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φ-CONTRACTIBILITY AND CHARACTER CONTRACTIBILITY OF FRÉCHET ALGEBRAS

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Abstract. Right φ-contractibility and right character contractibility of Banach algebras have been introduced and investigated. Here, we introduce and generalize these concepts for Fréchet algebras. We then verify available results about right φ-contractibility and right character contractibility of Banach algebras for Fréchet algebras. Moreover, we provide related results about Segal–Fréchet algebras.

1. Introduction

The notion of amenability of Fréchet algebras was introduced by Helemskii [9], [10] and studied by Pirkovskii [15]. Lawson and Read [13] also introduced and studied some notions about approximate amenability and approximate contractibility of Fréchet algebras. Furthermore in [2], the current authors and Rejali introduced and studied the notion of weak amenability of Fréchet algebras. Then recently in [3], according to the basic definition of Segal algebras in [16] and abstract Segal algebras in [5], the same three authors introduced the notion of a Segal–Fréchet algebra in the Fréchet algebra $(\mathcal{A}, p_\ell)$ and generalized many results in the field of abstract Segal algebras to the Segal–Fréchet algebras. They also showed that every continuous linear left multiplier of a Fréchet algebra $(\mathcal{A}, p_\ell)$ is also a continuous linear left multiplier of any Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Furthermore, they showed that if $\mathcal{A}$ is a commutative Fréchet Q-algebra with

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an approximate identity, then the space of all modular maximal closed ideals of $\mathcal{A}$ and any Segal–Fréchet algebra $(\mathcal{B}, q_m)$ in $(\mathcal{A}, p_\ell)$ are homeomorphic, and in particular obtaining that $(\mathcal{A}, p_\ell)$ is semisimple if and only if $(\mathcal{B}, q_m)$ is semisimple (see [1] for more information).

Let $\mathcal{A}$ be a Banach algebra, and let $\varphi \in \sigma(\mathcal{A})$, consisting of all nonzero characters on $\mathcal{A}$. The notions of right φ-contractibility and right character contractibility of Banach algebras were introduced and studied by Hu, Monfared, and Traynor [12]. Later in [4], character contractibility was characterized for abstract Segal algebras. In particular, it was shown that right φ-contractibility of $\mathcal{A}$ is equivalent to right $\varphi|_\mathcal{B}$-contractibility of $\mathcal{B}$, where $\mathcal{B}$ is an abstract Segal algebra in $\mathcal{A}$.

In this article, we introduce and study the concepts of right φ-contractibility and right character contractibility of the Fréchet algebras. Our definitions coincide with the ones presented in the Banach algebra case, whenever $\mathcal{A}$ is a Banach algebra. In addition, we study the Cartesian product of two Fréchet algebras, under a suitable fundamental system of seminorms, which makes it a Fréchet algebra. As the main result of Section 4, we prove that this Cartesian product is right character contractible if and only if both Fréchet algebras are. In the final section, we first recall from [1] the concept of a Segal–Fréchet algebra, and then we study the results of [4] for Segal–Fréchet algebras. For example, we characterize $\sigma(\mathcal{B})$, where $(\mathcal{B}, q_m)$ is a Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$. Similar to [4, Lemma 2.2], we show that

$$\sigma(\mathcal{B}) = \{\varphi|_\mathcal{B} : \varphi \in \sigma(\mathcal{A})\}.$$

As a main result, we show that Fréchet algebra $(\mathcal{A}, p_\ell)$ is right φ-contractible, for $\varphi \in \sigma(\mathcal{A})$, if and only if Segal Fréchet algebra $(\mathcal{B}, q_m)$ is right $\varphi|_\mathcal{B}$-contractible. Finally, we show that if $(\mathcal{A}, p_\ell)$ is right character contractible, then $(\mathcal{B}, q_m)$ is right $\varphi|_\mathcal{B}$-contractible, for each $\varphi \in \sigma(\mathcal{A})$. Moreover, we prove that $(\mathcal{B}, q_m)$ is right character contractible if and only if $\mathcal{A} = \mathcal{B}$, as two Fréchet algebras.

2. Preliminaries

In this section, we briefly mention some basic definitions related to locally convex spaces and also Fréchet algebras, which will be required throughout the article. (See [6], [8], and [14] for more information.)

A locally convex topological vector space E is a topological vector space in which the origin has a local base of absolutely convex absorbing sets. A collection $\mathcal{U}$ of zero neighborhoods in E is called a *fundamental system of zero neighborhoods* if for every zero neighborhood U, there exist $V \in \mathcal{U}$ and $\lambda > 0$ such that $\lambda V \subset U$. Throughout this article, all locally convex spaces are assumed to be Hausdorff. Recall that $S \subseteq E$ is called *balanced* if, for each $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, there is $\alpha S \subseteq S$. Moreover, S is called *absorbing* if, for each $x \in E$, there is the scalar λ such that $x \in \lambda S$.

A locally convex topological vector space is also defined to be a vector space E along with a family of seminorms $\{p_\alpha\}_{\alpha \in A}$ on E. The space E carries a natural topology, the initial topology of the seminorms. A family $(p_\alpha)_{\alpha \in A}$ of continuous seminorms on E is called a *fundamental system of seminorms* if the sets

$$U_\alpha = \big\{x \in E : p_\alpha(x) < 1\big\} \quad (\alpha \in A)$$

form a fundamental system of zero neighborhoods (see [14, p. 251] for more details). By [14, Lemmas 22.4 and 22.5], every locally convex space E has a fundamental system of seminorms $(p_\alpha)_{\alpha\in A}$; or equivalently, a family of seminorms satisfying the following properties:

(i) for every $x \in E$ with $x \neq 0$, there exists an $\alpha \in A$ with $p_\alpha(x) > 0$;
(ii) for all $\alpha, \beta \in A$, there exist $\gamma \in A$ and $C > 0$ such that

$$\max\big(p_\alpha(x), p_\beta(x)\big) \le C p_\gamma(x) \quad (x \in E).$$

We denote by (E, p_α) a locally convex space E with a fundamental system of seminorms $(p_\alpha)_\alpha$. A Fréchet space is a complete locally convex space whose topology is given by the countable fundamental system of seminorms $\{p_n\}_{n\in\mathbb{N}}$ (see [14] for more information). Note that by passing over to $(\max_{1\le j\le n} p_j)_{n\in\mathbb{N}}$, one may assume that $\{p_n\}_{n\in\mathbb{N}}$ is an increasing sequence.

