Ann. Funct. Anal. 8 (2017), no. 1, 51–62
http://dx.doi.org/10.1215/20088752-3749995
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

GEOMETRIC DESCRIPTION OF MULTIPLIER MODULES FOR HILBERT C^*-MODULES IN SIMPLE CASES

ZHU JINGMING

Communicated by J. Hamhalter

Abstract. In this article we suggest a vector bundle description for multiplier modules of vector bundles over noncompact spaces. We prove that the isomorphism classes of multiplier modules are dependent on the isomorphism classes of their underlying modules. This gives a way to evaluate the set of extensions of Hilbert modules in topological terms in simple cases.

1. Introduction

Multiplier modules of Hilbert C^*-modules are generalizations of multiplier algebras of C^*-algebras first studied in [2]. Instead of being all the adjointable operators from a C^*-algebra A to itself, multiplier modules of a Hilbert A-module E are the set of all adjointable operators from A to E. The set $M(E)$ of all multipliers of E is a Hilbert $M(A)$-module in a natural way. Similarly to the problem of classification of extensions of C^*-algebras, which uses the Busby invariant and can be described in KK-theory terms, classification of extensions of Hilbert C^*-modules uses an analog of the Busby invariant, which is a map into the outer multiplier module. But classification of extensions of Hilbert C^*-modules is much more difficult than that of C^*-algebras. One of the simplest cases was considered in [6], where extensions of a free singly generated Hilbert C^*-module were classified. This paper provides a tool for the next step: classification of extensions of finitely generated projective Hilbert C^*-modules over commutative C^*-algebras. We give a description of the multiplier modules for projective Hilbert C^*-modules of finite type over a nonunitial commutative C^*-algebra A. As is well known, a Hilbert

Received Mar. 27, 2016; Accepted Jun. 8, 2016.
2010 *Mathematics Subject Classification.* Primary 39B82; Secondary 44B20, 46C05.
Keywords. Hilbert modules, compactification, vector bundle.

A-module E of this type can be described by the cocycles for some open cover on the spectrum space of A. We show that the multiplier module of E can be described by the cocycles for a corresponding open cover on the spectrum space of the multiplier algebra $M(A)$.

2. Preliminaries

For a compact Hausdorff space Y and a locally compact Hausdorff path-connected space X, let $A = C_0(X \times Y)$. It is well known that all projections p in $K^0(Y)$ could be identified with isomorphism classes of vector bundles over Y, and any vector bundle V over Y could be identified with a pair (h_{ij}, Y_i) for some finite open cover $\{Y_i\}_i$ with $Y_{ij} = Y_i \cap Y_j$ with the transition matrix-valued functions $h_{ij} \in U_{n\times n}(C(Y_{ij}))$ for some $n \in \mathbb{N}$ acting on the local sections $f_i = \{f_i^{(1)}, f_i^{(2)}, \dots, f_i^{(n)}\} \in \bigoplus_{k=1}^n C(Y_i)$ of the bundle V. Because X is contractible, all projections p in $K^0(X \times Y)$ could be identified with isomorphism classes of vector bundles over $X \times Y$, and any vector bundle V over $X \times Y$ could be identified with a pair $(g_{ij}, X \times Y_i)$ for some finite open cover $\{X \times Y_i\}_i$ with $X \times Y_{ij} = (X \times Y_i) \cap (X \times Y_j)$ with the transition matrix-valued functions $g_{ij} \in U_{n\times n}(C(X \times Y_{ij}))$ for some $n \in \mathbb{N}$.

For every subset $X \times Y_i$, we say that $X \times Y_i$ is C^*-embedding into $X \times Y$ if every $f \in C_b(X \times Y_i)$ could extend to $C_b(X \times Y)$.

Remark 2.1. Although we usually use finite open cover $\{Y_i\}_i$ to describe the cocycles, in this paper we will use finite closed C^*-embedding cover instead to simplify the argument. From now on, the cover $\{Y_i\}_i$ will always be assumed to be finite, closed, and C^*-embedding.

For $[g_{ij}, X \times Y_i] \in K^0(X \times Y)$, let

$$V_{(g_{ij}, X\times Y_i)} = \Big\{\xi = (\xi_1, \dots, \xi_n) : \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \dots, \xi_i^{(n)}) \in \bigoplus_{k=1}^n C_0(X \times Y_i) \\ \text{s.t. } g_{ij} \cdot \xi_i|_{X\times Y_{ij}} = \xi_j|_{X\times Y_{ij}}\Big\},$$

where $g_{ij} \in U_{n\times n}(C(X \times Y_{ij}))$.

It is easy to see that V_p is a Hilbert A-module in a standard way. First, C^*-algebra A could be identified with

$$A \simeq \Big\{h = (h_1, \dots, h_n) : h_i = (h_i^{(1)}, h_i^{(2)}, \dots, h_i^{(n)}) \in \bigoplus_{k=1}^n C_0(X \times Y_i) \\ \text{s.t. } t_{ij} \cdot h_i|_{X\times Y_{ij}} = h_j|_{X\times Y_{ij}}\Big\},$$

and so V_p is a right A-module and its module structure is defined by

$$(\xi_i)_i \cdot (h_i)_i = (\xi_i \cdot h_i)_i \tag{2.1}$$

for $(\xi_i)_i \in V_p$ and $(h_i)_i \in A$ with $g_{ij}\xi_i \cdot h_i = g_{ij}\xi_i h_i$.

