Ann. Funct. Anal. 8 (2017), no. 1, 106–123
http://dx.doi.org/10.1215/20088752-3764507
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

BASKAKOV–SZÁSZ-TYPE OPERATORS BASED ON INVERSE PÓLYA–EGGENBERGER DISTRIBUTION

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Communicated by S. Barza

Abstract. The present article deals with the modified forms of the Baskakov and Szász basis functions. We introduce a Durrmeyer-type operator having the basis functions in summation and integration due to Stancu (1970) and Păltănea (2008). We obtain some approximation results, which include the Voronovskaja-type asymptotic formula, local approximation, error estimation in terms of the modulus of continuity, and weighted approximation. Also, the rate of convergence for functions with derivatives of bounded variation is established. Furthermore, the convergence of these operators to certain functions is shown by illustrative graphics using MAPLE algorithms.

1. Introduction

The inverse Pólya–Eggenberger distribution (see [11, pp. 229–232]) is defined by

$$
\begin{aligned}
&Pr(N=n+k)\\
&\quad=\binom{n+k-1}{k}\frac{u(u+s)\cdots(u+(n-1)s)v(v+s)\cdots(v+(k-1)s)}{(u+v)(u+v+s)\cdots(u+v+(n+k-1)s)},\\
&\qquad k=0,1,\dots.
\end{aligned}
$$

Received Apr. 27, 2016; Accepted Jul. 1, 2016.
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2010 *Mathematics Subject Classification.* Primary 26A15; Secondary 41A25, 41A28.
Keywords. Stancu operators, Baskakov operators, Szász operators, Pólya–Eggenberger distribution, modulus of continuity.

Stancu [19] proposed Baskakov operators based on inverse Pólya–Eggenberger distribution depending on a nonnegative parameter $\alpha = \alpha(n) \to 0$ as $n \to \infty$,

$$V_n^{[\alpha]}(f;x) = \sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x) f\Big(\frac{k}{n}\Big), \quad x \in [0,\infty), \tag{1.1}$$

where $v_{n,k}^{[\alpha]}(x) = \binom{n+k-1}{k} \frac{1^{[n,-\alpha]} x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}$ and $t^{[n,h]} = t(t-h)\cdots(t-(n-1)h)$.

In 1989, Razi [16] introduced a Bernstein–Kantorovich operator based on Pólya–Eggenberger distribution and studied the rate of convergence and degree of approximation for these operators. Very recently, Deo, Dhamija, and Miclăuş [4] considered a Stancu–Kantorovich operator based on inverse Pólya–Eggenberger distribution of the operators (1.1) and established some direct results. Păltănea [15] introduced a generalization of the well-known Phillips operators by considering the generalized basis functions under integration depending on a certain parameter $\rho > 0$. Gupta and Rassias [8] proposed a Durrmeyer modification of certain Szász-type operators and obtained some approximation properties, for example, asymptotic formula, weighted approximation, and error estimation in terms of modulus of continuity. Very recently, Goyal, Gupta, and Agrawal [6] defined a one-parameter family of hybrid operators and studied local, weighted, and simultaneous approximation properties for these operators. The rate of convergence for functions with derivatives of bounded variation is another important topic of research. In the literature, many authors have discussed the approximation behavior of different summation-integral-type operators (see [2], [7], [12], [18], [20], etc.).

Inspired by the above work, we consider a new sequence of summation-integral-type operators as follows.

For $\gamma > 0$ and $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \le N_f e^{\gamma t}, \text{for some } N_f > 0\}$, we define

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) = \sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x) \int_0^{\infty} s_{n,k}^{\rho}(t) f(t)\,dt + v_{n,0}^{[\alpha]}(x) f(0), \tag{1.2}$$

where $s_{n,k}^{\rho}(t) = n\rho e^{-n\rho t} \frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $v_{n,k}^{[\alpha]}(x)$ is defined as above. We note that the operators (1.2) preserve only the constant functions.

Special cases:

(1) For $\alpha = 0$ and $\rho = 1$, these operators include Baskakov–Szász operators (see, e.g., [1], [9]).
(2) For $\alpha = 0$ and $\rho \to \infty$, these operators reduce to well-known Baskakov operators [3].
(3) For $\alpha > 0$ and $\rho \to \infty$, these operators include Stancu operators [19].

Our aim is to study some approximation properties of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}$. We begin with an estimate of the rate of convergence in terms of the moduli of continuity and a Lipschitz-type space, and then we find the weighted approximation properties for these operators. Furthermore, we obtain the rate of convergence for unbounded functions with derivatives of bounded variation by these operators.

2. Direct results

Let $e_l(u)=u^l, l\in\mathbb{N}\cup\{0\}$. Applying the definition of gamma function, we get

$$\begin{aligned}\int_0^\infty s_{n,k}^{\rho}(t)t^l\,dt &= \int_0^\infty n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}t^l\,dt\\ &=\frac{\Gamma(k\rho+l)}{\Gamma(k\rho)(n\rho)^l}=\frac{(k\rho+l-1)\cdots(k\rho)}{(n\rho)^l}. \end{aligned}\tag{2.1}$$

For $f\in C_\gamma[0,\infty)$, we consider

$$L_n(f;t)=\sum_{k=0}^{\infty}b_{n,k}(t)f\Big(\frac{k}{n}\Big),$$

$$\text{with } b_{n,k}(t)=\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}},\tag{2.2}$$

$$K_{n,\rho}(f;t)=\sum_{k=1}^{\infty}b_{n,k}(t)\int_0^\infty s_{n,k}^{\rho}(u)f(u)\,du+b_{n,0}(t)f(0).\tag{2.3}$$

Then, we obtain the following representation of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}$.

Lemma 2.1. *For $\alpha>0$ and $x\in\mathbb{R}^+$, we have*

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)=\frac{1}{B(\frac{x}{\alpha},\frac{1}{\alpha})}\int_0^\infty\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}}K_{n,\rho}(f;t)\,dt,\tag{2.4}$$

where $B(r,s), r,s>0$ is the beta function.

