s e_{1,\text{non-}p} = \kappa(s) e_{1,\text{non-}p}, \qquad s e_{2,\text{non-}p} = e_{2,\text{non-}p}, \qquad s e_{3,\text{non-}p} = e_{3,\text{non-}p}$$

$(s \in \text{Gal}(\bar{k}/k))$, where κ is the cyclotomic character.

W is the increasing filtration on H defined by

$$0 = W_{-2} \subset \bigoplus_{j=1}^{2} (K_0 \times \mathbb{Q}^f_{A,\text{non-}p}) \cdot e_j = W_{-1} \subset H = W_0,$$

$$h_{w,i} \text{ are as above.}$$

$\mathfrak{N}$ is the set of all $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-linear maps $H \to H$ which send e_1 to zero and e_2 and e_3 into $\mathbb{Q}e_1$.

Then we have an isomorphism of analytic manifolds over K,

$$D \cong (\mathbb{Q} - \{0\}) \times \mathbb{Q} \times K^2,$$

where $(\mathbb{Q} - \{0\}) \times \mathbb{Q}$ is discrete. Here $(c, c', \tau, z) \in (\mathbb{Q} - \{0\}) \times \mathbb{Q} \times K^2$ corresponds to $(N_{c,c'}, F(\tau, z)) \in D$, where $N = N_{c,c'}$ is defined by

$$N(e_1) = 0, \qquad N(e_2) = ce_1, \qquad N(e_3) = c'e_1,$$

and where $F = F(\tau, z)$ is defined by

$$0 = F^1 \subset K(\tau e_{1,p} + e_{2,p}) + K(ze_{1,p} + e_{3,p}) = F^0 \subset H_K = F^{-1}.$$

Let L be the free $\mathbb{Q}^f_A$-module of rank 3 with basis $(e_{j,L})_{1 \le j \le 3}$ endowed with the weight filtration W defined by

$$0 = W_{-2}L \subset \bigoplus_{j=1}^{2} \mathbb{Q}^f_A \cdot e_j = W_{-1}L \subset L = W_0 L.$$

Let

$$\Gamma = \begin{pmatrix} \mathrm{GL}_2(\hat{\mathbb{Z}}) & * \\ 0 & \hat{\mathbb{Z}}^\times \end{pmatrix} \subset \mathrm{Aut}(L), \qquad * = \begin{pmatrix} \hat{\mathbb{Z}} \\ \hat{\mathbb{Z}} \end{pmatrix}.$$

Let ${}_\Gamma D^\circ$ be the subset of ${}_\Gamma D$ consisting of all elements whose restriction to W_{-1} belongs to the ${}_\Gamma D^\circ$ of Example b and whose level structure induces on gr^W_0 the standard level structure. Then ${}_\Gamma D^\circ$ is open and closed in ${}_\Gamma D$. We have an isomorphism of analytic manifolds over K,

$${}_\Gamma D^\circ \cong \Delta^* \times K^\times.$$

Here $(q, r) \in \Delta^* \times K^\times$ corresponds to $\big(N_{\mathrm{ord}_\xi(q), \mathrm{ord}_\xi(r)}, F(\log(q), \log(r)), \mu\big) \in {}_\Gamma D^\circ$, where μ is the class of $\tilde{\mu}$ defined by

$$\tilde{\mu}(e_{1,L,p}) = te_{1,p}, \qquad \tilde{\mu}(e_{j,L,p}) = \log\big(\tilde{q}\ (\text{resp., } \tilde{r})\big)e_{1,p} + e_{j,p} \quad (j = 2,\ \text{resp., } 3),$$

$$\tilde{\mu}(e_{1,L,\text{non-}p}) = e_{1,\text{non-}p}, \qquad \tilde{\mu}(e_{j,L,\text{non-}p}) = \log\big(\tilde{q}\ (\text{resp., } \tilde{r})\big)e_{1,\text{non-}p} + e_{j,\text{non-}p}$$
$$(j = 2,\ \text{resp., } 3).$$

Here $\tilde{q}$ (resp., $\tilde{r}$) denotes a lifting of q (resp., r) to $P(K)$ (see Section 5.5.1).

The space ${}_\Gamma D^\circ$ is a p-adic analogue of $\Gamma_2 \backslash D$ over $\mathbb{C}$ in Section 2.2.

A p-adic analogue of $\Gamma_3 \backslash D$ in Section 2.2 is the quotient $\Gamma_0 \backslash {}_\Gamma D^\circ$ of ${}_\Gamma D^\circ$, where

$$\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{Aut}(H).$$

This quotient $\Gamma_0 \backslash D^\circ$ is also an analytic space over K, and we have a commutative diagram

$$\begin{array}{ccc}
(\mathbb{Q}-\{0\})\times\mathbb{Q}\times K^2 & \cong & D \\
\uparrow & & \uparrow \\
\Delta^*\times K^\times & \cong & {}_\Gamma D^\circ \\
\downarrow & & \downarrow \\
\bigcup_{q\in\Delta^*} K^\times/q^{\mathbb{Z}} & \cong & \Gamma_0 \backslash {}_\Gamma D^\circ
\end{array}$$

which is similar to a diagram in Section 2.2.5. Thus $\Gamma_0 \backslash {}_\Gamma D^\circ$ is the universal Tate elliptic curve over Δ^*.

This action of the group Γ_0 is understood in terms of motives as follows. In general, in the formulation of Section 5.4, the group $\mathrm{Aut}(M, W)$ acts naturally on D_{motive} by $(\tilde{M}, \iota) \mapsto (\tilde{M}, a\iota)$ $(a \in \mathrm{Aut}(M, W))$. In this Example c, Γ_0 is a subgroup of $\mathrm{Aut}(M, W)$, and the action of Γ_0 on ${}_\Gamma D^\circ$ is compatible with the action of $\mathrm{Aut}(M, W)$ on D_{motive} (D in this case).

5.5.4. Example d

This is a p-adic analogue of Section 2.2, Example d.

In the formulation of Section 5.4, let $(M, W, (h_{w,i})_{w,i\in\mathbb{Z}})$ be as follows:

$$M = \mathbb{Q}(2) \oplus \mathbb{Q}(1) \oplus \mathbb{Q},$$
$$0 = W_{-5} \subset \mathbb{Q}(2) = W_{-4} = W_{-3} \subset W_{-3} + \mathbb{Q}(1) = W_{-2} = W_{-1} \subset M = W_0,$$
$$h_{0,0} = h_{-2,-1} = h_{-4,-2} = 1, \quad \text{other } h_{w,i} \text{ are zero.}$$

In the formulation in Section 5.1, we take a quadruple $(H, W, (h_{w,i}), \mathfrak{N})$ as follows.

H is a free $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-module of rank 3 with basis e_1, e_2, e_3; $\varphi : H_p \to H_p$ is defined by

$$\varphi(e_{1,p}) = p^{-2}e_{1,p}, \qquad \varphi(e_{2,p}) = p^{-1}e_{2,p}, \qquad \varphi(e_{3,p}) = e_{3,p};$$

the action of $\mathrm{Gal}(\bar{k}/k)$ on $H_{\text{non-}p}$ is given by

$$se_{1,\text{non-}p} = \kappa(s)^2 e_{1,\text{non-}p}, \qquad se_{2,\text{non-}p} = \kappa(s)e_{2,\text{non-}p}, \qquad se_{3,\text{non-}p} = e_{3,\text{non-}p}$$

($s \in \mathrm{Gal}(\bar{k}/k)$, κ is the cyclotomic character).

W is the increasing filtration on H defined by

$$0 = W_{-5} \subset (K_0 \times \mathbb{Q}^f_{A,\text{non-}p}) \cdot e_1 = W_{-4} = W_{-3}$$
$$\subset W_{-3} + (K_0 \times \mathbb{Q}^f_{A,\text{non-}p}) \cdot e_2 = W_{-2} = W_{-1} \subset H = W_0,$$

$h_{w,i}$ are as above.

$\mathfrak{N}$ is the set of all $(K_0 \times \mathbb{Q}^f_{A,\text{non-}p})$-linear maps $H \to H$ which send e_1 to zero, e_2 into $\mathbb{Q}e_1$, and e_3 into $\mathbb{Q}e_2$.