There is also a natural sesquilinear form defined on V_p by $\langle\cdot,\cdot\rangle : V_p \times V_p \to A$:

$$\begin{aligned}\big\langle(\xi_i)_i,(\eta_i)_i\big\rangle &= (\overline{\xi_i}\eta_i)_i = \big(\overline{(\xi_i^{(1)},\xi_i^{(2)},\dots,\xi_i^{(n)})}(\eta_i^{(1)},\eta_i^{(2)},\dots,\eta_i^{(n)})\big)_i\\ &= (\overline{\xi_i^{(1)}}\eta_i^{(1)},\overline{\xi_i^{(2)}}\eta_i^{(2)},\dots,\overline{\xi_i^{(n)}}\eta_i^{(n)})_i\end{aligned}$$

with $(\xi_i)_i,(\eta_i)_i \in V_p$ and $\overline{\xi_i}|_{X\times Y_{ij}} \cdot \overline{g_{ij}} \cdot g_{ij} \cdot \eta_i|_{X\times Y_{ij}} = \overline{\xi_j}\eta_j|_{X\times Y_{ij}}$.

We also have a norm $\|\cdot\|$ related to this sesquilinear form defined by

$$\big\|(\xi_i)_i\big\| = \max_{1\le i\le n}\Big\{\sup_{x\in X\times Y_i}\big|\big\langle\big(\xi_i(y)\big)_i,\big(\xi_i(y)\big)_i\big\rangle\big|^{\frac{1}{2}}\Big\}.$$

With all the structure above, V_p becomes a Hilbert A-module.

It is well known that the multiplier algebra $M(A)$ is the closure of A with respect to the strict topology defined by the seminorms

$$a \to \|ab\|, \qquad a \to \|ba\|, \quad \forall a \in A$$

for any $b \in A$.

The following theorem is from Pedersen's book [7].

Theorem 2.2 (see [7, p. 84])**.** *For a C^*-algebra A, its completion with respect to the strict topology is its multiplier algebra $M(A)$.*

Remark 2.3. Then any $m \in M(A)$ could be identified with a Cauchy net $\{m_\lambda\}$ in A such that

$$ma = \lim_{\lambda\to\infty} m_\lambda a, \quad \forall a \in A \tag{2.2}$$

because $A = C_0(X \times Y)$ is commutative. Corresponding to the same $m \in M(A)$, there could be many Cauchy nets satisfying the condition (2.2). Here we introduce the "standard" one. Let $\{e_\lambda\}_\lambda$ be a self-adjoint approximate identity in A. For any $m \in M(A)$, it is easy to see that $\{me_\lambda\}_\lambda$ is a Cauchy net in A with the condition (2.2).

Similarly to A, we can identify the multiplier algebra $M(A)$ as

$$\begin{aligned}M(A) = \Big\{f = (f_1,\dots,f_n) : f_i = (f_i^{(1)},f_i^{(2)},\dots,f_i^{(n)}) \in \bigoplus_{k=1}^{n} C_b(X\times Y_i)\\ \text{s.t. } f_i|_{X\times Y_{ij}} = f_j|_{X\times Y_{ij}}\Big\}.\end{aligned}$$

It is also worth noting that the multiplier algebra $M(A)$ could also be identified by the set of all the A-linear adjointable maps from A to itself. Similarly to the case of C^*-algebras, for the Hilbert A-module V_p, the multiplier module $M(V_p)$ could also be identified by the set of all the A-linear adjointable maps from A to V_p; that is, $L(A, V_p) \doteq M(V_p)$.

With all the preparation above, now we can give a geometric description of multiplier modules for some special Hilbert C^*-module.

3. Geometric description of multiplier modules

Since we have given the cocycle description for the module V_p in the previous section, our aim in this section is to give a cocycle description for the multiplier module $M(V_p)$. For this reason, we first give the following notation.

For the C^*-algebra A and Hilbert A-module V_p in Section 2, we denote

$$\beta V_p = \Big\{ \xi = (\xi_1, \dots, \xi_n) : \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \dots, \xi_i^{(n)}) \in \bigoplus_{k=1}^{n} C_b(X \times Y_i) \\ \text{s.t. } g_{ij} \cdot \xi_i|_{X \times Y_{ij}} = \xi_j|_{X \times Y_{ij}} \Big\} \tag{3.1}$$

in which $\{Y_i\}_i$ is a finite closed cover of Y with all the Y_i's contractible and $g_{ij} \in U_{n\times n}(C_b(X \times Y_{ij}))$.

Similarly to V_p, it is easy to see βV_p is a (right) $M(A)$-module with the module action defined as follows:

$$(\xi_i)_i \cdot (f_i)_i = (\xi_i \cdot f_i)_i \tag{3.2}$$

in which $\xi = (\xi_1, \dots, \xi_n) \in \beta V_p$ with $g_{ij} \cdot \xi_i|_{X \times Y_{ij}} = \xi_j|_{X \times Y_{ij}}$ and $(f_i)_i \in M(A)$ with trivial cocycle condition $h_i|_{X \times Y_{ij}} = h_j|_{X \times Y_{ij}}$. There is also a natural sesquilinear form defined on βV_p by $\langle \cdot, \cdot \rangle : \beta V_p \times \beta V_p \to M(A)$:

$$\begin{aligned} \big\langle (\xi_i)_i, (\eta_i)_i \big\rangle &= (\overline{\xi_i} \eta_i)_i = \big(\overline{(\xi_i^{(1)}, \xi_i^{(2)}, \dots, \xi_i^{(n)})} (\eta_i^{(1)}, \eta_i^{(2)}, \dots, \eta_i^{(n)}) \big)_i \\ &= \big((\overline{\xi_i^{(1)}} \eta_i^{(1)}), (\overline{\xi_i^{(2)}} \eta_i^{(2)}), \dots, (\overline{\xi_i^{(n)}} \eta_i^{(n)}) \big)_i \end{aligned} \tag{3.3}$$

for $(\xi_i)_i$, $(\eta_i)_i$ being the sections of βV_p, in which $\overline{\xi_i}|_{X \times Y_{ij}} \cdot \overline{g_{ij}} \cdot g_{ij} \cdot \eta_i|_{X \times Y_{ij}} = \overline{\xi_j} \eta_j|_{X \times Y_{ij}}$.

