Ann. Funct. Anal. 7 (2016), no. 4, 672–677
http://dx.doi.org/10.1215/20088752-3661116
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A NEW CHARACTERIZATION OF THE BOUNDED APPROXIMATION PROPERTY

JU MYUNG KIM[1] and KEUN YOUNG LEE[2*]

Communicated by J. Esterle

Abstract. We prove that a Banach space X has the bounded approximation property if and only if, for every separable Banach space Z and every injective operator T from Z to X, there exists a net (S_α) of finite-rank operators from Z to X with $\|S_\alpha\| \le \lambda_T$ such that $\lim_\alpha \|S_\alpha z - Tz\| = 0$ for every $z \in Z$.

1. Introduction

A Banach space X is said to have the *approximation property* (AP) if, for every compact subset K of X and every $\varepsilon > 0$, there exists a finite-rank and continuous linear map (operator) S from X to X such that $\sup_{x\in K} \|Sx - x\| \le \varepsilon$; briefly, $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_c}$, where id_X is the identity map on X, $\mathcal{F}(X,X)$ is the space of all finite-rank operators from X to X, and τ_c is the topology of uniformly compact convergence on the ideal $\mathcal{L}$ of all operators. If there exists a $\lambda \ge 1$ such that $\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X,X) : \|S\| \le \lambda\}}^{\tau_c}$, then we say that X has the *bounded approximation property* (BAP) or λ-BAP when we need to indicate the constant λ.

Lima, Nygaard, and Oja [5, Corollary 1.5] proved that X has the AP if and only if, for every Banach space Y and every $T \in \mathcal{W}(Y,X)$, the space of all weakly compact operators from Y to X, $T \in \overline{\{S \in \mathcal{F}(Y,X) : \|S\| \le \|T\|\}}^{\tau_c}$. A simple verification shows that X has the BAP if and only if for every Banach space Y and every $T \in \mathcal{L}(Y,X)$, there exists a $\lambda_T > 0$ such that $T \in \overline{\{S \in \mathcal{F}(Y,X) : \|S\| \le \lambda_T\}}^{\tau_c}$.

Received Mar. 22, 2016; Accepted Jun. 24, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B28; Secondary 47L20.
Keywords. bounded approximation property, bounded compact approximation property, separable Banach space.

Figiel and Johnson [1] proved that the AP does not imply the BAP in general. Consequently, the BAP cannot be characterized by all weakly compact operators in that criterion of the BAP. The purpose of this paper is to replace Banach spaces Y in that criterion of the BAP by separable Banach spaces.

Theorem 1.1. *A Banach space X has the BAP if and only if, for every separable Banach space Z and every injective operator J from Z to X, there exists a $\lambda_J > 0$ such that*

$$J \in \overline{\{S \in \mathcal{F}(Z,X) : \|S\| \le \lambda_J\}}^{\tau_c}.$$

2. Proof of Theorem 1.1

When the ideal of finite-rank operators in the definition of the BAP for a Banach space X is replaced by the ideal $\mathcal{K}$ of compact operators, we say that X has the *bounded compact approximation property* (BCAP). The following lemma gives an affirmative answer to a problem in [4].

Lemma 2.1. *Let $\lambda \ge 1$. A Banach space X has the λ-BAP (resp., λ-BCAP) if and only if for every separable closed subspace Y of X,*

$$J_Y \in \overline{\{S \in \mathcal{F}(Y,X) \ \big(\text{resp., } \mathcal{K}(Y,X)\big) : \|S\| \le \lambda\}}^{\tau_c},$$

where $J_Y : Y \to X$ is the inclusion map.

Proof. We only need to prove the "if" part. This proof comes from the one in [2, Lemma 3(b)]. Let F be a finite-dimensional subspace of X, and let $\varepsilon > 0$ be given. Then by [7, Lemma 1] there exists a separable closed subspace Y of X such that, for every finite-dimensional subspace E of X with $F \subset E$, there exists an operator $T_E : E \to Y$ satisfying $\|T_E\| \le 1 + 1/\dim E$ and such that the restriction $T_E|_F$ is the identity map.