Then we have an isomorphism of analytic spaces over K,

$$D \cong \mathbb{Q}^2 \times K^3,$$

where $\mathbb{Q}^2$ is discrete. Here $(c, c', z_1, z_2, z_3) \in \mathbb{Q}^2 \times K^3$ corresponds to $(N_{c,c'}, F(z_1, z_2, z_3)) \in D$, where $N = N_{c,c'}$ is defined by

$$N(e_1) = 0, \qquad N(e_2) = ce_1, \qquad N(e_3) = c'e_2,$$

and $F = F(z_1, z_2, z_3)$ is defined by

$$\begin{aligned} 0 = F^1 &\subset K(z_1 e_{1,p} + z_2 e_{2,p} + e_{3,p}) = F^0 \subset F^0 + K(z_3 e_{1,p} + e_{2,p}) \\ &= F^{-1} \subset H_K = F^{-2}. \end{aligned}$$

Let L be the free $\mathbb{Q}_A^f$-module of rank 3 with basis $(e_{j,L})_{1 \le j \le 3}$ endowed with the weight filtration

$$\begin{aligned} 0 = W_{-5}L \subset \mathbb{Q}_A^f \cdot e_1 = W_{-4}L = W_{-3}L \subset W_{-3}L + \mathbb{Q}_A^f \cdot e_2 = W_{-2}L = W_{-1}L \subset L \\ = W_0 L. \end{aligned}$$

Let

$$\Gamma = \begin{pmatrix} \hat{\mathbb{Z}}^\times & \hat{\mathbb{Z}} & \hat{\mathbb{Z}} \\ 0 & \hat{\mathbb{Z}}^\times & \hat{\mathbb{Z}} \\ 0 & 0 & \hat{\mathbb{Z}}^\times \end{pmatrix}.$$

Let ${}_\Gamma D^\circ$ be the subset of ${}_\Gamma D$ consisting of all points whose level structures induce on gr_w^W the standard level structures for all w. Then ${}_\Gamma D^\circ$ is open and closed in ${}_\Gamma D$. Since the data given on W_{-2} and on H/W_{-3} are isomorphic to the data given in Example a, the restriction to W_{-2} and the projection to H/W_{-3} give an analytic map over K,

$${}_\Gamma D^\circ \to K^\times \times K^\times.$$

Let $\mu(K)$ be the finite group of all roots of 1 in K, let $(\ ,\ )_{\mu(K)} : K^\times \times K^\times \to \mu(K)$ be Hilbert symbol map, and let $S = \{(x, y) \in K^\times \times K^\times \mid (x, y)_{\mu(K)} = 1\}$. Then S is an open set of $K^\times \times K^\times$. It can be shown that the image of ${}_\Gamma D^\circ \to K^\times \times K^\times$ coincides with S, and that as an analytic space over K, ${}_\Gamma D^\circ$ is an $H^1(K, \hat{\mathbb{Z}}(2))$-torsor over S. Here $H^1(K, \hat{\mathbb{Z}}(2))$ denotes the continuous Galois cohomology $H^1(\mathrm{Gal}(\bar{K}/K), \hat{\mathbb{Z}}(2))$, which is regarded as a compact Lie group over K by transporting the analytic structure of K via the local homeomorphism

$$H^1\big(K, \hat{\mathbb{Z}}(2)\big) \to H^1\big(K, \mathbb{Q}_p(2)\big) \cong K,$$

where the last isomorphism is given by the exponential map $\exp : K \xrightarrow{\cong} H^1(K, \mathbb{Q}_p(2))$ (see [4]). The action of $H^1(K, \hat{\mathbb{Z}}(2))$ on ${}_\Gamma D^\circ$ over S is characterized as follows. Let $x, y \in K^\times$. By Kummer theory, x and y determine extensions of representations of $\mathrm{Gal}(\bar{K}/K)$ over $\hat{\mathbb{Z}}$:

$$0 \to \hat{\mathbb{Z}}(2) \to T(x) \to \hat{\mathbb{Z}}(1) \to 0, \qquad 0 \to \hat{\mathbb{Z}}(1) \to T(y) \to \hat{\mathbb{Z}} \to 0.$$

The connecting map $H^1(K, \hat{\mathbb{Z}}(1)) \to H^2(K, \hat{\mathbb{Z}}(2))$ in the exact sequence

$$0 \to H^1\big(K, \hat{\mathbb{Z}}(2)\big) \to H^1\big(K, T(x)\big) \to H^1\big(K, \hat{\mathbb{Z}}(1)\big) \to H^2\big(K, \hat{\mathbb{Z}}(2)\big) \to \cdots$$

associated to the extension $T(x)$ sends the class of the extension $T(y)$ in $H^1(K, \hat{\mathbb{Z}}(1))$ to the Hilbert symbol $(x,y)_{\mu(K)} \in \mu(K) \cong H^2(K, \hat{\mathbb{Z}}(2))$. Hence the condition $(x,y) \in S$ is equivalent to the following condition. There exists a representation T of $\mathrm{Gal}(\bar{K}/K)$ over $\hat{\mathbb{Z}}$ having an increasing filtration by subrepresentations W_wT such that $W_{-5}T = 0$; $W_0T = T$; $\mathrm{gr}^W_w T$ is $\hat{\mathbb{Z}}(2)$, 0, $\hat{\mathbb{Z}}(1)$, 0, $\hat{\mathbb{Z}}$ for $w = -4, -3, -2, -1, 0$, respectively; and $W_{-2}T = T(x)$, $T/W_{-3}T = T(y)$. For $a \in H^1(K, \hat{\mathbb{Z}}(2))$, the action of a on ${}_\Gamma D^\circ$ sends $(N, F, \mu) \in {}_\Gamma D^\circ$ with $F = F(z_1, z_2, z_3)$ to (N, F', μ'), where $F' = F(z_1 + \exp^{-1}(a), z_2, z_3)$ and μ' is characterized by the following property. Let $T(\mu)$ be the $\hat{\mathbb{Z}}$-lattice in $V(N,F)$ corresponding to μ. Then $T(\mu)$ has the property of the above T, that is, $W_{-2}T(\mu) = T(x)$ and $T(\mu)/W_{-3}T(\mu) = T(y)$. The extension $0 \to T(x) \to T(\mu') \to \hat{\mathbb{Z}} \to 0$ is the Baer sum of the extension $0 \to T(x) \to T(\mu) \to \hat{\mathbb{Z}} \to 0$ and the pushout $0 \to T(x) \to\ ? \to \hat{\mathbb{Z}} \to 0$ of $0 \to \hat{\mathbb{Z}}(2) \to T(a) \to \hat{\mathbb{Z}} \to 0$ corresponding to a by the inclusion map $\hat{\mathbb{Z}}(2) \to T(x)$.

6. Toroidal partial compactifications ${}_\Gamma D_\Sigma$

Let the notation be as in Section 5.1. In particular, we assume that the residue field k of K is finite.

In this section, we construct the toroidal partial compactifications ${}_\Gamma D_\Sigma$ of ${}_\Gamma D$.

6.1. p-Adic nilpotent orbits in $\tilde{D}$

In Sections 6.1 and 6.2, we define the p-adic analogues (see Sections 6.1.5, 6.2.7) of the notion of nilpotent orbit in Hodge theory (see Section 2.3.4).

6.1.1
Let

$$\tilde{D} = \{(N, F) \mid N \in \mathfrak{N}, F \in {}_N D\}.$$

We have

$$D \subset \tilde{D} \subset \mathfrak{N} \times \check{D}.$$

6.1.2
We call a subset σ of $\mathfrak{N}$ a *nilpotent cone in* $\mathfrak{N}$ if the following conditions (i) and (ii) are satisfied:

(i) σ is a finitely generated $\mathbb{Q}_{\geq 0}$-cone; that is,

$$\sigma = \mathbb{Q}_{\geq 0}N_1 + \cdots + \mathbb{Q}_{\geq 0}N_n$$

for some $n \geq 0$ and $N_1, \ldots, N_n \in \mathfrak{N}$;

(ii) $NN' = N'N$ for any $N, N' \in \sigma$.

For a nilpotent cone σ in $\mathfrak{N}$, let $\sigma_{\mathbb{Q}}$ be the $\mathbb{Q}$-linear span of σ in $\mathfrak{N}$. That is, $\sigma_{\mathbb{Q}} = \{a - b \mid a, b \in \sigma\} \subset \mathfrak{N}$. For a commutative ring R over $\mathbb{Q}$, let $\sigma_R = R \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}} \subset R \otimes_{\mathbb{Q}} \mathfrak{N}$.

