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CHARACTER AMENABILITY AND CONTRACTIBILITY OF SOME BANACH ALGEBRAS ON LEFT COSET SPACES

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Abstract. Let H be a compact subgroup of a locally compact group G, and let μ be a strongly quasi-invariant Radon measure on the homogeneous space G/H. In this article, we show that every element of $\widehat{G/H}$, the character space of G/H, determines a nonzero multiplicative linear functional on $L^1(G/H,\mu)$. Using this, we prove that for all $\phi \in \widehat{G/H}$, the right ϕ-amenability of $L^1(G/H,\mu)$ and the right ϕ-amenability of $M(G/H)$ are both equivalent to the amenability of G. Also, we show that $L^1(G/H,\mu)$, as well as $M(G/H)$, is right ϕ-contractible if and only if G is compact. In particular, when H is the trivial subgroup, we obtain the known results on group algebras and measure algebras.

1. Introduction

Let A be a Banach algebra and let $\Delta(A)$ be the spectrum of A, consisting of all nonzero multiplicative linear functionals on A. Then for $\varphi \in \Delta(A)$, the Banach algebra A is called *right φ-amenable* if there exists an element $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A^*$. One may similarly define the left φ-amenable Banach algebras. The right φ-amenability, as a modification of Johnson's amenability, was introduced and studied by Kaniuth, Lau, and Pym [7] under the name of *φ-amenability*. This notion of amenability is a generalization of the left amenability of a class of Banach algebras studied by Lau in [8] known as *Lau algebras*.

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Monfared introduced the notion of character amenability for Banach algebras in [9]. He characterized the structure of (right) character amenable Banach algebras and showed that the right (resp., left) character amenability of A is equivalent to A being right (resp., left) φ-amenable for all $\varphi \in \Delta(A)$ and A having a bounded right (resp., left) approximate identity. For any locally compact group G, the (right) character amenability of the group algebra $L^1(G)$ is equivalent to the amenability of G and the (right) character amenability of the measure algebra $M(G)$ is equivalent to the discreteness and amenability of G (see [9]).

The Banach algebra A is called *right φ-contractible* if there exists an element $m \in A$ satisfying $\varphi(m) = 1$ and $am = \varphi(a)m$ for all $a \in A$. If A has a right identity and if it is right φ-contractible for all $\varphi \in \Delta(A)$, then A is called *right character contractible*. One may define the left φ-contractibility and the left character contractibility of Banach algebras similarly. These notions were introduced and studied by Hu, Monfared, and Traynor in [4]. (Several authors have studied the character amenability and the character contractibility of some Banach algebras; see, for example [7], [9], [4], [10].)

Among all locally compact Hausdorff spaces, it seems valuable to consider homogeneous spaces and to investigate the structures and the properties of their function spaces. The term *homogeneous space* refers to a transitive G-space which is topologically isomorphic to G/H, the space of all left cosets of a closed subgroup H in a locally compact Hausdorff topological group G. In [3] and [6], the authors introduced and investigated the Fourier algebra $A(G/H)$ and the Fourier–Stieltjes algebra $B(G/H)$, where H is compact. In [12], a bounded surjective linear map $T_p : L^p(G) \to L^p(G/H)$ was introduced using a compact subgroup H of G and equipping the homogeneous space G/H with a strongly quasi-invariant Radon measure. The authors also showed that the restriction of T_p to a special closed subspace $L^p(G : H)$ of $L^p(G)$ is an isometric isomorphism for all $1 \le p \le \infty$. In particular, it has been shown that $L^1(G : H)$ is a closed left ideal of $L^1(G)$ which has a bounded right approximate identity. The Banach space $L^1(G/H)$ is converted to a Banach algebra with a bounded right approximate identity, transferring the multiplication of $L^1(G : H)$ to $L^1(G/H)$. It is also shown that $L^1(G/H)$ has a bounded left approximate identity just when H is normal (see [5]).

