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EXTREMALLY RICH JB*-TRIPLES

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Abstract. We introduce and study the class of extremally rich JB*-triples. We establish new results to determine the distance from an element a in an extremally rich JB*-triple E to the set $\partial_e(E_1)$ of all extreme points of the closed unit ball of E. More concretely, we prove that

$$\operatorname{dist}\big(a, \partial_e(E_1)\big) = \max\big\{1, \|a\| - 1\big\},$$

for every $a \in E$ which is not Brown–Pedersen quasi-invertible. As a consequence, we determine the form of the λ-function of Aron and Lohman on the open unit ball of an extremally rich JB*-triple E by showing that $\lambda(a) = 1/2$ for every non-BP quasi-invertible element a in the open unit ball of E. We also prove that for an extremally rich JB*-triple E, the quadratic conorm $\gamma^q(\cdot)$ is continuous at a point $a \in E$ if and only if either a is not von Neumann regular (i.e., $\gamma^q(a) = 0$) or a is Brown–Pedersen quasi-invertible.

1. Introduction

This article presents new investigations which provide some answers to problems concerning the geometric structure of those complex Banach spaces included in the class of JB*-triples. In 1983, Kaup proved that the open unit ball of a complex Banach space X is a bounded symmetric domain (a pure holomorphic property) if and only if X is a JB*-triple (see [13]). That is, the holomorphic properties

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of the open unit ball of X determine when X satisfies certain algebraic-geometric axioms, which are listed below. In [9], Harris proved that the open unit ball of a C*-algebra A is a bounded symmetric domain; actually, A is a JB*-triple with triple product defined by

$$\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x). \tag{1.1}$$

JB*-triples have been intensively studied during the last three decades, and special attention has been paid to the geometric properties of these spaces. In several cases, the studies determine whether JB*-triples satisfy certain properties fulfilled by C*-algebras. For example, the quasi-invertible elements of C*-algebras studied by Brown and Pedersen in [3] gave rise to the introduction of the Brown–Pedersen quasi-invertible elements in a JB*-triple, which are found in [19] and [20]. Building on work of Aron and Lohman in [2], Brown and Pedersen also showed in [4] that quasi-invertible elements in C*-algebras play a crucial role in determining the form of the λ-function. The precise description of the λ-function is determined in [4]. A C*-algebra A is said to be extremally rich if the set A_q^{-1} of quasi-invertible elements in A is norm-dense in A (see [3, Section 3]). The class of extremally rich C*-algebras is strictly larger than the class of von Neumann algebras. From the geometric point of view, a unital C*-algebra A is extremally rich if and only if it has the (uniform) λ-property, that is, if the infimum of the values of the λ-function on the closed unit ball of A is greater than zero (see [3, Section 3] and [4, Theorem 3.7]).

In a recent paper [11], we proved that every JBW*-triple (i.e. a JB*-triple which is also a dual Banach space) satisfies the uniform λ-property. In that same paper we also determined the λ-function on the closed unit ball of a JBW*-triple and on the set E_q^{-1} of all Brown–Pedersen quasi-invertible elements in the closed unit ball of a general JB*-triple E. If we assume that the set $\partial_e(E_1)$ of all extreme points in the closed unit ball E_1 of E is nonempty, then we can only prove that the inequality

$$\lambda(a) \leq \frac{1}{2}\big(1 - \alpha_q(a)\big) \tag{1.2}$$

holds for every $a \in E_1 \backslash E_q^{-1}$, where $\alpha_q(a)$ is the distance from a to E_q^{-1} (see [11, Corollary 3.7]).

The question whether in (1.2) the inequality sign can be replaced with an equality symbol is one of the main open problems in the setting of JB*-triples. This question is related to the problem of determining the distance from an element a to the set $\partial_e(E_1)$. The best estimation follows from Theorem 3.6 in [11], where it is established that for every JB*-triple E with $\partial_e(E_1) \neq \emptyset$, the inequalities

$$1 + \|a\| \geq \operatorname{dist}\big(a, \partial_e(E_1)\big) \geq \max\big\{1 + \alpha_q(a), \|a\| - 1\big\}$$

hold for every a in $E \backslash E_q^{-1}$.

We introduce here the notion of extremally rich JB*-triples, with the aim of studying the above problems in more depth. We devote some effort to clarifying the relationships between the different uses of the notion of quasi-invertibility found in the literature. We also consider the notion of Brown–Pedersen quasi-invertible elements, introduced by Brown and Pedersen in the setting of C*-algebras and by the last two authors in the case of JB*-triples. We clarify the relationship between the concept of Brown–Pedersen quasi-invertible elements and the notion of Jordan quasi-invertibility developed in the monograph [15].

We say that a JB*-triple E is *extremally rich* if the set of Brown–Pedersen quasi-invertible elements in E is norm-dense in E. Several characterizations of extremally rich JB*-triples are provided in Proposition 2.4. Among the new results in this article, we prove that for an extremally rich JB*-triple E, we have

$$\operatorname{dist}\big(a, \partial_e(E_1)\big) = \max\big\{1, \|a\| - 1\big\}$$

for every $a \in E\backslash E_q^{-1}$ (see Theorem 2.6). As a consequence, we show that $\lambda(a) = 1/2$ for every non-BP quasi-invertible element a in the open unit ball of an extremally rich JB*-triple (see Corollary 2.7).

We also deal with another related open question. We recall that the *reduced minimum modulus* of a nonzero bounded linear or conjugate linear operator T between two normed spaces X and Y is defined by

$$\gamma(T) := \inf\big\{\big\|T(x)\big\| : \operatorname{dist}\big(x, \ker(T)\big) \geq 1\big\}. \tag{1.3}$$

Following [12], we set $\gamma(0) = \infty$. When X and Y are Banach spaces, we have that $\gamma(T) > 0 \Leftrightarrow T(X)$ is norm-closed (see [12, Theorem IV.5.2]).

