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# SCHÄFFER-TYPE CONSTANT AND UNIFORM NORMAL STRUCTURE IN BANACH SPACES

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Abstract. The exact value of the Schäffer-type constants are investigated under the absolute normalized norms on $\mathbb{R}^2$ by means of their corresponding continuous convex functions on $[0,1]$. Moreover, some sufficient conditions which imply uniform normal structure are presented. These results improve some known results.

## 1. Introduction and preliminaries

Throughout this article, $S_X$ and $B_X$ denote the unit sphere and the unit ball of a Banach space $X$, respectively. Let $C$ be a nonempty bounded closed convex subset of a Banach space $X$. A mapping $T : C \to C$ is said to be *nonexpansive*, provided that the inequality

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for every $x, y \in C$. A Banach space $X$ is said to have the *fixed point property* if every nonexpansive mapping $T : C \to C$ has a fixed point.

Recall that a Banach space $X$ is called *uniformly nonsquare* in the sense of Schäffer if there is a $\lambda > 1$ such that

$$\max\big(\|x + y\|, \|x - y\|\big) \geq \lambda$$

for all $x, y \in S_X$. The Schäffer constant, defined by

$$S(X) = \inf\big\{\max\big\{\|x + y\|, \|x - y\| : x, y \in S_X\big\}\big\},$$


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is introduced to characterize this concept: $X$ is uniformly nonsquare in the sense of Schäffer if and only if $S(X) > 1$.

A bounded convex subset $K$ of a Banach space $X$ is said to have *normal structure* if for every convex subset $H$ of $K$ that contains more than one point there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space $X$ is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any closed bounded convex subset $K$ of $X$ that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x - y\| : x, y \in K\}.$$

It was proved by W. A. Kirk that every reflexive Banach space with normal structure has the fixed point property.

In recent times, many constants in Banach space have been defined and/or studied, such as the James constant in [6], the Schäffer constant in [11], and the von Neumann–Jordan constant in [6]. It has been shown that these constants are very useful in the geometric theory of Banach spaces, which enables us to classify several important concepts of Banach spaces such as uniformly nonsquareness and uniform normal structure (see [6], [9], [14]). On the other hand, calculation of the constant for some concrete spaces is also of some interest (see [3], [4], [6], [8], [13]–[15]). In [14], Takahashi introduced the Schäffer-type constant $S_{X,t}(\tau)$ as a generalization of the Schäffer constant $S(X)$.

*Definition* 1.1. For $\tau \geq 0$ and $1 < t \leq \infty$ the constant $S_{X,t}(\tau)$ is defined to be

$$S_{X,t}(\tau) = \inf\{\mathcal{M}_t(\|x + \tau y\|, \|x - \tau y\|) : x, y \in S_X\}.$$

$\mathcal{M}_t(a, b)$ is the generalized mean defined by

$$\mathcal{M}_t(a, b) := \Big(\frac{a^t + b^t}{2}\Big)^{1/t},$$

where $a$ and $b$ are two positive real numbers. It is well known that $\mathcal{M}_t(a, b)$ is nondecreasing and $\mathcal{M}_\infty(a, b) := \lim_{t\to\infty} \mathcal{M}_t(a, b) = \max(a, b)$. Obviously $S_{X,t}(\tau)$ is an extension of $S(X)$, which also includes Gao's constant $f(X) = 2S^2_{X,2}(1)$ (see [3]) as a special case. Some basic properties of this new coefficient are investigated in [14] and [15]. In particular, Wang and Yang in [15] study the constant $S_{X,\tau}(1)$ extensively and get some useful results, such as

(1) $S_{X,\tau}(1) = \min\{\mathcal{M}_t(\epsilon, 2(1 - \varrho_X(\epsilon))) : 0 \leq \epsilon \leq S(X)\}$;

(2) $X$ is uniformly nonsquare $\Leftrightarrow S_{X,\tau}(1) > 1$ for some $t > 1$.

The modulus of smoothness $\varrho_X(\epsilon)$ is defined in [1] as

$$\varrho_X(\epsilon) = \sup\Big\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \leq \epsilon\Big\}.$$

By the formula (1), they compute $S_{X,\tau}(1)$ in some concrete spaces by the modulus of smoothness $\varrho_X(\epsilon)$. However, the value of $\varrho_X(\epsilon)$ in many some concrete Banach spaces, such as the space of absolute normalized norms, is not known, and therefore the formula (1) is invalid.

In the following pages we give a simple method to determine and estimate the $S_{X,\tau}(1)$ of absolute normalized norms on $\mathbb{R}^2$ which are complementary to results of Takahashi and of Wang and Yang (see [14] and [15]). Moreover, some sufficient conditions which imply uniform normal structure are presented.

Recall that a norm on $\mathbb{R}^2$ is called *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z,w \in \mathbb{R}$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Let $N_\alpha$ denote the family of all absolute normalized norms on $\mathbb{R}^2$, and let $\Psi$ denote the family of all continuous convex functions on $[0,1]$ such that $\psi(1) = \psi(0) = 1$ and $\max\{1-s, s\} \le \psi(s) \le 1$ $(0 \le s \le 1)$. It has been shown that $N_\alpha$ and $\Psi$ are in a one-to-one correspondence (see [2]).

