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THE GENERALIZED DRAZIN INVERSE OF THE SUM IN A BANACH ALGEBRA

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Abstract. In this article, we obtain new additive results on the generalized Drazin inverse of a sum of two elements in a Banach algebra. Applying these additive results, we also give explicit formulas for the generalized Drazin inverse of a block matrix in a Banach algebra.

1. Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. We denote the sets of all invertible, nilpotent, and quasinilpotent elements of $\mathcal{A}$ by $\mathcal{A}^{-1}$, $\mathcal{A}^{\text{nil}}$, and $\mathcal{A}^{\text{qnil}}$, respectively.

The generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a; see [12]) is the unique element $a^d \in \mathcal{A}$ which satisfies

$$a^d a a^d = a^d, \qquad a a^d = a^d a, \qquad a - a^2 a^d \in \mathcal{A}^{\text{qnil}}.$$

Recall that a^d exists if and only if $0 \notin \operatorname{acc}\sigma(a)$, where $\operatorname{acc}\sigma(a)$ is the set of all accumulation points of the spectrum of a. If the generalized Drazin inverse of a exists, then a is the generalized Drazin invertible. The set of all generalized Drazin invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^d$. For $a \in \mathcal{A}^d$, $a^\pi = 1 - aa^d$ is the spectral idempotent of a corresponding to the set $\{0\}$. If $a \in \mathcal{A}^{\text{qnil}}$, then $a^d = 0$.

If we suppose that $a - a^2 a^d \in \mathcal{A}^{\text{nil}}$ in the above definition, then $a^d = a^D$ is the ordinary Drazin inverse of a. A particular case of the Drazin inverse is the group inverse for which $a = aa^d a$ instead of $a - a^2 a^d \in \mathcal{A}^{\text{nil}}$. By $a^\#$ and $\mathcal{A}^\#$ we

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denote the group inverse of a and the set of all group invertible elements of $\mathcal{A}$, respectively.

The following auxiliary result gives a property of quasinilpotent elements.

Lemma 1.1 (see [10]). *Let $q \in \mathcal{A}$. Then q is quasinilpotent if and only if $1+xq \in \mathcal{A}^{-1}$ for all $x \in \mathcal{A}$ satisfying $xq = qx$.*

We state now one well-known additive result on the generalized Drazin inverse in a Banach algebra.

Lemma 1.2 ([4, Corollary 3.4]). *Let $b \in \mathcal{A}^d$ and $a \in \mathcal{A}^{\mathrm{qnil}}$. If $ab = 0$, then $a+b \in \mathcal{A}^d$ and*

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n.$$

For $p = p^2 \in \mathcal{A}$, any element $a \in \mathcal{A}$ can be expressed as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

Let us recall that if $a \in \mathcal{A}^d$, then

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

relative to $p = aa^d$, where $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{\mathrm{qnil}}$. In this case, the generalized Drazin inverse of a is given by

$$a^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We will use the next result related to the generalized Drazin inverse of a triangular block matrix.

Lemma 1.3 ([4, Theorem 2.3]). *Let $x = \left[\begin{smallmatrix} a & 0 \\ c & b \end{smallmatrix}\right] \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$, and let $y = \left[\begin{smallmatrix} b & c \\ 0 & a \end{smallmatrix}\right] \in \mathcal{A}$ relative to the idempotent $1-p$.*

(i) *If $a \in (p\mathcal{A}p)^d$ and $b \in ((1-p)\mathcal{A}(1-p))^d$, then $x, y \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix}, \qquad y^d = \begin{bmatrix} b^d & u \\ 0 & a^d \end{bmatrix},$$

where

$$u = \sum_{n=0}^{\infty} (b^d)^{n+2} c a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c (a^d)^{n+2} - b^d c a^d.$$

(ii) *If $x \in \mathcal{A}^d$ and $a \in (p\mathcal{A}p)^d$, then $b \in ((1-p)\mathcal{A}(1-p))^d$ and x^d is given as in part* (i).

One special topic concerning the generalized Drazin inverse is to find explicit expressions for the generalized Drazin inverse of a sum of two elements. Much has been written on this subject (see [4], [6], [9]), but the motivation for this article was Liu and Qin [13]. They presented a formula for the generalized Drazin inverse of the sum of two elements of a Banach algebra under some conditions which contain $a^k b = ab$ $(k > 1)$ and/or $ba = ab^2$ (or $a^r b = ba^t$, $r, t \in N$).

Under new conditions involving $a^\pi a^k b = a^\pi ab$ and $a^\pi ba^t = a^\pi a^r b^m$ (or $ba^\pi = b$ or $a^l ba^\pi = a^\pi ba^m$), $k, l, m, r, t \in N$, $k > 1$, we investigate the existence of the generalized Drazin inverse of the sum $a + b$ in a Banach algebra and give explicit representations for the generalized Drazin inverse of this sum. As an application of our results, we obtain several expressions for the generalized Drazin inverse of a block matrix.

2. Generalized Drazin inverse of the sum

First, we study the existence and present the formula for the generalized Drazin inverse of the sum $a+b$ under the assumptions $a^\pi a^k b = a^\pi ab$ and $a^\pi ba^t = a^\pi a^r b^m$ $(k, m, r, t \in N,\ k > 1)$.