Proof. The form (2.4) of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}$ can be obtained using the following relation:

$$B\Big(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\Big)=v_{n,k}^{[\alpha]}(x)\binom{n+k-1}{k}^{-1}B\Big(\frac{x}{\alpha},\frac{1}{\alpha}\Big).$$

□

Using (2.1)–(2.3), Lemma 2.1, and [17, Lemma 1] in the following result, we calculate the values of the first five moments of the operators given by (1.2).

Lemma 2.2. *For $\rho>0$, we have*

(i) $\mathcal{R}_{n,\rho}^{[\alpha]}(e_0;x)=1$;
(ii) $\mathcal{R}_{n,\rho}^{[\alpha]}(e_1;x)=\frac{x}{1-\alpha}$;
(iii) $\mathcal{R}_{n,\rho}^{[\alpha]}(e_2;x)=\frac{(n+1)x^2}{(1-\alpha)(1-2\alpha)n}+[\frac{\alpha(n+1)}{(1-\alpha)(1-2\alpha)n}+\frac{\rho+1}{n\rho(1-\alpha)}]x$.

Remark 2.1. By simple computations, from Lemma 2.2 we get

$$\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x)=\frac{\alpha x}{1-\alpha},$$

$$\mathcal{R}_{n,\rho}^{[\alpha]}((t-x)^2;x)=\frac{(1+n\alpha+2n\alpha^2)x^2}{n(1-\alpha)(1-2\alpha)}+\frac{(1-2\alpha+\rho+(n-1)\alpha\rho)x}{n\rho(1-\alpha)(1-2\alpha)}.$$

Lemma 2.3. *If $\alpha = \alpha(n) \to 0$, as $n \to \infty$ and $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$, then*

$$\lim_{n\to\infty} n\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x) = lx,$$

$$\lim_{n\to\infty} n\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big) = (1+l)x^2 + \frac{1+\rho+l\rho}{\rho}x, \tag{2.5}$$

$$\lim_{n\to\infty} n^2\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^4;x\big) = 3(l+1)^2x^4 + \frac{6(l+1)(\rho l+\rho+1)}{\rho}x^3 + \frac{3(\rho l+\rho+1)^2}{\rho^2}x^2. \tag{2.6}$$

Proof. This result is obtained by straightforward computation, but the details are omitted. □

3. Main results

Theorem 3.1. *Let $f \in C_\gamma[0,\infty)$, and let $\alpha = \alpha(n) \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \mathcal{R}_{n,\rho}^{[\alpha]}(f;x) = f(x)$, uniformly in each compact subset of $[0,\infty)$.*

Proof. Using Lemma 2.2 and the Bohman–Korovkin theorem, this result is proved. □

Example 3.1. The convergence of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)$ is illustrated in Figure 1, where $f(x) = -3xe^{-2x}$, $\alpha \in \{3, 1.8, 0.4\}$, $n = 20$, and $\rho = 5$. We can see that when the values of α are decreasing, the graph of operators $\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)$ are going to the graph of the function f.

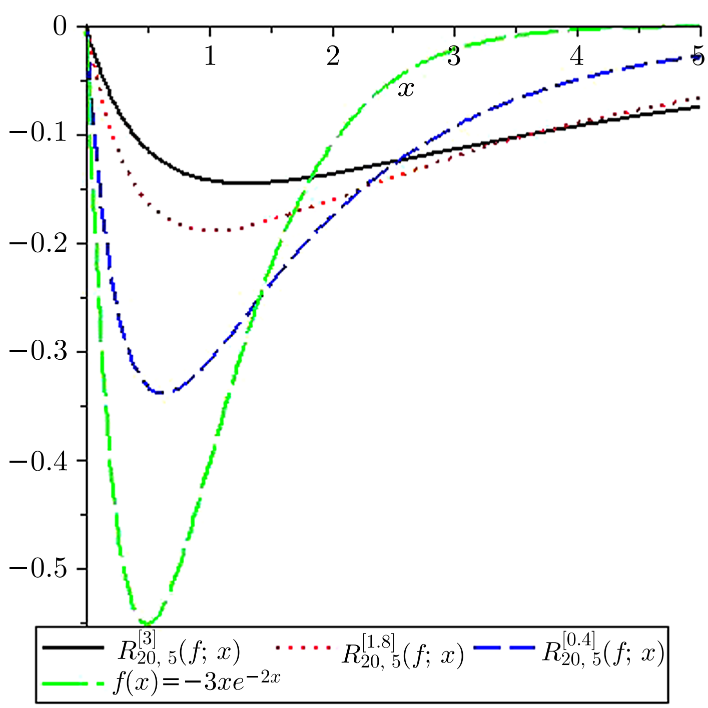

FIGURE 1. The convergence of $\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)$ to $f(x)$.

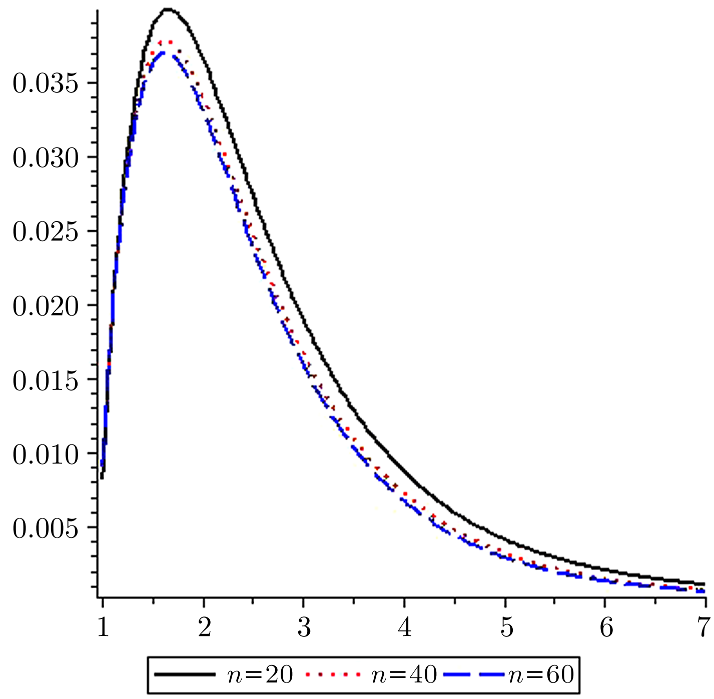

FIGURE 2. Graphics of the difference $x \to \mathcal{R}_{n,5}^{[0.4]}(f;x) - f(x)$, when $n = 20, 40, 60$.