In this paper, for a compact subgroup H of G, and with the homogeneous space G/H equipped with a strongly quasi-invariant Radon measure, we show that every element of $\widehat{G/H}$, the character space of G/H, determines an element of $\Delta(M(G/H))$ whose restriction to $L^1(G/H)$ belongs to $\Delta(L^1(G/H))$. Using this point, we prove that for all $\phi \in \widehat{G/H}$ the right ϕ-amenability of $L^1(G/H)$ and the right ϕ-amenability of $M(G/H)$ are both equivalent to the amenability of G. We also show that $L^1(G/H, \mu)$ as well as $M(G/H)$ is right ϕ-contractible if and only if G is compact. In particular, when H is the trivial subgroup, we obtain the related results on group and measure algebras.

2. Notation and preliminaries

In this section, for the reader's convenience, we provide a summary of the mathematical notation and definitions which will be used in the sequel. (For details, we refer the reader to the general reference [2], or any other standard book of harmonic analysis.)

Let X be a locally compact Hausdorff space. By $M(X)$ we mean the Banach space of all complex Borel measures on X, and if μ is a positive Borel measure on X, then we denote by $L^1(X)$ and $L^\infty(X)$, the Banach spaces of all equivalence classes of integrable complex-valued functions and all locally measurable and locally essentially bounded functions on X, respectively.

Let H be a closed subgroup of a locally compact topological group G. A Radon measure μ on G/H is called *strongly quasi-invariant* if there is a continuous function $\lambda : G \times (G/H) \to (0, +\infty)$ such that $d\mu_x(yH) = \lambda(x, yH)\, d\mu(yH)$ for all $x \in G$, where μ_x is defined by $\mu_x(E) = \mu(xE)$ for all Borel subsets E of G/H.

Let Δ_G and Δ_H be the modular functions of G and H,respectively. A rho-function for the pair (G, H) is a continuous function $\rho : G \to (0, +\infty)$ for which $\rho(x\xi) = (\Delta_H(\xi)/\Delta_G(\xi))\rho(x)$ for all $x \in G$ and $\xi \in H$. By [2, Proposition 2.54], the pair (G, H) always admits a rho-function, and each rho-function ρ induces a strongly quasi-invariant Radon measure μ on G/H for which the Mackey–Bruhat formula

$$\int_{G/H} \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi\, d\mu(xH) = \int_G f(x)\, dx \quad \big(f \in L^1(G)\big)$$

holds, where dx and $d\xi$ are the left Haar measures on G and H, respectively. Moreover, every strongly quasi-invariant Radon measure on G/H arises from a rho-function in this way (see [2, Section 2.6]).

It is well known that $L^1(G)$ is an involutive Banach algebra with a bounded approximate identity. Assuming that H is a compact subgroup of G, the operator $T_1 : L^1(G) \to L^1(G/H, \mu)$ is defined by $T_1 f(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi$ for almost all $xH \in G/H$. Then, it has been shown that $L^1(G/H, \mu)$ becomes a Banach algebra by multiplication $f * g = T_1(f_\rho * g_\rho)$, where $f_\rho, g_\rho \in L^1(G)$ are defined by $f_\rho(x) = \rho(x)f(xH)$ and $g_\rho(x) = \rho(x)g(xH)$ for almost all $x \in G$ (see [12]). Also, one can easily show that for each $f, g \in L^1(G/H)$ we have

$$(f * g)_\rho = f_\rho * g_\rho. \tag{2.1}$$

In [12], it has been also shown that $L^1(G/H)$ is isometrically isomorphic to the closed subalgebra $L^1(G : H) = \{f \in L^1(G); \forall \xi \in H, R_\xi f = f\}$ of $L^1(G)$ via T_1.

For a function f on G, we define the left and the right translations of f by $x \in G$ by $L_x f(y) = f(x^{-1}y)$ and $R_x f(y) = f(yx)$, $y \in G$, respectively. Using these, the left and the right translations of $f \in L^1(G/H)$ by $x \in G$ in $L^1(G/H)$ are defined by $L_x f = T_1(L_x f_\rho)$ and $R_x f = T_1(R_x f_\rho)$, respectively.