The quadratic-conorm, $\gamma^q(a)$, of an element a in a JB*-triple E is defined as the reduced minimum modulus of the conjugate linear operator $Q(a) : E \to E$, $x \mapsto Q(a)(x) := \{a, x, a\}$; that is, $\gamma^q(a) = \gamma(Q(a))$ (see [5]). Theorem 3.13 in [5] proves that $\gamma^q(\cdot)$ is upper semicontinuous on $E\backslash\{0\}$. It is also remarked, in the reference quoted above, that the continuity points of $\gamma^q(\cdot)$ are, in general, unknown. In the present article we shed some light on the question of determining the continuity points of the quadratic conorm, showing that for an extremally rich JB*-triple E, the quadratic conorm $\gamma^q(\cdot)$ is continuous at a point $a \in E$ if and only if either a is not von Neumann regular (i.e., $\gamma^q(a) = 0$) or a is BP quasi-invertible (see Theorem 3.3). We also explore the applications of this result to determine the continuity points of the conorm of an extremally rich C*-algebra in the sense introduced by Harte and Mbekhta in [10].

1.1. **Preliminaries.** A complex Banach space E is a JB*-triple if it can be equipped with a triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$, $(x, y, z) \mapsto \{x, y, z\}$, which is linear and symmetric in x and z, conjugate linear in y, and satisfies the following axioms:

(a) (Jordan identity)

$$\big\{x, y, \{a, b, c\}\big\} = \big\{\{x, y, a\}, b, c\big\} - \big\{a, \{y, x, b\}, c\big\} + \big\{a, b, \{xyc\}\big\},$$

for every $a, b, c \in E$;

(b) for each $a \in E$, the operator $x \mapsto L(a, a)(x) := \{a, a, x\}$ is hermitian with nonnegative spectrum;

(c) $\|\{x, x, x\}\| = \|x\|^3$, for all $x \in E$.

The class of JB*-triples includes all C*-algebras, all complex Hilbert spaces, and all spin factors. It is further known that every JB*-algebra is a JB*-triple with the triple product $\{x, y, z\} := (x \circ y^*) \circ z - (x \circ z) \circ y^* + (y^* \circ z) \circ x$.

A *JBW*-triple* is a JB*-triple which is also a dual Banach space. The second dual E^{**} of a JB*-triple E is a JBW*-triple (see [7, Corollary 3.3.5]). Every JBW*-triple W admits a unique isometric predual W_*, and its triple product is separately $\sigma(W, W_*)$-continuous (see [7, Theorem 3.3.9, p. 210]).

JB*-triples are stable by ℓ_∞-sums (see [13, p. 523]); that is, if (E_j) is a family of JB*-triples, then the ℓ_∞-sum $\oplus_j^\infty E_j$ is a JB*-triple with respect to the product

$$\big\{(a_j), (b_j), (c_j)\big\} := \big(\{a_j, b_j, c_j\}\big).$$

Following standard notation, given two elements x, y in a JB*-triple E, the conjugate linear operator $Q(x, y) : E \to E$ is defined by $Q(x, y)z := \{x, z, y\}$ for all $z \in E$; we usually write $Q(x)$ instead of $Q(x, x)$. The symbol $L(x, y)$ will stand for the linear operator on E given by $L(x, y)(z) = \{x, y, z\}$.

We recall that an element e in a JB*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. It is known that, for each tripotent e in E, the eigenvalues of the operator $L(e, e)$ are contained in the set $\{0, 1/2, 1\}$, and E decomposes in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where, for $i = 0, 1, 2$, $E_i(e)$ is the $\frac{i}{2}$-eigenspace of $L(e, e)$. This decomposition is called the *Peirce decomposition* of E with respect to e. The Peirce subspaces appearing in the above decomposition satisfy certain multiplication rules (called *Peirce rules*), which can be stated as follows:

$$\big\{E_i(e), E_j(e), E_k(e)\big\} \subseteq E_{i-j+k}(e)$$

if $i - j + k \in \{0, 1, 2\}$, and it is zero otherwise. In addition, $\{E_2(e), E_0(e), E\} = 0$. The projection of E onto $E_k(e)$ is denoted by $P_k(e)$, and it is called the *Peirce k-projection*. Peirce projections are contractive (see [7, Lemma 3.2.1]) and satisfy $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and $P_0(e) = \mathrm{Id}_E - 2L(e, e) + Q(e)^2$. A tripotent e in E is said to be *unitary* if $L(e, e)$ coincides with the identity map on E—that is, $E_2(e) = E$. If $E_0(e) = \{0\}$, then we say that e is *complete*.

The Peirce space $E_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x, e, y\}$, and involution $x^{*_e} := \{e, x, e\}$, respectively. Furthermore, the triple product on $E_2(e)$ is given by

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e} \quad \big(a, b, c \in E_2(e)\big).$$

Let a be an element in a JB*-triple E, and let E_a denote the JB*-subtriple of E generated by a. That is, E_a coincides with the closed linear span of the elements a, $a^{[3]} = \{a, a, a\}$, $a^{[2n+1]} := \{a, a, a^{[2n-1]}\}$ $(n \geq 2)$. It follows from the *commutative Gelfand theory* that there exist a locally compact Hausdorff space $L_a \subseteq (0, \|a\|]$, with $L_a \cup \{0\}$ compact, and a triple isomorphism $\Psi_a : E_a \to C_0(L_a)$, where $C_0(\Omega_x)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0, such that $\Psi_a(a)(t) = t$, $\forall t \in L_a$ (see [13, Section 1, Corollary 1.15]). Therefore, for each natural n, there exists a unique

$a^{[1/(2n-1)]} \in E_a$ satisfying $(a^{[1/(2n-1)]})^{[2n-1]} = a$. The sequence $(a^{[1/(2n-1)]})$ need not be convergent in the norm topology of E. However, when a is an element in a JBW*-triple W, the sequence $(a^{[1/(2n-1)]})$ converges in the weak* topology of W to a tripotent in W, which is denoted by $r(a)$ and is called the *range tripotent* of a. The tripotent $r(a)$ is the smallest tripotent e in W such that a is positive in the JBW*-algebra $W_2(e)$ (see [8, Section 3, Lemma 3.2]).

