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DENSE BANACH SUBALGEBRAS OF THE NULL SEQUENCE ALGEBRA WHICH DO NOT SATISFY A DIFFERENTIAL SEMINORM CONDITION

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Abstract. We construct dense Banach subalgebras A of the null sequence algebra c_0 which are spectral-invariant but do not satisfy the D_1-condition $\|ab\|_A \leq K(\|a\|_\infty \|b\|_A + \|a\|_A \|b\|_\infty)$ for all $a, b \in A$. The sequences in A vanish in a skewed manner with respect to an unbounded function $\sigma\colon \mathbb{N} \to [1, \infty)$.

1. Introduction

We say that A is a *dense Banach subalgebra* of a $C^\star$-algebra B if A is a dense subalgebra of B, and A is a Banach algebra in some norm $\|\cdot\|_A$, which is stronger than the restriction to A of the norm on B. We say that A is *spectral-invariant* in B if the quasi-invertible elements of A are precisely those elements of A which are quasi-invertible in B. Recall that for $a, b \in A$, $a \circ b = a + b - ab$, and b is a quasi-inverse for a if and only if $a \circ b = b \circ a = 0$ (see [4, Definition 2.1.1]). Dense subalgebras of a $C^\star$-algebra which are spectral-invariant or which satisfy a differential seminorm condition such as D_1 are used to give differential structure to the $C^\star$-algebra, and they also have applications in noncommutative differential geometry (see [1], [2]).

Let c_0 be the $C^\star$-algebra of complex-valued vanishing sequences, or null sequences, on the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$, with pointwise multiplication and involution. For $n \in \mathbb{N}$, let $e_n\colon \mathbb{N} \to \{0, 1\}$ be the unit step function at n, where

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$e_n(k) = 1$ if $k = n$ and $e_n(k) = 0$ for $k \in \mathbb{N} \setminus \{n\}$. Let c_f denote the linear span of $\{e_n\}_{n=0}^{\infty}$, which is a dense ideal in c_0. The following result shows that it is relatively easy for a dense subalgebra of c_0 to be spectral-invariant.

Theorem 1.1. *Let A be a dense Banach subalgebra of c_0. Assume that the subalgebra of finite support functions c_f is a dense subset of A. Then A is spectral-invariant in c_0.*

Proof. Let A_{qG} denote the group of quasi-invertible elements of A, with group operation $\circ$ and identity 0. Let $a \in A \setminus A_{qG}$. Then $A \circ a$ cannot intersect A_{qG}, or else a would have a quasi-inverse. So $(a + A(1-a)) \cap A_{qG} = \emptyset$. Note that $J = A(1-a)$ is an ideal in A, and is proper since $J \cap (A_{qG} - a) = \emptyset$. Since A is a Banach algebra, A_{qG} and its translate $A_{qG} - a$ are open sets in A. Hence J cannot be dense in A, and $c_f \nsubseteq J$, using the hypothesis. Since J is a linear space, there is some $n_0 \in \mathbb{N}$ for which $e_{n_0} \notin J$. Since J is a c_f-module not containing e_{n_0}, every element of J must vanish at n_0. This can only happen if $a(n_0) = 1$. Hence a is not quasi-invertible in c_0. $\square$

A dense Banach subalgebra A is a D_1-*subalgebra* of B if, for some constant $K_A > 0$, the D_1-*condition*

$$\|ab\|_A \le K_A \big\{ \|a\|_A \|b\|_B + \|a\|_B \|b\|_A \big\} \tag{1.1}$$

is satisfied for all $a, b \in A$, where $\|\cdot\|_A$ is the norm on A and $\|\cdot\|_B$ is the norm on B (see [3]). Being a D_1-subalgebra implies spectral invariance (see [3, Theorem 5 and Lemma 4]), which raises the question: Is every dense Banach subalgebra of the null sequence algebra, which satisfies the hypotheses of Theorem 1.1, also D_1? The purpose of the present paper is to provide a counterexample. To this end, we think of c_0 as $\mathbb{C}^2$-valued sequences vanishing at infinity, $c_0(\mathbb{N}, \mathbb{C}^2)$, where $\mathbb{C}^2$ denotes the 2-dimensional commutative $C^\star$-algebra, with coordinatewise multiplication and involution. We identify the two $C^\star$-algebras using the isomorphism $\theta \colon c_0 \cong c_0(\mathbb{N}, \mathbb{C}^2)$,

$$\theta(f)(n) = \big(f(2n), f(2n+1)\big), \tag{1.2}$$

for $f \in c_0$, $n \in \mathbb{N}$. If A is a dense subalgebra, D_1 is satisfied along each $\mathbb{C}^2$-summand because it is finite-dimensional. In Section 2, we construct submultiplicative norms $\|\cdot\|_n$ on the nth copy of $\mathbb{C}^2$, which make the D_1-constants K_n become unbounded as n increases. In Section 3, we define the Banach algebra norm $\|\cdot\|_A$ as the sup of these $\mathbb{C}^2$-norms to construct the counterexample.