Also, there is a bounded surjective linear map $T_\infty : L^\infty(G) \to L^\infty(G/H)$ defined by

$$T_\infty \varphi(xH) = \int_H \varphi(x\xi)\, d\xi \quad \text{(locally almost every } xH \in G/H\text{)},$$

where $\varphi \in L^\infty(G)$. The closed subspace

$$L^\infty(G : H) = \big\{\varphi \in L^\infty(G), \forall \xi \in H, R_\xi \varphi = \varphi\big\}$$

of $L^\infty(G)$ is also isometrically isomorphic to $L^\infty(G/H)$, via the mapping T_∞. Moreover, using the duality between L^1 and L^∞, for all $\psi \in L^\infty(G/H)$ and $f \in L^1(G/H)$, we have

$$\langle \psi, f\rangle = \langle \psi_q, f_\rho\rangle, \tag{2.2}$$

where $\psi_q \in L^\infty(G)$ is given by $\psi_q(x) = \psi(xH)$, for locally almost all $x \in G$.

If A is a Banach algebra and if X is a Banach A-bimodule, then the dual Banach space X^* of X is a Banach A-bimodule, with the dual actions given by

$$(a \cdot f)(x) = f(xa) \qquad \text{and} \qquad (f \cdot a)(x) = f(ax) \quad (f \in X^*, a \in A, x \in X).$$

In particular, A^* is a Banach A-bimodule. Using T_1 and T_∞, we may express the left and the right dual $L^1(G/H)$-module actions of $L^\infty(G/H)$ via corresponding the left and the right $L^1(G)$-module actions of $L^\infty(G)$. In detail, for all $\varphi \in L^\infty(G : H)$ and $f \in L^1(G : H)$, we have

$$T_\infty(\varphi \cdot f) = T_\infty(\varphi) \cdot T_1(f) \qquad \text{and} \qquad T_\infty(f \cdot \varphi) = T_1(f) \cdot T_\infty(\varphi).$$

In other words, if $\psi \in L^\infty(G/H)$ and $f \in L^1(G/H)$, then

$$\psi \cdot f = T_\infty(\psi_q \cdot f_\rho) \qquad \text{and} \qquad f \cdot \psi = T_\infty(f_\rho \cdot \psi_q). \tag{2.3}$$

3. Character amenability and contractibility

Let $\widehat{G}$ denote the dual group of G consisting of all continuous homomorphisms from G into the circle group $\mathbb{T}$. Every $\phi \in \widehat{G}$ defines a nonzero multiplicative linear functional on $L^1(G)$, which we denote by ϕ, that is,

$$\phi(f) = \int_G \phi(s) f(s)\, ds \quad \big(f \in L^1(G)\big).$$

It is well known that every element of $\Delta(L^1(G))$ arises from some $\phi \in \widehat{G}$ in this way. In other words,

$$\Delta\big(L^1(G)\big) = \widehat{G}.$$

At the beginning of this section, we offer a definition of a character of G/H.

Definition 3.1. Let H be a compact subgroup of G. A continuous function ϕ from G/H into the circle group $\mathbb{T}$ is called a *character* of G/H if $\phi(xyH) = \phi(xH)\phi(yH)$ for each $x, y \in G$. The set of all characters of G/H is denoted by $\widehat{G/H}$.

The next result, which is an extension of [2, Theorem 4.39], shows that $(\widehat{G : H})$ may be identified with $\widehat{G/H}$, where

$$(\widehat{G : H}) = \{\phi \in \widehat{G}, R_\xi \phi = \phi \ \forall \xi \in H\}.$$

Proposition 3.2. *Let H be a compact subgroup of G. Then $(\widehat{G : H})$ is isometrically isomorphic to $\widehat{G/H}$. More precisely, the restriction of T_∞ on $(\widehat{G : H})$ is an isometric isomorphism.*

Proof. First, note that if $\phi \in (\widehat{G:H})$, then $T_\infty\phi(xH) = \phi(x)$, for all $x \in G$. It follows that $T_\infty((\widehat{G:H})) \subseteq \widehat{G/H}$. The reverse inclusion follows obviously from the equality

$$\widehat{G/H} = \{\phi \in L^\infty(G/H), \phi_q \in \widehat{G}\}.$$

As $(\widehat{G:H}) \subseteq L^\infty(G:H)$, the restriction of T_∞ on $(\widehat{G:H})$ is isometry (see [12, Theorem 4.2]). ☐

Theorem 3.3. *If H is a compact subgroup of G, then $(\widehat{G:H}) \subseteq \Delta(L^1(G:H))$.*

