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HYPERRIGID OPERATOR SYSTEMS AND HILBERT MODULES

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Abstract. It is shown that, for an operator algebra A, the operator system $S = A + A^*$ in the C^*-algebra $C^*(S)$, and any representation ρ of $C^*(S)$ on a Hilbert space $\mathcal{H}$, the restriction $\rho_{|_S}$ has a unique extension property if and only if the Hilbert module $\mathcal{H}$ over A is both orthogonally projective and orthogonally injective. As a corollary we deduce that, when S is separable, the hyperrigidity of S is equivalent to the Hilbert modules over A being both orthogonally projective and orthogonally injective.

1. Introduction

The notion of boundary representations introduced by Arveson [1] proved to be a very important idea connecting various directions of research in noncommutative approximation theory. The other areas of operator algebras such as noncommutative convexity, peaking phenomena for operator systems, and Korovkin type properties for completely positive maps benefited from Averson's theory. Arveson's definition of boundary representation is in the context of an operator system and the generated C^*-algebra. An irreducible representation of a C^*-algebra is called a *boundary representation* for an operator system in it if the only completely positive extension of the restriction of the representation to the operator system is the given representation.

Muhly and Solel [9] in 1998 gave an algebraic characterization of boundary representations in terms of Hilbert modules, but used a generalized version of boundary representation by dropping the irreducibility condition. Muhly and

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Solel proved that boundary representations of operator algebras may be characterized as those completely contractive representations that determine modules that are simultaneously orthogonally projective and orthogonally injective.

By using the same generalized notion of boundary representation, Dritschel and McCullough [5] in 2005 showed the existence of C^*-envelopes for operator systems and operator algebras. Another important work to be mentioned here is that of Hamana [6] in 1979, where the existence of C^*-envelopes using methods other than that of boundary representations is established. It is to be noted that C^*-envelopes may exist without the existence of boundary representations.

In this article, for a (unital) operator algebra A and the operator system $S = A + A^*$, we show that the *unique extension property* of the restriction to S of a representation of $C^*(S)$ is equivalent to the Hilbert modules over A corresponding to the representation being simultaneously orthogonally projective and orthogonally injective. This result leads to an algebraic characterization of *hyperrigidity* of the operator system $A + A^*$ in terms of the orthogonality properties of Hilbert modules over A.

2. Preliminaries

2.1. **Operator systems and boundary representations.** Here we establish some definitions, conventions, and notation.

Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ be the bounded linear operators on $\mathcal{H}$. A (concrete) *operator system* S is a unital self-adjoint subspace of $B(\mathcal{H})$. We will view S as a subspace of the C^*-algebra it generates, namely $C^*(S) \subseteq B(\mathcal{H})$. There is a theory of abstract operator systems given by an axiomatic definition due to Choi and Effros [4] as opposed to the so-called *concrete operator system* defined above. This distinction is irrelevant due to the representation theorem for the abstract operator system established in [4].

A (concrete) *operator algebra* A is a unital subalgebra of a $B(\mathcal{H})$. Similar to the case of operator systems, there is a notion of abstract operator algebras and a corresponding representation theorem due to Blecher, Ruan, and Sinclair [3] showing that those can be represented completely isometrically as subalgebras of $B(\mathcal{H})$, thus making the distinction irrelevant here also.

A linear map φ from any subspace V of a C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ determines a family of maps $\varphi_n : M_n(V) \to M_n(\mathcal{B})$ given by $\varphi_n([a_{ij}]) = [\varphi(a_{ij})]$. We say that φ is *completely bounded* (CB) if $\|\varphi\|_{\mathrm{CB}} = \sup_{n\geq 1} \|\varphi_n\| < \infty$. We say that φ is *completely contractive* (CC) if $\|\varphi\|_{\mathrm{CB}} \leq 1$ and that φ is *completely isometric* if φ_n is isometric for all $n \geq 1$. If the domain of φ is an operator system S, then we say that φ is *completely positive* (CP) if φ_n is positive for all $n \geq 1$, and that φ is *unital completely positive* (UCP) if in addition $\varphi(1) = 1$. Since $\|\varphi\|_{\mathrm{CB}} = \|\varphi(1)\|$ for CP maps, we see that UCP maps are always completely contractive.