We also have a norm $\|\cdot\|$ related to this sesquilinear form defined by

$$\big\| (\xi_i)_i \big\| = \max_{1 \le i \le n} \Big\{ \sup_{y \in X \times Y_i} \big| \big\langle \big(\xi_i(y)\big)_i, \big(\xi_i(y)\big)_i \big\rangle \big|^{\frac{1}{2}} \Big\}.$$

With all the structure above, βV_p becomes a Hilbert $M(A)$-module.

Proposition 3.1. *The (right) $M(A)$-module βV_p is a Hilbert $M(A)$-module with respect to the sesquilinear form (3.3).*

Proof. This result follows from all the structure mentioned above if we can show that βV_p is closed with respect to the norm. Suppose $\xi_{(\alpha)} = (\xi_{1,(\alpha)}, \dots, \xi_{n,(\alpha)})$ is a Cauchy net in βV_p. Then, by definition, for each i, $\xi_{i,(\alpha)}$ is a Cauchy net with respect to the supremum norms in $\bigoplus_{k=1}^{n} C_b(X \times Y_i)$, respectively. But $\bigoplus_{k=1}^{n} C_b(X \times Y_i)$ is complete with respect to their own supremum norms, and so there are $(\xi_i)_i \in \bigoplus_{k=1}^{n} C_b(X \times Y_i)$ s.t. $\xi_{i,(\alpha)} \to \xi_i$ for any i, which means that $(\xi_{i,(\alpha)})_i \to (\xi_i)_i$ and βV_p is complete with respect to the norm above. $\square$

The first main result of this section is the following theorem, which claims that the module βV_p is just the multiplier module of V_p.

Theorem 3.2. *For the module βV_p, $M(V_p)$ defined above, $\beta V_p = M(V_p)$.*

In order to prove the theorem, we first show that $\beta V_p \subset M(V_p)$.

Proposition 3.3. *There is a natural inclusion $V_p \subset \beta V_p \subset M(V_p)$.*

Proof. The first inclusion is obvious, and it suffices to prove the second one. By the formula (2.1), the left multiplier of $(\xi_i)_i \in \beta V_p$ on $M(V_p)$ defines an A-linear map from A to V_p. It is easy to see it is a bounded operator, and so we only need to show adjointability. For any $(\xi_i)_i \in \beta V_p$, $h = (h_i)_i \in A$, and $(\mathfrak{h}_i)_i \in V_p$ with conditions $g_{ij} \cdot \xi_i|_{X\times Y_{ij}} = \xi_j|_{X\times Y_{ij}}$, $g_{ij} \cdot \mathfrak{h}_i|_{X\times Y_{ij}} = \mathfrak{h}_j|_{X\times Y_{ij}}$, and $h_i|_{X\times Y_{ij}} = h_j|_{X\times Y_{ij}}$,

$$\begin{aligned}\big\langle (\mathfrak{h}_i)_i, (\xi_i)_i(h_i)_i \big\rangle &= \big\langle (\mathfrak{h}_i)_i, (\xi_i h_i)_i \big\rangle \\ &= (\overline{\mathfrak{h}_i}\xi_i h_i)_i = \big\langle (\mathfrak{h}_i\overline{\xi_i})_i, (h_i)_i \big\rangle = \big\langle (\mathfrak{h}_i)_i\overline{(\xi_i)_i}, (h_i)_i \big\rangle \end{aligned} \tag{3.4}$$

with $\langle (\mathfrak{h}_i)\overline{(\xi_i)}, (h_i)\rangle \in C_0(X \times Y_i)$. Then $\langle (\mathfrak{h}_i)_i\overline{(\xi_i)_i}, (h_i)_i\rangle \in A$ and the adjoint operator of $(\xi_i)_i$ is $(\xi_i)_i^* = \overline{(\xi_i)}_i$, and hence there is a natural inclusion $\beta V_p \subset L(A, V_p) \doteq M(V_p)$. □

Remark 3.4. By the inclusion and the inner product on $M(V_p)$, we identify $r(v)$ as $\langle r, v\rangle$ for any $r \in M(V_p)$ and $v \in V_p$. And for $r \in M(V_p)$, $x \in \beta V_p$, and $v \in V_p$, the inner products $\langle r, x\rangle$, $\langle r, v\rangle$, and $\langle x, v\rangle$ are well defined.

Remark 3.5. It is worth pointing out that, after multiplication by A, the inclusion becomes the equality $M(V_p)A = \beta V_p A = V_p$. First, any $x \in M(V_p)$, and also xa for any $a \in A$, could be also viewed as an A-linear adjointable map from A to V_p. Then $xa(b) = x(ab) = x(a)b$ for any $b \in A$, and hence $M(V_p)A \subset V_p$. Finally, $M(V_p)A = \beta V_p A = V_p$ since $V_p A = V_p$ implies that $M(V_p)A = V_p$.