The deep geometric-algebraic connections appearing in the setting of JB*-triples materialize in many important properties, one of them ensures that complete tripotents in a JB*-triple E coincide with the extreme points of its closed unit ball (see [7, Theorem 3.2.3]). Throughout this article, the set of all extreme points of the closed unit ball X_1 of a Banach space X is denoted by $\partial_e(X_1)$.

1.2. **Quasi-invertibility.** Let E be a JB*-triple. The Bergmann operator, $B(x, y)$, associated with a couple of elements $x, y \in E$ is the mapping defined by $B(x, y) := \mathrm{Id} - 2L(x, y) + Q(x)Q(y)$, where Id is the identity operator on E. We observe that, for a tripotent $e \in E$, $B(e, e) = P_0(e)$. When a C*-algebra A is regarded as a JB*-triple with the product in (1.1), then the identity

$$B(x, y)(z) = (1 - xy^*)z(1 - y^*x) \tag{1.4}$$

holds for every $x, y, z \in A$.

Following the notation introduced by Brown and Pedersen in [3, Theorem 1.1 and p. 118], we say that an element a in a unital C*-algebra A is *quasi-invertible* if a belongs to the set $A^{-1}\partial_e(A_1)A^{-1}$, where A^{-1} denotes the group of invertible elements in A. The open subset of quasi-invertible elements in A is denoted by A_q^{-1}. It was shown in [3, Theorem 1.1] that $A_q^{-1} = \partial_e(A_1)A_+^{-1}$, and that an element a lies in A_q^{-1} if and only if there is a pair of closed ideals I, J of A, such that $IJ = \{0\}$, $a + I$ is left-invertible in A/I, and $a + J$ is right invertible in A/J.

A celebrated result (see [17, Theorem 1.6.4]), due to Kadison, proves that a norm 1 element $v \in A$ is an extreme point of A_1 if and only if v is a partial isometry such that $(1 - vv^*)A(1 - v^*v) = 0$. Suppose that $a = cvd \in A_q^{-1}$, where $v \in \partial_e(A_1)$ and $c, d \in A^{-1}$. Taking $b = (c^{-1})^*v(d^{-1})^* \in A$, we deduce from (1.4) that

$$B(a, b)(z) = (1 - ab^*)z(1 - b^*a) = c(1 - vv^*)c^{-1}zd^{-1}(1 - v^*v)\, d = 0$$

for every $z \in A$. Conversely, Ara, Pedersen, and Perera observed in [1, end of p. 611] that, for each $a \in A$, the existence of an element $b \in A$ such that $\{0\} = (1 - ab^*)A(1 - b^*a) = B(a, b)(A)$ implies that a is quasi-invertible. The last two authors of the present article show up this equivalence in [19, Theorem 3.1] by proving that an element a in a unital C*-algebra A is quasi-invertible if and only if there exists $b \in A$ with $B(a, b) = 0$. It should be also remarked here that for a suitable left-invertible element $a \in A$, we can find different elements $b_1 \neq b_2 \in A$ with $b_j^*a = 1$, and hence $B(a, b_j) = 0$ for every $j = 1, 2$.

The results in the above paragraphs motivated the last two authors to introduce the notion of Brown–Pedersen quasi-invertibility in the wider setting of JB*-triples. According to [19], [20], an element x in a JB*-triple E is called

Brown–Pedersen quasi-invertible (BP quasi-invertible for short) if there exists $y \in E$ satisfying $B(x,y) = 0$. In the conditions above, we say that y is a *BP quasi-inverse of* x. It is known that $B(x,y) = 0 \Rightarrow B(y,x) = 0$. A BP quasi-invertible element need not admit a unique BP quasi-inverse. It is established in [20] that an element x in E is BP quasi-invertible if and only if there exists a complete tripotent $v \in E$ ($v \in \partial_e(E_1)$) such that x is positive and invertible in the Peirce 2-space $E_2(v)$. Therefore, the set E_q^{-1} of all BP quasi-invertible elements in E contains all extreme points of the closed unit ball of E. When $E = \mathcal{J}$ is a JB*-algebra, $\mathcal{J}_q^{-1}$ contains the set $\mathcal{J}^{-1}$ of all invertible elements in E.

When a C*-algebra A is regarded as a JB*-triple, BP quasi-invertible elements in A are precisely the quasi-invertible elements of A in the sense defined by Brown and Pedersen in [3].

Remark 1.1. In the setting of Jordan algebras there exists another meaning for the term "quasi-invertible". Following Definition 1.3.1 in the monograph [15], an element x in a Jordan algebra $\mathcal{J}$ is called *quasi-invertible* (or *Jordan quasi-invertible* to avoid confusion) if $(\widehat{1} - x)$ is invertible (in the Jordan sense) in the unital hull $\widehat{\mathcal{J}}$ of $\mathcal{J}$; the element $w = \widehat{1} - (\widehat{1} - x)^{-1}$ is called the *Jordan quasi-inverse of* x, and it is denoted by $qi(x)$.

The unit of a JB*-algebra is not Jordan quasi-invertible. In the commutative C*-algebra $C[0,1]$, an element f is Jordan quasi-invertible if and only if 1 is not in the image of f. In $\mathcal{J} = M_n(\mathbb{C})$, a matrix a is Jordan quasi-invertible if and only if $\det(I_n - a) \neq 0$. In the last two C*-algebras there are examples of invertible elements which are not Jordan quasi-invertible, and examples of Jordan quasi invertible elements which are not invertible. In particular, there is no relation between the notions of Jordan quasi-invertibility and Brown–Pedersen quasi-invertibility in the setting of C*-algebras.