Proof. Let $\phi \in (\widehat{G:H})$. Then $\phi \in \Delta(L^1(G))$. It is enough to show that ϕ is nonzero on $L^1(G:H)$. For this, take $f_0 \in L^1(G)$ with $\langle \phi, f_0\rangle = 1$. Then $(T_1 f_0)_\rho \in L^1(G:H)$ and $\langle \phi, , (T_1 f_0)_\rho\rangle = 1$, as required. ☐

Corollary 3.4. *Let H be a compact subgroup of G. Then $\widehat{G/H} \subseteq \Delta(L^1(G/H))$.*

We recall from [1] that the homogeneous space G/H is considered amenable if there is a state $M \in L^\infty(G/H)^*$ such that $M(L_x\psi) = M(\psi)$ for all $x \in G$ and $\psi \in L^\infty(G/H)$, where the left translation on $L^\infty(G/H)$ is given by $L_x\psi = T_\infty(L_x(\psi_q))$. The topological group G is amenable if G/H, when H is a trivial subgroup, is amenable. Examples of amenable groups includes abelian groups and compact groups. It has been shown in [1, Section 3] that if H is amenable, then G/H is amenable if and only if G is amenable. In the next result, we give a different proof for this point under the assumption of compactness of H. This result also extends related results in [8].

Theorem 3.5. *Let H be a compact subgroup of G and let $\phi \in \widehat{G/H}$. Then the following are equivalent:*

(a) *$L^1(G/H)$ is right ϕ-amenable,*
(b) *G/H is amenable,*
(c) *G is amenable.*

Proof. (a) ⇒ (b): Let $L^1(G/H)$ be right ϕ-amenable. Then there exists some $M \in L^\infty(G/H)^*$ such that $M(\phi) = 1$ and $M(\psi \cdot f) = \langle \phi, f\rangle M(\psi)$ for all $\psi \in L^\infty(G/H)$ and $f \in L^1(G/H)$. Define $m \in L^\infty(G/H)^*$ by $m(\psi) = M(\phi\psi)$. Obviously, $m(\mathbf{1}) = 1$. Using (2.3), for all $f \in L^1(G/H)$ and $\psi \in L^\infty(G/H)$, we have

$$\begin{aligned}\phi\psi \cdot \bar{\phi} f &= T_\infty\big((\phi\psi)_q \cdot (\bar{\phi} f)_\rho\big)\\ &= \phi T_\infty(\psi_q \cdot f_\rho)\\ &= \phi(\psi \cdot f).\end{aligned}$$

It follows that

$$\begin{aligned}m(\psi \cdot f) &= M\big(\phi(\psi \cdot f)\big) = M(\phi\psi \cdot \bar{\phi} f)\\ &= \langle \phi, \bar{\phi} f\rangle M(\phi\psi) = \langle \mathbf{1}, f\rangle m(\psi).\end{aligned}$$

Since $L_x f_\rho = (L_x f)_\rho$ and $(L_x\psi)_q = L_x(\psi_q)$, a straightforward argument shows that

$$(L_x\psi)\cdot f = \psi\cdot(L_{x^{-1}}f),$$

for all $x\in G$, $f\in L^1(G/H)$ and $\psi\in L^\infty(G/H)$. Take $f\in L^1(G/H)$ with $\mathbf{1}(f)=1$. Then for all $x\in G$ and $\psi\in L^\infty(G/H)$, we have

$$\begin{aligned} m(L_x\psi) &= m\big((L_x\psi)\cdot f\big)\\ &= m\big(\psi\cdot(L_{x^{-1}}f)\big)\\ &= \langle\mathbf{1}, L_{x^{-1}}f\rangle m(\psi)\\ &= m(\psi). \end{aligned}$$

This implies that G/H is amenable (see [11, Proposition 2.2]).

(b) $\Rightarrow$ (c): Let G/H be amenable. Take a state $M\in L^\infty(G/H)^*$ such that $M(L_x\psi)=M(\psi)$ for all $x\in G$ and $\psi\in L^\infty(G/H)$. The equality $T_\infty(L_x\varphi) = L_x(T_\infty\varphi)$ for all $x\in G$ and $\varphi\in L^\infty(G)$ implies that $M\circ T_\infty$ is a state in $L^\infty(G)^*$ for which $(M\circ T_\infty)(L_x\varphi) = (M\circ T_\infty)(\varphi)$ for all $x\in G$ and $\varphi\in L^\infty(G)$. Therefore, G is amenable.