For a Hilbert B-module V and an essential ideal I in B, we denote the $V_I \doteq \{vb : v \in V, b \in I\}$. Recall the following definition.

Definition 3.6 (see [2, Definition 1.3]). Let V be a Hilbert B-module, let I be an essential ideal in B, and let V_I be the associated ideal submodule. The strict topology with respect to V_I (or the V_I-strict topology) on V is defined by two families of seminorms $v \to \|\langle v, x\rangle\|$ for any $x \in V_I$ and $v \to \|vb\|$ for any $b \in B$.

Similarly to the case of multiplier algebras for C^*-algebras, Bakić and Guljăs proved the following results.

Theorem 3.7 (see [2, Proposition 1.6])**.** *Let V be a Hilbert B-module, let I be an essential ideal in B, and let V_I be the essential ideal submodule. Each V_I-strict Cauchy net in V determines an adjointable map $v \in M(V_I)$.*

Remark 3.8. In Theorem 3.7, if we let $B = I$, then $V_I = V$, and hence each V-strict Cauchy net in V determines an adjointable map $v \in M(V)$.

Theorem 3.9 (see [2, Theorem 1.8])**.** *Let W be a full Hilbert I-module. $M(W)$ is a W-strict completion of W.*

In this article, for $I = A$, $B = M(A)$, $W = V_p$, $V = M(V_p)$, and hence $V_I = M(V_p)A$, we have the corresponding corollaries since A is an essential ideal in $M(A)$.

Corollary 3.10. *Each V_p-strict Cauchy net determines an adjointable map $v \in M(V_p)$.*

Corollary 3.11. *It holds that $M(V_p)$ is the V_p-strict completion of V_p.*

Remark 3.12. Since A is an essential ideal in $M(A)$, by Remark 3.4, $M(V_p)A = V_p$. Then the corollary above could be rewritten as "$M(V_p)$ is strict completion of V_p with respect to V_p strict topology."

By Corollary 3.10 and Definition 3.6, any $r \in M(V_p)$ could be identified as a V_p-strict Cauchy net $\{r_\lambda\}_\lambda$ such that

(1) $\langle r, x\rangle = \lim_\lambda \langle r_\lambda, x\rangle, \forall x \in V_p$,
(2) $rb = \lim_\lambda r_\lambda b, \forall b \in A$.

Corresponding to any $r \in M(A)$, there could be many V_p-strict Cauchy nets $\{r_\lambda\}_\lambda$ with the conditions above. Similarly to the case of $M(A)$, we introduce a "standard" one. Let $\{e_\lambda\}_\lambda$ be a self-adjoint approximate identity in A. For any $r \in M(V_p)$, it is easy to see that $\{re_\lambda\}_\lambda$ is a Cauchy net in V_p with the condition (1) and (2).

Lemma 3.13. *For any $r \in M(V_p)$, there exists a $w \in \beta V_p$ such that*

(1) $\langle w, x\rangle = \langle r, x\rangle$, $\forall x \in \beta V_p$
(2) $rb = wb$, $\forall b \in A$.

Proof. For each $b \in A$, $\{re_\lambda b\}_\lambda$ is a Cauchy net in V_p. After restricting on $X \times Y_i$, $\{(re_\lambda b)_i\}_\lambda$ is a Cauchy net in $\bigoplus_{k=1}^m C_0(X \times Y_i)$ for each i. Since $b \in A$ is arbitrarily chosen, $\{(re_\lambda)_i\}_\lambda$ strictly converges to an element $w_i \in \bigoplus_{k=1}^m C_b(X \times Y_i)$. By the definition, $re_\lambda \in V_p$ with $g_{ij} \cdot (re_\lambda)_i = (re_\lambda)_j$, and so $w = (w_1, \dots, w_n) \in \beta V_p$ with $g_{ij} \cdot w_i = w_j$, and hence $\lim_{\lambda\to\infty} re_\lambda b = wb, \forall b \in A$.

For each $x \in V_p$, $\{\langle re_\lambda, x\rangle\}_\lambda$ is a Cauchy net in A. Again after restricting on $X \times Y_i$, $\{(\langle re_\lambda, x\rangle)_i\}_\lambda = \{(\overline{re_\lambda})_i x_i\}_\lambda$ becomes a Cauchy net in $\bigoplus_{k=1}^m C_0(X \times Y_i)$ for each i. Since $x \in V_p$ is arbitrarily chosen, $\{(re_\lambda)_i\}_\lambda$ strictly converges to the same element $w_i \in \bigoplus_{k=1}^m C_b(X \times Y_i)$ as above. Then $\lim_{\lambda\to\infty} \langle re_\lambda, x\rangle = \langle w, x\rangle, \forall x \in V_p$. □

The lemma above could be rewritten as follows.

Lemma 3.14. *We have that βV_p is closed with respect to strict topology.*

By taking the closure of the inclusion $V_p \subset \beta V_p \subset M(V_p)$ with respect to the V_p-strict topology, it is easy to prove Theorem 3.2.

The second main result of this section is the following theorem, which declares that the module βV_p we constructed is a vector bundle.

Theorem 3.15. *The module βV_p is a vector bundle over $\beta(X \times Y)$.*

In order to prove the theorem, we need the following lemmas.