Example 3.2. For $x \in [1,7]$, $\alpha = 0.4$, $\rho = 5$, the convergence of the difference of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}(f;\cdot)$ to the function f, where $f(x) = 2xe^{-3x}$, for different values of n is illustrated in Figure 2.

3.1. **Voronovskaja-type theorem.** In this section, we establish a Voronvoskaja-type result for the $\mathcal{R}_{n,\rho}^{[\alpha]}$ operators.

Theorem 3.2. *Let $f \in C_\gamma[0,\infty)$, and let $\alpha = \alpha(n) \to 0$ as $n \to \infty$. If f'' exists at a point $x \in [0,\infty)$ and $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$, then we have*

$$\lim_{n\to\infty} n\big[\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)\big] = lxf'(x) + \frac{1}{2}\Big[(1+l)x^2 + \frac{1+\rho+l\rho}{\rho}x\Big]f''(x).$$

Proof. Applying Taylor's expansion, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2, \tag{3.1}$$

where $\lim_{t\to x}\varepsilon(t,x) = 0$. By using the linearity of the operator $\mathcal{R}_{n,\rho}^{[\alpha]}$, we get

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x) = \mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x);x\big)f'(x) + \frac{1}{2}\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)f''(x) + \mathcal{R}_{n,\rho}^{[\alpha]}\big(\varepsilon(t,x)(t-x)^2;x\big).$$

Applying the Cauchy–Schwarz inequality, we obtain

$$n\mathcal{R}_{n,\rho}^{[\alpha]}\big(\varepsilon(t,x)(t-x)^2;x\big) \le \sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big(\varepsilon^2(t,x);x\big)}\sqrt{n^2\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^4;x\big)}.$$

In view of Theorem 3.1, $\lim_{n\to\infty} \mathcal{R}^{[\alpha]}_{n,\rho}(\varepsilon^2(t,x);x)= \varepsilon^2(x,x) = 0$, since $\varepsilon(t,x) \to 0$ as $t \to x$, and using Lemma 2.3, we obtain $\lim_{n\to\infty} n\mathcal{R}^{[\alpha]}_{n,\rho}(\varepsilon(t,x)(t-x)^2;x) = 0$. Thus, the theorem is proved. □

3.2. **Local approximation.** Let $\tilde{C_B}[0,\infty)$ be the space of all real-valued bounded and uniformly continuous functions f on $[0,\infty)$ endowed with the norm

$$\|f\|_{\tilde{C_B}[0,\infty)} = \sup_{x\in[0,\infty)} \big|f(x)\big|.$$

For $f \in \tilde{C_B}[0,\infty)$, the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}} \big[2f(x+u+v) - f\big(x+2(u+v)\big)\big]\,du\,dv. \tag{3.2}$$

By simple computation, it is observed that

(a) $\|f_h - f\|_{\tilde{C_B}[0,\infty)} \le \omega_2(f,h)$,

(b) $f_h', f_h'' \in \tilde{C_B}[0,\infty)$ and $\|f_h'\|_{\tilde{C_B}[0,\infty)} \le \frac{5}{h}\omega(f,h), \|f_h''\|_{\tilde{C_B}[0,\infty)} \le \frac{9}{h^2}\omega_2(f,h)$,

where the second order modulus of continuity is defined as

$$\omega_2(f,\delta) = \sup_{x,u,v\ge 0}\ \sup_{|u-v|\le\delta} \big|f(x+2u) - 2f(x+u+v) + f(x+2v)\big|, \quad \delta \ge 0.$$

The usual modulus of continuity of $f \in \tilde{C_B}[0,\infty)$ is given by

$$\omega(f,\delta) = \sup_{x,u,v\ge 0}\ \sup_{|u-v|\le\delta} \big|f(x+u) - f(x+v)\big|.$$

Theorem 3.3. *Let $f \in \tilde{C_B}[0,\infty)$. Then for every $x \ge 0$, the following inequality holds:*

$$\big|\mathcal{R}^{[\alpha]}_{n,\rho}(f;x) - f(x)\big| \le 5\omega\big(f, \sqrt{\Theta^{\alpha}_{n,\rho}(x)}\big) + \frac{13}{2}\omega_2\big(f, \sqrt{\Theta^{\alpha}_{n,\rho}(x)}\big),$$

where $\Theta^{\alpha}_{n,\rho}(x) = \mathcal{R}^{[\alpha]}_{n,\rho}((t-x)^2;x)$.

Proof. For $x \ge 0$, and applying the Steklov mean f_h that is given by (3.2), we can write

$$\begin{aligned}\big|\mathcal{R}^{[\alpha]}_{n,\rho}(f;x) - f(x)\big| \le \mathcal{R}^{[\alpha]}_{n,\rho}\big(|f-f_h|;x\big) + \big|\mathcal{R}^{[\alpha]}_{n,\rho}\big(f_h - f_h(x);x\big)\big| \\ + \big|f_h(x) - f(x)\big|.\end{aligned} \tag{3.3}$$

From (1.2), for every $f \in \tilde{C}_B[0,\infty)$, we have

$$\big|\mathcal{R}^{[\alpha]}_{n,\rho}(f;x)\big| \le \|f\|_{\tilde{C_B}[0,\infty)}. \tag{3.4}$$

Using property (a) of the Steklov mean and (3.4), we get

$$\mathcal{R}^{[\alpha]}_{n,\rho}\big(|f-f_h|;x\big) \le \big\|\mathcal{R}^{[\alpha]}_{n,\rho}(f-f_h)\big\|_{\tilde{C_B}[0,\infty)} \le \|f-f_h\|_{\tilde{C_B}[0,\infty)} \le \omega_2(f,h).$$

By Taylor's expansion and the Cauchy–Schwarz inequality, we have

$$\left|\mathcal{R}_{n,\rho}^{[\alpha]}\big(f_h - f_h(x);x\big)\right| \leq \|f_h'\|_{\tilde{C_B}[0,\infty)}\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)} + \frac{1}{2}\|f_h''\|_{\tilde{C_B}[0,\infty)}\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big).$$