There is another connection between Jordan quasi-invertibility for pairs and Brown–Pedersen quasi-invertibility. Namely, by [15, Definition 1.4.1(1)], a pair (x,y) of elements in a Jordan algebra $\mathcal{J}$ is called a *Jordan quasi-invertible pair* if x is Jordan quasi-invertible in the homotope $J^{(y)}$. It is shown in the same reference that (x,y) is a Jordan quasi-invertible pair if and only if the Bergmann operator $B(x,y)$ is an invertible linear operator on the space $\mathcal{J}$. For each $x \in \mathcal{J}$, we have $B(x,0) = I_{\mathcal{J}}$, and hence the pair $(x,0)$ is Jordan quasi-invertible. When A is a unital C*-algebra, $B(1,\lambda 1) = (1-\overline{\lambda})^2 I_A$ is an invertible linear operator on A for every $\lambda \neq 1$. That is, the pair $(1,\lambda 1)$ is Jordan quasi-invertible for every $\lambda \neq 1$. Elements x in a JB*-triple E for which there exists $y \in E$ such that $B(x,y)$ is an invertible linear operator on E do not receive a special name in the literature, and there is no link between these elements and Brown–Pedersen quasi-invertible elements.

After introducing the basic results on quasi-invertibility, and clarifying the relationship between the different concepts established in the literature, we recall that an element a in a JB*-triple E is called *von Neumann regular* if and only if there exists $b \in E$ such that $Q(a)b = a$, $Q(b)a = b$, and $[Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0$ (see [14, Lemma 4.1]). For a von Neumann regular element a, there might exist many elements c in E such that $Q(a)c = a$. However, there exists a unique element $b \in E$ (denoted by $a^\dagger$) satisfying $Q(a)b = a$, $Q(b)a = b$ and $[Q(a), Q(b)] = 0$; this unique element b is called the *generalized inverse* of a in E. For an element a in a JB*-triple E, we can consider the range tripotent, $r(a)$, of a in E^{**}. It is known that a is von Neumann regular if and only if $r(a) \in E$ and a is positive and invertible in $E_2(r(a))$ (see [5, Section 2, pp. 191–192]).

2. Extremally Rich JB*-triples

In [3, Section 3], Brown and Pedersen introduced and studied the class of extremally rich C*-algebras. We recall that a unital C*-algebra A is *extremally rich* if the set A_q^{-1} of Brown–Pedersen quasi-invertible elements in A is (norm-) dense in A. When A is nonunital, it is extremally rich if its unitization enjoys this property. Every von Neumann algebra and every purely infinite (simple) C*-algebra is extremally rich (see [3, Section 3]). From the point of view of Banach space theory, a unital C*-algebra is extremally rich if and only if it has the (uniform) λ-property defined by Aron and Lohman in [2] (see also [3, Section 3] and [4, Theorem 3.7]).

We recall that, given a normed space X, $x, y \in X$, with $\|y\| \leq 1$, $e \in \partial_e(X_1)$, and $0 < \lambda \leq 1$, the ordered 3-tuple (e, y, λ) is said to be *amenable* to x if $x = \lambda e + (1 - \lambda)y$. The *$\lambda$-function* is defined by

$$\lambda(x) := \sup\big\{\lambda : (e, y, \lambda) \text{ is a 3-tuple amenable to } x\big\}.$$

The space X is said to have the *λ-property* if each element in its closed unit ball admits an amenable 3-tuple (see [2]).

The notion of Brown–Pedersen quasi-invertibility in JB*-triples was recently studied in [19], [20] and [21]. The study of the λ-function in JB*-triples was developed in [21] and [11], where it was proved that every JBW*-triple (i.e., a JB*-triple which is a dual Banach space) satisfies the (uniform) λ-property. We introduce the following definition with the aim of determining those JB*-triples satisfying the (uniform) λ-property.

Definition 2.1. A JB*-triple E is called *extremally rich* if the set E_q^{-1} of BP quasi-invertible elements in E is norm-dense in E.

Recall that an element u in a unital JB*-algebra $\mathcal{J}$ is called *unitary* if $u^* = u^{-1}$ (where u^{-1} denotes the inverse of u), or equivalently, if $\{u, u, z\} = z$, $\forall z \in \mathcal{J}$ (see [6, Definition 4.1.53, Propositions 4.1.54, 4.1.55]); that is, $L(u, u) = I_{\mathcal{J}}$ (the identity operator over $\mathcal{J}$).

Remark 2.2. (a) We recall that a unital C*-algebra A is of *topological stable rank 1* (tsr 1) if the subgroup A^{-1} of invertible elements in A is norm-dense in A (see [16]). A similar definition is introduced in the category of JB*-algebras in [18].

If $\mathcal{J}$ is a JB*-algebra of tsr 1, then $\mathcal{J} = \overline{\mathcal{J}^{-1}} \subseteq \overline{\mathcal{J}_q^{-1}}$. This shows that every JB*-algebra $\mathcal{J}$ of tsr 1 is extremally rich. There exist examples of extremally rich C*-algebras which are not of tsr 1. For example, suppose that A is a von Neumann algebra that contains a nonunitary, maximal partial isometry (say, v) which is a nonunitary extreme point of A_1. Then, $v \in \partial_e(A_1) \neq \mathcal{U}(A)$, which implies that A

is not of tsr 1 (see [18, Corollary 6.10]). On the other hand, every von Neumann algebra is extremally rich (see [3, p. 126]).

(b) It should be also noted that the von Neumann envelope of a JB*-algebra of tsr 1 need not be, in general, of tsr 1 (see [18, Theorems 3.1, 3.2]).