Theorem 2.1. *Let $a, b \in \mathcal{A}^d$, $a^\pi a^k b = a^\pi ab$, and $a^\pi ba^t = a^\pi a^r b^m$, for some $k, m, r, t \in N$ such that $k > 1$. If $a^\pi b$ (or ba^π or $a^\pi ba^\pi$) is generalized Drazin invertible, then*

$$a + b \in \mathcal{A}^d \Leftrightarrow e = (a + b)aa^d \in \mathcal{A}^d \Leftrightarrow aa^d(a + b) \in \mathcal{A}^d \Leftrightarrow aa^d(a + b)aa^d \in \mathcal{A}^d.$$

In this case,

$$\begin{aligned}(a + b)^d = e^d &+ \sum_{n=0}^{\infty}(e^d)^{n+2}ba^\pi(a + b)^n\Big(a^\pi - \sum_{j=0}^{t-1} a^\pi b(b^d)^{j+1}a^j\Big)\\ &+ \sum_{n=0}^{\infty} e^\pi e^n aa^d bx^{n+2} + (1 - e^d b)x, \qquad (2.1)\end{aligned}$$

where $x = \sum_{j=0}^{t-1} a^\pi (b^d)^{j+1}a^j$.

Proof. We have the following matrix representations of a and b relative to $p = aa^d$:

$$a = \begin{bmatrix} a_1 & 0\\ 0 & a_2\end{bmatrix}, \qquad b = \begin{bmatrix} b_1 & b_2\\ b_3 & b_4\end{bmatrix}, \qquad (2.2)$$

where $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1 - p)\mathcal{A}(1 - p))^{\text{qnil}}$.

Observe that, by

$$\begin{bmatrix} 0 & 0\\ a_2^k b_3 & a_2^k b_4\end{bmatrix} = a^\pi a^k b = a^\pi ab = \begin{bmatrix} 0 & 0\\ a_2 b_3 & a_2 b_4\end{bmatrix},$$

we conclude that $a_2^k b_3 = a_2 b_3$ and $a_2^k b_4 = a_2 b_4$. Since $a_2 \in ((1-p)\mathcal{A}(1-p))^{\text{qnil}}$, by Lemma 1.1, $(1 - p) - a_2^{k-1} \in ((1 - p)\mathcal{A}(1 - p))^{-1}$. From $((1 - p) - a_2^{k-1})a_2 b_3 = 0$

and $((1-p)-a_2^{k-1})a_2b_4 = 0$, we get $a_2b_3 = 0$ and $a_2b_4 = 0$. Hence, $a^\pi ab = 0$ which gives

$$0 = a^\pi a^r b^m = a^\pi b a^t = \begin{bmatrix} 0 & 0 \\ b_3a_1^t & b_4a_2^t \end{bmatrix},$$

that is, $b_3a_1^t = 0$ and $b_4a_2^t = 0$. Because a_1 is invertible, we deduce that $b_3 = 0$. Since

$$b = \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}$$

and $a^\pi b$ (or ba^π or $a^\pi ba^\pi$) are generalized Drazin invertible, by Lemma 1.3, $b_4 \in ((1-p)\mathcal{A}(1-p))^d$, $b_1 \in (p\mathcal{A}p)^d$,

$$b^d = \begin{bmatrix} b_1^d & v \\ 0 & b_4^d \end{bmatrix} \qquad \text{and} \qquad b^\pi = \begin{bmatrix} b_1^\pi & -b_1v - b_2b_4^d \\ 0 & b_4^\pi \end{bmatrix},$$

where

$$v = \sum_{n=0}^{\infty}(b_1^d)^{n+2}b_2b_4^nb_4^\pi + \sum_{n=0}^{\infty} b_1^\pi b_1^n b_2(b_4^d)^{n+2} - b_1^d b_2 b_4^d.$$

Using Lemma 1.2, note that $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and

$$(a_2+b_4)^d = \sum_{j=0}^{t-1}(b_4^d)^{j+1}a_2^j.$$

Thus,

$$\begin{aligned}(a_2+b_4)^\pi &= (1-p) - (a_2+b_4)\sum_{j=0}^{t-1}(b_4^d)^{j+1}a_2^j \\ &= (1-p) - \sum_{j=0}^{t-1} b_4(b_4^d)^{j+1}a_2^j.\end{aligned}$$

By Lemma 1.3, $a+b = \left[\begin{smallmatrix} a_1+b_1 & b_2 \\ 0 & a_2+b_4 \end{smallmatrix}\right]$ is generalized Drazin invertible if and only if

$$e = (a+b)aa^d = \begin{bmatrix} a_1+b_1 & 0 \\ 0 & 0 \end{bmatrix} = a_1 + b_1 = aa^d(a+b)aa^d$$

is generalized Drazin invertible if and only if $aa^d(a+b)$ is generalized Drazin invertible. In this case,

$$(a+b)^d = \begin{bmatrix} e^d & u \\ 0 & (a_2+b_4)^d \end{bmatrix}, \tag{2.3}$$

where

$$u = \sum_{n=0}^{\infty}(e^d)^{n+2}b_2(a_2+b_4)^n(a_2+b_4)^\pi + \sum_{n=0}^{\infty} e^\pi e^n b_2\left[(a_2+b_4)^d\right]^{n+2} - e^d b_2(a_2+b_4)^d.$$