Lemma 3.16. *We have*

$$\bigcup_{i=1}^n \beta(X \times Y_i) = \beta(X \times Y).$$

Proof. Since, in Section 2, all the $X \times Y_i$'s are assumed to be C^*-embedded into $X \times Y$, by [8], $\beta(X \times Y_i)$ equals the closure of $X \times Y_i$ in $\beta(X \times Y)$; that is,

$\beta(X \times Y_i) = cl_{\beta(X\times Y)}(X \times Y_i) \subset \beta(X \times Y)$. For the inverse, $\forall x \in \beta(X \times Y)$, $\exists\{x_\lambda\}_\lambda$ a Cauchy net in $X \times Y$ which converges to x. Since $\bigcup_{i=1}^n (X \times Y_i) = X \times Y$, there exists at least one of them, for example, $X \times Y_{i_0}$, that contains a subnet of $\{x_\lambda\}_\lambda$ which also converges to x. Then $x \in \beta(X \times Y_{i_0})$, and hence $\bigcup_{i=1}^n \beta(X \times Y_i) = \beta(X \times Y)$. □

As a special case, after taking the closure in $\beta(X \times Y)$, we have the corollary.

Corollary 3.17. *We have*

$$\beta(X \times Y_i) \cup \beta(X \times Y_j) = \beta\big((X \times Y_i) \cup (X \times Y_j)\big).$$

For the intersection, we have another lemma.

Lemma 3.18. *We have*

$$\beta(X \times Y_i) \cap \beta(X \times Y_j) = \beta\big((X \times Y_i) \cap (X \times Y_j)\big) = \beta(X \times Y_{ij}).$$

Proof. Since $\beta(X \times Y_i) \cup \beta(X \times Y_j) = \beta((X \times Y_i) \cup (X \times Y_j))$, $C(\beta(X \times Y_i) \cup \beta(X \times Y_j)) = C(\beta((X \times Y_i) \cup (X \times Y_j)))$. Because $C(\beta(X \times Y_i) \cup \beta(X \times Y_j))$ could be identified as

$$\big\{(f_i, f_j) : f_k \in C\big(\beta(X \times Y_k)\big), f_i|_{\beta(X\times Y_i)\cap\beta(X\times Y_j)} = f_j|_{\beta(X\times Y_i)\cap\beta(X\times Y_j)}, k = i, j\big\}$$

and $C(\beta((X \times Y_i) \cup (X \times Y_j)))$ could be identified as

$$\big\{(g_i, g_j) : g_k \in C_b(X \times Y_k), g_i|_{X\times Y_{ij}} = g_j|_{X\times Y_{ij}}, k = i, j\big\},$$

or in another form

$$\big\{(g_i, g_j) : g_k \in C\big(\beta(X \times Y_k)\big), g_i|_{X\times Y_{ij}} = g_j|_{X\times Y_{ij}}, k = i, j\big\},$$

then

$$g_i|_{X\times Y_{ij}} = g_j|_{X\times Y_{ij}} \Leftrightarrow g_i|_{\beta(X\times Y_i)\cap\beta(X\times Y_j)} = g_j|_{\beta(X\times Y_i)\cap\beta(X\times Y_j)}. \tag{3.5}$$

For $k = i, j$, because $g_k \in C(\beta(X \times Y_k))$ and $\beta(X \times Y_{ij}) \subset \beta(X \times Y_k)$,

$$g_i|_{X\times Y_{ij}} = g_j|_{X\times Y_{ij}} \Leftrightarrow g_i|_{\beta(X\times Y_{ij})} = g_j|_{\beta(X\times Y_{ij})}. \tag{3.6}$$

Combining (3.5) and (3.6) together, because the functions g_k's are arbitrarily chosen, the lemma follows. □

Then the space $\beta X \times Y$ has a (not necessarily open) cover $\{\beta(X \times Y_i)\}_i$ with the intersections $\beta(X \times Y_i) \cap \beta(X \times Y_j) = \beta(X \times Y_{ij})$. Compared with (3.1), the module βV_p could also be identified as

$$\beta V_p = \Big\{\xi = (\xi_1, \ldots, \xi_n) : \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(n)}) \in \bigoplus_{k=1}^n C\big(\beta(X \times Y_i)\big)$$
$$\text{s.t. } \beta g_{ij} \cdot \xi_i|_{\beta(X\times Y_{ij})} = \xi_j|_{\beta(X\times Y_{ij})}\Big\},$$

in which $\beta g_{ij} \in C(\beta(X \times Y_{ij}))$ is the unique function such that $\beta g_{ij}|_{X\times Y_{ij}} = g_{ij}$, and hence Theorem 3.15 follows.

4. The isomorphism classes of multiplier modules

The isomorphism class of βV_p defines an element $[\beta V_p]$ in $K^0(\beta(X\times Y))$. It is a natural question whether they are the same in $K^0(\beta(X\times Y))$, or, in another word, whether $[\beta V_p]=[\beta V_q]$ when $p\neq q$. In this section, we will give a negative answer as follows.