By Remark 2.1 and property (b) of the Steklov mean, we obtain

$$\left|\mathcal{R}_{n,\rho}^{[\alpha]}\big(f_h - f_h(x);x\big)\right| \leq \frac{5}{h}\omega(f,h)\sqrt{\Theta_{n,\rho}^{\alpha}(x)} + \frac{9}{2h^2}\omega_2(f,h)\Theta_{n,\rho}^{\alpha}(x).$$

Choosing $h = \sqrt{\Theta_{n,\rho}^{\alpha}(x)}$, and substituting the values of the above estimates in (3.3), we get the desired relation. □

Let $\beta_1 \geq 0, \beta_2 > 0$ be fixed. We consider the following Lipschitz-type space (see [14]):

$$\mathrm{Lip}_M^{(\beta_1,\beta_2)}(r) := \Big\{f \in C[0,\infty) : \left|f(t) - f(x)\right| \leq M\frac{|t-x|^r}{(t+\beta_1 x^2 + \beta_2 x)^{\frac{r}{2}}}; \; x,t \in (0,\infty)\Big\},$$

where $0 < r \leq 1$.

Theorem 3.4. *Let $f \in \mathrm{Lip}_M^{(\beta_1,\beta_2)}(r)$ and $r \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$\left|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq M\Big(\frac{\Theta_{n,\rho}^{\alpha}(x)}{\beta_1 x^2 + \beta_2 x}\Big)^{\frac{r}{2}},$$

where $\Theta_{n,\rho}^{\alpha}(x) = \mathcal{R}_{n,\rho}^{[\alpha]}((t-x)^2;x)$.

Proof. Applying Hölder's inequality with $p = \frac{2}{r}, q = \frac{2}{2-r}$, we find that

$$\begin{aligned}
&\left|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \\
&\leq \sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)\left|f(t)-f(x)\right|dt + v_{n,0}^{[\alpha]}\left|f(0)-f(x)\right| \\
&\leq \sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x)\Big(\int_0^{\infty} s_{n,k}^{\rho}(t)\left|f(t)-f(x)\right|^{\frac{2}{r}}dt\Big)^{\frac{r}{2}} + v_{n,0}^{[\alpha]}\left|f(0)-f(x)\right| \\
&\leq \Big\{\sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)\left|f(t)-f(x)\right|^{\frac{2}{r}}dt + v_{n,0}^{[\alpha]}\left|f(0)-f(x)\right|^{\frac{2}{r}}\Big\}^{\frac{r}{2}} \\
&\quad\times\Big(\sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x)\Big)^{\frac{2-r}{2}} \\
&= \Big\{\sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)\left|f(t)-f(x)\right|^{\frac{2}{r}}dt + v_{n,k}^{[\alpha]}(x)\left|f(0)-f(x)\right|^{\frac{2}{r}}\Big\}^{\frac{r}{2}}
\end{aligned}$$

$$\leq M\Big(\sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)\frac{(t-x)^2}{t+\beta_1x^2+\beta_2x}\,dt + v_{n,0}^{[\alpha]}(x)\frac{x^2}{\beta_1x^2+\beta_2x}\Big)^{\frac{r}{2}}$$
$$\leq \frac{M}{(\beta_1x^2+\beta_2x)^{\frac{r}{2}}}\Big(\sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)(t-x)^2\,dt + v_{n,0}^{[\alpha]}(x)x^2\Big)^{\frac{r}{2}}$$
$$= \frac{M}{(\beta_1x^2+\beta_2x)^{\frac{r}{2}}}\big(\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)\big)^{\frac{r}{2}} = \frac{M}{(\beta_1x^2+\beta_2x)^{\frac{r}{2}}}\big(\Theta_{n,\rho}^{\alpha}(x)\big)^{\frac{r}{2}}.$$

Thus, the proof is completed. □

Lenze [13] defined the Lipschitz-type maximal function of order r as follows:

$$\widetilde{\omega}_r(f,x) = \sup_{t\neq x, t\in[0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^r}, \quad x\in[0,\infty) \text{ and } r\in(0,1]. \tag{3.5}$$

Theorem 3.5. *Let $f\in \tilde{C}_B[0,\infty)$ and $r\in(0,1]$. Then, for all $x\in[0,\infty)$ we have*

$$\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\big| \leq \widetilde{\omega}_r(f,x)\big(\Theta_{n,\rho}^{\alpha}(x)\big)^{\frac{r}{2}}.$$

Proof. From (3.5), we have

$$\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\big| \leq \widetilde{\omega}_r(f,x)\mathcal{R}_{n,\rho}^{[\alpha]}\big(|t-x|^r;x\big).$$

Using Hölder's inequality with $p=\frac{2}{r}$ and $q=\frac{2}{2-r}$ and Remark 2.1, we obtain

$$\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\big| \leq \widetilde{\omega}_r(f,x)\big(\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)\big)^{\frac{r}{2}} = \widetilde{\omega}_r(f,x)\big(\Theta_{n,\rho}^{\alpha}(x)\big)^{\frac{r}{2}}. \quad □$$

Theorem 3.6. *For any $f\in \tilde{C}_B^1[0,\infty)$ and $x\in[0,\infty)$, we have*

$$\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\big| \leq \Big|\frac{\alpha x}{1-\alpha}\Big||f'(x)| + 2\sqrt{\Theta_{n,\rho}^{\alpha}(x)}\,\omega\big(f',\sqrt{\Theta_{n,\rho}^{\alpha}(x)}\big). \tag{3.6}$$

Proof. Let $f\in \tilde{C}_B^1[0,\infty)$. For any $t\in[0,\infty), x\in[0,\infty)$, we have

$$f(t)-f(x) = f'(x)(t-x) + \int_x^t \big(f'(u)-f'(x)\big)\,du.$$

Applying $\mathcal{R}_{n,\rho}^{[\alpha]}(\cdot;x)$ on both sides of the above relation, we get

$$\mathcal{R}_{n,\rho}^{[\alpha]}\big(f(t)-f(x);x\big) = f'(x)\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x) + \mathcal{R}_{n,\rho}^{[\alpha]}\Big(\int_x^t \big(f'(u)-f'(x)\big)\,du;x\Big).$$

Using the well-known property of modulus of continuity

$$\big|f(t)-f(x)\big| \leq \omega(f,\delta)\Big(\frac{|t-x|}{\delta}+1\Big), \quad \delta>0,$$

we obtain $|\int_x^t (f'(u)-f'(x))\,du| \leq \omega(f',\delta)(\frac{(t-x)^2}{\delta}+|t-x|)$.