Using the equalities

$$x = \sum_{j=0}^{t-1} a^\pi (b^d)^{j+1} a^j = \sum_{j=0}^{t-1} \begin{bmatrix} 0 & 0 \\ 0 & (b_4^d)^{j+1} a_2^j \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_4)^d \end{bmatrix},$$

$$e^d - e^d bx = \begin{bmatrix} e^d & -e^d b_2 (a_2+b_4)^d \\ 0 & 0 \end{bmatrix},$$

$$a^\pi - \sum_{j=0}^{t-1} a^\pi b (b^d)^{j+1} a^j = \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_4)^\pi \end{bmatrix},$$

$$\sum_{n=0}^{\infty} (e^d)^{n+2} b a^\pi (a+b)^n = \sum_{n=0}^{\infty} \begin{bmatrix} 0 & (e^d)^{n+2} b_2 (a_2+b_4)^n \\ 0 & 0 \end{bmatrix},$$

$$\sum_{n=0}^{\infty} e^\pi e^n a a^d b x^{n+2} = \sum_{n=0}^{\infty} \begin{bmatrix} 0 & e^\pi e^n b_2 [(a_2+b_4)^d]^{n+2} \\ 0 & 0 \end{bmatrix}$$

and (2.3), we get (2.1). □

Note that Theorem 2.1 generalizes [13, Theorem 8] which involves conditions $a, b, aa^d(a+b) \in \mathcal{A}^d$, $a^k b = ab$ and $a^r b = ba^t$ $(k, r, t \in N,\ k > 1)$.

In the case that $ba^t = a^\pi a^r b^m$ instead of $a^\pi b a^t = a^\pi a^r b^m$ in Theorem 2.1, we obtain a simpler expression for $(a+b)^d$.

Theorem 2.2. *Let $a, b \in \mathcal{A}^d$. If $a^\pi a^k b = a^\pi ab$ and $ba^t = a^\pi a^r b^m$, for some $k, m, r, t \in N$ such that $k > 1$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^d + \sum_{n=0}^{\infty} (a^d)^{n+2} b (a+b)^n \Big(a^\pi - \sum_{j=0}^{t-1} a^\pi b (b^d)^{j+1} a^j \Big) + (1 - a^d b) \sum_{j=0}^{t-1} a^\pi (b^d)^{j+1} a^j. \tag{2.4}$$

Proof. If we suppose that $a \in \mathcal{A}^{\text{qnil}}$, note that $a^k b = ab$, $ba^t = a^r b^m$ and, by Lemma 1.1, $1 - a^{k-1} \in \mathcal{A}^{-1}$. Then, by $(1 - a^{k-1})ab = 0$, we get $ab = 0$. So, $ba^t = 0$ and the formula (2.4) holds by Lemma 1.2. When $a \in \mathcal{A}^{-1}$, we have that $ba^t = 0$ yields $b = 0$ and the formula (2.4) is satisfied.

In the case that a is neither invertible nor quasinilpotent, we consider matrix representations of a and b relative to $p = aa^d$ given by (2.2). As in the proof of Theorem 2.1, notice that $a^\pi a^k b = a^\pi ab$ yields $a_2 b_3 = 0$ and $a_2 b_4 = 0$. From

$$0 = a^\pi a^r b^m = ba^t = \begin{bmatrix} b_1 a_1^t & b_2 a_2^t \\ b_3 a_1^t & b_4 a_2^t \end{bmatrix},$$

we get $b_1 = 0$, $b_3 = 0$, and $b_2 a_2^t = b_4 a_2^t = 0$, that is,

$$b = \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}.$$

Now, by Lemma 1.3, $b_4 \in ((1-p)\mathcal{A}(1-p))^d$,

$$b^d = \begin{bmatrix} 0 & b_2(b_4^d)^2 \\ 0 & b_4^d \end{bmatrix} \quad \text{and} \quad b^\pi = \begin{bmatrix} p & -b_2 b_4^d \\ 0 & b_4^\pi \end{bmatrix}. \tag{2.5}$$

Applying Lemma 1.2, $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and we represent $(a_2+b_4)^d$ and $(a_2+b_4)^\pi$ as in the proof of Theorem 2.1. Then, by Lemma 1.3, $a+b \in \mathcal{A}^d$ and

$$(a+b)^d = \begin{bmatrix} a_1 & b_2 \\ 0 & a_2+b_4 \end{bmatrix}^d = \begin{bmatrix} a_1^{-1} & u \\ 0 & (a_2+b_4)^d \end{bmatrix}, \tag{2.6}$$

where

$$u = \sum_{n=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n (a_2+b_4)^\pi - a_1^{-1} b_2 (a_2+b_4)^d.$$

The equalities

$$a^d b = \begin{bmatrix} 0 & a_1^{-1} b_2 \\ 0 & 0 \end{bmatrix},$$

$$\sum_{j=0}^{t-1} a^\pi (b^d)^{j+1} a^j = \sum_{j=0}^{t-1} \begin{bmatrix} 0 & 0 \\ 0 & (b_4^d)^{j+1} a_2^j \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_4)^d \end{bmatrix},$$

$$a^\pi - \sum_{j=0}^{t-1} a^\pi b (b^d)^{j+1} a^j = \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_4)^\pi \end{bmatrix},$$

$$\sum_{n=0}^{\infty} (a^d)^{n+2} b (a+b)^n = \begin{bmatrix} 0 & \sum_{n=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n \\ 0 & 0 \end{bmatrix}$$

and (2.6) imply that (2.4) holds. □

If we suppose that $t = 1$ in Theorem 2.2, we have the following consequence.