6.1.3
Let σ be a nilpotent cone in $\mathfrak{N}$. Let

$$\tilde{D}(\sigma) \subset \tilde{D}$$

be the subset of $\tilde{D}$ consisting of all elements (N, F) which satisfy the following conditions (i) and (ii):

(i) $NN' = N'N$ for any $N' \in \sigma$;
(ii) $N'F^r \subset F^{r-1}$ for all $N' \in \sigma$ and all $r \in \mathbb{Z}$ (Griffiths transversality).

Note that for $(N, F) \in \tilde{D}$, $(N, F) \in \tilde{D}(\sigma)$ if and only if as a set of K_0-linear maps $H_p \to H_p$, $\sigma_{\mathbb{Q}_p}$ is contained in the space $\mathfrak{M}$ of (H_p, N, F) in Section 3.5.1.

PROPOSITION 6.1.4
Let σ be a nilpotent cone in $\mathfrak{N}$. Let $(N, F) \in \tilde{D}(\sigma)$. Then for any $b \in \sigma_K$ and $c \in \sigma_{\mathbb{Q}}$, we have $(N + c, \exp(b)F) \in \tilde{D}(\sigma)$.

This follows from Propositions 3.5.2 and 3.5.4.

6.1.5. p-adic nilpotent orbit in $\tilde{D}$
Let σ be a nilpotent cone in $\mathfrak{N}$, and let Z be a subset of $\tilde{D}$. We say that Z is a *σ-nilpotent orbit* if the following conditions (i)–(iv) are satisfied.

(i) We have $Z \subset \tilde{D}(\sigma)$.
(ii) Let $(N, F) \in Z$. Then $Z = \{(N + c, \exp(b)F) \mid c \in \sigma_{\mathbb{Q}}, b \in \sigma_K\}$.
(iii) Write $\sigma = \mathbb{Q}_{\geq 0}N'_1 + \cdots + \mathbb{Q}_{\geq 0}N'_n$. Let $(N, F) \in Z$. Then $N + \sum_{j=1}^n y_j \times N'_j \in C$ if $y_j \in \mathbb{Q}$ and $y_j \gg 0$.
(iv) For any $N' \in \sigma$, the relative monodromy filtration $M(N', W)$ exists. (This is a condition on σ.)

REMARK 6.1.6 (COMPARISON WITH NILPOTENT ORBIT OVER $\mathbb{C}$)
Condition (i) (resp., (ii), (iii), (iv)) in Section 6.1.5 is an analogue of condition (ii) (resp., (i), (iii), (iv)) in Section 2.3.4.

PROPOSITION 6.1.7
Let σ be a nilpotent cone in $\mathfrak{N}$, and let $(N, F) \in \tilde{D}(\sigma)$. Let $N' \in \sigma$, and assume that the relative monodromy filtration $M(N', W)$ exists. Denote $M(N', W)$ by W'. Then for any $i \in \mathbb{Z}$, the subobject $W'_i(H_p, N, F)$ of (H_p, N, F) in MF_K is admissible.

Proof
Let $V = V_p(N, F)$. Let t be the generator of $\mathbb{Z}_p(1) \subset B_{\mathrm{crys}}$ given in Section 3.2.

Then the filtration $B_{\mathrm{st}} \otimes_{K_0} W'_\bullet H_p$ on $B_{\mathrm{st}} \otimes_{K_0} H_p$ is the relative monodromy filtration of $tN' : B_{\mathrm{st}} \otimes_{K_0} H_p \to B_{\mathrm{st}} \otimes_{K_0} H_p$ with respect to $B_{\mathrm{st}} \otimes_{K_0} W_\bullet H_p$. Hence the map $tN' : V \to V$ (see Section 3.5.1; see also the proof of Claim 1 in the proof of Proposition 3.5.2) has a relative monodromy filtration with respect to $W_\bullet V$, which we denote by $W'_\bullet V$, and $W'_\bullet V$ induces the above filtration on $B_{\mathrm{st}} \otimes_{K_0} H_p = B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$. Since the map $N' : V \to V(-1)$ is compatible with the action of $\mathrm{Gal}(\bar{K}/K)$, the filtration $W'_\bullet V$ is stable under the action of $\mathrm{Gal}(\bar{K}/K)$. Since $W'_i(H_p, N, F)$ corresponds to the semistable Galois representation $W'_i V$, $W'_i(H_p, N, F)$ is admissible. □

6.2. p-Adic nilpotent orbits in ${}_\Gamma\tilde{D}$

6.2.1

For a compact open subgroup Γ of $G(\mathbb{Q}_A^f)$, let ${}_\Gamma\tilde{D}$ be the set of all triples (N, F, μ), where $(N, F) \in \tilde{D}$ (see Section 6.1.3) and where μ is a Γ-level structure on $V(N, F) = V_p(N, F) \times V_{\text{non-}p}(N)$.

6.2.2

Let σ be a nilpotent cone in $\mathfrak{N}$, let Γ be a compact open subgroup of $G(\mathbb{Q}_A^f)$, and let $\alpha = (N, F, \mu) \in {}_\Gamma\tilde{D}(\sigma)$. Then we define a submonoid

$$\sigma(\alpha) \subset \sigma$$

as follows. By Section 3.5.1, for $a \in \mathbb{Q}_p(1) \times \mathbb{Q}^f_{A,\text{non-}p}$ and $N' \in \sigma_{\mathbb{Q}}$, we have $\exp(aN') : V(N, F) \to V(N, F)$. Let $\sigma(\alpha)$ be the subset of σ consisting of all elements N' such that for any $a \in \mathbb{Z}_p(1) \times \hat{\mathbb{Z}}_{\text{non-}p}$, $\exp(aN') : V(N, F) \to V(N, F)$ does not change the level structure μ. Here the last condition means that if $\tilde{\mu} : L \to V(N, F)$ is a representative of μ, then for each $a \in \mathbb{Z}_p(1) \times \hat{\mathbb{Z}}_{\text{non-}p}$, there is $\gamma \in \Gamma$ such that $\exp(aN')\tilde{\mu} = \tilde{\mu}\gamma$.

LEMMA 6.2.3

The $\mathbb{Z}$-linear span $\sigma(\alpha)_{\mathbb{Z}} = \{a - b \mid a, b \in \sigma(\alpha)\}$ of $\sigma(\alpha)$ in $\mathfrak{N}$ is a finitely generated $\mathbb{Z}$-module, $\sigma(\alpha) = \sigma(\alpha)_{\mathbb{Z}} \cap \sigma$, and $\sigma_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$.

Proof

Take a representative $\tilde{\mu} : L \to V(N, F)$ of μ. We have an injective homomorphism

$$\mathbb{Q}_A^f \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}} \to G(\mathbb{Q}_A^f), \qquad a \mapsto \tilde{\mu}^{-1} \exp(a_p t, a_{\text{non-}p})\tilde{\mu}.$$

This is a continuous homomorphism and is a closed map. Hence the inverse image I of Γ under this homomorphism is a compact open subgroup of $\mathbb{Q}_A^f \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}}$. Hence the intersection $\sigma_{\mathbb{Q}} \cap I$ is a finitely generated $\mathbb{Z}$-module which generates $\sigma_{\mathbb{Q}}$ over $\mathbb{Q}$. Since $\sigma(\alpha) = \sigma \cap I$, we have the lemma. □

6.2.4

Let the notation be as in Section 6.2.2. Let

$$\mathrm{ord}_\xi : K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}} \to \sigma_{\mathbb{Q}}, \qquad \log : K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}} \to \sigma_K$$

be the homomorphisms induced by $\mathrm{ord}_\xi : K^\times \to \mathbb{Q}$ and $\log : K^\times \to K$, respectively.

6.2.5
Let the notation be as in Section 6.2.2. For $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, we define

$$a\alpha = \big(N + \mathrm{ord}_\xi(a), \exp(\log(a))F, a\mu\big) \in {}_\Gamma\tilde{D}(\sigma),$$

where $a\mu$ is as follows.