(c) Let A be a C*-algebra. Then A is extremally rich as a C*-algebra if and only if A is extremally rich when it is regarded as a JB*-triple with the product in (1.1).

Since the class of extremally rich C*-algebras is strictly bigger than the class of von Neumann algebras, we can immediately confirm that the class of extremally rich JB*-triples is agreeably large, strictly bigger than the class of JBW*-triples. In our next result we establish some characterizations of extremally rich JB*-triples along the lines set down by Brown and Pedersen for C*-algebras in [3, Theorem 3.3]. To that end, we recall a result taken from [11]. First, we recall that for each element a in a JB*-triple E, $m_q(a) := \operatorname{dist}(a, E\backslash E_q^{-1})$ coincides with the square root of the quadratic conorm of a, whenever a is in E_q^{-1} (see [11, Theorem 3.1]).

Proposition 2.3 ([11, Proposition 4.4])**.** *Let a and b be elements in a JB*-triple E. Suppose that $\|a-b\| < \beta$ and $b \in E_q^{-1}$. Then $a+\beta r(b) \in E_q^{-1}$ and the inequality*

$$m_q\big(a + \beta r(b)\big) \geq \beta - \|b - a\|$$

holds. Furthermore, under the above conditions, the element $P_2(r(b))(a) + \beta r(b)$ is invertible in the JB-algebra $E_2(r(b))$.*

The promised characterization of extremally rich JB*-triples reads as follows.

Proposition 2.4. *For a JB*-triple E with $\partial_e(E_1) \neq \emptyset$, the following conditions are equivalent.*

(i) *E is extremally rich.*
(ii) *For every $a \in E$ and any $\beta > 0$, there is an element $b \in E_q^{-1}$, with range tripotent $r(b) \in \partial_e(E_1)$, such that $a + \beta r(b) \in E_q^{-1}$.*
(iii) *For every $a \in E$ and any $\beta > 0$, there is an element $b \in E_q^{-1}$ such that $P_2(r(b))(a) + \beta r(b)$ is invertible in the JB*-algebra $E_2(r(b))$.*

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) follow from the above Proposition 2.3 (see [11, Proposition 4.4] and [20, Theorem 6]). The implication (ii) ⇒ (i) is clear from the definition of extremal richness and the arbitrariness of β. For (iii) ⇒ (ii), fix $a \in E$ and $\beta > 0$. By assumption there exists $b \in E_q^{-1}$ such that $P_2(r(b))(a) + \beta r(b) \in E_2(r(b))$ is invertible in the JB*-algebra $E_2(r(b))$. Since $r(b)$ is an extreme point of E and $P_2(r(b))(a + \beta r(b)) = P_2(r(b))(a) + \beta r(b)$ is invertible in the JB*-algebra $E_2(r(b))$, it follows from [11, Lemma 2.2] that $a + \beta r(b) \in E_q^{-1}$, and clearly $\|a - (a + \beta r(b))\| = \beta$. □

We explore next the stability of the property of being extremally rich under ℓ_∞-sums, ideals, and quotients. We recall that a (closed) subtriple I of a JB*-triple E is said to be an *ideal* of E if $\{E, E, I\} + \{E, I, E\} \subseteq I$. It is known that I is an ideal if and only if $\{E, E, I\} \subseteq I$ or $\{E, I, E\} \subseteq I$.

Theorem 2.5. *Every quotient of an extremally rich JB*-triple is extremally rich. Let (E_j) be a family of JB*-triples. Then an element $a = (a_j) \in E = \oplus_j^{\infty} E_j$ is BP quasi-invertible if and only if a_j is BP quasi-invertible in E_j for every j. Consequently, the ℓ_∞-sum $E = \oplus_j^{\infty} E_j$ is an extremally rich JB*-triple if and only if each E_j is extremally rich.*

Proof. Let E be an extremally rich JB*-triple, and let J be a closed ideal of E. Let $\pi : E \to E/J$, $\pi(x) = x + J$, denote the canonical projection of E onto E/J. Since π is a surjective triple homomorphism, it follows that $\pi(E_q^{-1}) \subseteq (E/J)_q^{-1}$ (see [20, Theorem 3]). Let us take an element $x \in E$. By hypothesis, there exists a sequence (x_n) in E_q^{-1} converging to x in the norm topology of E. Since π is continuous, $\pi(x_n) \to \pi(x)$ in norm, which proves that $\pi(x) \in \overline{(E/J)_q^{-1}}$, and hence $\overline{(E/J)_q^{-1}} = E/J$.

Now, let $(E_j)_{j\in I}$ be an indexed family of JB*-triples. We set $E = \oplus_j^{\infty} E_j$. Suppose that $a = (a_j) \in E$ is BP quasi-invertible. Since for each j_0, the canonical projection $\pi_{j_0} : \oplus_{j\in I}^{\infty} E_j \to E_{j_0}$ is a surjective triple homomorphism, we deduce that $a_{j_0} = \pi_{j_0}(a) \in (E_{j_0})_q^{-1}$. Suppose now that a_j is BP quasi-invertible in E_j for every j. Consider $b_j \in E_j$ satisfying $B(a_j, b_j) = 0$ on E_j and $(b_j^{\dagger}) \in E$ (cf. Proposition 2.3). Then $B((a_j),(b_j)) = 0$ on E, which shows that $a = (a_j) \in E$ is BP quasi-invertible in E. The final statement follows from the above. □

Theorem 3.6 in [11] establishes that, for every JB*-triple E with $\partial_e(E_1) \neq \emptyset$, the inequalities

$$1 + \|a\| \geq \operatorname{dist}\big(a, \partial_e(E_1)\big) \geq \max\big\{1 + \alpha_q(a), \|a\| - 1\big\}$$

hold for every a in $E \backslash E_q^{-1}$. Under the additional hypothesis that E is extremally rich, we can obtain an optimal computation of the distance from an element in E to the set $\partial_e(E_1)$ of extreme points of E_1.