Theorem 4.1. *We have*

$$[\beta V_p]\neq[\beta V_q] \quad \text{in } K^0\big(\beta(X\times Y)\big) \text{ when } p\neq q \text{ in } K^0(Y).$$

We postpone the proof until the end of this section. For a cover $\{Y_i\}_{i=1}^n$ of Y and any two projections $p\neq q\in K^0(Y)$, as in Section 2, p and q could be identified as two isomorphism classes of vector bundles $[Z_p]$ and $[Z_q]$ with the cocycle conditions (p_{ij},Y_i) and (q_{ij},Y_i), respectively; that is,

$$\begin{aligned} Z_p=\Big\{&f=(f_1,\dots,f_n): f_i=(f_i^{(1)},f_i^{(2)},\dots,f_i^{(n)})\in\bigoplus_{k=1}^n C(Y_i),\\ &\text{s.t. } p_{ij}f_i|_{Y_{ij}}=f_j|_{Y_{ij}}\Big\}, \end{aligned}\tag{4.1}$$

$$\begin{aligned} Z_q=\Big\{&f=(f_1,\dots,f_n): f_i=(f_i^{(1)},f_i^{(2)},\dots,f_i^{(n)})\in\bigoplus_{k=1}^n C(Y_i),\\ &\text{s.t. } q_{ij}f_i|_{Y_{ij}}=f_j|_{Y_{ij}}\Big\}. \end{aligned}\tag{4.2}$$

Thus if $p\neq q$ in $K^0(Y)$, then $[Z_p]\neq[Z_q]$.

Corresponding to the same cover $\{Y_i\}_{i=1}^n$ of Y and the cocycles (p_{ij},Y_i) and (q_{ij},Y_i), there is a cover $\{\beta X\times Y_i\}_{i=1}^n$ of $\beta X\times Y$ and the cocycles $(\widehat{p_{ij}},\beta X\times Y_i)$ and $(\widehat{q_{ij}},\beta X\times Y_i)$ with $\widehat{p_{ij}}\in C(\beta X\times Y_{ij})$ defined by $\widehat{p_{ij}}(x,y)=p_{ij}(y)$ and $\widehat{q_{ij}}\in C(\beta X\times Y_{ij})$ defined by $\widehat{q_{ij}}(x,y)=q_{ij}(y)$.

Then corresponding to Z_p and Z_q, over $\beta X\times Y$, there are vector bundles

$$\begin{aligned} W_p=\Big\{&f=(f_1,\dots,f_n): f_i=(f_i^{(1)},f_i^{(2)},\dots,f_i^{(n)})\in\bigoplus_{k=1}^n C(\beta X\times Y_i),\\ &\text{s.t. } \widehat{p_{ij}}f_i|_{\beta X\times Y_{ij}}=f_j|_{\beta X\times Y_{ij}}\Big\}, \end{aligned}\tag{4.3}$$

$$\begin{aligned} W_q=\Big\{&f=(f_1,\dots,f_n): f_i=(f_i^{(1)},f_i^{(2)},\dots,f_i^{(n)})\in\bigoplus_{k=1}^n C(\beta X\times Y_i),\\ &\text{s.t. } \widehat{q_{ij}}f_i|_{\beta X\times Y_{ij}}=f_j|_{\beta X\times Y_{ij}}\Big\}. \end{aligned}\tag{4.4}$$

Since $X\times Y_{ij}$ is dense in $\beta X\times Y_{ij}$, W_p and W_q could be identified as

$$\begin{aligned} W_p=\Big\{&f=(f_1,\dots,f_n): f_i=(f_i^{(1)},f_i^{(2)},\dots,f_i^{(n)})\in\bigoplus_{k=1}^n C(\beta X\times Y_i),\\ &\text{s.t. } \widehat{p_{ij}}f_i|_{X\times Y_{ij}}=f_j|_{X\times Y_{ij}}\Big\}, \end{aligned}\tag{4.5}$$

$$W_q = \Big\{ f = (f_1, \ldots, f_n) : f_i = (f_i^{(1)}, f_i^{(2)}, \ldots, f_i^{(n)}) \in \bigoplus_{k=1}^{n} C(\beta X \times Y_i),$$
$$\text{s.t. } \widehat{q_{ij}} f_i|_{X \times Y_{ij}} = f_j|_{X \times Y_{ij}} \Big\}. \tag{4.6}$$

Since X path connected and so is βX, we can choose any point $x_0 \in \beta X$ as a base point, and we can identify Y as $\{x_0\} \times Y$, the cover $\{Y_i\}_i$ as $\{\{x_0\} \times Y_i\}_i$, and hence $K^0(Y)$ as $K^0(\{x_0\} \times Y)$. If we denote the natural surjective map from $\beta X \times Y$ onto $\{x_0\} \times Y$ by h_1, it induces an injective map h_1^* from $Vect(\{x_0\} \times Y)$, the set of vector bundles over $\{x_0\} \times Y$, to $Vect(\beta X \times Y)$, and hence an injective homomorphism from $K^0(\{x_0\} \times Y) \simeq K^0(Y)$ to $K^0(\beta X \times Y)$ which is also denoted by h_1^* when there is no confusion.

Lemma 4.2. *We have $h_1^*([Z_p]) = [W_p]$ and $h_1^*([Z_q]) = [W_q]$.*

Proof. By the definition of h_1^*, $h_1^*(Z_p)$ is the pullback of Z_p to the base space $\beta X \times Y$. Then, according to (4.1), after identifying $\{x_0\} \times Y$ as Y and $\{\{x_0\} \times Y_i\}_i$ as $\{Y_i\}$,

$$h_1^*(Z_p) = \Big\{ h_1^*(f) = \big(h_1^*(f_1), \ldots, h_1^*(f_n)\big) :$$
$$h_1^*(f_i) = \big(h_1^*(f_i^{(1)}), h_1^*(f_i^{(2)}), \ldots, h_1^*(f_i^{(n)})\big) \in \bigoplus_{k=1}^{n} C(\beta X \times Y_i)$$
$$\text{s.t. } h_1^*(p_{ij}) h_1^*(f_i)|_{\beta X \times Y_{ij}} = h_1^*(f_j)|_{\beta X \times Y_{ij}} \Big\}. \tag{4.7}$$

Because $h_1^*([Z_p]) = [h_1^*(Z_p)]$, by (4.3), we have $h_1^*([Z_p]) = [W_p]$. Replacing p by q in the argument above, the second result follows. □

Because h_1^* is injective and $p \neq q \in K^0(Y)$, we have the following.