Let $P(K)$ be as in Section 5.5.1, let $\log : P(K) \to B_{\mathrm{st}}^{\varphi=p}$ be the logarithm (see [7]) as in Section 5.5.1, let $\log_{\text{non-}p} : P(K) \to \mathbb{Q}^f_{A,\text{non-}p}$ be as in Section 5.5.1, and let

$$\log : P(K) \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}} \to B_{\mathrm{st}} \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}}, \qquad \log_{\text{non-}p} : P(K) \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}} \to \mathbb{Q}^f_{A,\text{non-}p} \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}}$$

be the induced homomorphisms, respectively. Take an element $\tilde{a}$ of $P(K) \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ whose image in $K^\times$ is a. Then define $a\mu$ to be the Γ-equivalence class of $\tilde{a} \circ \tilde{\mu} : L \to V\big(N + \mathrm{ord}_\xi(a), \exp(\log(a))F\big)$, where $\tilde{a}$ denotes the automorphism $\big(\exp(\log(\tilde{a})), \exp(\log_{\text{non-}p}(a))\big)$ of $(B_{\mathrm{st}} \otimes_{K_0} H_p) \times H_{\text{non-}p}$ and where $\tilde{\mu} : L \to V(N, F)$ is a representative of μ (see Propositions 3.5.2(2), 3.5.4(2)). By the definition of $\sigma(\alpha)$, $a\mu$ is independent of the choices of $\tilde{a}$ and $\tilde{\mu}$.

LEMMA 6.2.6
Let the notation be as in Section 6.2.2.

(1) *For $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, we have $\sigma(a\alpha) = \sigma(\alpha)$.*
(2) *For $a, b \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, we have $(ab)\alpha = a(b\alpha)$.*

This is shown easily.

6.2.7. p-adic nilpotent orbits in ${}_\Gamma\tilde{D}$
Let σ be a nilpotent cone in $\mathfrak{N}$, and let Γ be an compact open subgroup of $G(\mathbb{Q}^f_A)$.

By a (σ, Γ)-*nilpotent orbit*, we mean a nonempty subset Z of ${}_\Gamma\tilde{D}(\sigma)$ satisfying the following conditions (i) and (ii):

(i) $Z = \{a\alpha \mid a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}\}$ for some (and hence for any) $\alpha \in Z$;
(ii) let $(N, F, \mu) \in Z$; then $\{(N + c, \exp(b)F) \mid c \in \sigma_{\mathbb{Q}}, b \in \sigma_K\}$ is a σ-nilpotent orbit (see Section 6.1.5).

PROPOSITION 6.2.8
Let σ be a nilpotent cone in $\mathfrak{N}$, and let Γ be an compact open subgroup of $G(\mathbb{Q}^f_A)$. Let $Z \subset {}_\Gamma\tilde{D}(\sigma)$ be a (σ, Γ)-nilpotent orbit, and let $\alpha, \beta \in Z$. Then the element a of $K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ such that $\beta = a\alpha$ is unique.

Proof
Assume that $a' \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ also has the property of a. Then $\exp(\log(a/a')) \times F = F$, and hence $\log(a/a') = 0$ by Proposition 3.5.5. Since $N + \mathrm{ord}_\xi(a) = N + \mathrm{ord}_\xi(a')$, we have $\mathrm{ord}_\xi(a/a') = 0$. Hence a/a' is a root of 1. Next, since $a\alpha = a'\alpha$, a/a' does not change the level structure μ of α. Let u be an element of

$P(K)\otimes_{\mathbb{Z}}\sigma(\alpha)_{\mathbb{Z}}$ whose image in $K^\times$ is a/a'. Since a/a' is a root of 1, u belongs to $\hat{\mathbb{Z}}(1)\otimes_{\mathbb{Z}}\sigma_{\mathbb{Q}}$ in $P(K)\otimes_{\mathbb{Z}}\sigma_{\mathbb{Q}}$. It is sufficient to prove that $u\in\hat{\mathbb{Z}}(1)\otimes_{\mathbb{Z}}\sigma(\alpha)_{\mathbb{Z}}$. Let $(N'_j)_{1\le j\le n}$ be a $\mathbb{Z}$-basis of $\sigma(\alpha)_{\mathbb{Z}}$, and write u in the form $u_1N'_1+\cdots+u_nN'_n$ with $u_j\in\mathbb{Q}^f_A(1)$. If there is a j such that $u_j\notin\hat{\mathbb{Z}}(1)$, there are j and a nonzero integer m such that $mu_j\notin\hat{\mathbb{Z}}(1)$ and $mu_{j'}\in\hat{\mathbb{Z}}(1)$ for $j'\neq j$. Then exp of the image of $mu_jN'_j$ in $(\mathbb{Q}_p(1)\times\mathbb{Q}^f_{A,\text{non-}p})\otimes_{\mathbb{Z}}\sigma(\alpha)_{\mathbb{Z}}$ does not change μ. This shows that $cN'_j\in\sigma(\alpha)_{\mathbb{Z}}$ for some $c\in\mathbb{Q}$ which does not belong to $\mathbb{Z}$, a contradiction. □

6.3. The space ${}_\Gamma D_\Sigma$

6.3.1

In Sections 6.3–6.5, Σ denotes a nonempty set of nilpotent cones in $\mathfrak{N}$ satisfying the following conditions (i) and (ii).

(i) Any element σ of Σ is sharp (that is, $\sigma\cap(-\sigma)=\{0\}$).
(ii) If $\sigma\in\Sigma$, then all faces of σ belong to Σ.

6.3.2

Let Γ be a compact open subgroup of $G(\mathbb{Q}^f_A)$, and let Σ be as in Section 6.3.1. We define

$$ {}_\Gamma D_\Sigma=\big\{(\sigma,Z)\bigm| \sigma\in\Sigma,\ Z \text{ is a } (\sigma,\Gamma)\text{-nilpotent orbit}\big\}. $$

We denote an element (σ,Z) of ${}_\Gamma D_\Sigma$, as $(\sigma,\text{class}(\alpha))$ for $\alpha\in Z$.

6.3.3

We have an embedding

$$ {}_\Gamma D\overset{\subset}{\to}{}_\Gamma D_\Sigma,\qquad \alpha\mapsto(\{0\},\{\alpha\}). $$

In fact, ${}_\Gamma D$ is identified with the subset of ${}_\Gamma D_\Sigma$ consisting of all elements (σ,Z) such that $\sigma=\{0\}$.

6.4. The analytic structure over K of ${}_\Gamma D_\Sigma$

Let Γ be an compact open subgroup of $G(\mathbb{Q}^f_A)$, and let Σ be as in Section 6.3.1. We endow ${}_\Gamma D_\Sigma$ with a topology, a sheaf of analytic functions over K, and a log structure (see Section 6.4.8).

In Sections 6.4.1–6.4.7, we fix $\sigma\in\Sigma$ and $\alpha=(N,F,\mu)\in{}_\Gamma\tilde{D}(\sigma)$.

6.4.1

Let the affine toric variety $\text{toric}_{\sigma(\alpha)}$ be the set of all homomorphisms $\sigma(\alpha)^\vee\to K$, where K is regarded as a multiplicative monoid. Here $\sigma(\alpha)^\vee$ denotes the dual monoid $\text{Hom}(\sigma(\alpha),\mathbb{N})$ of $\sigma(\alpha)$. ($\mathbb{N}$ is regarded as an additive monoid.) The torus $K^\times\otimes_{\mathbb{Z}}\sigma(\alpha)_{\mathbb{Z}}=\text{Hom}(\sigma^\vee,K^\times)$ is contained in $\text{toric}_{\sigma(\alpha)}$ as a dense open subset and acts on $\text{toric}_{\sigma(\alpha)}$.

For a face τ of σ, let 0_τ be the element of $\text{toric}_{\sigma(\alpha)}$ which sends an element h of $\sigma(\alpha)^\vee$ to $1\in K$ if $h:\sigma(\alpha)\to\mathbb{N}$ kills $\sigma(\alpha)\cap\tau$, and to $0\in K$ otherwise. Any element of $\text{toric}_{\sigma(\alpha)}$ is written in the form $a0_\tau$ for some face τ of σ and for some

$a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$. (Here we have $a0_\tau$ by the action of $K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ on $\mathrm{toric}_{\sigma(\alpha)}$.) For $a, b \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, $a0_\tau = b0_\tau$ if and only if $a^{-1}b \in K^\times \otimes_{\mathbb{Z}} (\sigma(\alpha) \cap \tau)_{\mathbb{Z}}$.