Let (E, p_α) $(\alpha \in A)$ and (F, q_β) $(\beta \in B)$ be locally convex spaces. Then [14, Proposition 22.6] shows that for every linear mapping $T : E \to F$, the following assertions are equivalent:

(i) T is continuous;
(ii) for each $\beta \in B$ there exist $\alpha \in A$ and $C > 0$ such that

$$q_\beta\big(T(x)\big) \le C p_\alpha(x),$$

for all $x \in E$.

It should be noted that by [8, p. 24], if (E, p_μ), (F, q_λ), and (G, r_ν) are locally convex spaces, and $\theta : E \times F \to G$ is a bilinear map, then θ is jointly continuous if and only if for any ν_0 there exist μ_0 and λ_0 such that the bilinear map

$$\theta : (E, p_{\mu_0}) \times (F, q_{\lambda_0}) \longrightarrow (G, r_{\nu_0})$$

is jointly continuous. In other words, there exists $C > 0$ such that

$$r_{\nu_0}\big(\theta(x, y)\big) \le C p_{\mu_0}(x) q_{\lambda_0}(y),$$

for all $x \in E$ and $y \in F$. Recall from [17] that a bilinear map f from $E \times F$ into G is said to be *separately continuous* if all partial maps $f_x : F \to G$ and $f_y : E \to G$ defined by $y \mapsto f(x, y)$ and $x \mapsto f(x, y)$, respectively, are continuous for each $x \in E$ and $y \in F$. By [17, Chapter III.5.1], both types of continuity coincide in the class of Fréchet spaces and, in particular, Banach spaces. In such a situation, we use only continuous phrase representations.

A topological algebra $\mathcal{A}$ is an algebra which is a topological vector space, and the multiplication

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}, \qquad (a, b) \mapsto ab$$

is a separately continuous mapping (see [15]). A Fréchet algebra is a complete topological algebra, whose topology is given by the countable family of increasing submultiplicative seminorms (see [6] for more information).

A closed subalgebra of a Fréchet algebra $(\mathcal{A}, p_\ell)$ is clearly a Fréchet algebra. Moreover, if I is a proper closed ideal of $\mathcal{A}$, then $\frac{\mathcal{A}}{I}$ endowed with the quotient topology is a Fréchet space and the topology is defined by the seminorms

$$\hat{p}_\ell(a+I) = \inf\big\{p_\ell(a+b) : b \in I\big\}.$$

It is easy to show that the multiplication is continuous and that $\frac{\mathcal{A}}{I}$ is a topological algebra. Moreover, each $\hat{p}_\ell$ is submultiplicative and so $(\frac{\mathcal{A}}{I}, \hat{p}_\ell)$ is a Fréchet algebra (see [6, Section 3.2.10, p. 81]).

Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. A locally convex $\mathcal{A}$-bimodule is a locally convex topological vector space X together with the structure of an $\mathcal{A}$-bimodule such that the corresponding mappings are separately continuous. We call it a *Fréchet $\mathcal{A}$-bimodule* in the case where X is a Fréchet space. If I is a closed ideal of $\mathcal{A}$, then $(\frac{\mathcal{A}}{I}, \hat{p}_\ell)$ endowed with the quotient topology is a Fréchet $\mathcal{A}$-bimodule with the following module actions

$$b.(a+I) = ba + I, \qquad (a+I).b = ab + I,$$

for all $a, b \in \mathcal{A}$ (see [20] for more details).

Now consider X^*, the dual space of X, with the module actions given by

$$\langle a.f, x\rangle = \langle f, x.a\rangle, \qquad \langle f.a, x\rangle = \langle f, a.x\rangle,$$

for all $a \in \mathcal{A}$, $x \in X$, and $f \in X^*$. As is mentioned in [15], [20, Definition 3.1], and [7, Section 3], if X is a locally convex $\mathcal{A}$-bimodule, then so is X^*. Note that in this case, X^* is always considered with the strong topology on bounded subsets of X. Indeed, the net $(f_\alpha)_\alpha$ in X^* is convergent to $f \in X^*$ in the strong topology of X^*, if for every bounded subset B of X

$$\sup_{x\in B}\big|\langle f_\alpha - f, x\rangle\big| \longrightarrow_\alpha 0.$$

In particular, $\mathcal{A}^*$ is a locally convex $\mathcal{A}$-bimodule with module actions of $\mathcal{A}$ on $\mathcal{A}^*$ given by

$$\langle a.f, b\rangle = \langle f, ba\rangle, \qquad \langle f.a, b\rangle = \langle f, ab\rangle,$$

for all $a, b \in \mathcal{A}$ and $f \in \mathcal{A}^*$.

Finally, we also recall the projective tensor product of a Fréchet algebra $(\mathcal{A}, p_\ell)$, which was introduced in [17]. The construction is similar to the Banach algebra case. It will be denoted by $(\mathcal{A}\widehat{\otimes}\mathcal{A}, r_\ell)$, where

$$r_\ell(u) = \inf\Big\{\sum_{n\in\mathbb{N}} p_\ell(a_n)p_\ell(b_n) : u = \sum_{n\in\mathbb{N}} a_n \otimes b_n\Big\},$$

for each $u \in \mathcal{A}\widehat{\otimes}\mathcal{A}$. By [17] and also [21, Theorem 45.1], $(\mathcal{A}\widehat{\otimes}\mathcal{A}, r_\ell)$ is again a Fréchet algebra. Also, $(\mathcal{A}\widehat{\otimes}\mathcal{A}, r_\ell)$ is a Fréchet $\mathcal{A}$-bimodule by the following module actions:

$$a.(b\otimes c) = ab\otimes c, \qquad (b\otimes c).a = b\otimes ca \quad (a,b,c\in\mathcal{A}).$$

3. Right φ-contractibility and character contractibility

Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra, and let X be a locally convex $\mathcal{A}$-bimodule. Following [8], a continuous derivation of $\mathcal{A}$ into X is a continuous linear mapping D from $\mathcal{A}$ into X such that

$$D(ab) = a.D(b) + D(a).b,$$

for all $a, b \in \mathcal{A}$. For each $x \in X$, the mapping $\delta_x : \mathcal{A} \to X$ defined by

$$\delta_x(a) = a.x - x.a \quad (a \in \mathcal{A}),$$

is a continuous derivation and is called the *inner derivation associated with x.*

We first introduce the concepts of right φ-contractibility and right φ-diagonal for Fréchet algebras, according to their definitions for Banach algebras.