(c) $\Rightarrow$ (a): Let G be amenable. Then, $L^1(G)$ is amenable and hence it is right ϕ_q-amenable (see [8, Theorem 1.1]). So, there is $m\in L^\infty(G)^*$ such that

$$m(\phi_q)=1 \qquad\text{and}\qquad m(\varphi\cdot f) = \langle\phi_q, f\rangle m(\varphi)$$

for all $\varphi\in L^\infty(G)$ and $f\in L^1(G)$. Define $M\in L^\infty(G/H)^*$ by $M(\psi)=m(\psi_q)$. Then, $M(\phi)=1$. By using (2.2) and (2.3), we have

$$\begin{aligned} M(\psi\cdot f) &= M\big(T_\infty(\psi_q\cdot f_\rho)\big)\\ &= m\big(T_\infty(\psi_q\cdot f_\rho)_q\big)\\ &= m(\psi_q\cdot f_\rho)\\ &= \langle\phi_q, f_\rho\rangle m(\psi_q)\\ &= \langle\phi, f\rangle M(\psi), \end{aligned}$$

for all $f\in L^1(G/H)$ and $\psi\in L^\infty(G/H)$. So, $L^1(G/H)$ is right ϕ-amenable and the proof is complete. □

The next result is an immediate consequence of Theorem 3.5, when H is a trivial subgroup.

Corollary 3.6. *Let $\phi\in\Delta(L^1(G))$. Then $L^1(G)$ is right ϕ-amenable if and only if G is amenable.*

In the next result we show that the converse of Lemma 3.1 in [7] is also true.

Theorem 3.7. *Let A be a Banach algebra and let J be a closed two-sided ideal of A. If $\phi\in\Delta(A)$ such that $\phi|_J\neq 0$, then A is right ϕ-amenable if and only if J is right $\phi|_J$-amenable.*

Proof. While the necessity follows from [7, Lemma 3.1], we now give a different proof. Let A be right ϕ-amenable. Then there exists a bounded net $\{u_\alpha\}$ in A such that $\phi(u_\alpha) = 1$ for all α and $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ for all $a \in A$. Take $j \in J$ with $\phi(j) = 1$ and set $v_\alpha = u_\alpha j$. Then $\{v_\alpha\}$ is a bounded net in J such that $\phi(v_\alpha) = 1$ for all α and

$$\big\|bv_\alpha - \phi(b)v_\alpha\big\| \le \big\|bu_\alpha - \phi(b)u_\alpha\big\|\|j\| \to 0,$$

for all $b \in J$. By [7, Theorem 1.4], J is right $\phi|_J$-amenable.

For the converse, let $\{v_\alpha\}$ be a bounded net in J such that $\phi(v_\alpha) = 1$ for all α and $\|bv_\alpha - \phi(b)v_\alpha\| \to 0$ for all $b \in J$. Take $j \in J$ with $\phi(j) = 1$ and set $u_\alpha = jv_\alpha$. Then $\{u_\alpha\}$ is a bounded net in A such that $\phi(u_\alpha) = 1$ for all α and

$$\big\|au_\alpha - \phi(a)u_\alpha\big\| \le \big\|ajv_\alpha - \phi(a)v_\alpha\big\| + \big|\phi(a)\big|\|jv_\alpha - v_\alpha\| \to 0$$

for all $a \in A$. So, A is right ϕ-amenable. □

It is worthwhile to mention that there is a multiplication $*$ on $M(G/H)$ which makes it a Banach algebra containing the Banach algebra $L^1(G/H,\mu)$ as a closed two-sided ideal (see [5]). Moreover, for all $\mu,\nu \in M(G/H)$,

$$(\mu * \nu)(G/H) = \mu(G/H)\nu(G/H). \tag{3.1}$$

For every $\phi \in \widehat{G/H}$, we may define $\tilde{\phi} \in \Delta(M(G/H))$ by

$$\tilde{\phi}(\nu) := \int_{G/H} \phi(xH)\, d\nu(xH) \quad \big(\nu \in M(G/H)\big).$$

As a consequence of Theorems 3.5 and 3.7, we have the next result.