Therefore, it follows that

$$\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\big| \leq \big|f'(x)\big|\big|\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x)\big| + \omega(f',\delta)\Big\{\frac{1}{\delta}\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)+\mathcal{R}_{n,\rho}^{[\alpha]}\big(|t-x|;x\big)\Big\}.$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\right| &\le \left|f'(x)\right|\left|\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x)\right| \\ &\quad + \omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)}+1\right\}\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)}.\end{aligned}$$

Choosing $\delta=\sqrt{\Theta_{n,\rho}^{\alpha}(x)}$, the required result follows. □

Let $B_\sigma[0,\infty)$ be the space of all real-valued functions on $[0,\infty)$ satisfying the condition $|f(x)|\le M_f\sigma(x)$, where M_f is a positive constant depending only on f and $\sigma(x)=1+x^2$ is a weight function. Let $C_\sigma[0,\infty)$ be the space of all continuous functions in $B_\sigma[0,\infty)$ endowed with the norm

$$\|f\|_\sigma := \sup_{x\in[0,\infty)}\frac{|f(x)|}{\sigma(x)},$$

and let

$$C_\sigma^0[0,\infty) := \left\{f\in C_\sigma[0,\infty) : \lim_{x\to\infty}\frac{|f(x)|}{\sigma(x)} \text{ exists and is finite}\right\}.$$

The usual modulus of continuity of f on $[0,b]$ is defined as

$$\omega_b(f,\delta)=\sup_{0<|t-x|\le\delta}\ \sup_{x,t\in[0,b]}\left|f(t)-f(x)\right|.$$

Theorem 3.7. *Let $f\in C_\sigma[0,\infty)$. Then, we have*

$$\left|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\right|\le 4M_f(1+x^2)\Theta_{n,\rho}^{\alpha}(x)+2\omega_{b+1}\big(f,\sqrt{\Theta_{n,\rho}^{\alpha}(x)}\big). \tag{3.7}$$

Proof. From [10], for $x\in[0,b]$ and $t\ge 0$, we have

$$\left|f(t)-f(x)\right|\le 4M_f(1+x^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta),\quad \delta>0.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\left|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\right| \\ &\quad\le 4M_f(1+x^2)\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\mathcal{R}_{n,\rho}^{[\alpha]}\big(|t-x|;x\big)\right) \\ &\quad\le 4M_f(1+x^2)\Theta_{n,\rho}^{\alpha}(x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\Theta_{n,\rho}^{\alpha}(x)}\right).\end{aligned}$$

Now, choosing $\delta=\sqrt{\Theta_{n,\rho}^{\alpha}(x)}$, we get (3.7). □

3.3. **Rate of convergence of Baskakov–Szász-type operators for functions with derivatives of bounded variation.** In this section, we discuss the approximation of functions with a derivative of bounded variation.

Let $DBV[0,\infty)$ be the class of all functions $f\in B_\sigma[0,\infty)$ having a derivative of bounded variation on every finite subinterval of $[0,\infty)$. The function $f\in$

$DBV[0,\infty)$ has the following representation:

$$f(x) = \int_0^x g(t) + f(0),$$

where g is a function of bounded variation on each finite subinterval of $[0,\infty)$.

In order to study the convergence of the operators $\mathcal{R}_{n,\rho}^{[\alpha]}$ for functions having a derivative of bounded variation, we rewrite the operators (1.2) as follows:

$$\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) = \int_0^\infty \mathcal{K}_{n,\rho}^{[\alpha]}(x,t) f(t)\,dt, \tag{3.8}$$
$$\mathcal{K}_{n,\rho}^{[\alpha]}(x,t) = \sum_{k=1}^{\infty} v_{n,k}^{[\alpha]}(x) s_{n,k}^{\rho}(t) + v_{n,0}^{[\alpha]}(x)\delta(t),$$

with $\delta(t)$ being the Dirac delta function.

Lemma 3.1. *Let $\alpha = \alpha(n) \to 0$ as $n \to \infty$, and let $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. For all $x \in (0,\infty)$ and sufficiently large n, we have*

(i) $\xi_{n,\rho}^{[\alpha]}(x,t) = \int_0^t \mathcal{K}_{n,\rho}^{[\alpha]}(x,u)\,du \le \frac{N(l,\rho)}{(x-t)^2}\frac{1+x^2}{n}, 0 \le t < x,$

(ii) $1 - \xi_{n,\rho}^{[\alpha]}(x,t) = \int_t^\infty \mathcal{K}_{n,\rho}^{[\alpha]}(x,u)\,du \le \frac{N(l,\rho)}{(t-x)^2}\frac{1+x^2}{n}, x < t < \infty,$

where $N(l,\rho)$ is a positive constant depending on l and ρ.

Proof. For sufficiently large n, it follows from (2.5) that

$$\mathcal{R}_{n,\rho}^{[n]}\big((u-x)^2;x\big) < N(l,\rho)\frac{1+x^2}{n}. \tag{3.9}$$

Applying Lemma 2.2, we get

$$\begin{aligned}\xi_{n,\rho}^{[\alpha]}(x,t) &= \int_0^t \mathcal{K}_{n,\rho}^{[\alpha]}(x,u)\,du \le \int_0^t \Big(\frac{x-u}{x-t}\Big)^2 \mathcal{K}_{n,\rho}^{[\alpha]}(x,u)\,du \\ &\le \frac{1}{(x-t)^2}\mathcal{R}_{n,\rho}^{[\alpha]}\big((u-x)^2;x\big) \le \frac{N(l,\rho)}{(x-t)^2}\frac{1+x^2}{n}.\end{aligned}$$