2. Some norms on $\mathbb{C}^2$

Let $r \in \mathbb{R}$, and let σ be a constant in $[1, \infty)$. Let $\|\vec{v}\|_{\max} = \max\{|x|, |y|\}$, $\vec{v} = (x, y) \in \mathbb{C}^2$, denote the $C^\star$-norm on $\mathbb{C}^2$. Define a seminorm on $\mathbb{C}^2$ by

$$\|\vec{v}\|_{r,\sigma} = \|T_{r,\sigma}\vec{v}\|_{\max} = \max\big\{|x + ry|, \sigma|y - rx|\big\}, \tag{2.1}$$

where $T_{r,\sigma}$ is the (2×2)-matrix

$$\begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix}. \tag{2.2}$$

Note that $\Delta = \det(T_{r,\sigma}) = \sigma(1 + r^2) > 0$, so $T_{r,\sigma}$ is invertible, and $\|\cdot\|_{r,\sigma}$ is a norm on $\mathbb{C}^2$.

We want to find the smallest constant $D > 0$ which satisfies $\|\vec{v}\|_{\max} \le D\|\vec{v}\|_{r,\sigma}$ for all $\vec{v} \in \mathbb{C}^2$. Then D is the norm of

$$T_{r,\sigma}^{-1} = \frac{1}{\Delta}\begin{pmatrix} \sigma & -r \\ \sigma r & 1 \end{pmatrix}$$

as an operator on $\mathbb{C}^2$ with max-norm,

$$\begin{aligned} D = \|T_{r,\sigma}^{-1}\|_{\mathrm{op}} &= \max\big\{\big\|(\sigma, -r)\big\|_1, \big\|(\sigma r, 1)\big\|_1\big\}/\Delta \\ &= \frac{1}{\sigma(1+r^2)}\max\big\{\sigma + |r|, \sigma|r| + 1\big\} \\ &= \frac{1}{1+r^2}\max\big\{1 + |r|/\sigma, |r| + 1/\sigma\big\} \\ &\le \frac{1}{1+r^2}\max\big\{1 + |r|, |r| + 1\big\} \quad \text{since } \sigma \ge 1 \\ &= \frac{1+|r|}{1+r^2}, \end{aligned} \tag{2.3}$$

since the dual of the max-norm is the ℓ^1-norm. As r ranges over all real numbers, the last expression in (2.3) is bounded by 1.21.

Next we want to find a constant $C > 0$ satisfying $\|\vec{v}_1 * \vec{v}_2\|_{r,\sigma} \le C\|\vec{v}_1\|_{r,\sigma}\|\vec{v}_2\|_{r,\sigma}$, for all $\vec{v}_1, \vec{v}_2 \in \mathbb{C}^2$, where $\vec{v}_1 * \vec{v}_2$ denotes pointwise multiplication. This is equivalent to finding the norm of the operator $\vec{u}_1 \otimes \vec{u}_2 \mapsto T_{r,\sigma}(T_{r,\sigma}^{-1}\vec{u}_1 * T_{r,\sigma}^{-1}\vec{u}_2)$ from $\mathbb{C}^2 \otimes \mathbb{C}^2$ to $\mathbb{C}^2$. The operator is $\frac{1}{\Delta^2}$ times

$$\begin{aligned} &\begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix}\left(\begin{pmatrix} \sigma & -r \\ \sigma r & 1 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} * \begin{pmatrix} \sigma & -r \\ \sigma r & 1 \end{pmatrix}\begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix}\begin{pmatrix} (\sigma u_{11} - r u_{12})(\sigma u_{21} - r u_{22}) \\ (\sigma r u_{11} + u_{12})(\sigma r u_{21} + u_{22}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & r \\ -\sigma r & \sigma \end{pmatrix}\begin{pmatrix} \sigma^2 & -r\sigma & -r\sigma & r^2 \\ \sigma^2 r^2 & \sigma r & \sigma r & 1 \end{pmatrix}\begin{pmatrix} u_{11}u_{21} \\ u_{11}u_{22} \\ u_{12}u_{21} \\ u_{12}u_{22} \end{pmatrix} \\ &= \begin{pmatrix} \sigma^2(1+r^3) & \sigma(r^2 - r) & \sigma(r^2 - r) & r^2 + r \\ \sigma^3(r^2 - r) & \sigma^2(r^2 + r) & \sigma^2(r^2 + r) & \sigma(1 - r^3) \end{pmatrix}\begin{pmatrix} u_{11}u_{21} \\ u_{11}u_{22} \\ u_{12}u_{21} \\ u_{12}u_{22} \end{pmatrix}. \end{aligned}$$