For $q \in \mathrm{toric}_{\sigma(\alpha)}$, the face τ of σ such that $q = a0_\tau$ for some $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ is uniquely determined by q and is denoted by $\sigma(q)$.

LEMMA 6.4.2

Write $\sigma = \mathbb{Q}_{\geq 0} N_1' + \cdots + \mathbb{Q}_{\geq 0} N_n'$. For $c > 0$, let $A_c = \left\{ \sum_{j=1}^n y_j N_j' \;\middle|\; y_j \geq c \right\} \subset \sigma$. Then we have the following.

(1) *Let U be a neighborhood of 0_σ in $\mathrm{toric}_{\sigma(\alpha)}$. Then there exists $c > 0$ such that any element a of $K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ such that $\mathrm{ord}_\xi(a) \in A_c$ belongs to U.*

(2) *Let $c > 0$. Then there exists a neighborhood U of 0_σ in $\mathrm{toric}_{\sigma(\alpha)}$ such that $\mathrm{ord}_\xi(a) \in A_c$ for any $a \in U \cap (K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}})$.*

This is proved easily. (This lemma is essentially [20, Lemma 2.5.6].)

6.4.3

For an open set U of $\check{D}$, we define sets

$$E_{\sigma,\alpha}(U) \subset \tilde{E}_{\sigma,\alpha}(U) \subset \mathrm{toric}_{\sigma(\alpha)} \times \check{D}$$

as follows. Let $\tilde{E}_{\sigma,\alpha}(U)$ be the subset of $\mathrm{toric}_{\sigma(\alpha)} \times U$ consisting of all elements (q, F') satisfying the following condition (i).

(i) F' satisfies Griffiths transversality for all elements of the cone $\sigma(q)$. That is, $N'(F')^r \subset (F')^{r-1}$ for any $N' \in \sigma(q)$ and $r \in \mathbb{Z}$.

Let $E_{\sigma,\alpha}(U)$ be the subset $\tilde{E}_{\sigma,\alpha}(U)$ consisting of all elements (q, F') satisfying the following condition (ii).

(ii) Write $q = 0_\tau a$ with $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, and write $\tau = \mathbb{Q}_{\geq 0} N_1' + \cdots + \mathbb{Q}_{\geq 0} N_n'$. Then $N + \mathrm{ord}_\xi(a) + \sum_{j=1}^n y_j N_j' \in C$ if $y_j \in \mathbb{Q}$ and $y_j \gg 0$ $(1 \leq j \leq n)$.

This condition does not depend on the choice of a such that $q = 0_\tau a$.

Condition (ii) shows that $\{b\beta \mid b \in \mathrm{toric}_{\tau(\beta)}\}$ with $\beta = a\alpha$ is a (τ, Γ)-nilpotent orbit.

Endow $\tilde{E}_{\sigma,\alpha}(U)$ with the topology as a subspace of $\mathrm{toric}_{\sigma(\alpha)} \times \check{D}$, with the pullback of the sheaf of analytic functions over K of $\mathrm{toric}_{\sigma(\alpha)} \times \check{D}$, and with the pullback of the canonical log structure of the affine toric variety $\mathrm{toric}_{\sigma(\alpha)}$.

Note that $E_{\sigma,\alpha}(U)$ is an open set of $\tilde{E}_{\sigma,\alpha}(U)$. We endow $E_{\sigma,\alpha}(U)$ with the restrictions of these structures of $\tilde{E}_{\sigma,\alpha}(U)$.

Our method to endow ${}_\Gamma D_\Sigma$ with a topology, a sheaf of analytic functions, and a log structure is to relate ${}_\Gamma D_\Sigma$ to the spaces $E_{\sigma,\alpha}(U)$ for various (σ, α).

PROPOSITION 6.4.4

Let I be a compact (additive) subgroup of σ_K, and let J be a compact subgroup of $\sigma_{\mathbb{Q}_p}$. Then there exists a neighborhood U of F in $\check{D}$ such that $\exp(a)F' \in {}_{N+c}D$ for any $F' \in U$, $a \in I$, and $c \in J$.

Proof
This is proved by modifying the proof of Proposition 4.2.2 slightly as follows.

We may and do assume that there is $w \in \mathbb{Z}$ such that $W_w = H$ and $W_{w-1} = 0$, and that $t_N(H_p) = \sum_{w,i} ih_{w,i}$. Let $K_0' = W(\bar{k}) \otimes_{W(k)} K_0$, and let $K' = K \otimes_{K_0} K_0'$; so K_0' (resp., K') is the completion of the maximal unramified extension of K_0 (resp., K). Let $H' = K_0' \otimes_{K_0} H$. For integers $r \geq 0$ and m, let

$$\tilde{P}_{r,m} = \Big\{(x,c) \in \Big(\bigwedge_{K_0'}^{r} H'\Big) \times J \;\Big|\; \varphi(x) = p^m x, (N+c)(x) = 0\Big\},$$

$$P_{r,m} = (\tilde{P}_{r,m} - \{0\})/\mathbb{Q}_p^\times.$$

Here $\mathbb{Q}_p^\times$ acts on $\tilde{P}_{r,m}$ by $(x,c) \mapsto (zx,c)$ $(z \in \mathbb{Q}_p^\times)$. Then $P_{r,m}$ is compact. Note that $P_{r,m}$ are empty for almost all (r,m). Let

$$S_{r,m} = \Big\{(x,c,F',a) \in P_{r,m} \times \check{D} \times I \;\Big|\; \exp(a)\tilde{x} \in (F')^{m+1}\Big(\bigwedge_{K'}^{r} H_{K'}\Big)\Big\}.$$

Here $\tilde{x}$ denotes a lifting of x to $\big(\bigwedge_{K_0'}^r H'\big) - \{0\}$, and $(F')^{m+1}\big(\bigwedge_{K'}^r H_{K'}\big)$ is induced by F'. Since $P_{r,m} \times I$ is compact and $S_{r,m}$ is closed in $P_{r,m} \times \check{D} \times I$, the map $S_{r,m} \to \check{D}; (x,c,F',a) \mapsto F'$ is proper. Hence the image of this map is closed in $\check{D}$. Let $U \subset \check{D}$ be the complement of the union of the images of this map for all (r,m). Then U is open. Furthermore, since $\exp(a)F \in {}_{N+c}D$ for any $a \in \sigma_K$ and $c \in \sigma_{\mathbb{Q}_p}$ by Propositions 3.5.2 and 3.5.4, we have $F \in U$. □

PROPOSITION 6.4.5

Fix a finitely generated $\mathbb{Z}_p$-submodule $L_{\mathbb{Z}_p}$ of L which satisfies $L = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_{\mathbb{Z}_p}$ and which is stable under the action of Γ via $\Gamma \to G(\mathbb{Q}_p)$. Denote the automorphism group of $(L_{\mathbb{Z}_p} \times L_{\text{non-}p}, W_\bullet(L_{\mathbb{Z}_p} \times L_{\text{non-}p}))$ regarded as a smooth group scheme over $\mathbb{Z}_p \times \mathbb{Q}^f_{A,\text{non-}p}$ by G. Fix a representative $\tilde{\mu} : L \to V(N,F)$ of μ. Fix integers $c \geq 2$ and $n \geq 0$. Let $U_{\sigma,\alpha,c,n}$ be the subset of $\check{D}$ consisting of all elements F' of $\check{D}$ satisfying the following condition (i).

(i) *For any $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, we have $\exp(\log(a))F' \in U_{a\alpha,c,n}$, where $U_{a\alpha,c,n}$ is as in Section 4.4.6.*

Then $U_{\sigma,\alpha,c,n}$ is an open neighborhood of F in $\check{D}$.