Definition 3.1. Let $\mathcal{A}$ be a Fréchet algebra.

(i) For $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$, $\mathcal{A}$ is called *right φ-contractible* if each continuous derivation of $\mathcal{A}$ into every locally convex $\mathcal{A}$-bimodule X, with the right module action

$$X \times \mathcal{A} \to X, \qquad (x, a) \mapsto x.a = \varphi(a)x \quad (a \in \mathcal{A}, x \in X),$$

is inner. Furthermore, $\mathcal{A}$ is called *right character contractible* if it is right φ-contractible, for all $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$.

(ii) Let $\varphi \in \sigma(\mathcal{A})$. A *right φ-diagonal for* $\mathcal{A}$ is an element $m \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\varphi(\pi(m)) = 1$ and $a.m = \varphi(a)m$, for all $a \in \mathcal{A}$, where $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the canonical mapping, defined as $a \otimes b \mapsto ab$.

Note 3.2. If m is a right φ-diagonal for $\mathcal{A}$, then the element $u = \pi(m) \in \mathcal{A}$ satisfies the following conditions

$$\varphi(u) = 1 \qquad \text{and} \qquad au = \varphi(a)u, \tag{3.1}$$

for all $a \in \mathcal{A}$. Conversely, if there exists $u \in \mathcal{A}$ satisfying (3.1), then $u \otimes u$ is clearly a right φ-diagonal for $\mathcal{A}$.

In this section, we generalize most of the results of [12] for Fréchet algebras. We commence with the following result, which is a generalization of [12, Theorem 6.3] to the Fréchet case.

Proposition 3.3. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. Then the following assertions hold.*

(i) *$\mathcal{A}$ is right 0-contractible if and only if $\mathcal{A}$ has a right identity.*

(ii) *For $\varphi \in \sigma(\mathcal{A})$, $\mathcal{A}$ is right φ-contractible if and only if $\mathcal{A}$ has a right φ-diagonal.*

(iii) *$\mathcal{A}$ is right character contractible if and only if $\mathcal{A}$ has a right identity and also has a right φ-diagonal for every $\varphi \in \sigma(\mathcal{A})$.*

Proof. (i). First, suppose that $\mathcal{A}$ is 0-contractible. Let $X = \mathcal{A} \times \mathcal{A}$ with the fundamental system of seminorms defined as

$$r_\ell\big((u, v)\big) = p_\ell(u) + p_\ell(v) \quad (u, v \in \mathcal{A}),$$

for each $\ell \in \mathbb{N}$. Then X is a Fréchet (and so locally convex) $\mathcal{A}$-bimodule with the following module actions:

$$\mathcal{A} \times X \to X, \qquad \big(a,(b,c)\big) \mapsto a.(b,c) = (ab, ac),$$
$$X \times \mathcal{A} \to X, \qquad \big((b,c),a\big) \mapsto (b,c).a = (0,0) \quad (a,b,c \in \mathcal{A}).$$

Now define $D : \mathcal{A} \to X$ by $D(a) = (a,a)$, for each $a \in \mathcal{A}$. Thus D is a continuous derivation and by the assumption it is inner. So there exists $(a_0, b_0) \in X$ such that

$$D(a) = (a,a) = a.(a_0,b_0) - (a_0,b_0).a = (aa_0, ab_0) \quad (a \in \mathcal{A}).$$

It follows that both a_0 and b_0 are right identities for $\mathcal{A}$. Conversely, suppose that $\mathcal{A}$ has a right identity, denoted by e. Let X be a locally convex $\mathcal{A}$-bimodule with the right module action $x.a = 0$, and let $D : \mathcal{A} \to X$ be a continuous derivation. Thus for each $a \in \mathcal{A}$

$$D(ae) = D(a).e + a.D(e) = a.D(e) = a.D(e) - D(e).a.$$

It follows that D is an inner derivation, and so $\mathcal{A}$ is right 0-contractible.

(ii). Let $\mathcal{A}$ be right φ-contractible, and consider the Fréchet (and so locally convex) $\mathcal{A}$-bimodule $\mathcal{A}\widehat{\otimes}\mathcal{A}$ with the module actions defined as

$$\mathcal{A} \times (\mathcal{A}\widehat{\otimes}\mathcal{A}) \to \mathcal{A}\widehat{\otimes}\mathcal{A}, \qquad a.(b \otimes c) = ab \otimes c,$$

and

$$(\mathcal{A}\widehat{\otimes}\mathcal{A}) \times \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}, \qquad (b \otimes c).a = \varphi(a) b \otimes c \quad (a,b,c \in \mathcal{A}).$$

It is easily verified that the map $\varphi \otimes \varphi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathbb{C}$, defined as

$$(\varphi \otimes \varphi)(a \otimes b) = \varphi(a)\varphi(b) \quad (a,b \in \mathcal{A}),$$

is a nonzero continuous homomorphism. Choose $m_0 \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$(\varphi \otimes \varphi)(m_0) = 1,$$

and consider the following inner derivation

$$\delta_{m_0} : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}, \qquad a \mapsto a.m_0 - m_0.a = a.m_0 - \varphi(a)m_0 \quad (a \in \mathcal{A}).$$

Some easy calculations show that $\delta_{m_0}(\mathcal{A}) \subset \ker(\varphi \otimes \varphi)$. Therefore there exists $m_1 \in \ker(\varphi \otimes \varphi)$ such that $\delta_{m_0} = \delta_{m_1}$. Thus $m_0 - m_1$ is a right φ-diagonal for $\mathcal{A}$. Conversely, suppose that m is a right φ-diagonal for $\mathcal{A}$ and that X is a locally convex $\mathcal{A}$-bimodule with the right module action defined by

$$X \times \mathcal{A} \to X, \qquad (x,a) \mapsto x.a = \varphi(a)x \quad (x \in X, a \in \mathcal{A}).$$

Let $D : \mathcal{A} \to X$ be a continuous derivation, and let $x_0 = -D(\pi(m)) \in X$. Then arguments exactly similar to the proof of the Banach case imply that

$$D(a) = a.x_0 - x_0.a \quad (a \in \mathcal{A}),$$

and so D is inner. Therefore (ii) is obtained.