Corollary 4.3. *For $p \neq q \in K^0(Y)$, $[W_p] \neq [W_q]$.*

Since both $\beta X \times Y$ and $\beta(X \times Y)$ are compact and contain $X \times Y$ as a dense subset, by the property of Stone–Čech compactification, there is a continuous surjective map h_2 from $\beta(X \times Y)$ to $\beta X \times Y$ with $h_2|_{X \times Y} = id$, which induces an injective homomorphism h^* from $C(\beta X \times Y)$ to $C(\beta(X \times Y))$. In fact, h_2 also induces another injective homomorphism h_2^* from $Vect(\beta X \times Y)$, the set of complex bundles over $\beta X \times Y$, to $Vect(\beta(X \times Y))$, and hence a homomorphism from $K^0(\beta X \times Y)$ to $K^0(\beta(X \times Y))$. These two maps would be still denoted by h_2^* when there is no confusion.

Lemma 4.4. *Given that $X \times Y_i$ is dense in $h_2^{-1}(\beta X \times Y_i)$, we have $h_2^{-1}(\beta X \times Y_i) = \beta(X \times Y_i)$.*

Proof. By the definition of h_2 and the property of the Stone–Čech compactification, there are surjective continuous maps f_i from $\beta X \times Y_i$ onto $X \times Y_i$, $\forall i$ and the commutative diagrams:

$$\begin{array}{ccc} X \times Y \to \beta(X \times Y) & & X \times Y_i \to \beta(X \times Y_i) \\ \searrow \quad \downarrow{\scriptstyle h_2} & \text{and} & \searrow \quad \downarrow{\scriptstyle f_i} \\ \beta X \times Y & & \beta X \times Y_i \end{array} \tag{4.8}$$

If we denote by t the inclusion $\beta(X\times Y_i)(=cl_{\beta(X\times Y)}(X\times Y_i))\hookrightarrow\beta(X\times Y)$ and denote by t' the inclusion $\beta X\times Y_i\hookrightarrow\beta X\times Y$, together with the diagrams above, we have their corresponding commutative diagrams of algebras:

$$\begin{array}{ccccccc} C_0(X\times Y) & \twoheadleftarrow & C_b(X\times Y) & \overset{t^*}{\twoheadrightarrow} & C_b(X\times Y_i) & \twoheadrightarrow & C_0(X\times Y_i) \\ \| & & \uparrow h_2^* & & \uparrow h_2^*|_{C_b(X)\otimes C(Y_i)} & & \| \\ C_0(X\times Y) & \twoheadleftarrow & C_b(X)\otimes C(Y) & \overset{t'^*}{\twoheadrightarrow} & C_b(X)\otimes C(Y_i) & \twoheadrightarrow & C_0(X\times Y_i) \end{array} \tag{4.9}$$

The right square in the diagram above gives a commutative diagram of spaces

$$\begin{array}{ccc} X\times Y_i & \rightarrow & \beta(X\times Y_i) \\ & \searrow & \downarrow l \\ & & \beta X\times Y_i \end{array}$$

in which $l^*=h_2^*|_{C_b(X)\otimes C(Y_i)}$. Then, by (4.9) and $\beta(X\times Y_i)=cl_{\beta(X\times Y)}(X\times Y_i)$, we have the commutative diagram of spaces

$$\begin{array}{ccccccc} X\times Y & \hookrightarrow & \beta(X\times Y) & \overset{t}{\hookleftarrow} & cl_{\beta(X\times Y)}(X\times Y_i) & \hookleftarrow & X\times Y_i \\ \| & & \downarrow h_2 & & \downarrow l & & \| \\ X\times Y & \hookrightarrow & \beta X\times Y & \overset{t'}{\hookleftarrow} & \beta X\times Y_i & \hookleftarrow & X\times Y_i \end{array}$$

Then $h_2|_{cl_{\beta(X\times Y)}}(X\times Y_i)=l$. After identifying $\beta X\times Y_i$ as $t'(\beta X\times Y_i)$, we have $h_2^{-1}(\beta X\times Y_i)=l^{-1}(\beta X\times Y_i)=\beta(X\times Y_i)$, and hence $X\times Y_i$ is dense in $h_2^{-1}(\beta X\times Y_i)$. □

Lemma 4.5. *We have*

$$h_2^*(W_p)\in[\beta V_p] \qquad \textit{and} \qquad h_2^*(W_q)\in[\beta V_q].$$

Proof. By the definition of h_2^*, $h_2^*(W_p)$ is the pullback of W_p to the base space $\beta(X\times Y)$. Then, according to (4.5),

$$\begin{aligned} h_2^*(W_p)=\Big\{ & h_2^*(f)=\big(h_2^*(f_1),\dots,h_2^*(f_n)\big): \\ & h_2^*(f_i)=\big(h_2^*(f_i^{(1)}),h_2^*(f_i^{(2)}),\dots,h_2^*(f_i^{(n)})\big)\in\bigoplus_{k=1}^n C\big(h_2^{-1}(\beta X\times Y_i)\big) \\ & \text{s.t. } h_2^*(p_{ij})h_2^*(f_i)|_{h_2^{-1}(X\times Y_{ij})}=h_2^*(f_j)|_{h_2^{-1}(X\times Y_{ij})}\Big\}. \end{aligned} \tag{4.10}$$