Corollary 2.3. *Let $a, b \in \mathcal{A}^d$. If $a^\pi a^k b = a^\pi a b$ and $ba = a^\pi a^r b^m$, for some $k, m, r \in N$ such that $k > 1$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^\pi b^d + \sum_{n=0}^{\infty} (a^d)^{n+1} b^n b^\pi. \tag{2.7}$$

Proof. The assumption $ba = a^\pi a^r b^m$ gives $a^d b^j a = 0$ for all $j \in N$. Now, by Theorem 2.2, we obtain (2.7). □

If we define the reverse multiplication in a Banach algebra $\mathcal{A}$ by $a \circ b = ba$, we obtain a Banach algebra $(\mathcal{A}, \circ)$. Applying Theorem 2.2 and Corollary 2.3 to the new algebra $(\mathcal{A}, \circ)$, we get the next result.

Corollary 2.4. *Let $a, b \in \mathcal{A}^d$ and $ba^k a^\pi = baa^\pi$ for some $k \in N$ such that $k > 1$.*

(i) *If* $a^t b = b^m a^r a^\pi$, *for some* $m, r, t \in N$, *then* $a + b \in \mathcal{A}^d$ *and*

$$(a+b)^d = a^d + \sum_{n=0}^{\infty}\Big(a^\pi - \sum_{j=0}^{t-1} a^j (b^d)^{j+1} b a^\pi\Big)(a+b)^n b (a^d)^{n+2} + \sum_{j=0}^{t-1} a^j (b^d)^{j+1} a^\pi (1 - b a^d).$$

(ii) *If* $ab = b^m a^r a^\pi$, *for some* $m, r \in N$, *then* $a + b \in \mathcal{A}^d$ *and*

$$(a+b)^d = b^d a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

If we replace the hypothesis $a^\pi b a^t = a^\pi a^r b^m$ of Theorem 2.1 with $ba = ab^m$ or $a^\pi ba = a^\pi ab^m$, we show the following theorem (cf. [13, Theorem 6] where the representation for $(a+b)^d$ was given when $a^k b = ab$ and $ba = ab^2$).

Theorem 2.5. *Let* $a, b \in \mathcal{A}^d$, $a^\pi a^k b = a^\pi ab$, *and* ($ba = ab^m$ *or* $a^\pi ba = a^\pi ab^m$), *for some* $k, m \in N$ *such that* $k > 1$. *If* $a^\pi b$ (*or* ba^π *or* $a^\pi ba^\pi$) *is generalized Drazin invertible, then*

$$a + b \in \mathcal{A}^d \Leftrightarrow e = (a+b)aa^d \in \mathcal{A}^d \Leftrightarrow aa^d(a+b) \in \mathcal{A}^d \Leftrightarrow aa^d(a+b)aa^d \in \mathcal{A}^d.$$

In this case,

$$(a+b)^d = e^d + a^\pi b^d + (e^d)^2 b a^\pi b^\pi + \sum_{n=1}^{\infty} (e^d)^{n+2} b a^\pi (a^n + b^n b^\pi) + \sum_{n=0}^{\infty} e^\pi e^n b a^\pi (b^d)^{n+2} - e^d b a^\pi b^d. \tag{2.8}$$

Proof. Let a and b be represented as in (2.2) relative to $p = aa^d$. The equality $a^\pi a^k b = a^\pi ab$ gives $a_2 b_3 = 0 = a_2 b_4$ as in the proof of Theorem 2.1. Then, by $ba = ab^m$ (or $a^\pi ba = a^\pi ab^m$), we deduce that $b_3 = 0$ and $b_4 a_2 = 0$. So, b, b^d, and b^π are represented as in the proof of Theorem 2.1. By Lemma 1.2, we deduce that $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$, $(a_2 + b_4)^d = b_4^d$, and $(a_2 + b_4)^\pi = b_4^\pi$.

Using Lemma 1.3, $a + b = \left[\begin{smallmatrix} a_1 + b_1 & b_2 \\ 0 & a_2 + b_4 \end{smallmatrix}\right]$ is generalized Drazin invertible if and only if $e (= (a+b)aa^d = aa^d(a+b)aa^d) = a_1 + b_1$ is generalized Drazin invertible if and only if $aa^d(a+b)$ is generalized Drazin invertible. In this case,

$$(a+b)^d = \begin{bmatrix} e^d & u \\ 0 & b_4^d \end{bmatrix}, \tag{2.9}$$

where

$$u = (e^d)^2 b_2 b_4^\pi + \sum_{n=1}^{\infty} (e^d)^{n+2} b_2 (a_2^n + b_4^n b_4^\pi) + \sum_{n=0}^{\infty} e^\pi e^n b_2 (b_4^d)^{n+2} - e^d b_2 b_4^d.$$