(iii). This is immediately obtained by (i) and (ii). $\square$

Proposition 3.3 and Note 3.2 yield the following immediate result.

Corollary 3.4. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. Then the following assertions hold.*

(i) *For $\varphi \in \sigma(\mathcal{A})$, $\mathcal{A}$ is right φ-contractible if and only if there exists $u \in \mathcal{A}$ satisfying (3.1).*
(ii) *$\mathcal{A}$ is right character contractible if and only if $\mathcal{A}$ has a right identity, and for each $\varphi \in \sigma(\mathcal{A})$ there exists $u \in \mathcal{A}$ satisfying (3.1).*

Remark 3.5. We mention some remarkable points, which are important in this discussion.

(1) Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra, and let $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. It is worth noting that the proof of Proposition 3.3 remains valid even if in the definition of right φ-contractibility of Fréchet algebras, we replace the term "locally convex $\mathcal{A}$-bimodule" with "Fréchet $\mathcal{A}$-bimodule." In fact, $\mathcal{A}$ is right φ-contractible if and only if each continuous derivation of $\mathcal{A}$ into every Fréchet $\mathcal{A}$-bimodule X, with the right module action
$$X \times \mathcal{A} \to X, \qquad (x, a) \mapsto x.a = \varphi(a)x \quad (a \in \mathcal{A}, x \in X),$$
is inner.
(2) Let $\mathcal{A}$ be a Banach algebra, and let $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. It follows from Corollary 3.4 that the notion of right φ-contractibility of $\mathcal{A}$ coincides with Definition 3.1, when $\mathcal{A}$ is considered as a Fréchet algebra. Moreover, the concept of right character contractibility of $\mathcal{A}$ also coincides with the Fréchet algebra case. In fact the following assertions are equivalent.
   (i) $\mathcal{A}$ is right character contractible, in the sense of Banach algebra.
   (ii) $\mathcal{A}$ is right character contractible, in the sense of Fréchet algebra.
   (iii) For each $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$ and all locally convex $\mathcal{A}$-bimodules X, each continuous derivation $D : \mathcal{A} \to X$ with the right module action $x.a = \varphi(a)x$ is inner.
   (iv) For each $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$ and all Fréchet $\mathcal{A}$-bimodules X, each continuous derivation $D : \mathcal{A} \to X$ with the right module action $x.a = \varphi(a)x$ is inner.
   (v) For each $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$ and all Banach $\mathcal{A}$-bimodules X, each continuous derivation $D : \mathcal{A} \to X$ with the right module action $x.a = \varphi(a)x$ is inner.

The following theorem is a generalization of [12, Theorem 6.4] for Fréchet algebras.

Theorem 3.6. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra, and let $\varphi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is right φ-contractible and has a right identity if and only if $\ker \varphi$ has a right identity.*

Proof. First, suppose that $\mathcal{A}$ is right φ-contractible and has a right identity, denoted by e. By Note 3.2, there is $u \in \mathcal{A}$, satisfying (3.1). Then $e - u$ is a right identity for $\ker \varphi$. In fact $e - u \in \ker \varphi$, and for each $b \in \ker \varphi$ we have
$$b(e-u) = be - bu = b - \varphi(b)u = b.$$
Conversely, suppose that $\ker \varphi$ has a right identity, denoted by e'. Let a_0 be an element of $\mathcal{A}$ such that $\varphi(a_0) = 1$, and let $e = a_0 + e' - a_0 e'$. Since for each $a \in \mathcal{A}$,

$a - aa_0 \in \ker\varphi$, we have

$$ae - aa_0 = ae' - aa_0e' = (a - aa_0)e' = a - aa_0,$$

which implies that e is a right identity for $\mathcal{A}$. Now we show that $u = e - ee' \in \mathcal{A}$ satisfies the condition (3.1). In fact

$$\varphi(u) = \varphi(e - ee') = \varphi(e) - \varphi(ee') = 1.$$

Moreover, since for each $a \in \mathcal{A}$, $a - \varphi(a)e \in \ker\varphi$ thus

$$\begin{aligned} au - \varphi(a)u &= a(e - ee') - \varphi(a)(e - ee') \\ &= a - ae' - \varphi(a)e + \varphi(a)ee' \\ &= a - \varphi(a)e - \big(a - \varphi(a)e\big)e' \\ &= a - \varphi(a)e - \big(a - \varphi(a)e\big) \\ &= 0, \end{aligned}$$

and consequently (3.1) is satisfied. Now Corollary 3.4 implies that $\mathcal{A}$ is right φ-contractible. □

In the next result, we prove [12, Lemma 6.8] for Fréchet algebras.

Proposition 3.7. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be Fréchet algebras, and let I be a closed two-sided ideal of $\mathcal{A}$. Then the following assertions hold.*

(i) *Suppose that $\varphi \in \sigma(\mathcal{B})$ and that $h : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism with dense range. If $\mathcal{A}$ is right $\varphi \circ h$-contractible, then $\mathcal{B}$ is right φ-contractible. Moreover, if $\mathcal{A}$ is right character contractible, then $\mathcal{B}$ is right character contractible as well.*

(ii) *If $\mathcal{A}$ is right character contractible, then $\frac{\mathcal{A}}{I}$ is right character contractible.*

(iii) *If $\mathcal{A}$ is right character contractible, then I is right character contractible if and only if I has a right identity.*