So the constant C is

$$\begin{aligned} C = \max\big\{&\big\|\big(\sigma^2(1+r^3), \sigma(r^2 - r), \sigma(r^2 - r), r^2 + r\big)\big\|_1, \\ &\big\|\big(\sigma^3(r^2 - r), \sigma^2(r^2 + r), \sigma^2(r^2 + r), \sigma(1 - r^3)\big)\big\|_1\big\}/\Delta^2 \\ = \frac{\sigma\max\{\frac{|1+r^3|}{\sigma} + \frac{2|r^2-r|}{\sigma^2} + \frac{|r^2+r|}{\sigma^3}, |r^2 - r| + \frac{2|r^2+r|}{\sigma} + \frac{|1-r^3|}{\sigma^2}\}}{(1+r^2)^2} \end{aligned} \tag{2.4}$$

$$\leq \frac{\sigma \max\{|1+r^3| + 2|r^2 - r| + |r^2 + r|, |r^2 - r| + 2|r^2 + r| + |1 - r^3|\}}{(1+r^2)^2},$$

where the first step used the fact that the dual of the max-norm is the ℓ^1-norm, and the last step used that $\sigma \geq 1$. As r ranges over all real numbers, the last expression in (2.4) is bounded by 2σ.

It follows that $2\sigma\|\vec{v}_1 * \vec{v}_2\|_{r,\sigma} \leq (2\sigma\|\vec{v}_1\|_{r,\sigma})(2\sigma\|\vec{v}_2\|_{r,\sigma})$, for $\vec{v}_1, \vec{v}_2 \in \mathbb{C}^2$. By the preceding paragraph, $\|\vec{v}\|_{\max} \leq D\|\vec{v}\|_{r,\sigma} < 2\|\vec{v}\|_{r,\sigma} = \frac{1}{\sigma}(2\sigma\|\vec{v}\|_{r,\sigma})$, for $\vec{v} \in \mathbb{C}^2$.

3. The Banach algebras $A_{r,\sigma}$

In this section, we pass from the case of a single copy of $\mathbb{C}^2$ (see Section 2) to infinitely many copies of $\mathbb{C}^2$. Let $r \in \mathbb{R}$, and let σ be any unbounded function from $\mathbb{N}$ to $[1, \infty)$. Define an infinite matrix $S_{r,\sigma}$ as the direct sum of (2×2)-matrices,

$$S_{r,\sigma} = \bigoplus_{n=0}^{\infty} 2\sigma(n) T_{r,\sigma(n)},$$

where each $T_{r,\sigma(n)}$ is defined as in (2.2). Let $A_{r,\sigma} = \{f \in c_0 \mid S_{r,\sigma}\theta f \in c_0(\mathbb{N}, \mathbb{C}^2)\}$, where θ was defined in the Introduction (see (1.2)). By [5, Theorem 4.3.1], $A_{r,\sigma}$ is complete in the norm

$$\|f\|_{r,\sigma} = \|S_{r,\sigma}\theta f\|_{c_0(\mathbb{N},\mathbb{C}^2)} = \sup_{n=0}^{\infty}\{2\sigma(n)\big\|\big(f(2n), f(2n+1)\big)\big\|_{r,\sigma(n)}\}, \tag{3.1}$$

where each $\|\cdot\|_{r,\sigma(n)}$ is defined as in (2.1), and where by the final remarks of Section 2, $\|fg\|_{r,\sigma} \leq \|f\|_{r,\sigma}\|g\|_{r,\sigma}$ and $\|f\|_\infty \leq \|f\|_{r,\sigma}$, for $f, g \in A_{r,\sigma}$. Since $S_{r,\sigma}\theta f \in c_0(\mathbb{N}, \mathbb{C}^2)$ for $f \in A_{r,\sigma}$, then

$$0 = \lim_{n\to\infty} \big\|(S_{r,\sigma}\theta f)(n)\big\|_{\mathbb{C}^2} = \lim_{n\to\infty} 2\sigma(n)\big\|\big(f(2n), f(2n+1)\big)\big\|_{r,\sigma(n)}.$$