From

$$X_1 = e^d + a^\pi b^d - e^d b a^\pi b^d = \begin{bmatrix} e^d & -e^d b_2 b_4^d \\ 0 & b_4^d \end{bmatrix},$$

$$X_2 = (e^d)^2 b a^\pi b^\pi + \sum_{n=1}^{\infty} (e^d)^{n+2} b a^\pi (a^n + b^n b^\pi)$$

$$= \begin{bmatrix} 0 & (e^d)^2 b_2 b_4^\pi \\ 0 & 0 \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & (e^d)^{n+2} b_2 (a_2^n + b_4^n b_4^\pi) \\ 0 & 0 \end{bmatrix},$$

$$X_3 = \sum_{n=0}^{\infty} e^\pi e^n b a^\pi (b^d)^{n+2} = \sum_{n=0}^{\infty} \begin{bmatrix} 0 & e^\pi e^n b_2 (b_4^d)^{n+2} \\ 0 & 0 \end{bmatrix}$$

and (2.9), we get (2.8). □

In the case that $a^k b = ab$ $(k > 1)$ and $b a^\pi = b$, the expression for the generalized Drazin inverse $(a+b)^d$ was proved in [13, Theorem 4]. Now, using conditions $a^\pi a^k b = a^\pi ab$ and $b a^\pi = b$, we obtain the same formula for $(a+b)^d$.

Theorem 2.6. *Let $a, b \in \mathcal{A}^d$. If $a^\pi a^k b = a^\pi ab$ and $b a^\pi = b$, for some $k \in N$ such that $k > 1$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^d + a^\pi \sum_{n=0}^{\infty} (b^d)^{n+1} a^n + \sum_{n=0}^{\infty} (a^d)^{n+2} b (a+b)^n b^\pi$$
$$- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (a^d)^{n+2} b (a+b)^n (b^d)^{k+1} a^{k+1}$$
$$- a^d b \sum_{n=0}^{\infty} (b^d)^{n+1} a^n. \tag{2.10}$$

Proof. In the case that $a \in \mathcal{A}^{\text{qnil}}$, by $a^k b = ab$, we get $ab = 0$ as in the proof of Theorem 2.2. Thus, using Lemma 1.2, the formula (2.10) is satisfied. When $a \in \mathcal{A}^{-1}$, $b = 0$ and (2.10) holds.

If a is neither invertible nor quasinilpotent, we assume that a and b have matrix representations as in (2.2) relative to $p = aa^d$. The hypothesis $b a^\pi = b$ implies $b_1 = 0$ and $b_3 = 0$. Hence,

$$b = \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}$$

and so b^d and b^π are represented by (2.5).

From

$$\begin{bmatrix} 0 & 0 \\ 0 & a_2^k b_4 \end{bmatrix} = a^\pi a^k b = a^\pi ab = \begin{bmatrix} 0 & 0 \\ 0 & a_2 b_4 \end{bmatrix},$$

we deduce that $a_2^k b_4 = a_2 b_4$. Because $a_2 \in ((1-p)\mathcal{A}(1-p))^{\text{qnil}}$ and $(1-p) - a_2^{k-1} \in ((1-p)\mathcal{A}(1-p))^{-1}$, then $a_2 b_4 = 0$. Using Lemma 1.2, $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and

$$(a_2 + b_4)^d = \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^n.$$

By Lemma 1.3, observe that $a+b \in \mathcal{A}^d$ and

$$(a+b)^d = \begin{bmatrix} a_1 & b_2 \\ 0 & a_2+b_4 \end{bmatrix}^d = \begin{bmatrix} a_1^{-1} & u \\ 0 & (a_2+b_4)^d \end{bmatrix}, \tag{2.11}$$

where

$$u = \sum_{n=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n (a_2+b_4)^\pi - a_1^{-1} b_2 (a_2+b_4)^d.$$

The equality $a_2 b_4 = 0$ yields $a_2 b_4^d = 0$ and

$$\begin{aligned}(a_2+b_4)^\pi &= (1-p) - (a_2+b_4)(a_2+b_4)^d = (1-p) - b_4 \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^n \\ &= b_4^\pi - \sum_{n=0}^{\infty} (b_4^d)^{n+1} a_2^{n+1}.\end{aligned}$$

Thus,

$$\begin{aligned}u = &\sum_{n=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n b_4^\pi - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n (b_4^d)^{k+1} a_2^{k+1} \\ &- a_1^{-1} b_2 (a_2+b_4)^d.\end{aligned}$$

Now, from (2.11),

$$\begin{aligned}X_1 &= a^d + a^\pi \sum_{n=0}^{\infty} (b^d)^{n+1} a^n = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & (b_4^d)^{n+1} a_2^n \end{bmatrix} \\ &= \begin{bmatrix} a_1^{-1} & 0 \\ 0 & (a_2+b_4)^d \end{bmatrix}, \\ X_2 &= \sum_{n=0}^{\infty} (a^d)^{n+2} b (a+b)^n b^\pi = \begin{bmatrix} 0 & \sum_{n=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n b_4^\pi \\ 0 & 0 \end{bmatrix}, \\ X_3 &= -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (a^d)^{n+2} b (a+b)^n (b^d)^{k+1} a^{k+1} \\ &= \begin{bmatrix} 0 & -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_1^{-(n+2)} b_2 (a_2+b_4)^n (b_4^d)^{k+1} a_2^{k+1} \\ 0 & 0 \end{bmatrix}, \\ X_4 &= -a^d b \sum_{n=0}^{\infty} (b^d)^{n+1} a^n = \begin{bmatrix} cc0 & -a_1^{-1} b_2 (a_2+b_2)^d \\ 0 & 0 \end{bmatrix},\end{aligned}$$

we obtain

$$(a+b)^d = X_1 + X_2 + X_3 + X_4,$$

that is, the formula (2.10) is satisfied. □

If we apply Theorem 2.6 to the algebra $(\mathcal{A}, \circ)$, we obtain the following result as a consequence.