Proof. (i). Let $\varphi \in \sigma(\mathcal{B})$, and let $\mathcal{A}$ be right $\varphi \circ h$-contractible. By Corollary 3.4, there exists $u \in \mathcal{A}$ such that

$$\varphi \circ h(u) = 1 \qquad \text{and} \qquad au = \varphi \circ h(a)u, \tag{3.2}$$

for all $a \in \mathcal{A}$. We show that $h(u)$ is an element of $\mathcal{B}$ satisfying the condition (3.1). By (3.2), we have $\varphi(h(u)) = 1$. Moreover, for b in the range of h, there is $a \in \mathcal{A}$ such that $b = h(a)$. Thus by (3.2)

$$bh(u) = h(a)h(u) = h(au) = h\big(\varphi \circ h(a)u\big) = \varphi \circ h(a)h(u) = \varphi(b)h(u).$$

Density of the range h in $\mathcal{B}$ and also the continuity of h implies that for each $b \in \mathcal{B}$, $bh(u) = \varphi(b)h(u)$. Thus, $h(u)$ satisfies the condition (3.1) and so $\mathcal{B}$ is right φ-contractible by Corollary 3.4. Now suppose that $\mathcal{A}$ is right character contractible. Thus it has a right identity e and $\mathcal{A}$ is right $\varphi \circ h$-contractible, for all $\varphi \in \sigma(\mathcal{B})$. It is easily verified that $h(e)$ is a right identity for $\mathcal{B}$. Moreover, $\mathcal{B}$ is right φ-contractible, for all $\varphi \in \sigma(\mathcal{B})$. Thus $\mathcal{B}$ is right character contractible by Proposition 3.3.

(ii). Consider the quotient map $q : \mathcal{A} \to \frac{\mathcal{A}}{I}$ defined by $a \mapsto a + I$, which is a continuous epimorphism. Thus $\frac{\mathcal{A}}{I}$ is right character contractible, by (i).

(iii). Suppose that I has a right identity e and that $\varphi \in \sigma(I)$. By Theorem 3.6 it suffices to show that $\ker\varphi$ has a right identity. Take $u \in I$ with $\varphi(u) = 1$, and define

$$\bar{\varphi} : \mathcal{A} \to \mathbb{C}, \qquad \bar{\varphi}(a) = \varphi(ua).$$

Since φ is continuous on I, there exist $C > 0$ and $\ell \in \mathbb{N}$ such that

$$\big|\bar{\varphi}(a)\big| = \big|\varphi(ua)\big| \leq Cp_\ell(ua) \leq Cp_\ell(u)p_\ell(a),$$

and so $\bar{\varphi}$ is continuous. On the other hand, for all $a, b \in \mathcal{A}$, $(uau - ua)$ and so $(uau - ua)b$ belong to $\ker\varphi$. Thus

$$\bar{\varphi}(ab) = \varphi(uab) = \varphi(uaub) = \varphi(ua)\varphi(ub) = \bar{\varphi}(a)\bar{\varphi}(b).$$

It follows that $\bar{\varphi} \in \sigma(\mathcal{A})$. Since $\mathcal{A}$ is right character contractible, Theorem 3.6 implies that $\ker\bar{\varphi}$ has a right identity denoted by $\bar{e}$. Then $e\bar{e}$ is a right identity for $\ker\varphi$. Indeed,

$$\begin{aligned}\varphi(e\bar{e}) &= \varphi(e\bar{e})\varphi(u)\\ &= \varphi(ue\bar{e}) = \bar{\varphi}(e\bar{e})\\ &= \bar{\varphi}(e)\bar{\varphi}(\bar{e}) = 0.\end{aligned}$$

It follows that $e\bar{e} \in \ker\varphi$. Moreover, for each $b \in \ker\varphi$,

$$b(e\bar{e}) = b\bar{e} = b,$$

and so $e\bar{e}$ is a right identity for $\ker\varphi$. The converse is obvious by Theorem 3.3(iii). $\square$

Example 3.8. Let P be a family of real-valued sequences such that for each $i \in \mathbb{N}$, there exists $p \in P$ with $p_i > 0$. Suppose also that P is directed; that is, for each $p, q \in P$ there exists $r \in P$ such that $r_i \geq \max\{p_i, q_i\}$ for all $i \in \mathbb{N}$. Recall from [15] that the Köthe sequence space $\lambda(P)$ is defined as follows:

$$\lambda(P) = \Big\{a = (a_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|a\|_p = \sum_i |a_i|p_i < \infty \ \forall p \in P\Big\}.$$

By [15, Lemma 10.1], there exists a unique continuous product on $\lambda(P)$ such that $e_ie_j = e_{\min\{i,j\}}$ for all $i, j \in \mathbb{N}$, where for each $i \in \mathbb{N}$, $e_i = (0, \ldots, 0, 1, 0, \ldots)$, where the single nonzero entry is in the ith slot. Now define P exactly as in the classical Köthe–Grothendieck example. Namely, for each $k \in \mathbb{N}$ we define an infinite matrix $\alpha^{(k)} = (\alpha^{(k)}_{ij})_{i,j\in\mathbb{N}}$ by setting

$$\alpha^{(k)}_{i,j} = \begin{cases} j^k & i < k,\\ i^k & i \geq k.\end{cases}$$

Fix a bijection $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that $\varphi(i, j + 1) < \varphi(i, j)$ for all $i, j \in \mathbb{N}$. For each $k \in \mathbb{N}$, define a sequence $p^{(k)} = (p^{(k)}_n)_{n\in\mathbb{N}}$ by $p^{(k)}_n = \alpha^{(k)}_{\varphi^{-1}(n)}$. Finally, set $P = \{p^{(k)} : k \in \mathbb{N}\}$. Since P is countable, $\lambda(P)$ is a Fréchet algebra, and in fact a commutative Fréchet algebra. By [15, Lemma 10.2], $\lambda(P)$ has no identity. Thus Theorem 3.3 implies that $\lambda(P)$ is not right character contractible.