Theorem 2.6. *Let E be an extremally rich JB*-triple and let $x \in E\backslash E_q^{-1}$. Then* $\operatorname{dist}(x, \partial_e(E_1)) = \max\{1, \|x\| - 1\}$. *In particular, if $x \in E_1$, then* $\operatorname{dist}(x, \partial_e(E_1)) = 1$. *Consequently, for each x in E we have*

$$\operatorname{dist}\big(x, \partial_e(E_1)\big) = \begin{cases} \max\{1 - m_q(x), \|x\| - 1\} & \text{if } x \in E_q^{-1}, \\ \max\{1, \|x\| - 1\} & \text{if } x \notin E_q^{-1}, \end{cases}$$

where $m_q(x) = \operatorname{dist}(x, E\backslash E_q^{-1})$.

Proof. Since the JB*-triple E is extremally rich, $\alpha_q(x) = \operatorname{dist}(x, E_q^{-1}) = 0$ for all $x \in E$. Theorem 3.6 in [11] implies that

$$\operatorname{dist}\big(x, \partial_e(E_1)\big) \geq \max\big\{1 + \alpha_q(x), \|x\| - 1\big\} = \max\big\{1, \|x\| - 1\big\}.$$

Applying [20, Theorem 27], we obtain $\operatorname{dist}(x, \partial_e(E_1)) \leq \max\{1, \|x\| - 1\}$ for all $x \in \overline{E_q^{-1}} = E$. Combining the above inequalities, we have

$$\operatorname{dist}\big(x, \partial_e(E_1)\big) = \max\big\{1, \|x\| - 1\big\}$$

for all $x \in E\backslash E_q^{-1}$. The second statement of the theorem follows immediately when $\|x\| \leq 1$. The final statement follows form the first estimation and from [11, Proposition 3.2]. □

We have already noted that from the geometric point of view of Banach space theory, a C*-algebra is extremally rich if and only if it has the (uniform) λ-property (see [3, Section 3] and [4, Theorem 3.7]). We do not know if this statement remains true for JB*-triples. We know that every JBW*-triple satisfies the uniform λ-property (see [11]). For an element a in a JB*-triple E, we also know that $\lambda(a) = \frac{1+m_q(a)}{2}$ whenever a is a BP quasi-invertible element in E_1 (see [11, Theorem 3.4]). If we also assume that $\partial_e(E_1) \neq \emptyset$, then the inequalities

$$1 + \|a\| \geq \operatorname{dist}\big(a, \partial_e(E_1)\big) \geq \max\big\{1 + \alpha_q(a), \|a\| - 1\big\}$$

hold for every a in $E\backslash E_q^{-1}$ (see [11, Theorem 3.6]), and hence

$$\lambda(a) \leq \frac{1}{2}\big(1 - \alpha_q(a)\big), \tag{2.1}$$

for every $a \in E_1\backslash E_q^{-1}$. We now prove that the λ-function takes only values greater than or equal to 1/2 on the open unit ball of an extremally rich JB*-triple.

Corollary 2.7. *Let a be an element in the open unit ball of an extremally rich JB*-triple. Suppose that a is not BP quasi-invertible. Then $\lambda(a) = 1/2$.*

Proof. Let us pick a real number $t < 1/2$. Clearly $\beta = 1/t > 2$. Since $\|a\| < 1$ and $a \in E\backslash E_q^{-1}$, we deduce, via Theorem 2.6, that

$$\operatorname{dist}\big(\beta a, \partial_e(E_1)\big) = \max\big\{1, \beta\|a\| - 1\big\} < \beta - 1.$$

Therefore, there exists $e \in \partial_e(E_1)$ satisfying $\|\beta a - e\| < \beta - 1$. The element $y = \frac{1}{\beta-1}(\beta a - e)$ lies in the open unit ball of E, and we can write $\beta a = e + (\beta - 1)y$, and hence $a = \frac{1}{\beta}e + \frac{\beta-1}{\beta}y = te + (1-t)y$, which proves that $\lambda(a) \geq t$. The arbitrariness of t shows that $1/2 \leq \lambda(a)$. (The final statement follows from [11, Corollary 3.7].) □

Remark 2.8. Let E be a JB*-triple satisfying the uniform λ-property of Aron and Lohman with $1/2 \leq \inf\{\lambda(x) : x \in E_1\}$. We can assure, via (2.1), that $\alpha_q(a) = 0$ for every $a \in E_1\backslash E_q^{-1}$. This shows that E is extremally rich.

In [21, Section 4], the authors introduce the so-called Λ-*condition* in JB*-triples. A JB*-triple E satisfies the Λ-condition if, for every $v \in \partial_e(E_1)$ and every $y \in (E_2(v))_1\backslash E_q^{-1}$ with $\lambda(y) = 0$, we have $\alpha_q(y) = 1$. We will consider the following stronger variant: A JB*-triple E satisfies the *strong*-Λ-*condition* if, for each $y \in E_1\backslash E_q^{-1}$ with $\lambda(y) = 0$, we have $\alpha_q(y) = 1$. Every C*-algebra A satisfies $\lambda(a) = (1/2)(1 - \alpha_q(a))$ for every $a \in A_1\backslash A_q^{-1}$ (see [4, Theorem 3.7]). Therefore every C*-algebra fulfills the strong-Λ-condition. A similar identity and statement is also valid for every JBW*-triple (see [11, Theorem 4.2]).

Clearly, if E satisfies the strong-Λ-condition, then $\lambda(a) > 0$ for every $a \in E_1\backslash E_q^{-1}$ with $\alpha_q(a) < 1$. The following result is a consequence of this fact, Corollary 2.7, and the comments preceding it (see [11, Theorems 3.4, 3.6]).