The C^*-*envelope* of an operator algebra A, denoted by $C_e^*(A)$, is the essentially unique smallest C^*-algebra among those C^*-algebras $\mathcal{C}$ for which there is a completely isometric homomorphism $\phi : A \to \mathcal{C}$.

Definition 2.1. A UCP map $\pi : S \to B(\mathcal{H})$ is said to have *unique extension property* (UEP) if

(i) π has a unique completely positive extension $\widetilde{\pi} : C^*(S) \to B(\mathcal{H})$, and
(ii) $\widetilde{\pi}$ is a representation of $C^*(S)$ on $\mathcal{H}$.

If the extension $\widetilde{\pi}$ of such a map π to $C^*(S)$ is an irreducible representation, then the extension is called a *boundary representation*.

Two theorems due to Arveson [1], which we quote below, concerning extensions of contractive linear maps on unital subspaces of C^*-algebras are crucial to the proof of our main result.

The following theorem from [1, Proposition 1.2.8] shows that every unital completely contractive linear map from a unital subspace V of a C^*-algebra can be extended uniquely in a completely positive way to the operator system $V + V^*$.

Theorem 2.2. *Let V be a linear subspace of a C^*-algebra $\mathcal{A}$ such that identity $e \in V$, and let S be the norm-closure of $V+V^*$. Then every contractive linear map φ of V into $B(\mathcal{H})$, for which $\varphi(e) = I$, has a unique bounded self-adjoint linear extension φ_1 to S. φ_1 is positive, and it is completely positive if φ is completely contractive.*

The following theorem from [1, Theorem. 1.2.9] shows that every unital completely contractive linear map from a unital subspace V of a C^*-algebra can be extended to a completely positive map on the C^*-algebra.

Theorem 2.3. *Let V be a linear subspace of a C^*-algebra $\mathcal{A}$ such that identity $e \in V$, and let $\mathcal{H}$ be a Hilbert space. Let φ be a completely contractive linear map V into $B(\mathcal{H})$ such that $\varphi(e) = I$. Then φ has a completely positive extension to $\mathcal{A}$.*

In connection with the fundamental work related to the noncommutative approximation theory, Arveson [2] introduced the notion of noncommutative Korovkin sets which he called *hyperrigid sets*.

Definition 2.4 ([2, Definition 1.1]). A finite or countably infinite set G of generators of a C^*-algebra $\mathcal{A}$ is said to be *hyperrigid* if, for every faithful representation $\mathcal{A} \subseteq B(\mathcal{H})$ of $\mathcal{A}$ on a Hilbert space and every sequence of unit-preserving completely positive (UCP) maps $\phi_n : B(\mathcal{H}) \to B(\mathcal{H})$, $n = 1, 2, \ldots,$

$$\lim_{n\to\infty} \big\|\phi_n(g) - g\big\| = 0, \quad \forall g \in G \quad \Rightarrow \quad \lim_{n\to\infty} \big\|\phi_n(a) - a\big\| = 0, \quad \forall a \in \mathcal{A}.$$

Arveson's still not completely resolved "hyperrigidity conjecture" relates boundary representations of C^*-algebras for operator systems with hyperrigidity of operator systems. It states that, if every irreducible representation of a C^*-algebra $\mathcal{A}$ is a boundary representation for a separable operator system $S \subseteq \mathcal{A}$, then S is hyperrigid. Arveson [2] proved the conjecture for C^*-algebras having a countable spectrum, while Kleski [7] established the conjecture for all type-I C^*-algebras.

2.2. **Hilbert modules over operator algebras.** Here we recall basic definitions of Hilbert modules and related concepts relevant to our discussion.

As far as representations of algebras are concerned, we are interested in contractive representations here. One reason for focusing on contractive representations is that they coincide with C^*-representations when the operator algebra is a C^*-algebra. Representations of algebras on Hilbert spaces give rise to Hilbert modules over algebras and vice versa. We will assume that all given representations are nondegenerate.