Now, by the assumption there exists an $S \in \mathcal{F}(Y,X)$ (resp., $\mathcal{K}(Y,X)$) with $\|S\| \le \lambda$ such that

$$\|Sf - f\| \le \varepsilon \|f\|$$

for every $f \in F$. We define the map $S_E : X \to X$ by

$$S_E x = ST_E x \quad \text{if } x \in E, \qquad S_E = 0 \quad \text{otherwise},$$

for every finite-dimensional subspace E of X with $F \subset E$. Let us consider the product topological space

$$\Pi := \prod_{x \in X} \overline{S\big(2\|x\| B_Y\big)}$$

equipped with the product topology of norms. Then Π is a compact Hausdorff space. We see that $(\prod_{x\in X} S_E x)$ is a net in Π, where $E_2 \succeq E_1$ if and only if $E_2 \supset E_1$. Then there exists a subnet $(\prod_{x\in X} S_G x)$ of $(\prod_{x\in X} S_E x)$ such that

$$\widetilde{S}x := \lim_G S_G x \in \overline{S\big(2\|x\| B_Y\big)}$$

exists for each $x \in X$. We see that $\widetilde{S} : X \to X$ is a finite-rank linear operator (resp., compact operator), $\|\widetilde{S}\| \le \lambda$, and $\|\widetilde{S}f - f\| \le \varepsilon\|f\|$ for every $f \in F$. Hence it follows that X has the λ-BAP (resp., λ-BCAP). $\square$

We note that recently, Oja proved Lemma 2.1 differently in [10, Proposition 2.2], basing that proof on ideals and a Hahn–Banach extension operator.

The argument in the proof of the following lemma is a symmetric version of the argument in [3, proof of Proposition 2.1].

Lemma 2.2. *Let X be a Banach space, and let $\mathcal{A}(X,X)$ be a convex subset of $\mathcal{L}(X,X)$. The following statements are equivalent.*

(a) *For every separable Banach space Y and every $T \in \mathcal{L}(Y,X)$, there exists a $\lambda_T > 0$ such that*

$$T \in \overline{\{ST : S \in \mathcal{A}(X,X), \|ST\| \le \lambda_T\}}^{\tau_c}.$$

(b) *For every separable Banach space Y and every $T \in \mathcal{L}(Y,X)$, there exists a $\lambda_T > 0$ such that*

$$\mathrm{id}_X \in \overline{\{S \in \mathcal{A}(X,X) : \|ST\| \le \lambda_T\}}^{\tau_c}.$$

(c) *There exists a $\lambda > 0$ such that, for every separable Banach space Y and every $T \in \mathcal{L}(Y,X)$,*

$$T \in \overline{\{ST : S \in \mathcal{A}(X,X), \|ST\| \le \lambda\|T\|\}}^{\tau_c}.$$

(d) *There exists a $\lambda > 0$ such that, for every separable Banach space Y and every $T \in \mathcal{L}(Y,X)$,*

$$\mathrm{id}_X \in \overline{\{S \in \mathcal{A}(X,X) : \|ST\| \le \lambda\|T\|\}}^{\tau_c}.$$

Proof. (a) $\Rightarrow$ (b) Let Y be a separable Banach space, and let $T \in \mathcal{L}(Y,X)$. We may assume that $\|T\| \le 1$. Suppose that for every $\lambda > 0$, $\mathrm{id}_X \notin \overline{\{S \in \mathcal{A}(X,X) : \|ST\| \le \lambda\}}^{\tau_c}$. Then by the separation theorem, for each $m \in \mathbb{N}$, there exists a $g_m \in (\mathcal{L}(X,X),\tau_c)^*$ such that

$$\operatorname{Re} g_m(\mathrm{id}_X) > \sup\{\operatorname{Re} g_m(S) : S \in \mathcal{A}(X,X), \|ST\| \le m\}.$$

By [8, Proposition 1.e.3], for each m, there exist sequences $(x_{n,m})_{n=1}^\infty$ and $(x^*_{n,m})_{n=1}^\infty$ in X and X^*, respectively, with $\sum_{n=1}^\infty \|x_{n,m}\|\|x^*_{n,m}\| < \infty$ such that

$$g_m(R) = \sum_{n=1}^\infty x^*_{n,m}(Rx_{n,m})$$

for all $R \in \mathcal{L}(X,X)$. We may assume that for every n and m, $\|x_{n,m}\| \le 1$ and $\sum_{n=1}^\infty \|x^*_{n,m}\| < \infty$.

Let us consider the balanced and closed convex hull C of

$$\Big(\bigcup_{m=1}^\infty \{x_{n,m}\}_{n=1}^\infty\Big) \cup T(B_Y),$$

which is a separable subset of B_X. By [5, Lemmas 1.1, 2.1, Theorem 2.2] there exists a separable Banach space Z, which is a linear subspace of X, such that $C \subset B_Z$ and the inclusion map $J : Z \to X$ has norm 1. By (a) there exists a $\lambda_J > 0$ such that

$$J \in \overline{\{SJ : S \in \mathcal{A}(X,X), \|SJ\| \le \lambda_J\}}^{\tau_c}.$$