4. Cartesian product of Fréchet algebras

Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be Fréchet algebras. Consider the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, and define

$$r_n(a,b) = \max\big\{p_n(a), q_n(b)\big\} \quad (a \in \mathcal{A}, b \in \mathcal{B}),$$

for each $n \in \mathbb{N}$. It is easily verified that (r_n) is a fundamental system of seminorms on $\mathcal{A}\times\mathcal{B}$ under which $(\mathcal{A}\times\mathcal{B}, r_n)$ is a Fréchet algebra (see also [14, pp. 276, 294]). For all $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$, define

$$\Theta_{(f,g)} : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$$

by

$$\Theta_{(f,g)}(a,b) = f(a) + g(b) \quad \big((a,b) \in \mathcal{A} \times \mathcal{B}\big).$$

It is easily verified that the dual space of $(\mathcal{A} \times \mathcal{B}, r_n)$ has the following form:

$$(\mathcal{A} \times \mathcal{B})^* = \{\Theta_{(f,g)} : f \in \mathcal{A}^*, g \in \mathcal{B}^*\}.$$

In this section, we first characterize $\sigma(\mathcal{A} \times \mathcal{B})$. Then we show that $\mathcal{A} \times \mathcal{B}$ is character contractible if and only if $\mathcal{A}$ and $\mathcal{B}$ are. It is in fact a generalization of [12, Lemma 6.8] for the Fréchet algebra case.

Proposition 4.1. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be two Fréchet algebras. Then*

$$\sigma(\mathcal{A} \times \mathcal{B}) = \big\{\Theta_{(\varphi,0)} : \varphi \in \sigma(\mathcal{A})\big\} \cup \big\{\Theta_{(0,\psi)} : \psi \in \sigma(\mathcal{B})\big\}.$$

Proof. It is clear that

$$\big\{\Theta_{(\varphi,0)} : \varphi \in \sigma(\mathcal{A})\big\} \cup \big\{\Theta_{(0,\psi)} : \psi \in \sigma(\mathcal{B})\big\} \subseteq \sigma(\mathcal{A} \times \mathcal{B}).$$

We prove the reverse of the inclusion. Suppose that $\theta \in \sigma(\mathcal{A} \times \mathcal{B})$, and define $\varphi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathcal{B} \to \mathbb{C}$ by $\varphi(a) = \theta(a,0)$ $(a \in \mathcal{A})$ and $\psi(b) = \theta(0,b)$ $(b \in \mathcal{B})$. Then

$$\theta(a,b) = \varphi(a) + \psi(b) \quad \big((a,b) \in \mathcal{A} \times \mathcal{B}\big).$$

Since θ is multiplicative, we thus obtain, for all $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$,

$$\varphi(a)\psi(b) + \varphi(c)\psi(d) = 0.$$

Choosing $d = 0$, we obtain $\varphi(a)\psi(b) = 0$, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If there is $b \in \mathcal{B}$ with $\psi(b) \neq 0$, then $\varphi = 0$ and so $\theta = \Theta_{(0,\psi)}$. Also, if there is $a \in \mathcal{A}$ with $\varphi(a) \neq 0$, then $\psi = 0$ and so $\theta = \Theta_{(\varphi,0)}$. Therefore the result is obtained. □

Proposition 4.2. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be two Fréchet algebras, and let $\varphi \in \sigma(\mathcal{A})$. Then $(\mathcal{A} \times \mathcal{B}, r_n)$ is right $\Theta_{(\varphi,0)}$-contractible if and only if $\mathcal{A}$ is right φ-contractible.*

Proof. First let $\mathcal{A} \times \mathcal{B}$ be right $\Theta_{(\varphi,0)}$-contractible, and define $h : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ by $(a,b) \to a$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then h is clearly a continuous epimorphism. Moreover, $\varphi \circ h = \Theta_{(\varphi,0)}$. By the hypothesis, $\mathcal{A} \times \mathcal{B}$ is right $(\varphi \circ h = \Theta_{(\varphi,0)})$-contractible. Then by Proposition 3.7, $\mathcal{A}$ is right φ-contractible. Conversely, suppose that $\mathcal{A}$ is right φ-contractible. By Corollary 3.4, there exists $u \in \mathcal{A}$ such that $\varphi(u) = 1$, and $au = \varphi(a)u$, for each $a \in \mathcal{A}$. It is easily verified that $(u, 0)$ is an element of $\mathcal{A} \times \mathcal{B}$, satisfying the condition (3.1). Therefore $\mathcal{A} \times \mathcal{B}$ is right $\Theta_{(\varphi,0)}$-contractible, by Corollary 3.4. □

Analogously, we have the following result.

Proposition 4.3. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be two Fréchet algebras, and let $\psi \in \sigma(\mathcal{B})$. Then $(\mathcal{A} \times \mathcal{B}, r_n)$ is right $\Theta_{(0,\psi)}$-contractible if and only if $\mathcal{B}$ is right ψ-contractible.*

We state here the main result of the present section, which provides an equivalent condition to right character contractibility of $\mathcal{A} \times \mathcal{B}$.

Theorem 4.4. *Let $(\mathcal{A}, p_\ell)$ and $(\mathcal{B}, q_m)$ be two Fréchet algebras. Then $(\mathcal{A} \times \mathcal{B}, r_n)$ is right character contractible if and only if $\mathcal{A}$ and $\mathcal{B}$ are.*

Proof. It is known that $\mathcal{A} \times \mathcal{B}$ has a right identity if and only if $\mathcal{A}$ and $\mathcal{B}$ have. Now the result is immediately obtained from Propositions 3.3, 4.1, 4.2, and 4.3. □

5. Results on Segal–Fréchet algebras

Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. According to [1], Fréchet algebra $(\mathcal{B}, q_m)$ is a Segal–Fréchet algebra in $(\mathcal{A}, p_\ell)$ if the following conditions are satisfied:

(i) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$;

(ii) the map

$$i : (\mathcal{B}, q_m) \longrightarrow (\mathcal{A}, p_\ell), \qquad a \mapsto a \quad (a \in \mathcal{B}) \tag{5.1}$$

is continuous;

(iii) the map

$$(\mathcal{B}, p_\ell) \times (\mathcal{B}, q_m) \longrightarrow (\mathcal{B}, q_m), \qquad (a,b) \mapsto ab \quad (a, b \in \mathcal{B}) \tag{5.2}$$

is jointly continuous.

It is not hard to see that the implication (5.2) implies that the map

$$(\mathcal{A}, p_\ell) \times (\mathcal{B}, q_m) \longrightarrow (\mathcal{B}, q_m), \qquad (a,b) \mapsto ab \quad (a \in \mathcal{A}, b \in \mathcal{B}) \tag{5.3}$$

is also jointly continuous. Note that the concept of Segal Fréchet algebra coincides with the concept of abstract Segal algebras, in the case where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras.