Let $\rho : A \to B(\mathcal{H})$ be a representation for an operator algebra A on a Hilbert space $\mathcal{H}$. A (left) *Hilbert module* over A is simply the Hilbert space $\mathcal{H}$ viewed as an algebraic (left) module over A via the module action $a\xi := \rho(a)\xi$.

The advantage of using the language of Hilbert modules over operator algebras and their representations is that we can pass from one to the other when it is convenient. If $\rho : A \to B(\mathcal{H})$ is a representation, then the associated module will be written as ${}_A\mathcal{H}$ or $\mathcal{H}_\rho$. If $\mathcal{H}$ is a Hilbert module, then the representation associated will be written as $\rho_{\mathcal{H}}$.

A *contractive Hilbert module* is one where the associated representation is contractive. A Hilbert module is called *completely bounded* (*completely contractive*) if the associated representation is completely bounded (completely contractive) as a linear operator-valued map on the operator algebra. *Module maps* are bounded linear intertwining operators for the representations, and *Hilbert module isomorphisms* are unitary module maps. We write $\mathrm{Hom}(\mathcal{H}, \mathcal{K})$ for the module maps from $\mathcal{H}$ to $\mathcal{K}$, and if $\mathcal{H} = \mathcal{K}$, then we write $\mathrm{Hom}(\mathcal{H})$.

A sequence of Hilbert modules over an operator algebra A,

$$\ldots \mathcal{H}_{i-1} \xrightarrow{\Phi_{i-1}} \mathcal{H}_i \xrightarrow{\Phi_i} \mathcal{H}_{i+1} \to \ldots,$$

where the Φ_i's are module maps, is called *exact* at $\mathcal{H}_i$ if the kernel of Φ_i coincides with the range of Φ_{i-1}. It is called *isometric* if each of the Φ_i's is a partial isometry as a Hilbert space map.

A sequence of Hilbert modules over an operator algebra A,

$$0 \to \mathcal{K} \xrightarrow{\Psi} \mathcal{M} \xrightarrow{\Phi} \mathcal{N} \to 0,$$

is said to be a *short exact isometric sequence* if the map Ψ has a zero kernel, the range of Ψ is the kernel of Φ, the range of Φ is all of $\mathcal{N}$, Ψ is an isometry, and Φ is a co-isometry (Φ^* is an isometry).

To say that the short exact sequence is isometric is to say that, as a Hilbert space, $\mathcal{M}$ is the orthogonal direct sum $\mathcal{K} \oplus \mathcal{N}$, and that in matrix form we may write $\rho_{\mathcal{M}}$ as

$$\begin{bmatrix} \rho_{\mathcal{K}} & D \\ 0 & \rho_{\mathcal{N}} \end{bmatrix},$$

where the map D carries A into the bounded operators mapping $\mathcal{N}$ into $\mathcal{K}$ and satisfies the equation $D(ab) = D(a)\rho_{\mathcal{N}}(b) + \rho_{\mathcal{K}}(a)D(b)$; that is, D is a derivation.

In pure algebra, a short exact sequence is said to *split* if there is a module map $\Phi' : \mathcal{N} \to \mathcal{M}$ with the property that $\Phi \circ \Phi'$ is the identity on $\mathcal{N}$. In this event,

$\mathcal{M}$ is isomorphic to the algebraic direct sum $\mathcal{K} \oplus \mathcal{N}$. In our theory, being at the Hilbert space level, we want direct sums to be orthogonal direct sums.

A short exact isometric sequence is *orthogonally split* if there is a contractive module map Φ' such that $\Phi \circ \Phi'$ is the identity on $\mathcal{N}$. It is easy to see that this happens if and only if Φ^* is a module map, which is equivalent to the initial space of Φ being a submodule of $\mathcal{M}$, in which case D is the zero map.