Corollary 2.7. *Let $a, b \in \mathcal{A}^d$. If $ba^k a^\pi = baa^\pi$ and $a^\pi b = b$, for some $k \in N$ such that $k > 1$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^d + \sum_{n=0}^{\infty} a^n (b^d)^{n+1} a^\pi + b^\pi \sum_{n=0}^{\infty} (a+b)^n b (a^d)^{n+2} \\ - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a^{k+1}(b^d)^{k+1}(a+b)^n b (a^d)^{n+2} - \sum_{n=0}^{\infty} a^n (b^d)^{n+1} b a^d.$$

Under conditions $a^\pi a^k b = a^\pi ab$ and $a^l b a^\pi = a^\pi b a^m$ ($k, l, m \in N$ and $k > 1$), we will now give the representation for the generalized Drazin inverse of $a + b$. The following result recovers [13, Theorem 8], where the conditions $a^k b = ab$ and $a^l b = ba^m$ were considered.

Theorem 2.8. *Let $a, b \in \mathcal{A}^d$, $a^\pi a^k b = a^\pi ab$, and $a^l b a^\pi = a^\pi b a^m$, for some $k, l, m \in N$ such that $k > 1$. Then*

$$a + b \in \mathcal{A}^d \Leftrightarrow e = (a+b)aa^d \in \mathcal{A}^d \Leftrightarrow aa^d(a+b) \in \mathcal{A}^d \Leftrightarrow aa^d(a+b)aa^d \in \mathcal{A}^d.$$

In this case,

$$(a+b)^d = e^d + \sum_{n=0}^{m-1} (b^d)^{n+1} a^n a^\pi. \tag{2.12}$$

Proof. Suppose that a and b are given by (2.2) relative to $p = aa^d$. From $a^\pi a^k b = a^\pi ab$, we have $a_2 b_3 = 0 = a_2 b_4$. The hypothesis $a^l b a^\pi = a^\pi b a^m$ gives $a_1^l b_2 = b_3 a_1^m = b_4 a_2^m = 0$, that is, $b_2 = b_3 = 0$. Since

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}$$

is generalized Drazin invertible, we deduce that $b_1 \in (p\mathcal{A}p)^d$, $b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and $b^d = \left[\begin{smallmatrix} b_1^d & 0 \\ 0 & b_4^d \end{smallmatrix}\right]$. By Lemma 1.2, $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and

$$(a_2 + b_4)^d = \sum_{n=0}^{m-1} (b_4^d)^{n+1} a_2^n.$$

Therefore, $a + b = \left[\begin{smallmatrix} a_1+b_1 & 0 \\ 0 & a_2+b_4 \end{smallmatrix}\right]$ is generalized Drazin invertible if and only if $e(= (a+b)aa^d = aa^d(a+b) = aa^d(a+b)aa^d) = a_1 + b_1$ is generalized Drazin invertible. Then

$$(a+b)^d = \begin{bmatrix} e^d & 0 \\ 0 & (a_2+b_4)^d \end{bmatrix}$$

implies that (2.12) holds. □

Replacing the condition $a^l b a^\pi = a^\pi b a^m$ of Theorem 2.8 with $a^l b a^\pi = ba^m$, we obtain the following theorem.

Theorem 2.9. *Let $a, b \in \mathcal{A}^d$, $a^\pi a^k b = a^\pi ab$, and $a^l ba^\pi = ba^m$, for some $k, l, m \in N$ such that $k > 1$. Then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^d + \sum_{n=0}^{m-1} (b^d)^{n+1} a^n. \tag{2.13}$$

Proof. Using the representations of a and b as in (2.2) relative to $p = aa^d$, by $a^\pi a^k b = a^\pi ab$ and $a^l ba^\pi = ba^m$, we obtain $a_2 b_3 = a_2 b_4 = 0$, $b_1 = b_3 = b_4 a_2^m = 0$, and $a_1^l b_2 = b_2 a_2^m$. For any $s \in N$, by $a_2 \in ((1-p)\mathcal{A}(1-p))^{\text{qnil}}$, the last equality yields

$$0 \le \|b_2\|^{\frac{1}{sm}} = \|a_1^{-sl} b_2 a_2^{sm}\|^{\frac{1}{sm}} \le \|a_1^{-sl}\|^{\frac{1}{sm}} \|b_2\|^{\frac{1}{sm}} \|a_2^{sm}\|^{\frac{1}{sm}},$$

that is, $b_2 = 0$. So,

$$b = \begin{bmatrix} 0 & 0 \\ 0 & b_4 \end{bmatrix} \qquad \text{and} \qquad b^d = \begin{bmatrix} 0 & 0 \\ 0 & b_4^d \end{bmatrix}.$$

Also, we have $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^d$ and $(a_2 + b_4)^d = \sum_{n=0}^{m-1} (b_4^d)^{n+1} a_2^n$. Hence, $a + b \in \mathcal{A}^d$,

$$(a+b)^d = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 + b_4 \end{bmatrix}^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & (a_2 + b_4)^d \end{bmatrix},$$

and (2.13) is satisfied. □

As in Theorem 2.9, we can verify the next result.