Now choose an $N \in \mathbb{N}$ such that $N \geq \lambda_J$. Since $h_N = \sum_{n=1}^{\infty} x^*_{n,N}(\cdot x_{n,N}) \in (\mathcal{L}(Z,X),\tau_c)^*$, we have

$$\begin{aligned}\operatorname{Re} g_N(\mathrm{id}_X) &= \operatorname{Re} h_N(J)\\ &\leq \sup\bigl\{\operatorname{Re} h_N(SJ) : S \in \mathcal{A}(X,X), \|SJ\| \leq \lambda_J\bigr\}\\ &= \sup\bigl\{\operatorname{Re} g_N(S) : S \in \mathcal{A}(X,X), \|SJ\| \leq \lambda_J\bigr\}.\end{aligned}$$

If $S \in \mathcal{A}(X,X)$ with $\|SJ\| \leq \lambda_J$, then

$$\|ST\| = \sup_{y \in B_Y} \|SJTy\| \leq \|SJ\| \leq \lambda_J.$$

Hence we have

$$\operatorname{Re} g_N(\mathrm{id}_X) \leq \sup\bigl\{\operatorname{Re} g_N(S) : S \in \mathcal{A}(X,X), \|ST\| \leq \lambda_J\bigr\},$$

which is a contradiction.

(b) $\Rightarrow$ (c) Suppose that (c) fails. Then for every $m \in \mathbb{N}$ there exist a separable Banach space Y_m and $T_m \in \mathcal{L}(Y_m, X)$ such that

$$T_m \notin \overline{\bigl\{ST_m : S \in \mathcal{A}(X,X), \|ST_m\| \leq m\|T_m\|\bigr\}}^{\tau_c}.$$

We may assume that $\|T_m\| = 1$ for all m. Let us define the map $T : (\sum_m \oplus Y_m)_{\ell_1} \to X$ by

$$T(y_m)_{m=1}^{\infty} = \sum_{n=1}^{\infty} T_m y_m.$$

Then the map is well defined and linear, and $\|T\| \leq 1$. Thus by (b) there exists a $\lambda_T > 0$ such that

$$\mathrm{id}_X \in \overline{\bigl\{S \in \mathcal{A}(X,X) : \|ST\| \leq \lambda_T\bigr\}}^{\tau_c}.$$

Hence for every m, we have

$$\begin{aligned}T_m &\in \overline{\bigl\{ST_m : S \in \mathcal{A}(X,X), \|ST\| \leq \lambda_T\bigr\}}^{\tau_c}\\ &\subset \overline{\bigl\{ST_m : S \in \mathcal{A}(X,X), \|ST_m\| \leq \lambda_T\bigr\}}^{\tau_c},\end{aligned}$$

which is a contradiction.

(c) $\Rightarrow$ (d) Let Y be a separable Banach space, and let $T \in \mathcal{L}(Y,X)$. We may assume that $\|T\| \leq 1$. Let K be a compact subset of B_X, and let $\varepsilon > 0$ be given. Let us consider the balanced and closed convex hull C of

$$K \cup T(B_Y),$$

which is a separable subset of B_X. Then by [5, Lemmas 1.1, 2.1, Theorem 2.2] there exists a separable Banach space Z, which is a linear subspace of X, such that $C \subset B_Z$ and the inclusion map $J : Z \to X$ has norm 1. Moreover, K is also compact in Z. By (c), we have

$$J \in \overline{\bigl\{SJ : S \in \mathcal{A}(X,X), \|SJ\| \leq \lambda\bigr\}}^{\tau_c}.$$

Hence there exists an $S \in \mathcal{A}(X,X)$ with $\|ST\| \le \|SJ\| \le \lambda$ such that

$$\sup_{x\in K}\|Sx - x\| = \sup_{x\in K}\|SJx - Jx\| \le \varepsilon.$$

(d) $\Rightarrow$ (a) This is clear. $\square$

Now, we prove Theorem 1.1. Let Y be a separable Banach space, and let $T \in \mathcal{L}(Y,X)$. Then by [5, Lemmas 1.1, 2.1, Theorem 2.2], there exist a separable Banach space $Z \subset X$ and an operator $R : Y \to Z$ such that $T = JR$, where the inclusion map $J : Z \to X$ has norm 1. By the assumption and [4, Lemma 3.5], there exists a $\lambda_J > 0$ such that

$$J \in \overline{\{S \in \mathcal{F}(Z,X) : \|S\| \le \lambda_J\}}^{\tau_c} = \overline{\{SJ : S \in \mathcal{F}(X,X), \|SJ\| \le \lambda_J\}}^{\tau_c}.$$

Thus there exists a net (S_α) in $\mathcal{F}(X,X)$ such that $\|S_\alpha J\| \le \lambda_J$ for all α and

$$S_\alpha J \xrightarrow{\tau_c} J.$$

Then for every $y \in Y$, we have

$$\|S_\alpha Ty - Ty\| = \|S_\alpha JRy - JRy\| \longrightarrow 0$$

and $\|S_\alpha T\| = \|S_\alpha JR\| \le \lambda_J\|R\|$. It follows that

$$T \in \overline{\{ST : S \in \mathcal{F}(X,X), \|ST\| \le \lambda_T\}}^{\tau_c}$$

for some $\lambda_T > 0$. Hence by Lemmas 2.1 and 2.2, X has the BAP.