Proof
This is proved by modifying slightly the proof of Proposition 4.4.7. Take a representative $\tilde{\mu} : L \to V(N,F)$ of μ. By the proof of Proposition 4.4.7 given in Section 4.6, it is sufficient to prove that there is a neighborhood U of F in $\check{D}$ such that for any $F' \in U$, any $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, and any lifting $\tilde{a} \in P(K) \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ of a, $\exp(\log(a))F' \in {}_{N+\mathrm{ord}_\xi(a)}D$ and the image of $h_{\tilde{a}\tilde{\mu}}(F')$ in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ belongs to the image of $G(B_c)_{\equiv 1 \bmod p^n}$. Here $\tilde{a}\tilde{\mu}$ denotes $\exp(\log(\tilde{a}))\tilde{\mu}$. Let ν and δ be as in Lemma 4.6.7. As is easily seen, if $g \in G(K)$ and if $F' = \nu^{-1} g \nu F \in {}_{N+\mathrm{ord}_\xi(a)}D$, the image of $h_{\tilde{a}\tilde{\mu}}(F')$ in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ is equal to $P_{\tilde{a}} g P_{\tilde{a}}^{-1} \bmod G(B^0_{\mathrm{dR}})$, where

$P_{\tilde{a}} = \tilde{\mu}^{-1}\exp(-\log(\tilde{a}))\tilde{\mu}\delta^{-1}\nu\exp(\log(a))\nu^{-1}$. (Here $\log(\tilde{a})$ is understood as an element of $B_{\mathrm{st}} \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}}$, and $\log(a)$ is understood as an element of σ_K.) There is a compact subset of $B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}} \sigma_{\mathbb{Q}}$ (resp., σ_K) which contains the elements $\log(\tilde{a})$ (resp., $\log(a)$) for all $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$. Hence by Proposition 6.4.4 and by the proof of Lemma 4.6.6, there exists a neighborhood U of 1 in $G(B^0_{\mathrm{dR}})$ such that, for any $g \in U$, any $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, and any lifting $\tilde{a}$ of a to $P(K) \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, the class of $P_{\tilde{a}} g P_{\tilde{a}}^{-1}$ in $G(B_{\mathrm{dR}})/G(B^0_{\mathrm{dR}})$ belongs to the image of $G(B_c)_{\equiv 1 \bmod p^n}$. This proves Proposition 6.4.5. □

6.4.6

Take n such that $G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma \cap G(\mathbb{Q}_p)$. Take $c \geq 2$, and let $U_{\sigma,\alpha,c,n}$ be as in Proposition 6.4.5. We have a map

$$e_{\sigma,\alpha} : E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_\Gamma D_\Sigma,$$

$$(a0_\tau, F') \mapsto \big(\tau, \mathrm{class}(N + \mathrm{ord}_\xi(a), e_{a\alpha}(\exp(\log(a))F'))\big)$$

$$(a \in \mathrm{torus}_{\sigma(a)} = K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}),$$

where $e_{a\alpha} : U_{a\alpha,c,n} \to {}_{N+\mathrm{ord}_\xi(a),\Gamma}D$ is as in Section 4.4.8.

The following properties of this map $e_{\sigma,\alpha}$ are proved easily.

LEMMA 6.4.7

(1) *Assume that $N \in C$. For the unit element $1 \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}} \subset \mathrm{toric}_{\sigma(\alpha)}$ and for $F' \in U_{\sigma,\alpha,c,n}$, we have*

$$e_{\sigma,\alpha}(1, F') = \big(N, e_\alpha(F')\big) \in {}_\Gamma D.$$

(2) *Let $a \in K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$, and let $F' \in U_{\sigma,\alpha,c,n}$. Then $\exp(\log(a))F' \in U_{\sigma,a\alpha,c,n}$. For $q \in \mathrm{toric}_{\sigma(\alpha)}$, $(qa, F') \in E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n})$ if and only if $\big(q, \exp(\log(a))F'\big) \in E_{\sigma,a\alpha}(U_{\sigma,a\alpha,c,n})$. If these equivalent conditions are satisfied, we have*

$$e_{\sigma,a\alpha}\big(q, \exp(\log(a))F'\big) = e_{\sigma,\alpha}(qa, F').$$

(3) *Let τ be a face of σ, let $F' \in U_{\sigma,\alpha,c,n}$, and assume that $\beta := (N, e_\alpha(F')) \in {}_\Gamma\tilde{D}$ belongs to ${}_\Gamma\tilde{D}(\tau)$. Then $\tau(\beta) = \sigma(\alpha) \cap \tau$, $U_{\sigma,\alpha,c,n} \subset U_{\tau,\beta,c,n}$, and on $E_{\tau,\beta}(U_{\sigma,\alpha,c,n})$, the restriction of $e_{\sigma,\alpha} : E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_\Gamma D_\Sigma$ and the restriction of $e_{\tau,\beta} : E_{\tau,\beta}(U_{\tau,\beta,c,n}) \to {}_\Gamma D_\Sigma$ coincide.*

6.4.8

We define the topology of ${}_\Gamma D_\Sigma$, the sheaf of analytic functions on it, and the log structure on it as follows.

Endow ${}_\Gamma D_\Sigma$ with the following topology. A subset S of ${}_\Gamma D_\Sigma$ is open if and only if for any $\sigma \in \Sigma$, any $\alpha = (N, F, \mu) \in {}_\Gamma\tilde{D}(\sigma)$, and any $c \geq 2$ and $n \geq 0$ such that $G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma \cap G(\mathbb{Q}_p)$, the inverse image of S under the map $e_{\sigma,\alpha} : E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_\Gamma D_\Sigma$ (see Section 6.4.6) is open. We define the sheaf of analytic functions on ${}_\Gamma D_\Sigma$ as follows. For a K-valued function f on an open set S of ${}_\Gamma D_\Sigma$, f is analytic if and only if for any σ, α, c, n as above, the pullback of f on the inverse image of S in $E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n})$ is analytic. The log structure of ${}_{\Gamma}D_{\Sigma}$ is defined as the subsheaf of the sheaf of analytic functions as follows. For an open set S of ${}_{\Gamma}D_{\Sigma}$ and for an analytic function f on S, f belongs to the log structure if and only if f does not have value zero on $S \cap {}_{\Gamma}D$.

The topology, the sheaf of analytic functions, and the log structure of ${}_{\Gamma}D_{\Sigma}$ are independent of the choice of $L_{\mathbb{Z}_p}$ (see Section 4.4).

6.4.9
The definitions in Section 6.4.8 are similar to the case over $\mathbb{C}$ in [20] and [14] III. In the theory over $\mathbb{C}$, we define a subset E_{σ} of the product space (a toric variety) $\times \check{D}$ for each $\sigma \in \Sigma$, we define maps $E_{\sigma} \to \Gamma \setminus D_{\Sigma}$, and we define the topology, the sheaf of analytic functions, and the log structure of $\Gamma \setminus D_{\Sigma}$ by using these maps. The above $e_{\sigma,\alpha} : E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_{\Gamma}D_{\Sigma}$ are the p-adic analogues of $E_{\sigma} \to \Gamma \setminus D_{\Sigma}$ over $\mathbb{C}$.

6.4.10
Let $\sigma \in \Sigma$, and let $\alpha = (N, F, \mu) \in {}_{\Gamma}\tilde{D}(\sigma)$. Then when $a \in K^{\times} \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}}$ tends to $0_{\sigma} \in \mathrm{toric}_{\sigma(\alpha)}$, the element $(\sigma, \mathrm{class}(\alpha))$ of ${}_{\Gamma}D_{\Sigma}$ is the limit of $a\alpha \in {}_{\Gamma}D$.

Here $0_{\sigma} \in \mathrm{toric}_{\sigma}$ is the homomorphism $\sigma^{\vee} \to K$ which sends the unit element of $\sigma^{\vee}$ to 1 and all other elements to zero.

This is proved easily.

This is the p-adic analogue of Section 2.3.10.

6.4.11
We have

$$ {}_{\Gamma}D_{\Sigma} = \bigcup_{\sigma \in \Sigma} {}_{\Gamma}D_{\sigma}, $$

where ${}_{\Gamma}D_{\sigma} := {}_{\Gamma}D_{\mathrm{face}(\sigma)}$. This is an open covering. In particular, ${}_{\Gamma}D = {}_{\Gamma}D_{\{0\}}$ is a dense open subset of ${}_{\Gamma}D_{\Sigma}$.

6.5. p-Adic log manifolds

6.5.1
We define the notion of a "log manifold" over K. This is the p-adic version of the notion of a log manifold in the complex analytic case (see Section 2.3.11).