Theorem 2.10. *Let $a, b \in \mathcal{A}^d$, $a^\pi a^k b = a^\pi ab$, and $a^l b = a^\pi ba^m$, for some $k, l, m \in N$ such that $k > 1$. Then $a + b \in \mathcal{A}^d$ and $(a+b)^d$ is represented as in (2.13).*

As a consequence of Theorem 2.9 and Theorem 2.10 in $(\mathcal{A}, \circ)$, we get the following expression for $(a+b)^d$.

Corollary 2.11. *Let $a, b \in \mathcal{A}^d$, $ba^k a^\pi = baa^\pi$, and ($a^\pi ba^l = a^m b$ or $ba^l = a^m ba^\pi$), for some $k, l, m \in N$ such that $k > 1$. Then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^d + \sum_{n=0}^{m-1} a^n (b^d)^{n+1}.$$

We remark that if $a \in \mathcal{A}^\#$ in the previous results, then the conditions $a^\pi a^k b = a^\pi ab$ and $ba^k a^\pi = baa^\pi$, for $k > 1$, are satisfied and can be omitted.

3. Applications

Generalized inverses of block matrices have important applications in automatics, probability, statistics, mathematical programming, numerical analysis, game theory, econometrics, control theory, and so on (see [1], [2]). Campbell and Meyer [2] proposed the problem of finding a formula for the Drazin inverse of a 2×2 matrix in terms of its various blocks. Until now, no complete solution was known to this problem, but some particular cases can be found in [3], [5], [7], [11], [15], [14], and [17].

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{3.1}$$

relative to the idempotent $p \in \mathcal{A}$, where $a \in (p\mathcal{A}p)^d$ and $d \in ((1-p)\mathcal{A}(1-p))^d$. In this section, applying Corollary 2.3 and Theorem 2.6, we present new expressions for the generalized Drazin inverse of a block matrix x.

Theorem 3.1. *Let x be defined as in (3.1), and let $a^\pi ab = a^\pi a^2 b$ and $(-ca^d - du)ab + d^\pi cb = (-ca^d - du)a^2b + d^\pi(cab + dcb)$, where*

$$u = \sum_{n=0}^{\infty}(d^d)^{n+2}ca^n a^\pi + \sum_{n=0}^{\infty} d^\pi d^n c(a^d)^{n+2} - d^d ca^d. \tag{3.2}$$

If

(i) *$bc = 0$, $bd = a^\pi ab$ and $(-ca^d - du)ab + d^\pi cb = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ u & d^d + ua^d b + d^d ub \end{bmatrix}; \tag{3.3}$$

(ii) *$bdd^d = 0$ and $\sum_{n=0}^{\infty} bd^n c(a^d)^{n+1} = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & (a^d)^{n+2}b \\ 0 & \sum_{k=0}^{n+1}(d^d)^k u(a^d)^{n-k+1}b \end{bmatrix} x^n; \tag{3.4}$$

(iii) *$bd = 0$ and $bca^d = 0$, then $x \in \mathcal{A}^d$ and x^d is represented as in (3.4).*

Proof. We can write

$$x = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} := y + z. \tag{3.5}$$

Then, by $z^2 = 0$, $z^d = 0$ and $z^\pi = 1$. Also, by Lemma 1.3, we have $y \in \mathcal{A}^d$,

$$y^d = \begin{bmatrix} a^d & 0 \\ u & d^d \end{bmatrix} \qquad \text{and} \qquad y^\pi = \begin{bmatrix} a^\pi & 0 \\ -ca^d - du & d^\pi \end{bmatrix},$$

where u is defined as in (3.2).

Since

$$y^\pi yz = \begin{bmatrix} 0 & a^\pi ab \\ 0 & (-ca^d - du)ab + d^\pi cb \end{bmatrix}$$

and

$$y^\pi y^2 z = \begin{bmatrix} 0 & a^\pi a^2 b \\ 0 & (-ca^d - du)a^2b + d^\pi(cab + dcb) \end{bmatrix},$$

we deduce that $y^\pi yz = y^\pi y^2 z$.

(i) By $zy = y^\pi yz$ and Corollary 2.3, $x \in \mathcal{A}^d$ and $x^d = y^d + (y^d)^2 z$, which implies (3.3).

(ii) From $bdd^d = 0$ and

$$bca^d + bdu = bca^d + \sum_{n=0}^{\infty} bd^{n+1}c(a^d)^{n+2} = \sum_{n=0}^{\infty} bd^n c(a^d)^{n+1} = 0,$$

we have $zy^\pi = z$. Applying Theorem 2.6, $x \in \mathcal{A}^d$,

$$x^d = y^d + \sum_{n=0}^{\infty} (y^d)^{n+2} z x^n$$

and (3.4) holds.

(iii) This part follows by (ii).

□

The assumptions $a^\pi ab = a^\pi a^2 b$ and $(-ca^d - du)ab + d^\pi cb = (-ca^d - du)a^2 b + d^\pi(cab + dcb)$ of Theorem 3.1 can be replaced with $ab = a^2 b$ and $d^\pi cb = d^\pi(cab + dcb)$ (or $ab = a^2 b$ and $d^\pi c = 0$).