For $\epsilon > 0$, let N_ϵ be large enough so that the argument of the limit is smaller than ϵ if $n > N_\epsilon$. Then for $n \geq N_\epsilon$,

$$\begin{aligned}\big\|f - \big(f(0), \ldots, f(2n+1), 0, 0, \ldots\big)\big\|_{r,\sigma} &= \big\|\big(\underbrace{0, \ldots, 0}_{2n+2 \text{ zeros}}, f(2n+2), \ldots\big)\big\|_{r,\sigma} \\ &= \sup_{k>n} 2\sigma(k)\big\|\big(f(2k), f(2k+1)\big)\big\|_{r,\sigma(k)} \\ &< \epsilon,\end{aligned}$$

using the definition of $\|\cdot\|_{r,\sigma}$ (3.1). It follows that c_f is dense in $A_{r,\sigma}$, and we can apply Theorem 1.1 to see that $A_{r,\sigma}$ is spectral-invariant in c_0.

Theorem 3.1. *The dense Banach subalgebra $A_{r,\sigma}$ of c_0 is not a D_1-subalgebra of c_0 for $r \in \mathbb{R} \setminus \{0, 1\}$.*

Proof. For $n \in \mathbb{N}$, define $a_n \in A_{r,\sigma}$ by

$$a_n = (\underbrace{0, 0, \ldots, 0, 0}_{2n \text{ zeros}}, 1, r, 0, 0, \ldots),$$

where $a_n(2n) = 1$ and $a_n(2n+1) = r$ are the only nonzero components. Then

$$\begin{aligned} \|a_n\|_{r,\sigma} &= 2\sigma(n)(1+r^2), \\ \|a_n^2\|_{r,\sigma} &= 2\sigma(n)\max\bigl(|1+r^3|, \sigma(n)|r^2-r|\bigr), \\ \|a_n\|_\infty &= \max\bigl(1, |r|\bigr). \end{aligned}$$

If $K > 0$ were a constant satisfying the D_1-condition (1.1) for $A_{r,\sigma}$ in c_0, then

$$\begin{aligned} K &\geq \frac{\|a_n^2\|_{r,\sigma}}{2\|a_n\|_\infty \|a_n\|_{r,\sigma}} = \frac{\max(|1+r^3|, \sigma(n)|r^2-r|)}{2(1+r^2)\max(1,|r|)} \\ &\geq \frac{\sigma(n)|r^2-r|}{2(1+r^2)\max(1,|r|)} \end{aligned}$$

must hold for each n for which $\sigma(n)|r^2-r| \geq |1+r^3|$. No such constant K can exist if $r \neq 1$ or 0, since σ is unbounded. $\square$

Remark 3.2. Note that $A_{r,\sigma}$ is a Banach $\star$-algebra. The norms defined on $\mathbb{C}^2$ (2.1) and the norm on $A_{r,\sigma}$ (3.1) are both left unchanged by the $\star$-operation of pointwise complex conjugation.

Remark 3.3. In the cases $r = 0$ and $r = 1$, it can be shown that $A_{r,\sigma}$ is a D_1-subalgebra of c_0. Further, $A_{r,\sigma}$ is a dense Banach ideal in c_0 if $r = 0$, but not if $r = 1$.

References

1. S. J. Bhatt, *Smooth Fréchet subalgebras of $C^\star$-algebras defined by first order differential seminorms*, Proc. Indian Acad. Sci. Math. Sci. **126** (2016), no. 1, 125–141. Zbl 06563136. MR3470819. DOI 10.1007/s12044-016-0265-8. 686
2. A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994. Zbl 0818.46076. MR1303779. 686
3. E. Kissin and V. S. Shulman, *Differential properties of some dense subalgebras of $C^\star$-algebras*, Proc. Edinb. Math. Soc. (2) **37** (1994), no. 3, 399–422. Zbl 0808.46066. MR1297311. DOI 10.1017/S0013091500018873. 687
4. T. W. Palmer, *Banach Algebras and the General Theory of $\star$-algebras, I: Algebras and Banach Algebras*, Encyclopedia Math. Appl. **49**, Cambridge Univ. Press, Cambridge, 1994. Zbl 0809.46052. MR1270014. DOI 10.1017/CBO9781107325777. 686
5. A. Wilansky, *Summability through Functional Analysis*, North-Holland Math. Stud. **85**, North-Holland, Amsterdam, 1984. Zbl 0531.40008. MR0738632. 689

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