By a *log manifold over* K we mean a local ringed space over K endowed with an fs log structure which has an open covering $(U_{\lambda})_{\lambda}$ with the following property. For each λ, there exist an affine toric variety Z_{λ} endowed with the canonical log structure, a finite subset I_{λ} of $\Gamma(Z_{\lambda}, \Omega^1_{Z_{\lambda}}(\log))$, and an isomorphism of local ringed spaces over K with log structures between U_{λ} and an open subset of

$$ S_{\lambda} = \left\{ z \in Z_{\lambda} \,\middle|\, \text{the image of } I_{\lambda} \text{ in } \Omega^1_z(\log) \text{ is zero} \right\}, $$

where S_{λ} is endowed with the induced topology in Z_{λ}, with the inverse image $\mathcal{O}_{Z_{\lambda}}$ of the sheaf of analytic functions $\mathcal{O}_{Z_{\lambda}}$ on Z_{λ}, and with the inverse image $M_{S_{\lambda}}$ of the log structure $M_{Z_{\lambda}}$ of Z_{λ}.

Here an affine toric variety means an analytic space over K of the form $\mathrm{Hom}(\mathcal{S}, K)$, where $\mathcal{S}$ is the intersection of a finitely generated $\mathbb{Q}_{\geq 0}$-cone τ and a finitely generated $\mathbb{Z}$-submodule of $\tau_{\mathbb{Q}}$, and Hom is the set of homomorphisms of monoids where K is regarded as a multiplicative monoid. We regard $\mathrm{Hom}(\mathcal{S}, K)$ as an analytic space over K in the natural way. An affine toric variety is endowed with a canonical log structure.

In the complex analytic case, we used the *strong topology* (that was important for having a good theory). Here we do not need such strong topology.

PROPOSITION 6.5.2

Let the notation be as in Section 6.4.3. Then $E_{\sigma,\alpha}(U)$ and $\tilde{E}_{\sigma,\alpha}(U)$ are log manifolds over K.

Proof

The statement for $\tilde{E}_{\sigma,\alpha}(U)$ is proved easily just as in the proof of [20, Section 3.5.10], where we proved that a space $\tilde{E}_\sigma$ over $\mathbb{C}$, whose definition is similar to that of $\tilde{E}_{\sigma,\alpha}(U)$, is a log manifold. The statement for $E_{\sigma,\alpha}(U)$ follows from that for $\tilde{E}_{\sigma,\alpha}(U)$ because $E_{\sigma,\alpha}(U)$ is an open set of $\tilde{E}_{\sigma,\alpha}(U)$. □

6.5.3

Consider $(\sigma, \mathrm{class}(\alpha)) \in {}_\Gamma D_\Sigma$, $\alpha = (N, F, \mu)$. Take a K-linear subspace S of $\mathrm{End}_K(H_K, W_K)$ such that

$$\mathrm{End}_K(H_K, W_K) = F^0\mathrm{End}_K(H_K, W_K) \oplus \sigma_K \oplus S.$$

Here $F^0\mathrm{End}_K(H_K, W_K)$ is the set of all K-linear maps $H_K \to H_K$ which respect W and F. The existence of such S follows from Proposition 3.5.5. For an open neighborhood U of 0_σ in $\mathrm{toric}_{\sigma(\alpha)}$ and for an open neighborhood U' of zero in S such that the exponential map into $\mathrm{Aut}_K(H_K, W_K)$ converges on U', let $X(U, U') \subset U \times U'$ be the set of all elements (q, x) of $U \times U'$ such that $(N, \exp(x)F) \in \tilde{D}(\sigma(q))$ (i.e., such that $\exp(x)F$ satisfies Griffiths transversality for $\sigma(q)$). Endow $X(U, U')$ with the sheaf of analytic functions which is the pullback of that of $U \times U'$, and define the log structure of $X(U, U')$ to be the pullback of the canonical log structure of $\mathrm{toric}_{\sigma(\alpha)}$.

PROPOSITION 6.5.4

Let the notation be as in Section 6.5.3. Then $X(U, U')$ is a log manifold over K.

Proof

This can be seen easily also just as in the proof of [20, Section 3.5.10]. □

PROPOSITION 6.5.5

Let the notation be as in Section 6.5.3. Let $c \geq 2$, let $n \geq 0$, and assume that $G(\mathbb{Z}_p)_{\equiv 1 \bmod p^n} \subset \Gamma \cap G(\mathbb{Q}_p)$. Then if U and U' are sufficiently small, $(q,$

$\exp(x)F) \in E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n})$ *for any* $(q,x) \in X(U,U')$, *and the composition*

$$X(U,U') \to E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_\Gamma D_\Sigma$$

is an open immersion of local ringed spaces over K *with log structures. Here the first arrow is* $(q,x) \mapsto (q, \exp(x)F)$ *and the second is* $e_{\sigma,\alpha}$.

This can be deduced from Lemma 6.4.7.

Proposition 6.5.5 proves the following.

THEOREM 6.5.6

${}_\Gamma D_\Sigma$ *is a log manifold over* K.

6.5.7

In the theory over $\mathbb{C}$, it is proved that the toroidal partial compactification associated to a weak fan in Section 2.3 is Hausdorff. Here in Theorem 6.5.6, we do not tell whether ${}_\Gamma D_\Sigma$ is Hausdorff. The Hausdorffness will be discussed in the later part of this series of papers.

6.6. Examples

Let the notation be as in Section 5.5. In Section 5.5, in each Example a–d, we defined an open closed subset ${}_\Gamma D^\circ$ of ${}_\Gamma D$. In this section, in each Example a–d, we define a subset ${}_\Gamma D^\circ_\Sigma$ of ${}_\Gamma D_\Sigma$, where Σ is as below, as the set of all elements (σ, Z) of ${}_\Gamma D_\Sigma$ such that Z contains an element of ${}_\Gamma D^\circ$. It is an open and closed subset of ${}_\Gamma D_\Sigma$.

6.6.1. Example a

This is a p-adic analogue of Section 2.3, Example a. The compactification $\mathbb{P}^1(K)$ of $K^\times$ appears here.

Let the notation be as in Section 5.5.1.

Let $\Sigma = \{\{0\}, \sigma, -\sigma\}$, where σ (resp., $-\sigma$) denotes the cone of all elements of $\mathfrak{N}$ which send e_1 to zero and e_2 into $\mathbb{Q}_{\geq 0} \cdot e_1$ (resp., $\mathbb{Q}_{\leq 0} \cdot e_1$). The isomorphism $K^\times \cong {}_\Gamma D^\circ$ in Section 5.5.1 extends uniquely to an isomorphism of analytic manifolds over K,

$$\mathbb{P}^1(K) \cong {}_\Gamma D^\circ_\Sigma.$$

In this isomorphism, $0 \in \mathbb{P}^1(K)$ (resp., $\infty \in \mathbb{P}^1(K)$) corresponds to $(\sigma, Z) \in {}_\Gamma D^\circ_\Sigma$ (resp., $(-\sigma, Z) \in {}_\Gamma D^\circ_\Sigma$), where $Z = {}_\Gamma D^\circ$.

Let $\alpha \in {}_\Gamma D^\circ$ be the element corresponding to $1 \in K^\times$. We describe $e_{\sigma,\alpha}$: $E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_\Gamma D_\Sigma$. The monoid $\sigma(\alpha)$ is generated by N, $K^\times \otimes_{\mathbb{Z}} \sigma(\alpha)_{\mathbb{Z}} = K^\times$, where $a \otimes N$ ($a \in K^\times$) is identified with a, this identification is extended to the identification $\mathrm{toric}_{\sigma(\alpha)} = K$, $U_{\sigma,\alpha,c,n} = \{F(z) \mid z \in p^{c+n} O_K\}$, $E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) = K \times U_{\sigma,\alpha,c,n}$, $e_{\sigma,\alpha} : E_{\sigma,\alpha}(U_{\sigma,\alpha,c,n}) \to {}_\Gamma D^\circ_\Sigma$ sends $(a, F(z))$ with $a \neq 0$ to the element of ${}_\Gamma D^\circ$ corresponding to $a \exp(z) \in K^\times$, and sends $(0, F(z))$ to (σ, Z), where $Z = {}_\Gamma D^\circ$.

6.6.2. Example b

This is a p-adic analogue of Section 2.3, Example b. The p-adic unit disc Δ (with the origin) appears here.

Let the notation be as in Section 5.5.2.

Let $\Sigma = \{\{0\}, \sigma\}$, where σ denotes the cone of all elements of $\mathfrak{N}$ which send e_1 to zero and e_2 into $\mathbb{Q}_{\geq 0} \cdot e_1$. The isomorphism $\Delta^* \cong {}_\Gamma D^\circ$ in Section 5.5.2 extends uniquely to an isomorphism of analytic manifolds over K,

$$\Delta \cong {}_\Gamma D^\circ_\Sigma.$$

In this isomorphism, $0 \in \Delta$ corresponds to $(\sigma, \mathrm{class}(\alpha)) \in {}_\Gamma D^\circ_\Sigma$ with $\alpha \in {}_\Gamma D^\circ$, which is independent of the choice of α.