As the first result of this section, similar to [4, Lemma 2.2], we characterize $\sigma(\mathcal{B})$, whenever $(\mathcal{B}, q_m)$ is a Segal–Fréchet algebra in a Fréchet algebra $(\mathcal{A}, p_\ell)$.

Lemma 5.1. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra, and let $(\mathcal{B}, q_m)$ be a Segal–Fréchet algebra in $\mathcal{A}$. Then*

$$\sigma(\mathcal{B}) = \{\varphi|_{\mathcal{B}} : \varphi \in \sigma(\mathcal{A})\}.$$

Proof. First suppose that $\varphi \in \sigma(\mathcal{A})$, and let $\psi = \varphi|_{\mathcal{B}}$. By the density of $\mathcal{B}$ in $\mathcal{A}$, ψ is nonzero. Since $\varphi : (\mathcal{A}, p_\ell) \mapsto \mathbb{C}$ is continuous, there exist $\ell \in \mathbb{N}$ and $C > 0$ such that $|\psi(b)| = |\varphi(b)| \leq C p_\ell(b)$, for each $b \in \mathcal{B}$. On the other hand, by (5.1), for $\ell \in \mathbb{N}$, there exist $K_\ell > 0$ and $m_\ell \in \mathbb{N}$ such that $p_\ell(b) \leq K_\ell q_{m_\ell}(b)$, for each $b \in \mathcal{B}$. Thus

$$\big|\psi(b)\big| = \big|\varphi(b)\big| \leq C K_\ell q_{m_\ell}(b) \quad (b \in \mathcal{B}),$$

which implies that ψ is continuous on $(\mathcal{B}, q_m)$ and so $\psi \in \sigma(\mathcal{B})$, as claimed. Conversely, let $\psi \in \sigma(\mathcal{B})$. Then there exists $b_0 \in \mathcal{B}$ such that $\psi(b_0) = 1$. Thus for each $b \in \mathcal{B}$, $\psi(b) = \psi(bb_0)$. On the other hand, by continuity of ψ and also (5.3), for each $m \in \mathbb{N}$, there exist $C_m > 0$ and $\ell_m, m_0 \in \mathbb{N}$ such that

$$\big|\psi(b)\big| = \big|\psi(bb_0)\big| \leq C_m p_{\ell_m}(b) q_{m_0}(b_0),$$

for each $b \in \mathcal{B}$. It follows that $\psi : (\mathcal{B}, p_\ell) \to \mathbb{C}$ is continuous. Since $\mathcal{B}$ is dense in $\mathcal{A}$, [14, Lemma 22.19] implies that ψ can be extended to a continuous linear functional φ. Since $\mathcal{B}$ is dense in $\mathcal{A}$, it follows that $\varphi \in \sigma(\mathcal{A})$ and the proof is completed. □

The next result is a generalization of [4, Proposition 2.5] for the Fréchet case.

Proposition 5.2. *Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra, let $(\mathcal{B}, q_m)$ be a Segal–Fréchet algebra in $\mathcal{A}$, and let $\varphi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is right φ-contractible if and only if $\mathcal{B}$ is right $\varphi|_{\mathcal{B}}$-contractible.*

Proof. First suppose that $\mathcal{A}$ is right φ-contractible. Then by Corollary 3.4, there exists $u \in \mathcal{A}$ such that $\varphi(u) = 1$ and $au = \varphi(a)u$, for each $a \in \mathcal{A}$. Since $\mathcal{B}$ is a dense left ideal of $\mathcal{A}$, there exists $b_0 \in \mathcal{B}$ such that $\varphi(b_0) = 1$. Let $b_1 = ub_0 \in \mathcal{B}$. Now for each $b \in \mathcal{B}$, we have

$$\varphi(b_1) = \varphi|_{\mathcal{B}}(b_1) = \varphi|_{\mathcal{B}}(ub_0) = \varphi(u)\varphi(b_0) = 1.$$

Moreover, for each $b \in \mathcal{B}$, we have

$$bb_1 = bub_0 = \varphi(b)ub_0 = \varphi(b)b_1.$$

Thus $b_1 \in \mathcal{B}$ satisfies the condition (3.1), and so $\mathcal{B}$ is right $\varphi|_{\mathcal{B}}$-contractible by Corollary 3.4. Conversely, suppose that $\mathcal{B}$ is right $\varphi|_{\mathcal{B}}$-contractible. By Corollary 3.4, there exists $v \in \mathcal{B}$ such that $\varphi|_{\mathcal{B}}(v) = 1$ and $bv = \varphi|_{\mathcal{B}}(b)v$, for all $b \in \mathcal{B}$. Density of $\mathcal{B}$ in $\mathcal{A}$ implies that $av = \varphi(a)v$, for all $a \in \mathcal{A}$. It follows that v is an element in $\mathcal{A}$, satisfying the condition (3.1). Therefore $\mathcal{A}$ is right φ-contractible by Corollary 3.4. □

Remark 5.3. One side of Proposition 5.2 can be obtained from Proposition 3.7, immediately. Indeed, consider the identity map $\iota : \mathcal{B} \to \mathcal{A}$ which is a continuous homomorphism with dense range. Now Proposition 3.7 implies that for $\varphi \in \sigma(\mathcal{A})$, if $\mathcal{B}$ is right $(\varphi \circ \iota = \varphi|_{\mathcal{B}})$-contractible, then $\mathcal{A}$ is right φ-contractible.