Theorem 3.8. *Let H be a compact subgroup of G and let $\phi \in \widehat{G/H}$. Then $M(G/H)$ is right $\tilde{\phi}$-amenable if and only if G is amenable.*

A Banach algebra A is called a *Lau algebra* if the dual space A^* of A is a W*-algebra and the identity element of A^* belongs to $\Delta(A)$. The subject of this large class of Banach algebras originated in [8]. The relations (2.2) and (3.1) show that $L^1(G/H)$ and $M(G/H)$ are examples of Lau algebras. A Lau algebra A is considered left amenable if there exists a state $m \in A^{**}$ such that $a \cdot m = \langle \mathbf{1}, a\rangle m$ for all $a \in A$, where $\mathbf{1}$ denotes the identity of A^* (see [8]). The next result follows from Theorems 3.5 and 3.8 and this fact that the left amenability of a Lau algebra is equivalent to its right $\mathbf{1}$-amenability (see [11, Proposition 2.2]).

Corollary 3.9. *Let H be a compact subgroup of G. Then the following are equivalent:*

(a) *G is amenable,*
(b) *$L^1(G/H)$ is left amenable,*
(c) *$M(G/H)$ is left amenable.*

As for the left character amenability of $L^1(G/H)$, it is worthwhile to mention that $L^1(G/H)$ has a bounded left approximate identity if and only if H is normal (see [5]). So, the left character amenability of $L^1(G/H)$ is equivalent to the fact that H is normal and G is amenable.

In the following, we characterize the right ϕ-contractibility of $L^1(G/H)$, which is an extension of Theorem 6.1 in [10].

Theorem 3.10. *Let H be a compact subgroup of G and let $\phi \in \widehat{G/H}$. Then $L^1(G/H)$ is right ϕ-contractible if and only if G is compact.*

Proof. Let $L^1(G/H)$ be right ϕ-contractible. Then there exists some $f_0 \in L^1(G/H)$ such that $\langle \phi, f_0 \rangle = 1$ and $f * f_0 = \langle \phi, f \rangle f_0$ for each $f \in L^1(G/H)$. Put $g_0 = \phi f_0$. Then $g_0 \in L^1(G/H)$, $\langle \mathbf{1}, g_0 \rangle = 1$, and $f * g_0 = \langle \mathbf{1}, f \rangle g_0$, for all $f \in L^1(G/H)$. So for all $x \in G$, we can write

$$g_0 = (L_x g_0) * g_0 = L_x g_0,$$

which implies that

$$g_0(xH) = k \frac{\rho(e)}{\rho(x)}$$

for some constant $k \in \mathbb{C}$. Thus $g_{0\rho} = k\rho(e) \in L^1(G)$. Hence G is compact.

Conversely, let G be compact. Then we have $\phi_q \in L^1(G) \cap \widehat{(G : H)}$. Set $g_0 = T_1(\overline{\phi_q})$. Then

$$f_\rho * \overline{\phi_q} = \langle \phi_q, f_\rho \rangle \overline{\phi_q} = \langle \phi, f \rangle \overline{\phi_q}.$$

It follows that

$$\begin{aligned} f * g_0 &= f * T_1(\overline{\phi_q}) = T_1(f_\rho * \overline{\phi_q}) \\ &= T_1\big(\langle \phi, f \rangle \overline{\phi_q}\big) \\ &= \langle \phi, f \rangle g_0, \end{aligned}$$

for all $f \in L^1(G/H)$. Also, by the compactness of G, we have

$$\langle \phi, g_0 \rangle = \big\langle \phi, T_1(\overline{\phi_q}) \big\rangle = \langle \phi_q, \overline{\phi_q} \rangle = 1.$$

So, $L^1(G/H)$ is right ϕ-contractible. □

We conclude with the following result on right $\tilde{\phi}$-contractibility of $M(G/H)$, which follows from [10, Proposition 3.8] and Theorem 3.10.

Corollary 3.11. *Let H be a compact subgroup of G and let $\phi \in \widehat{G/H}$. Then $M(G/H)$ is right $\tilde{\phi}$-contractible if and only if G is compact.*

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