6.6.3. Example c

This is a p-adic analogue of Section 2.3, Example c. A model of the universal Tate elliptic curve with degenerate fiber on $0 \in \Delta$ appears here.

Let the notation be as in Section 5.5.3.

For $n \in \mathbb{Z}$, let $N_n \in \mathfrak{N}$ be the element which sends e_1 to zero, e_2 to e_1, and e_3 to ne_1. Let $\Sigma = \{\{0\}, \sigma_n (n \in \mathbb{Z}), \sigma_{n,n+1} (n \in \mathbb{Z})\}$, where $\sigma_n = \mathbb{Q}_{\geq 0} N_n$, $\sigma_{n,n+1} = \mathbb{Q}_{\geq 0} N_n + \mathbb{Q}_{\geq 0} N_{n+1}$.

Then ${}_\Gamma D^\circ_\Sigma$ is an analytic manifold over K and is the p-adic analogue of $\Gamma_2 \setminus D_\Sigma$ over $\mathbb{C}$ in Section 2.3, Example c. The quotient $\Gamma_0 \setminus {}_\Gamma D^\circ_\Sigma$ of ${}_\Gamma D^\circ_\Sigma$ is still an analytic manifold over K. It is a p-adic analogue of $\Gamma_3 \setminus D_\Sigma$ over $\mathbb{C}$ in Section 2.3, Example c, and is identified with the set of all K-rational points of a proper model over Δ of the universal elliptic curve over Δ^*. The fiber on $q \in \Delta^*$ of $\Gamma_0 \setminus {}_\Gamma D^\circ_\Sigma \to \Delta$ is the Tate elliptic curve $K^\times / q^{\mathbb{Z}}$ (see Section 5.5.3), and the fiber on $0 \in \Delta$ is $\mathbb{P}^1(K)/(0 \sim \infty)$, the quotient of $\mathbb{P}^1(K)$ obtained by identifying zero and ∞.

Let S be the fiber on $0 \in \Delta$ in ${}_\Gamma D^\circ_\Sigma$. Then S is an infinite chain of $\mathbb{P}^1(K)$. More precisely, for each $n \in \mathbb{Z}$, we have an open immersion

$$u_n : K^\times \to S, a \mapsto \big(\sigma_n, \mathrm{class}(\alpha_a)\big),$$

where α_a is the image of $(q, aq^n) \in \Delta^* \times K^\times$ in ${}_\Gamma D^\circ$. (Then the class of the (σ_n, Γ)-nilpotent orbit containing α_a is independent of the choice of $q \in \Delta^*$.) We have $u_m(K^\times) \cap u_n(K^\times) = \emptyset$ if $m \neq n$. This u_n extends uniquely to a closed immersion $\bar{u}_n : \mathbb{P}^1(K) \to S$, and $S = \bigcup_{n \in \mathbb{Z}} \bar{u}_n(\mathbb{P}^1(K))$. If $m, n \in \mathbb{Z}$ and $n \notin \{m-1, m, m+1\}$, then $\bar{u}_m(\mathbb{P}^1(K)) \cap \bar{u}_n(\mathbb{P}^1(K)) = \emptyset$. We have $\bar{u}_n(\mathbb{P}^1(K)) \cap \bar{u}_{n+1}(\mathbb{P}^1(K)) = \{\bar{u}_n(0)\} = \{\bar{u}_{n+1}(\infty)\}$, and this point $\bar{u}_n(0)$ is $(\sigma_{n,n+1}, \mathrm{class}(\alpha))$, where α is any element of ${}_\Gamma D^\circ$. (The $(\sigma_{n,n+1}, \Gamma)$-nilpotent orbit containing α is independent of the choice of such α.) The action of the standard generator Γ_0 sends $\bar{u}_n(a)$ $(a \in \mathbb{P}^1(K))$ to $\bar{u}_{n+1}(a)$. The fiber on $0 \in \Delta$ of $\Gamma_0 \setminus {}_\Gamma D^\circ_\Sigma \to \Delta$ is the quotient of S by this action.

6.6.4. Example d

This is a p-adic analogue of Section 2.3, Example d. Let the notation be as in Section 5.5.4.

Let Σ be the set of the cones $\mathbb{Q}_{\geq 0}N$, where N ranges over all elements of $\mathfrak{N}$.

Then a remarkable fact is that ${}_\Gamma D_{\Sigma^\circ}$ has a slit and is not an analytic space. We describe it. Let $\sigma = \mathbb{Q}_{\geq 0}N$ with $N(e_1) = N(e_2) = 0$, $N(e_3) = e_2$. For $u \in K^\times$, let $\alpha(u) = \bigl(\mathrm{ord}_\xi(u)N, F(0, \log(u), 0), \mu_u\bigr) \in {}_\Gamma\tilde{D}(\sigma) \cap {}_\Gamma D^\circ$, where μ_u is the Γ-equivalence class of $\tilde{\mu}_u$ which is defined by

$$\tilde{\mu}_u(e_{1,L,p}) = t^2 e_{1,p}, \qquad \tilde{\mu}_u(e_{2,L,p}) = te_{2,p}, \qquad \tilde{\mu}_u(e_{3,L,p}) = \log(\tilde{u})e_{2,p} + e_{3,p},$$

$$\tilde{\mu}_u(e_{j,L,\text{non-}p}) = e_{j,\text{non-}p} \quad \text{for } j = 1, 2,$$

$$\tilde{\mu}_u(e_{3,L,\text{non-}p}) = \log_{\text{non-}p}(\tilde{u})e_{2,\text{non-}p} + e_{3,\text{non-}p}.$$

Here $\tilde{u}$ is a lifting of u to $P(K)$ (see Section 5.5.1). Then there is an open immersion of log manifolds over K from a sufficiently small open neighborhood U of $(0,0,0)$ in

$$\bigl\{(z_1, u, z_3) \in K^3 \bigm| \text{if } u = 0, \text{ then } z_3 = 0\bigr\}$$

to ${}_\Gamma D^\circ_\Sigma$ which sends $(z_1, u, z_3) \in U$ with $u \neq 0$ to $e_{\alpha(u)}\bigl(F(z_1, \log(u), z_3)\bigr)$ and sends $(z, 0, 0) \in U$ to $\bigl(\sigma, \mathrm{class}(e_{\alpha(1)}(F(z, 0, 0)))\bigr)$. Here the log structure of U is defined by the divisor $u = 0$ on K^3. The slit appears by Griffiths transversality: $(N, F(z_1, z_2, z_3))$ satisfies Griffiths transversality if and only if $z_3 = 0$.

The p-adic theory of dilog sheaves (a special case of the p-adic theory of polylog sheaves) shows that there is a unique morphism $\mathbb{P}^1(K) \to {}_\Gamma D^\circ_\Sigma$ of log manifolds over K (here $\mathbb{P}^1(K)$ is endowed with the log structure defined by $\{0, 1, \infty\}$) which sends $u \in K^\times \subset \mathbb{P}^1(K)$ with $|u|_p < 1$ to

$$e_{\alpha(u)}\biggl(F\Bigl(-\sum_{n=1}^{\infty}\frac{u^n}{n^2}, \log(u), \log(1-u)\Bigr)\biggr).$$

The image of $0 \in \mathbb{P}^1(K)$ under this morphism is $\bigl(\sigma, \mathrm{class}(\alpha(1))\bigr)$. We plan to discuss this in a later part of this series.

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The following papers, the subjects of which are related to the current paper, were published after this manuscript was submitted.

Ash, D. Mumford, M. Rapoport, and Y. S. Tai, *Smooth Compactifications of Locally Symmetric Varietes*, ed. 2, Cambridge Math. Library, Cambridge Univ. Press, Cambridge, 2010.

J.-F. Dat, S. Orlik, and M. Rapoport, *Period Domains over Finite and* p*-adic Fields*, Cambridge Tracts Math. **183**, Cambridge Univ. Press, Cambridge, 2010.

The author has also found that papers by Minhyong Kim on Diophantine geometry include studies on p-adic period domains.

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