Corollary 3.2. *Let x be defined as in (3.1), and let $ab = a^2 b$ and $d^\pi c = 0$. If $u_1 = \sum_{n=0}^{\infty} (d^d)^{n+2} ca^n a^\pi - d^d ca^d$,*

(i) *$bc = 0$, $bd = a^\pi ab$, and $(-ca^d - du_1)ab = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ u_1 & d^d - d^d c(a^d)^2 b + d^d u_1 b \end{bmatrix};$$

(ii) *$bdd^d = 0$ and $\sum_{n=0}^{\infty} bd^n c(a^d)^{n+1} = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u_1 & d^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & (a^d)^{n+2} b \\ 0 & -\sum_{k=0}^{n+1} (d^d)^{k+1} c(a^d)^{n-k+2} b \end{bmatrix} x^n; \tag{3.6}$$

(iii) *$bd = 0$ and $bca^d = 0$, then $x \in \mathcal{A}^d$ and x^d is represented as in (3.6).*

Theorem 3.3. *Let x be defined as in (3.1), and let u be defined as in (3.2). If*

(i) *$bc = 0$ and $bd = 0$, then $x \in \mathcal{A}^d$ and x^d is represented as in (3.3);*

(ii) *$bc = 0$, $bd = 0$, and $dc = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ c(a^d)^2 & d^d + c(a^d)^3 b \end{bmatrix}.$$

Proof. Suppose that x is given by (3.5).

(i) Since $yz^\pi = y$ and $z^\pi zy = 0 = z^\pi z^m y$, for $m \geq 2$, by Theorem 2.6, we check this part.

(ii) This is proved as a consequence of part (i).

□

Theorem 3.3(ii) recovers [8, Theorem 5.3] for operator matrices.

Theorem 3.4. *Let x be defined as in (3.1), and let $d^\pi dc = d^\pi d^2 c$ and $a^\pi bc + (-av - bd^d)dc = a^\pi(abc + bdc) + (-av - bd^d)d^2 c$, where*

$$v = \sum_{n=0}^{\infty} (a^d)^{n+2} bd^n d^\pi + \sum_{n=0}^{\infty} a^\pi a^n b(d^d)^{n+2} - a^d bd^d. \tag{3.7}$$

If

(i) $cb = 0$, $ca = d^\pi dc$ *and* $a^\pi bc + (-av - bd^d)dc = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d + a^d vc + vd^d c & v \\ (d^d)^2 c & d^d \end{bmatrix}; \tag{3.8}$$

(ii) $caa^d = 0$ *and* $\sum_{n=0}^{\infty} ca^n b(d^d)^{n+1} = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d & v \\ 0 & d^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} \sum_{k=0}^{n+1} (a^d)^k v (d^d)^{n-k+1} c & 0 \\ (d^d)^{n+2} c & 0 \end{bmatrix} x^n; \tag{3.9}$$

(iii) $ca = 0$ *and* $cbd^d = 0$, *then* $x \in \mathcal{A}^d$ *and* x^d *is represented as in (3.9).*

Proof. We prove this result as Theorem 3.1, using the representation

$$x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} := y + z.$$

□

Observe that Theorem 3.1 and Theorem 3.4 recover parts (iii) and (iv) of Corollaries 3.1 and 3.3 in [16].

Similarly as in Theorem 3.3, we check the next theorem by the representation of x as in the proof of Theorem 3.4.

Theorem 3.5. *Let x be defined as in (3.1), and let v be defined as in (3.7). If*

(i) $ca = 0$ *and* $cb = 0$, *then* $x \in \mathcal{A}^d$ *and* x^d *is represented as in (3.8);*

(ii) $ca = 0$, $cb = 0$ *and* $ab = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d + b(d^d)^3 c & b(d^d)^2 \\ (d^d)^2 c & d^d \end{bmatrix}.$$

If we suppose that $a \in (p\mathcal{A}p)^\#$ and $d \in ((1-p)\mathcal{A}(1-p))^\#$ in Theorem 3.1(iii) and Theorem 3.4(iii), we get the following result.

Corollary 3.6. *Let x be defined as in (3.1), and let $a \in (p\mathcal{A}p)^\#$ and $d \in ((1-p)\mathcal{A}(1-p))^\#$.*

(i) *If* $d^\pi ca^\pi b = 0$, $bd = 0$ *and* $bca^\# = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$\begin{aligned} x^d = &\begin{bmatrix} a^\# & 0 \\ (d^\#)^2 ca^\pi + d^\pi c (a^\#)^2 - d^\# ca^\# & d^\# \end{bmatrix} \\ &+ \sum_{n=0}^{\infty} \begin{bmatrix} 0 & (a^\#)^{n+2} b \\ 0 & d^\pi c (a^\#)^{n+3} b - \sum_{k=0}^{n+1} (d^\#)^{k+1} c (a^\#)^{n-k+2} b + (d^\#)^{n+3} ca^\pi b \end{bmatrix} x^n. \end{aligned}$$

(ii) *If* $a^\pi bd^\pi c = 0$, $ca = 0$ *and* $cbd^\# = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$\begin{aligned} x^d = &\begin{bmatrix} a^\# & (a^\#)^2 bd^\pi + a^\pi b (d^\#)^2 - a^\# bd^\# \\ 0 & d^\# \end{bmatrix} \\ &+ \sum_{n=0}^{\infty} \begin{bmatrix} a^\pi b (d^\#)^{n+3} c - \sum_{k=0}^{n+1} (a^\#)^{k+1} b (d^\#)^{n-k+2} c + (a^\#)^{n+3} bd^\pi c & 0 \\ (d^\#)^{n+2} c & 0 \end{bmatrix} x^n. \end{aligned}$$

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