Definition 2.5. A Hilbert module ${}_A\mathcal{P}$ over an operator algebra A is called *orthogonally projective* (or *orthoprojective*) in case every short exact isometric sequence

$$0 \to_A \mathcal{K} \to_A \mathcal{M} \to_A \mathcal{P} \to 0$$

is orthogonally split.

A Hilbert module ${}_A\mathcal{I}$ is called *orthogonally injective* (or *orthoinjective*) in case every short exact isometric sequence

$$0 \to_A \mathcal{I} \to_A \mathcal{M} \to_A \mathcal{N} \to 0$$

is orthogonally split.

Just as isometries and co-isometries are adjoints of one another, the same, essentially, is true of orthogonally projective Hilbert modules and orthogonally injective Hilbert modules. If $\mathcal{H}$ is a Hilbert module over operator algebra A with associated representation $\rho_{\mathcal{H}}$, then defining σ by the formula $\sigma(a) = (\rho_{\mathcal{H}}(a^*))^*$, $a \in A^*$, where the adjoint on elements of A is calculated in $C_e^*(A)$, yields a representation of A^* and a Hilbert module over A^*. It is easy to see that $\mathcal{H}$ is orthogonally projective if and only if the Hilbert space associated with σ is orthogonally injective.

The algebraic characterization of boundary representations first made by Muhly and Solel [9], which appeared in 1998, characterizes boundary representations of a C^*-algebra for an operator algebra in terms of orthogonally projective and orthogonally injective modules over the operator algebra. It is as follows.

Theorem 2.6. *Let $\mathcal{H}$ be a contractive Hilbert module over an operator algebra A, and let ρ be the associated representation. Then ρ is the restriction to A of a boundary representation of $C_e^*(A)$ for A if and only if $\mathcal{H}$ is both orthogonally projective and orthogonally injective.*

3. Main results

In this section, we establish a characterization of a unique extension property for representations in the context of a C^*-algebra generated by an operator system in terms of the orthogonal projectivity and orthogonal injectivity of the Hilbert modules over the operator algebra underlying the operator system. In the proof of the theorem below, we crucially make use of two extension theorems by Arveson in the context of operator systems and generated C^*-algebras given in the previous section. The theorem leads to a corollary characterizing hyperrigidity of operator systems in terms of orthogonality properties of Hilbert modules.

Theorem 3.1. *Let A be an operator algebra, and consider the operator system $S = A + A^*$. Let $C^*(S)$ be the C^*-algebra generated by S. For any representation*

ρ of $C^*(S)$ on a Hilbert space $\mathcal{H}$, the restriction $\rho_{|S}$ has UEP if and only if $\mathcal{H}$ as a Hilbert module over A is both orthogonally projective and orthogonally injective.

Proof. Assume that the Hilbert module $\mathcal{H}$ over A is both orthogonally projective and orthogonally injective. To show that $\rho_{|S}$ has UEP to $C^*(S)$, let σ be a unital completely positive extension of $\rho_{|S}$ to all of $C^*(S)$, and $\sigma(\cdot) = \Phi^*\pi(\cdot)\Phi$ be the minimal Stinespring dilation of σ. Thus π is a representation of $C^*(S)$ on a Hilbert space $\mathcal{K}$, and $\Phi : \mathcal{H} \to \mathcal{K}$ is a Hilbert space isometry such that $\sigma(a) = \Phi^*\pi(a)\Phi$ for all $a \in C^*(S)$ with the minimality assumption implying that the smallest reducing subspace for $\pi(C^*(S))$ containing $\Phi\mathcal{H}$ is all of $\mathcal{K}$. In particular, for $a \in S$, $\rho(a) = \sigma(a) = \Phi^*\pi(a)\Phi$.

We will establish the UEP of $\rho_{|S}$ by showing that σ is unitarily equivalent to the restriction of π to the range of Φ where the equivalence implementing unitary map is Φ. To prove that Φ is unitary, it is enough to prove that $\Phi\mathcal{H} = \mathcal{K}$ for which it is sufficient to show that $\Phi\mathcal{H}$ is invariant under $\pi(S)$. Then the self-adjointness of S will imply that Φ is reducing for $\pi(S)$, and hence for $\pi(C^*(S))$. Now, the minimality assumption above will show that $\Phi\mathcal{H} = \mathcal{K}$.