The proof of (ii) is similar and hence the details are omitted. □

Theorem 3.8. *Let $f \in DBV[0,\infty)$, let $\alpha = \alpha(n) \to 0$ as $n \to \infty$, and let $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}&\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)\big| \\ &\quad \le \frac{\alpha x}{1-\alpha}\Big|\frac{f'(x+)+f'(x-)}{2}\Big| + \sqrt{N(l,\rho)\frac{1+x^2}{n}}\Big|\frac{f'(x+)-f'(x-)}{2}\Big| \\ &\qquad + N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x-\frac{x}{k}}^{x} f'_x\Big) + \frac{x}{\sqrt{n}}\Big(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\Big) \\ &\qquad + \Big(4M_f + \frac{M_f + |f(x)|}{x^2}\Big)N(l,\rho)\frac{1+x^2}{n}\end{aligned}$$

$$+\left|f'(x+)\right|\sqrt{N(l,\rho)\frac{1+x^2}{n}}+N(l,\rho)\frac{1+x^2}{nx^2}\left|f(2x)-f(x)-xf'(x+)\right|$$
$$+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x+N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}} f'_x,$$

where $N(l,\rho)$ is a positive constant depending on l and ρ, $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$, and f'_x is defined by

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\le t<x,\\ 0, & t=x,\\ f'(t)-f'(x+), & x<t<\infty. \end{cases} \tag{3.10}$$

Proof. For any $f\in DBV[0,\infty)$, from (3.10) we can write

$$\begin{aligned} f'(u)=\frac{1}{2}\big(f'(x+)+f'(x-)\big)+f'_x(u)+\frac{1}{2}\big(f'(x+)-f'(x-)\big)\operatorname{sgn}(u-x)\\ +\delta_x(u)\Big(f'(u)-\frac{1}{2}\big(f'(x+)+f'(x-)\big)\Big), \end{aligned} \tag{3.11}$$

where

$$\delta_x(u)=\begin{cases} 1, & u=x,\\ 0, & u\neq x. \end{cases}$$

Since $\mathcal{R}_{n,\rho}^{[\alpha]}(e_0;x)=1$, using (3.8), for every $x\in(0,\infty)$ we get

$$\begin{aligned} \mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x) &= \int_0^\infty \mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\big(f(t)-f(x)\big)\,dt\\ &= \int_0^\infty \mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\int_x^t f'(u)\,du\,dt\\ &= -\int_0^x \Big(\int_t^x f'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\quad+\int_x^\infty \Big(\int_x^t f'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt. \end{aligned} \tag{3.12}$$

Denote

$$I_1:=\int_0^x \Big(\int_t^x f'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt,\qquad I_2:=\int_x^\infty \Big(\int_x^t f'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt.$$

Since $\int_x^t \delta_x(u)\,du=0$, and using relation (3.11), we get

$$\begin{aligned} I_1 = &\int_0^x \Big\{\int_t^x \Big(\frac{1}{2}\big(f'(x+)+f'(x-)\big)+f'_x(u).\\ &+\frac{1}{2}\big(f'(x+)-f'(x-)\big)\operatorname{sgn}(u-x)\Big)\,du\Big\}\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ =&\frac{1}{2}\big(f'(x+)+f'(x-)\big)\int_0^x (x-t)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt+\int_0^x\Big(\int_t^x f'_x(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &-\frac{1}{2}\big(f'(x+)-f'(x-)\big)\int_0^x (x-t)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt. \end{aligned} \tag{3.13}$$

In a similar way, we find that

$$\begin{aligned}I_2 &= \frac{1}{2}\big(f'(x+)+f'(x-)\big)\int_x^{\infty}(t-x)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\quad+\int_x^{\infty}\Big(\int_x^t f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\quad+\frac{1}{2}\big(f'(x+)-f'(x-)\big)\int_x^{\infty}(t-x)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt. \end{aligned}\tag{3.14}$$

Combining the relations (3.12)–(3.14), we get

$$\begin{aligned}\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x) &\leq \frac{1}{2}\big(f'(x+)+f'(x-)\big)\int_0^{\infty}(t-x)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\quad+\frac{1}{2}\big(f'(x+)-f'(x-)\big)\int_0^{\infty}|t-x|\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\quad-\int_0^{x}\Big(\int_t^x f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\quad+\int_x^{\infty}\Big(\int_x^t f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\big|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x)-f(x)\big| &= \left|\frac{f'(x+)+f'(x-)}{2}\right|\big|\mathcal{R}_{n,\rho}^{[\alpha]}(t-x;x)\big|\\ &\quad+\left|\frac{f'(x+)-f'(x-)}{2}\right|\mathcal{R}_{n,\rho}^{[\alpha]}\big(|t-x|;x\big)\\ &\quad+\left|\int_0^{x}\Big(\int_t^x f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\right|\\ &\quad+\left|\int_x^{\infty}\Big(\int_x^t f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\right|.\end{aligned}\tag{3.15}$$

Now, let

$$A_{n,\rho}^{[\alpha]}(f_x',x)=\int_0^{x}\Big(\int_t^x f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt$$

and

$$B_{n,\rho}^{[\alpha]}(f_x',x)=\int_x^{\infty}\Big(\int_x^t f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt.$$

Our problem is reduced to the calculation of the estimates of the terms $A_{n,\rho}^{[\alpha]}(f_x',x)$ and $B_{n,\rho}^{[\alpha]}(f_x',x)$. From the definition of $\xi_{n,\rho}^{[\alpha]}$ given in Lemma 3.1, applying the integration by parts, we can write

$$A_{n,\rho}^{[\alpha]}(f_x',x)=\int_0^{x}\Big(\int_t^x f_x'(u)\,du\Big)\frac{\partial}{\partial t}\xi_{n,\rho}^{[\alpha]}(x,t)\,dt=\int_0^{x}f_x'(t)\xi_{n,\rho}^{[\alpha]}(x,t)\,dt.$$

Thus,

$$\begin{aligned}\left|A_{n,\rho}^{[\alpha]}(f_x',x)\right| &\leq \int_0^x \left|f_x'(t)\right|\xi_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\leq \int_0^{x-\frac{x}{\sqrt{n}}} \left|f_x'(t)\right|\xi_{n,\rho}^{[\alpha]}(x,t)\,dt + \int_{x-\frac{x}{\sqrt{n}}}^{x} \left|f_x'(t)\right|\xi_{n,\rho}^{[\alpha]}(x,t)\,dt.\end{aligned}$$