By Lemma 4.4, $h_2^*(W_p)$ could be identified as

$$\begin{aligned} h_2^*(W_p)=\Big\{ & h_2^*(f)=\big(h_2^*(f_1),\dots,h_2^*(f_n)\big): \\ & h_2^*(f_i)=\big(h_2^*(f_i^{(1)}),h_2^*(f_i^{(2)}),\dots,h_2^*(f_i^{(n)})\big)\in\bigoplus_{k=1}^n C\big(\beta(X\times Y_i)\big) \\ & \text{s.t. } h_2^*(p_{ij})h_2^*(f_i)|_{X\times Y_{ij}}=h_2^*(f_j)|_{X\times Y_{ij}}\Big\}. \end{aligned} \tag{4.11}$$

Because $C_b(X\times Y_i)$ could be identified as $C(\beta(X\times Y_i))$, compared with (3.1), we have $h_2^*(W_p)\in[\beta V_p]$ and similarly $h_2^*(W_q)\in[\beta V_q]$. □

Remark 4.6. It is well known that, for the cover $\{\beta(X \times Y_i)\}_i$, the vector bundles determined by two cocycles $(h_2^*(p_{ij}), \beta(X \times Y_i))$ and $(h_2^*(q_{ij}), \beta(X \times Y_i))$ are isomorphic if and only if they are equivalent systems of transition functions; that is, for each i, there exists $r_i \in U_n(\beta(X \times Y_i))$ such that

$$h_2^*(p_{ij})|_{\beta(X\times Y_{ij})} = r_i h_2^*(q_{ij}) r_i^{-1}|_{\beta(X\times Y_{ij})}. \tag{4.12}$$

Now we can begin to prove Theorem 4.1.

Proof of Theorem 4.1. If $[\beta V_p] = [\beta V_q]$, by Lemma 4.5, the vector bundle $h_2^*(W_p)$ is isomorphic to $h_2^*(W_q)$. By Remark 4.6, there exists $r_i \in U_n(\beta(X \times Y_i))$ with the condition (4.12). But if $X \times Y_{ij}$ is dense in $\beta(X \times Y_{ij})$, then the condition (4.12) could be identified as

$$h_2^*(p_{ij})|_{X\times Y_{ij}} = r_i h_2^*(q_{ij}) r_i^{-1}|_{X\times Y_{ij}}. \tag{4.13}$$

By the definition of h_2, $\widehat{p_{ij}}$, and $\widehat{q_{ij}}$, $h_2^*(\widehat{p_{ij}})|_{X\times Y_{ij}} = p_{ij}$ and $h_2^*(\widehat{q_{ij}})|_{X\times Y_{ij}} = q_{ij}$. If we denote $r_i' = r_i|_{X\times Y_{ij}} \in U_n(C_b(X \times Y_{ij}))$, then we have

$$p_{ij}|_{X\times Y_{ij}} = r_i' q_{ij} r_i'^{-1}|_{X\times Y_{ij}}, \tag{4.14}$$

which induces the equivalence of the cocyles $(p_{ij}, X \times Y_i)$ and $(q_{ij}, X \times Y_i)$. Again by Remark 4.6, we have the isomorphism between Z_p and Z_q, and hence $p = q$, which is a contradiction to the assumption in the theorem. □

Remark 4.7. In the argument above, we use a map h_2 from $\beta(X \times Y)$ to $\beta X \times Y$ with $h_2|_{X\times Y} = id_{X\times Y}$. Generally, the map h_2 is not surjective, or, in other words, $\beta(X \times Y)$ and $\beta X \times Y$ might not be the same [4].

Acknowledgments. The author wishes to thank the reviewers for their careful reading and valuable comments.

This work was supported in part by National Natural Science Foundation of China (NSFC) grant no. 11326104 and by the Fundamental Research Funds of Jiaxing University grant no. 70513016. The author also thanks the Harbin Institute of Technology, where most of this article was written, for its warm hospitality.

References

1. D. Bakić and B. Guljaš, *Extensions of Hilbert C^*-modules II*, Glas. Mat. Ser. III **38** (2003), 341–357. Zbl 1057.46046. MR2052751. DOI 10.3336/gm.38.2.12.
2. D. Bakić and B. Guljaš, *Extensions of Hilbert C^*-modules*, Houston J. Math. **30** (2004), 537–558. Zbl 1069.46032. MR2084917. 51, 55
3. C. H. Dowker, *Mapping theorems for noncompact spaces*, Amer. J. Math. **69** (1947), 200–242. Zbl 0037.10101. MR0020771.
4. I. Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), no. 3, 369–382. Zbl 0089.38702. MR0105667. 61
5. D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966. Zbl 0144.44804. MR0229247.
6. V. M. Manuilov and J. Zhu, *Extensions of Hilbert C^*-modules: Classification in simple cases*, Russ. J. Math. Phys. **19** (2012), 197–202 Zbl 1285.46045. MR2926324. DOI 10.1134/S1061920812020069. 51
7. G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, Academic Press, New York, 1979. MR0548006. 53
8. R. Walker, *The Stone-Čech Compactification*, Ergeb. Math. Grenzgeb. **83**, Springer, Berlin, 1974. Zbl 0292.54001. MR0380698. 56

College of Mathematics, Physics and Information Engineering, Jiaxing Universitiy, Jiaxing 314000, Zhejiang Province, P.R. China.
E-mail address: zhujingming_math@163.com