In any case, ρ being a representation of A, the range of Φ is a semi-invariant subspace for $\pi(A)$. We will use Sarason's representation [10, Lemma 0] for the semi-invariant subspace $\Phi\mathcal{H}$ to proceed.

Let P be the projection of $\mathcal{K}$ onto $\Phi\mathcal{H}$. We have $P = \Phi\Phi^*$. Let $\mathcal{K}_1$ be the smallest invariant subspace for $\pi(A)$ containing $\Phi\mathcal{H}$. Then $\mathcal{K}_1 = \overline{\pi(A)\Phi\mathcal{H}}$ as we assume π to be nondegenerate. Let P_1 be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_1$. We write π_1 for the representation of A obtained by restricting $\pi(A)$ to $\mathcal{K}_1$ so that $\pi_1(a) = \pi(a)_{|\mathcal{K}_1}$ for all a in A. In other words, we may think of $\pi_1(a) = \pi(a)P_1$ for $a \in A$. Also, we set $\Phi_1 = P_1\Phi$; that is, Φ_1 is in fact Φ viewed as a map from $\mathcal{H}$ to $\mathcal{K}_1$ and gives $\Phi_1^* = \Phi^*P_1$. Further, let $\mathcal{K}_2 = \mathcal{K}_1 \ominus (\Phi\mathcal{H})$, and let P_2 be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_2$. Sarason's theory of semi-invariant subspaces gives that $\mathcal{K}_2$ is invariant for $\pi_1(A)$ (and for $\pi(A)$). By construction, $P_1 = P + P_2$.

We will show that $\Phi_1^* : \mathcal{K}_1 \to \mathcal{H}$ is a module map; that is, $\rho(a)\Phi_1^* = \Phi_1^*\pi_1(a)$ for all $a \in A$. Indeed, for $a \in A$, $\rho(a)\Phi_1^* = \rho(a)\Phi^*P_1 = \Phi^*\pi(a)\Phi\Phi^*P_1 = \Phi^*\pi(a)PP_1$. Now, $\mathcal{K}_2$ is invariant for every $\pi(a)$, and so $\pi(a)P_2 = P_2\pi(a)P_2$ for all $a \in A$. Furthermore, $\Phi^*P_2 = 0$ since the initial projection of Φ^*, namely P, is orthogonal to P_2. Thus we find that, for $a \in A$, $\Phi^*\pi(a)PP_1 = \Phi^*\pi(a)(P_1 - P_2)P_1 = \Phi^*\pi(a)P_1 = \Phi^*P_1\pi(a)P_1$ as $\mathcal{K}_2$ is invariant for $\pi(A)$ and $\Phi^*P_2 = 0$. But $(\Phi^*P_1)(\pi(a)P_1) = \Phi_1^*\pi_1(a)$ for all $a \in A$. Thus $\rho(a)\Phi_1^* = \Phi_1^*\pi_1(a)$ for all $a \in A$, and hence Φ_1^* is a module map.

Since $\mathcal{H}$ is orthogonally projective and Φ_1^* is co-isometric, we get that Φ_1 is a module map too; that is, $\Phi_1\rho(a) = \pi_1(a)\Phi_1$ for all $a \in A$, which can be rewritten as $P_1\Phi\rho(a) = \pi(a)P_1\Phi$. Then $\Phi\rho(a) = \pi(a)\Phi$ for all $a \in A$ by dropping P_1 as the range of Φ is contained in $\mathcal{K}_1 = \text{range}(P_1)$.

For all $a \in A$,

$$\begin{aligned}\pi(a)P &= \pi(a)PP_1 \\ &= \pi(a)\Phi\Phi^*P_1 \\ &= \Phi\rho(a)\Phi^*P_1\end{aligned}$$

$$
\begin{aligned}
&= \Phi\rho(a)\Phi_1^* \\
&= \Phi\Phi^*\pi(a)PP_1 \\
&= P\pi(a)P.
\end{aligned}
$$

This shows that $\Phi\mathcal{H}$ is invariant for $\pi(A)$.