Theorem 5.4. *Suppose that $(\mathcal{A}, p_\ell)$ is a Fréchet algebra and that $(\mathcal{B}, q_m)$ is a Segal–Fréchet algebra in $\mathcal{A}$. If $\mathcal{A}$ is right character contractible, then $\mathcal{B}$ is right $\varphi|_{\mathcal{B}}$-contractible, for all $\varphi \in \sigma(\mathcal{A})$. Moreover, $\mathcal{B}$ is right character contractible if and only if $\mathcal{B} = \mathcal{A}$, as two Fréchet algebras.*

Proof. The first part of the theorem is obtained by the definition of right character contractibility of $\mathcal{A}$ together with Proposition 5.2. Now suppose that $\mathcal{B}$ is right character contractible. By Theorem 3.3, $\mathcal{B}$ has a right identity. Some easy calculations imply that $\mathcal{B} = \mathcal{A}$, as two sets. Now the open mapping theorem (see [14, Theorem 8.5]) implies that $\mathcal{B} = \mathcal{A}$, as two Fréchet algebras. □

Example 5.5. Let G be a compact connected lie group, and let $C^\infty(G)$ be the algebra of infinitely differentiable functions on G. (We refer readers to [19] for full information about the construction of $C^\infty(G)$.) By [18, Example 8.4], $C^\infty(G)$ is a symmetric Segal–Fréchet algebra in the convolution algebra $L^1(G)$. On the other hand, let $\hat{G}$ denote the dual of the group G consisting of all continuous homomorphisms ρ from G into the circle group $\mathbb{T}$, and define $\varphi_\rho \in \sigma(L^1(G))$ to be the character induced by ρ on $L^1(G)$, that is,

$$\varphi_\rho(f) = \int_G \overline{\rho(x)} f(x)\, dx \quad \big(f \in L^1(G)\big).$$

It is known that there is no other character on $L^1(G)$, that is,

$$\sigma\big(L^1(G)\big) = \{\varphi_\rho : \rho \in \hat{G}\}$$

(see, e.g., [11, Theorem 23.6]). Moreover, by Lemma 5.1,

$$\sigma\big(C^\infty(G)\big) = \{\varphi_\rho|_{C^\infty(G)} : \rho \in \hat{G}\}.$$

Then by [4, Theorem 3.3], for each $\rho \in \hat{G}$, $m = \rho \otimes \rho$ is a right φ_ρ-diagonal for $L^1(G)$. Thus $L^1(G)$ is right φ_ρ-contractible, and hence by Proposition 5.2, $C^\infty(G)$ is a right φ_ρ-contractible Fréchet algebra.

Example 5.6. We explain here [18, Example 3.3(b)], which is a nice example in this field. Let X be a infinite countable set. A function $\ell : X \to [1, \infty)$ is a scale on X. We say that a scale ℓ on X is *proper* if the inverse image ℓ^{-1} takes bounded subsets of $[1, \infty)$ to finite subsets of X. For the family of scales $\ell = \{\ell^n\}_{n=0}^\infty$ on X, define

$$\mathcal{S}_\ell^\infty(X) = \big\{\varphi : X \to \mathbb{C}, \|\varphi\|_n^\infty < \infty, \forall n \in \mathbb{N}\big\},$$

where

$$\|\varphi\|_n^\infty = \sup_{x\in X}\big\{\ell^n(x)\big|\varphi(x)\big|\big\}.$$

Then $\mathcal{S}_\ell^\infty(X)$ is called the sup-norm ℓ-rapidly vanishing functions on X. The family $(\ell^n)_{n=0}^\infty$ will satisfy $\ell^0 \le \ell^1 \le \cdots \ell^n \le \cdots$, so that the families of norms $\{\|\cdot\|_n^\infty\}_{n=0}^\infty$ are increasing. Moreover, it is easy to see that all of them are submultiplicative under pointwise product. In fact $\mathcal{S}_\ell^\infty(X)$ is a Fréchet algebra. Now consider $c_0(X)$, the commutative Banach algebra of complex-valued sequences which vanish at infinity, with pointwise multiplication and sup-norm $\|\cdot\|_\infty$. We show that $\mathcal{S}_\ell^\infty(X) \subseteq c_0(X)$. Suppose on the contrary that there exists $\varphi \in \mathcal{S}_\ell^\infty(X)$ such that $\varphi \notin c_0(X)$. Thus there is $\varepsilon > 0$ such that for each finite subset F of X, there exists $x_F \notin F$ with $|\varphi(x_F)| \ge \varepsilon$. Since ℓ is a proper scale, thus for each $n \in \mathbb{N}$, there exists $x_n \notin \ell^{-1}([1, n])$ such that $|\varphi(x_n)| \ge \varepsilon$. It follows that

$$\sup_{x\in X}\big\{\ell(x)\big|\varphi(x)\big|\big\} = \infty,$$

which contradicts the assumption of $\varphi \in \mathcal{S}_\ell^\infty(X)$. Therefore $\mathcal{S}_\ell^\infty(X) \subseteq c_0(X)$. It is easy to see that the inequalities $\|\varphi\psi\|_n^\infty \leq \|\varphi\|_n^\infty \|\psi\|_\infty$ are satisfied, for all $\varphi, \psi \in \mathcal{S}_\ell^\infty(X)$. Since $\mathcal{S}_\ell^\infty(X)$ contains the space of finite support functions denoted by $c_{00}(X)$, it follows that $\mathcal{S}_\ell^\infty(X)$ is a dense Fréchet ideal in $c_0(X)$, and so $\mathcal{S}_\ell^\infty(X)$ is a Segal–Fréchet algebra in $c_0(X)$. It is known that

$$\sigma\big(c_0(X)\big) = \{\varphi_x : x \in X\},$$

where $\varphi_x(f) = f(x)$, for each $x \in X$. By Lemma 5.1,

$$\sigma\big(\mathcal{S}_\ell^\infty(X)\big) = \{\varphi_x|_{\mathcal{S}_\ell^\infty(X)} : x \in X\}.$$

Suppose that $x_0 \in X$, and take $f \in c_0(X)$ defined as $f(x_0) = 1$, and $f(x) = 0$ otherwise. Then $\varphi_{x_0}(f) = 1$. Moreover, $gf = \varphi_{x_0}(g)f$, for all $g \in c_0(X)$. In fact $f \in c_0(X)$ satisfies condition (3.1). In fact, Corollary 3.4 implies that $c_0(X)$ is right φ_x-contractible, for each $x \in X$. By Proposition 5.2, $\mathcal{S}_\ell^\infty(X)$ is right φ_x-contractible for all $x \in X$. Furthermore, since $c_0(X)$ does not have an identity, Proposition 3.3 implies that $c_0(X)$ is not right character contractible. In fact, by Theorem 5.4, the following assertions are equivalent:

(i) $c_0(X)$ is right character contractible,
(ii) $\mathcal{S}_\ell^\infty(X)$ is right character contractible, and
(iii) X is finite.

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