Since $f_x'(x)=0$ and $\xi_{n,\rho}^{[\alpha]}(x,t)\leq 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^{x} \left|f_x'(t)\right|\xi_{n,\rho}^{[\alpha]}(x,t)\,dt &= \int_{x-\frac{x}{\sqrt{n}}}^{x} \left|f_x'(t)-f_x'(x)\right|\xi_{n,\rho}^{[\alpha]}(x,t)\,dt\\ &\leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \Big(\bigvee_t^x f_x'\Big)\,dt \leq \Big(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\Big)\int_{x-\frac{x}{\sqrt{n}}}^{x} dt\\ &= \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'.\end{aligned}$$

By applying Lemma 3.1 and considering $t=x-\frac{x}{u}$, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} \left|f_x'(t)\right|\xi_{n,\rho}^{[\alpha]}(x,t)\,dt &\leq N(l,\rho)\frac{1+x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\left|f_x'(t)\right|\frac{dt}{(x-t)^2}\\ &\leq N(l,\rho)\frac{1+x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\Big(\bigvee_t^x f_x'\Big)\frac{dt}{(x-t)^2}\\ &= N(l,\rho)\frac{1+x^2}{nx}\int_1^{\sqrt{n}}\Big(\bigvee_{x-\frac{x}{u}}^{x} f_x'\Big)\,du\\ &\leq N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x-\frac{x}{k}}^{x} f_x'\Big).\end{aligned}$$

Therefore,

$$\left|A_{n,\rho}^{[\alpha]}(f_x',x)\right| \leq N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x-\frac{x}{k}}^{x} f_x'\Big) + \frac{x}{\sqrt{n}}\Big(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\Big). \tag{3.16}$$

Also, using integration by parts in $B_{n,\rho}^{[\alpha]}(f_x',x)$ and applying Lemma 3.1, we have

$$\begin{aligned}\left|B_{n,\rho}^{[\alpha]}(f_x',x)\right| &\leq \left|\int_x^{2x}\Big(\int_x^t f_x'(u)\,du\Big)\frac{\partial}{\partial t}\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt\right|\\ &\quad+\left|\int_{2x}^{\infty}\Big(\int_x^t f_x'(u)\,du\Big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\right|\\ &\leq \left|\int_x^{2x} f_x'(u)\,du\right|\left|1-\xi_{n,\rho}^{[\alpha]}(x,2x)\right| + \int_x^{2x}\left|f_x'(t)\right|\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt\end{aligned}$$

$$+\Big|\int_{2x}^{\infty}\big(f(t)-f(x)\big)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\Big| + \big|f'(x+)\big|\Big|\int_{2x}^{\infty}(t-x)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt\Big|.$$

We have

$$\int_x^{2x}\big|f_x'(t)\big|\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt = \int_x^{x+\frac{x}{\sqrt{n}}}\big|f_x'(t)\big|\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt + \int_{x+\frac{x}{\sqrt{n}}}^{2x}\big|f_x'(t)\big|\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt = J_1+J_2 \quad \text{(say)}. \tag{3.17}$$

Since $f_x'(x)=0$ and $1-\xi_{n,\rho}^{[\alpha]}\le 1$, we get

$$J_1=\int_x^{x+\frac{x}{\sqrt{n}}}\big|f_x'(t)-f_x'(x)\big|\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt \le \int_x^{x+\frac{x}{\sqrt{n}}}\Big(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\Big)\,dt = \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'.$$

Applying Lemma 3.1 and considering $t=x+\frac{x}{u}$, we obtain

$$\begin{aligned}
J_2 &\le N(l,\rho)\frac{1+x^2}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}\big|f_x'(t)-f_x'(x)\big|\,dt\\
&\le N(l,\rho)\frac{1+x^2}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}\Big(\bigvee_x^{t} f_x'\Big)\,dt\\
&= N(l,\rho)\frac{1+x^2}{nx}\int_1^{\sqrt{n}}\Big(\bigvee_x^{x+\frac{x}{u}} f_x'\Big)\,du\\
&\le N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\Big(\bigvee_x^{x+\frac{x}{u}} f_x'\Big)\,du \le N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_x^{x+\frac{x}{k}} f_x'\Big).
\end{aligned}$$

Putting the values of J_1 and J_2 in (3.17), we get

$$\int_x^{2x}\big|f_x'(t)\big|\big(1-\xi_{n,\rho}^{[\alpha]}(x,t)\big)\,dt \le \frac{x}{\sqrt{n}}\Big(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\Big)+N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_x^{x+\frac{x}{k}} f_x'\Big).$$

Therefore, applying the Cauchy–Schwarz inequality and Lemma 3.1, we get

$$\big|B_{n,\rho}^{[\alpha]}(f_x',x)\big| \le M_f\int_{2x}^{\infty}(t^2+1)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt + \big|f(x)\big|\int_{2x}^{\infty}\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt + \big|f'(x+)\big|\sqrt{N(l,\rho)\frac{1+x^2}{n}}$$

$$+ N(l,\rho)\frac{1+x^2}{nx^2}\big|f(2x)-f(x)-xf'(x+)\big| + \frac{x}{\sqrt{n}}\Big(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f_x'\Big) + N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x}^{x+\frac{x}{k}} f_x'\Big). \tag{3.18}$$

Since $t \leq 2(t-x)$ and $x \leq t-x$ when $t \geq 2x$, we obtain

$$\begin{aligned} &M_f\int_{2x}^{\infty}(t^2+1)\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt + \big|f(x)\big|\int_{2x}^{\infty}\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt \\ &\leq \big(M_f+\big|f(x)\big|\big)\int_{2x}^{\infty}\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt + 4M_f\int_{2x}^{\infty}(t-x)^2\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt \\ &\leq \frac{M_f+|f(x)|}{x^2}\int_{0}^{\infty}(t-x)^2\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt + 4M_f\int_{0}^{\infty}(t-x)^2\mathcal{K}_{n,\rho}^{[\alpha]}(x,t)\,dt \\ &\leq \Big(4M_f+\frac{M_f+|f(x)|}{x^2}\Big)N(l,\rho)\frac{1+x^2}{n}. \end{aligned} \tag{3.19}$$