Since ρ is a C^*-representation, using the fact that $\mathcal{H}$ is an orthogonally injective module for A if and only if $\mathcal{H}$ is orthogonally projective for A^*, and arguing as above, we can show that $\Phi\mathcal{H}$ is invariant for $\pi(A^*)$.

As $\Phi\mathcal{H}$ is invariant for $\pi(A)$ and $\pi(A^*)$, we have $\Phi\mathcal{H}$ is invariant for $\pi(S)$. Thus $\rho_{|S}$ has UEP.

To prove the converse, suppose that $\rho_{|S}$ has UEP. We will show that $\mathcal{H}$ is both orthogonally projective and orthogonally injective over A. Let

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{H} \to 0$$

be a short exact isometric sequence determined by Hilbert modules $\mathcal{N}$ and $\mathcal{M}$, where $\Phi : \mathcal{M} \to \mathcal{H}$ is a co-isometric module map.

Since $\mathcal{M}$ is a completely contractive module over A, let $\rho_{\mathcal{M}}$ be the completely contractive representation of A corresponding to $\mathcal{M}$. By Theorem 2.3, $\rho_{\mathcal{M}}$ has a completely positive linear extension to $C^*(S)$. Let η be the completely positive linear extension of $\rho_{\mathcal{M}}$ of A to $C^*(S)$; that is, $\rho_{\mathcal{M}} = \eta_{|A}$.

By Stinespring dilation, there is a representation π of $C^*(S)$ on a Hilbert space $\mathcal{K}$ and a co-isometry $\Psi : \mathcal{K} \to \mathcal{M}$ such that $\eta(a) = \Psi\pi(a)\Psi^*$ $\forall a \in C^*(S)$. In particular, $\rho_{\mathcal{M}}(a) = \eta(a) = \Psi\pi(a)\Psi^*$ $\forall a \in A$. By Theorem 2.2, there exists a unique completely positive extension $\widetilde{\rho}_{\mathcal{M}}$ of $\rho_{\mathcal{M}}$ to S so that

$$\widetilde{\rho}_{\mathcal{M}}(s) = \Psi\pi(s)\Psi^* \quad \forall s \in S.$$

But then, since $\Phi\rho_{\mathcal{M}}(a) = \rho(a)\Phi$ $\forall a \in A$, we find that $\Phi\rho_{\mathcal{M}}(a)\Phi^* = \rho(a)$ $\forall a \in A$, and hence $\Phi\widetilde{\rho}_{\mathcal{M}}(s)\Phi^* = \rho(s)$ $\forall s \in S$.

Substituting for $\widetilde{\rho}_M$, we get

$$\Phi\Psi\pi(s)(\Phi\Psi)^* = \rho(s) \quad \forall s \in S,$$

where $\Phi\Psi$ is a co-isometry. On $C^*(S)$, $\Phi\Psi\pi(\cdot)(\Phi\Psi)^*$ is a completely positive map that agrees with ρ on S. Since $\rho_{|S}$ has UEP, we conclude that

$$\Phi\Psi\pi(a)(\Phi\Psi)^* = \rho(a) \quad \forall a \in C^*(S).$$

Thus the initial space of $\Phi\Psi$ reduces π, and $\Phi\Psi$ implements an equivalence between ρ and π restricted to this initial space. Let P and Q be the initial projections of Φ and $\Phi\Psi$, respectively.