3. Open problems

Considering a symmetric version of Theorem 1.1, we ask the following question.

Problem 1. Let X be a Banach space. If, for every separable Banach space Z and every $T \in \mathcal{L}(X,Z)$, there exists a $\lambda_T > 0$ such that $T \in \overline{\{S \in \mathcal{F}(X,Z) : \|S\| \le \lambda_T\}}^{\tau_c}$, then does X have the BAP?

Considering a symmetric version of Lemma 2.1, we ask the following question.

Problem 2. Let X be a Banach space, and let $\lambda \ge 1$. If, for every separable Banach space Z and every $T \in \mathcal{L}(X,Z)$, $T \in \overline{\{S \in \mathcal{F}(X,Z) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$, then does X have the λ-BAP?

Lima and Oja [6] introduced a weaker notion of the BAP. A Banach space X is said to have the *weak* BAP if there exists a $\lambda \ge 1$ such that for every Banach space Y and every $T \in \mathcal{W}(X,Y)$, $\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X,X) : \|TS\| \le \lambda\|T\|\}}^{\tau_c}$. Oja [9, Theorem 2] proved that X has the weak λ-BAP if and only if for every Banach space Y whose dual space has the Radon–Nikodým property and every $T \in \mathcal{L}(X,Y)$, $\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X,X) : \|TS\| \le \lambda\|T\|\}}^{\tau_c}$. One may naturally ask whether the weak λ-BAP is equivalent to the λ-BAP. Lima and Oja [6] conjectured that the weak λ-BAP is strictly weaker than the λ-BAP. We ask the following question.

Problem 3. Let X be a Banach space, and let $\lambda \geq 1$. If, for every separable Banach space Z and every $T \in \mathcal{L}(X, Z)$, $\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\| \leq \lambda \|T\|\}}^{\tau_c}$, then does X have the λ-BAP?

At the present time, we do not know whether the assumptions in the above problems would be equivalent.

Acknowledgment. Kim's work was partially supported by National Research Foundation of Korea grant NRF-2013R1A1A2A10058087.

References

1. T. Figiel and W. B. Johnson, *The approximation property does not imply the bounded approximation property*, Proc. Amer. Math. Soc. **41** (1973), 197–200. Zbl 0289.46015. MR0341032. 673
2. W. B. Johnson, *A complementary universal conjugate Banach space and its relation to the approximation problem*, Israel J. Math. **13** (1972), 301–310. Zbl 0252.46024. MR0326356. 673
3. J. M. Kim and B. Zheng, *The strong approximation property and the weak bounded approximation property*, J. Funct. Anal. **266** (2014), no. 8, 5439–5447. Zbl 1330.46024. MR3177343. DOI 10.1016/j.jfa.2013.12.016. 674
4. K. Y. Lee, *The separable weak bounded approximation property*, Bull. Korean Math. Soc. **52** (2015), no. 1, 69–83. Zbl 1319.46015. MR3313425. DOI 10.4134/BKMS.2015.52.1.069. 673, 676
5. Å. Lima, O. Nygaard, and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. **119** (2000), 325–348. Zbl 0983.46024. MR1802659. DOI 10.1007/BF02810673. 672, 674, 675, 676
6. Å. Lima and E. Oja, *The weak metric approximation property*, Math. Ann. **333** (2005), no. 3, 471–484. Zbl 1097.46012. MR2198796. DOI 10.1007/s00208-005-0656-0. 676
7. J. Lindenstrauss, *On nonseparable reflexive Banach spaces*, Bull. Amer. Math. Soc. (N.S.) **72** (1966), 967–970. Zbl 0156.36403. MR0205040. 673
8. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I: Sequence Spaces*, Ergeb. Math. Grenzgeb (3) **92**, Springer, Berlin, 1977. Zbl 0362.46013. MR0500056. 674
9. E. Oja, *The impact of the Radon-Nikodým property on the weak bounded approximation property*, Rev. R. Acad. Cien. Exactas Fís. Nat. Ser. A. Mat. RACSAM **100** (2006), no. 1–2, 325–331. Zbl 1112.46017. MR2267414. 676
10. E. Oja, *On a separable weak version of the bounded approximation property*, Arch. Math. (Basel) **107** (2016), no. 2, 185–189. Zbl 0661.9006. MR3528390. 674

[1]Department of Mathematical Sciences, Seoul National University, Seoul, 151-747, Korea.

E-mail address: kjm21@kaist.ac.kr

[2]Institute for Ubiquitous Information Technology and Applications, Konkuk University, Seoul, 143-701, Korea.

E-mail address: northstar@kaist.ac.kr