Corollary 2.9. *Every extremally rich JB*-triple satisfying the strong-Λ-condition satisfies the Λ-property of Aron and Lohman.*

3. Quadratic conorm in extremally rich JB*-triples

As mentioned in the Introduction, the quadratic conorm, $\gamma^q(a)$, of an element a in a JB*-triple E is defined as the reduced minimum modulus of the conjugate linear operator $Q(a)$ (see [5, Definition 3.1]); that is,

$$\gamma^q(a) := \gamma\big(Q(a)\big) = \inf\big\{\big\|Q(a)(x)\big\| : \operatorname{dist}\big(x, \ker\big(Q(a)\big)\big) \geq 1\big\}.$$

In [5], the authors established that $\gamma^q(a) = \frac{1}{\|a^\dagger\|^2}$ whenever a is von Neumann regular (where $a^\dagger$ is the unique generalized inverse of a) and $\gamma^q(a) = 0$ otherwise (see [5, Theorem 3.4 and its proof]).

Theorem 8 in [20] asserts that the set E_q^{-1} of all BP quasi-invertible elements in a JB*-triple E is open in the norm topology. A more explicit measure of this fact is given in the next result.

Proposition 3.1. *Let a be a BP quasi-invertible element in a JB*-triple E. Suppose that b is an element in E satisfying $\|a-b\| < \gamma^q(a)^{1/2}$. Then b is BP quasi-invertible.*

Proof. We recall that a being BP quasi-invertible implies that $e = r(a) \in \partial_e(E_1)$, and a is positive and invertible in the JB*-algebra $(E_2(e), \circ_e, *_e)$. We further know that it is von Neumann regular and its generalized inverse $a^\dagger \in E_2(e)$ coincides with its inverse in this JB*-algebra (cf. [5, proof of Theorem 3.4]). Let $c = a^{1/2}$ denote the square root of a in $E_2(e)$. We observe that $a^{1/2}$ is positive and invertible in $E_2(e)$. Moreover, the inverse of $a^{1/2}$, $(a^{1/2})^{-1}$, coincides with $(a^{1/2})^\dagger = (a^\dagger)^{1/2}$, where the latter is the square root of $a^\dagger$ in $E_2(e)$.

Since $\|a-b\| < \gamma^q(a)^{1/2}$, by Peirce rules $\{c^\dagger, P_1(e)(b), c^\dagger\} = 0$, so

$$\begin{aligned}\big\|e - Q(c^\dagger)\big(P_2(e)(b)\big)\big\| &= \big\|Q(c^\dagger)(a-b)\big\| \leq \big\|Q(c^\dagger)\big\|\|a-b\| < \|c^\dagger\|^2\big(\gamma^q(a)\big)^{1/2} \\ &= \|a^\dagger\|\gamma^q(a)^{1/2} \\ &= \big(\text{see [5, Theorem 3.4 and its proof]}\big) = 1.\end{aligned}$$

Since e is the unit of $E_2(e)$, we deduce that $Q(c^\dagger)(P_2(e)(b))$ is invertible in $E_2(e)$. It is well known from the theory of invertible elements in JB*-algebras that $Q(c^\dagger)|_{E_2(e)} : E_2(e) \to E_2(e)$ is invertible as a mapping from $E_2(e)$ into itself with inverse $Q(c)|_{E_2(e)} : E_2(e) \to E_2(e)$ (see [6, Section 4.1.1]). Since $Q(c^\dagger)(P_2(e)(b))$ is invertible, we deduce that $P_2(e)(b) = Q(c)Q(c^\dagger)(P_2(e)(b))$ is invertible in $E_2(e)$ (see [6, Theorem 4.1.3]). Finally, Lemma 2.2 in [11] implies that $b \in E_q^{-1}$, as we desired. ☐

The next lemma gathers some consequences of results in [11, Section 3].

Lemma 3.2. *Let E be a JB*-triple. Then the inequality*

$$\big|\gamma^q(a) - \gamma^q(b)\big| < \big(\|a\| + \|b\|\big)\|a-b\|,$$

holds for all a and b in E_q^{-1}.

Proof. It is known that $\gamma^q(x) = m_q(x)^2 \leq \|x\|^2$ and that $|m_q(x) - m_q(y)| \leq \|x-y\|$ for every $x, y \in E_q^{-1}$ (see [11, Theorem 3.1 and subsequent comments]). Therefore,

$$\begin{aligned} \big|\gamma^q(a) - \gamma^q(b)\big| &= \big|m_q(a)^2 - m_q(b)^2\big| \\ &= \big|m_q(a) - m_q(b)\big|\big|m_q(a) + m_q(b)\big| \\ &< \|a-b\|\big(\|a\| + \|b\|\big). \end{aligned}$$ □

It is proved in [5, Theorem 3.13] that the quadratic conorm, $\gamma^q(\cdot)$, in a JB*-triple E is upper semicontinuous on $E\backslash\{0\}$. In the setting of extremally rich JB*-triples, we can characterize now the precise points at which $\gamma^q(\cdot)$ is continuous.

Theorem 3.3. *Let E be an extremally rich JB*-triple. Then the quadratic conorm $\gamma^q(\cdot)$ is continuous at a point $a \in E$ if and only if either a is not von Neumann regular (i.e., $\gamma^q(a) = 0$) or a is BP quasi-invertible.*

Proof. The upper semicontinuity of $\gamma^q(\cdot)$ implies that it is continuous at every point $a \in E$ which is not von Neumann regular. If $a \in E_q^{-1}$, the continuity of $\gamma^q(\cdot)$ at a follows from Proposition 3.1 and Lemma 3.2.