Using the inequality (3.19), it follows that

$$\begin{aligned} \big|B_{n,\rho}^{[\alpha]}(f_x',x)\big| &\leq \Big(4M_f+\frac{M_f+|f(x)|}{x^2}\Big)N(l,\rho)\frac{1+x^2}{n} \\ &+ \big|f'(x+)\big|\sqrt{N(l,\rho)\frac{1+x^2}{n}} \\ &+ N(l,\rho)\frac{1+x^2}{nx^2}\big|f(2x)-f(x)-xf'(x+)\big| \\ &+ \frac{x}{\sqrt{n}}\Big(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f_x'\Big) + N(l,\rho)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\Big(\bigvee_{x}^{x+\frac{x}{k}} f_x'\Big). \end{aligned} \tag{3.20}$$

From (3.15), (3.16), and (3.20), we get the required result. □

3.4. **Weighted approximation.**

Theorem 3.9. *Let $f \in C_\sigma^0[0,\infty)$, and let $\alpha = \alpha(n) \to 0$ as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty}\big\|\mathcal{R}_{n,\rho}^{[\alpha]}(f)-f\big\|_\sigma = 0. \tag{3.21}$$

Proof. In order to prove this result, it is sufficient to verify the following three relations (see [5]):

$$\lim_{n\to\infty}\big\|\mathcal{R}_{n,\rho}^{[\alpha]}(t^m;x)-x^m\big\|_\sigma = 0, \quad m=0,1,2. \tag{3.22}$$

Since $\mathcal{R}_{n,\rho}^{[\alpha]}(1;x)=1$, the condition in (3.22) holds true for $m=0$.

By Lemma 2.2, we have

$$\big\|\mathcal{R}_{n,\rho}^{[\alpha]}(t;x)-x\big\|_\sigma = \sup_{x\geq 0}\frac{1}{1+x^2}\Big|\frac{x}{1-\alpha}-x\Big| = \sup_{x\geq 0}\Big(\frac{x}{1+x^2}\Big)\frac{\alpha}{1-\alpha}. \tag{3.23}$$

Thus, $\lim_{n\to\infty} \|\mathcal{R}_{n,\rho}^{[\alpha]}(t;x) - x\|_\sigma = 0$. Finally, we obtain

$$\left\|\mathcal{R}_{n,\rho}^{[\alpha]}(t^2;x) - x^2\right\|_\sigma \leq \sup_{x\geq 0} \frac{x^2}{1+x^2} \frac{|1+3n\alpha-2n\alpha^2|}{n|(1-\alpha)(1-2\alpha)|} + \sup_{x\geq 0} \frac{x}{1+x^2} \frac{|1-2\alpha+\rho-\alpha\rho+n\alpha\rho|}{n\rho|(1-\alpha)(1-2\alpha)|},$$

which implies that $\lim_{n\to\infty} \|\mathcal{R}_{n,\rho}^{[\alpha]}(t^2;x) - x^2\|_\sigma = 0$. □

Let $f \in C_\sigma^0[0,\infty)$. We will consider the weighted modulus of continuity defined by Yüksel and Ispir [20] as follows:

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

Lemma 3.2 ([20, Section 3])**.** *Let $f \in C_\sigma^0[0,\infty)$, then:*

(i) *$\Omega(f;\delta)$ is a monotone increasing function of δ;*
(ii) *$\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$;*
(iii) *for each $m \in \mathbb{N}, \Omega(f,m\delta) \leq m\Omega(f;\delta)$;*
(iv) *for each $\lambda \in [0,\infty), \Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$.*

Theorem 3.10. *Let $f \in C_\sigma^0[0,\infty)$. If $\alpha = \alpha(n) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$, then there exists $\tilde{n} \in \mathbb{N}$ and a positive constant $N(l,\rho)$ depending on l and ρ such that*

$$\sup_{x\in(0,\infty)} \frac{|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq N(l,\rho)\Omega(f;n^{-1/2}), \quad \textit{for } n > \tilde{n}. \tag{3.24}$$

Proof. For $t > 0, x \in (0,\infty)$ and $\delta > 0$, by the definition of $\Omega(f;\delta)$ and Lemma 3.2, we can write

$$\left|f(t) - f(x)\right| \leq 2(1+x^2)\big(1+(t-x)^2\big)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).$$

Since the operator $\mathcal{R}_{n,\rho}^{[\alpha]}$ is linear and positive, we have

$$\left|\mathcal{R}_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq 2(1+x^2)\Omega(f;\delta)\Big\{1+\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big) + \mathcal{R}_{n,\rho}^{[\alpha]}\Big(\big(1+(t-x)^2\big)\frac{|t-x|}{\delta};x\Big)\Big\}. \tag{3.25}$$

From the relation (2.5), it follows that there is $n_1 \in \mathbb{N}$ such that

$$\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big) \leq N_1(l,\rho)\frac{(1+x^2)}{n}, \quad \text{for } n > n_1, \tag{3.26}$$

where $N_1(l,\rho)$ is a positive constant depending on l and ρ. Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\mathcal{R}_{n,\rho}^{[\alpha]}\Big(\big(1+(t-x)^2\big)\frac{|t-x|}{\delta};x\Big)\\
&\quad\leq \frac{1}{\delta}\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)}\\
&\qquad+\frac{1}{\delta}\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^4;x\big)}\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^2;x\big)}. \qquad (3.27)
\end{aligned}$$

From the relation (2.6), it follows that there is $n_2 \in \mathbb{N}$ such that

$$\sqrt{\mathcal{R}_{n,\rho}^{[\alpha]}\big((t-x)^4;x\big)} \leq N_2(l,\rho)\frac{(1+x^2)}{n}, \quad \text{for } n > n_2, \qquad (3.28)$$

where $N_2(l,\rho)$ is a positive constant depending on l and ρ.

Let $\tilde{n} = \max\{n_1, n_2\}$. Collecting the estimates (3.25)–(3.28) and taking

$$N(l,\rho) = 2\big(1 + N_1(l,\rho) + \sqrt{N_1(l,\rho)} + N_2(l,\rho)\sqrt{N_1(l,\rho)}\big), \quad \delta = \frac{1}{\sqrt{n}}, \text{for } n > \tilde{n},$$

we get the required result (3.24). □

Acknowledgments. Kajla would like to thank the University Grant Commission (UGC) of India for financial support to carry out this research work. Acu's work was partially financed by Lucian Blaga University of Sibiu research grant LBUS-IRG-2015-01 no. 2032/7.

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