Then, for $s \in S$, we have

$$
\begin{aligned}
\widetilde{\rho}_{\mathcal{M}}(s)P &= \big(\Psi\pi(s)\Psi^*\big)(\Phi^*\Phi) \\
&= \big(\Psi\pi(s)\Psi^*\big)(\Phi^*\Phi)(\Psi\Psi^*) \\
&= \Psi\pi(s)(\Psi^*\Phi^*\Phi\Psi)\Psi^* \\
&= \Psi\pi(s)Q\Psi^* \\
&= \Psi Q\pi(s)\Psi^*
\end{aligned}
$$

$$
\begin{aligned}
&= \Psi(\Psi^*\Phi^*\Phi\Psi)\pi(s)\Psi^* \\
&= (\Psi\Psi^*)(\Phi^*\Phi)\big(\Psi\pi(s)\Psi^*\big) \\
&= P\widetilde{\rho}_{\mathcal{M}}(s),
\end{aligned}
$$

crucially using the fact that Ψ is a co-isometry with range $\mathcal{M}$.

From above, in particular for $a \in A$,

$$\rho_{\mathcal{M}}(a)P = P\rho_{\mathcal{M}}(a).$$

This will imply by [8, Proposition 2.3] that $\mathcal{H}$ is orthogonally projective over A.

Also, for $a \in A^*$,

$$\rho_{\mathcal{M}}(a)P = P\rho_{\mathcal{M}}(a),$$

which shows that $\mathcal{H}$ is orthogonally projective over A^*; from this we conclude that $\mathcal{H}$ is orthogonally injective over A. □

The following corollary of the above theorem characterizes hyperrigidity of separable operator systems of the form $A + A^*$, where A is an operator algebra in terms of orthogonality properties of Hilbert modules over A.

Corollary 3.2. *For a separable operator algebra A, the operator system $S = A + A^*$, and the C^*-algebra $C^*(S)$, the following are equivalent:*

(i) *S is hyperrigid.*

(ii) *For every nondegenerate representation $\pi : C^*(S) \to B(\mathcal{H}_\pi)$ on a separable Hilbert space, $\pi_{|S}$ has a unique extension property.*

(iii) *The Hilbert module $\mathcal{H}_\pi$ over A is both orthogonally projective and orthogonally injective.*

Proof. The equivalence of (i) and (ii) follows from Arveson [2], and the equivalence of (ii) and (iii) follows from the above theorem. □

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References

1. W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224. Zbl 0194.15701. MR0253059. 133, 135
2. W. B. Arveson, *The noncommutative Choquet boundary II: Hyperrigidity*, Israel J. Math. **184** (2011), 349–385. Zbl 1266.46045. MR2823981. DOI 10.1007/s11856-011-0071-z. 135, 140
3. D. Blecher, Z. J. Ruan, and A. Sinclair, *A characterization of operator algebras*, J. Funct. Anal. **89** (1990), 188–201. Zbl 0714.46043. MR1040962. DOI 10.1016/0022-1236(90)90010-I. 134
4. M. D. Choi and E. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209. Zbl 0341.46049. MR0430809. 134

5. M. Dritschel and S. McCullough, *Boundary representations for families of representations of operator algebras and spaces*, J. Operator Theory **53** (2005), no. 1, 159–167. Zbl 1119.47311. MR2132691. 134
6. M. Hamana, *Injective envelopes of operator systems*, Publ. Res. Inst. Math. Sci. **15** (1979), 773–785. Zbl 0436.46046. MR0566081. DOI 10.2977/prims/1195187876. 134
7. C. Kleski, *Korovkin-type properties for completely positive maps*, Illinois J. Math. **58** (2014), no. 4, 1107–1116. Zbl 1331.41029. MR3421602. 135
8. P. Muhly and B. Solel, *Hilbert Modules Over Operator Algebras*, Mem. Amer. Math. Soc. **117**, Amer. Math. Soc., Providence, RI, 1995. Zbl 0857.46031. MR1271693. DOI 10.1090/memo/0559. 140
9. P. Muhly and B. Solel, "An algebraic characterization of boundary representations" in *Nonselfadjoint Operator Algebras, Operator Theory, and Related Topics*, Oper. Theory Adv. Appl. **104**, Birkhauser, Basel, 1998, 189–196. Zbl 0899.46038. MR1639657. 133, 137
10. D. Sarason, *On spectral sets having connected complement*, Acta Sci. Math. (Szeged) **26** (1965), 289–299. Zbl 0145.39302. MR0188797. 138

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