Suppose that $\gamma^q(\cdot)$ is continuous at a and that a is von Neumann regular (i.e., $\gamma^q(a) > 0$). In this case, $Q(a)(E)$ is norm-closed, or equivalently, $\gamma^q(a) = \gamma(Q(a)) > 0$ (see [5, Corollary 2.4 and proof of Theorem 3.4]). The mapping $x \mapsto \gamma^q(x)^{1/2}$ is continuous at a. So there exists $\delta > 0$ such that

$$\|a-b\| < \delta \Rightarrow \big|\gamma^q(a)^{1/2} - \gamma^q(b)^{1/2}\big| < \frac{\gamma^q(a)^{1/2}}{2};$$

that is, $\gamma^q(b)^{1/2} > \frac{\gamma^q(a)^{1/2}}{2}$, whenever $\|a-b\| < \delta$. Extremal richness of E implies that $\overline{E_q^{-1}} = E$. Thus, there is $c \in E_q^{-1}$ with $\|a-c\| < \min\{\delta, \frac{\gamma^q(a)^{1/2}}{2}\}$. In particular $\|a-c\| < \delta$, that is, $\gamma^q(c)^{1/2} > \frac{\gamma^q(a)^{1/2}}{2} > \|a-c\|$. Proposition 3.1 above proves that $a \in E_q^{-1}$. □

Remark 3.4. In [5, Remark 3.18] it is shown that the quadratic conorm $\gamma^q(\cdot)$ of a JB*-triple E is continuous at every element $a \in E$ for which $Q(a)$ is left- or right-invertible in $B(E)$. In the same remark it is also asked whether these points are the only nontrivial continuity points of $\gamma^q(\cdot)$. Theorem 3.3 characterizes the continuity points of the quadratic conorm in the class of extremally rich JB*-triples (a class that contains all JBW*-triples). Theorem 3.3 shows the existence of points x satisfying that the quadratic conorm is continuous at x, but $Q(x)$ is neither left- nor right-invertible. For example, when E is an extremally rich JB*-triple and e is a complete tripotent with $E_1(e) \neq \{0\}$, then the quadratic conorm is continuous at e, but $Q(e)$ is neither left- nor right-invertible.

The arguments in the second part of the proof of Theorem 3.3 are also valid and prove the following.

Proposition 3.5. *Let (a_n) be a sequence of BP quasi-invertible elements in a JB*-triple E. Suppose that (a_n) converges in norm to some element a in E, and let $\gamma^q(a_n) \to \gamma^q(a) > 0$. Then a is BP quasi-invertible.*

Our next result is a consequence of [5, Theorem 3.16, Corollary 3.17] and the previous Proposition 3.5.

Corollary 3.6. *Let* (a_n) *be a sequence of BP quasi-invertible elements in a JB*-triple* E*. Suppose that* (a_n) *converges in norm to some element* a *in* E*. Then the following assertions are equivalent:*

(a) $(a_n^\dagger)$ *is a bounded sequence in* E*,*
(b) $\gamma^q(a_n) \to \gamma^q(a) > 0$.

Furthermore, if any of the above statements holds, then a *is BP quasi-invertible and* $\|a_n^\dagger - a^\dagger\| \to 0$.

Proof. (a) ⇒ (b) Suppose that $(a_n^\dagger)$ is a bounded sequence in E. Corollary 3.17 in [5] implies that a is von Neumann regular (i.e., $\gamma^q(a) > 0$). It follows from [5, Theorem 3.16] (d) ⇒ (c) that $\gamma^q(a_n) \to \gamma^q(a) > 0$.

(b) ⇒ (a) Suppose that $\gamma^q(a_n) \to \gamma^q(a) > 0$. In particular, a is von Neumann regular. The desired statement follows from [5, Theorem 3.16] (c) ⇒ (d).

The final statement is a consequence of Proposition 3.5 and [5, Theorem 3.16]. □

The result in Theorem 3.3 is new, even in the case of C*-algebras. According to the notation of Harte and Mbekhta, who introduced the notions of left and right conorms for C*-algebras in [10], the left conorm, $\gamma(a)$, of an element a in a C*-algebra A is given by

$$\gamma(a) = \gamma^{\text{left}}(a) = \gamma(L_a) = \inf\left\{ \frac{\|ax\|}{\mathrm{d}(x, \ker(L_a))} : x \notin \ker(L_a) \right\},$$

where L_a is the left multiplication mapping by a; that is, $L_a(x) = ax$ $(x \in A)$. The right conorm is similarly defined. Theorem 4 in [10] shows that

$$\gamma(a)^2 = \gamma(aa^*) = \gamma(a^*a) = \gamma^{\text{right}}(a)^2 = \inf\{t : t \in \sigma(aa^*)\backslash\{0\}\},$$

where $\sigma(aa^*)$ denotes the spectrum of aa^*.

While Harte and Mbekhta established that the conorm $\gamma(\cdot)$ of a C*-algebra is upper semicontinuous (see [10, Theorem 7]), they also showed in [10, Theorem 9] that the reduced minimum modulus is always continuous on the open set of all bounded-below operators (resp., the set of all almost-open operators) between a pair of normed spaces. By the upper semicontinuity of $\gamma(\cdot)$, the conorm is continuous at elements with no generalized inverses (i.e., at elements a with $\gamma(a) = 0$). When $A = B(H)$, the C*-algebra of all bounded linear operators on a complex Hilbert space H, then these results cover all continuity points. The general case is left as an open problem. For a general C*-algebra A, Corollary 4.1 in [5] proves that $\gamma^q(a) = \gamma(a)^2$, for all $a \in A$. Theorem 3.3 particularizes in the following result, which provides additional information to the problem left open by Harte and Mbekhta.

Corollary 3.7. *Let* A *be an extremally rich C*-algebra. Then the conorm of* A *is continuous at a point* $a \in A$ *if and only if* a *is not von Neumann regular (i.e.,* $\gamma(a) = 0$*) or* a *is quasi-invertible.*

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