**Proposition 1.2.** *If $\|\cdot\| \in N_\alpha$, then $\psi(s) = \|(1-s,s)\| \in \Psi$. On the other hand, if $\psi(s) \in \Psi$, defining the norm $\|\cdot\|_\psi$ as*

$$\big\|(z,\omega)\big\|_\psi := \begin{cases} (|z|+|\omega|)\psi(\frac{|\omega|}{|z|+|\omega|}), & (z,\omega) \neq (0,0), \\ 0, & (z,\omega) = (0,0), \end{cases}$$

*then the norm $\|\cdot\|_\psi \in N_\alpha$.*

A simple example of the absolute normalized norm is the usual $l_r$ $(1 \le r \le \infty)$ norm. From Proposition 1.2, one can easily get the corresponding function of the $l_r$ norm:

$$\psi_r(s) = \begin{cases} \{(1-s)^r + s^r\}^{1/r}, & 1 \le r < \infty, \\ \max\{1-s, s\}, & r = \infty. \end{cases}$$

Also, the above correspondence enables us to get many non-$l_r$ norms on $\mathbb{R}^2$. One of the properties of these norms is stated in the following result.

**Proposition 1.3.** *Let $\psi, \varphi \in \Psi$, and let $\varphi \le \psi$. Put $M = \max_{0 \le s \le 1} \frac{\psi(s)}{\varphi(s)}$. Then*

$$\|\cdot\|_\varphi \le \|\cdot\|_\psi \le M\|\cdot\|_\varphi.$$

The Cesàro sequence space was defined by Shue [12] in 1970. It is very useful in the theory of matrix operators and others. Let $l$ be the space of real sequences. For $1 < p < \infty$, the Cesàro sequence space $\text{ces}_p$ is defined by

$$\text{ces}_p = \Big\{x \in l : \|x\| = \big\|\big(x(i)\big)\big\| = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\big|x(i)\big|\Big)^p\Big)^{1/p} < \infty\Big\}.$$

The geometry of Cesàro sequence spaces has been extensively studied in [7] and [10]. Let us restrict ourselves to the 2-dimensional Cesàro sequence space $\text{ces}_p^{(2)}$, which is just $\mathbb{R}^2$ equipped with the norm defined by

$$\big\|(x,y)\big\| = \Big(|x|^p + \Big(\frac{|x|+|y|}{2}\Big)^p\Big)^{1/p}.$$

## 2. Schäffer-type constant $S_{X,t}(1)$ and absolute normalized norm

For a norm $\|\cdot\|$ on $\mathbb{R}^2$, we write $S_{X,\tau}(1)(\|\cdot\|)$ for $S_{X,\tau}(1)(\mathbb{R}^2, \|\cdot\|)$. It follows from the property of $S_{X,\tau}(t)$ that one can easily get the following result.

**Proposition 2.1.** *Let $X$ be a nontrivial Banach space, and let $1 < t < \infty$. Then*

$$S_{X,\tau}(t) = \inf\Big\{\Big(\frac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2\min(\|x\|^t, \|y\|^t)}\Big)^{1/t} : x, y \in X, \|x\| + \|y\| \neq 0\Big\}.$$

**Proposition 2.2.** *Let $X$ be the space $l_r$ or $L_r[0,1]$ with $\dim X \geq 2$ (see [14]).*

(1) *Let $1 < r \leq 2$, and let $1/r + 1/r' = 1$. If $r \leq t \leq \infty$, then*

$$S_{X,t}(\tau) = \Big(\frac{(1+\tau)^r + |1-\tau|^r}{2}\Big)^{1/r} \quad \text{for all } \tau \geq 0.$$

(2) *Let $2 \leq r \leq \infty$, and let $1/r + 1/r' = 1$. If $r' \leq t \leq \infty$, then*

$$S_{X,t}(\tau) = (1+\tau^r)^{1/r} \quad \text{for all } \tau \geq 0.$$

**Proposition 2.3.** *Let $\varphi \in \Psi$, and let $\psi(s) = \varphi(1-s)$. Then*

$$S_{X,t}(\tau)\big(\|\cdot\|_\varphi\big) = S_{X,t}(\tau)\big(\|\cdot\|_\psi\big).$$

Since $\|\cdot\|_\psi$ and $\|\cdot\|_\varphi$ are isometric, the proof of Proposition 2.3 is trivial, and so we omit it.

**Lemma 2.4.** *Let $\|\cdot\|$ and $|\cdot|$ be two equivalent norms on a Banach space. If $a|\cdot| \leq \|\cdot\| \leq b|\cdot|$ $(b \geq a > 0)$, then*

$$\frac{a}{b}S_{X,t}(\tau)\big(|\cdot|\big) \leq S_{X,t}(\tau)\big(\|\cdot\|\big) \leq \frac{b}{a}S_{X,t}(\tau)\big(|\cdot|\big).$$

We now consider the constant $S_{X,t}(1)$ of a class of absolute normalized norms on $\mathbb{R}^2$. Now let us put

$$M_1 = \max_{0\leq s\leq 1}\frac{\psi_r(s)}{\psi(s)} \qquad \text{and} \qquad M_2 = \max_{0\leq s\leq 1}\frac{\psi(s)}{\psi_r(s)}.$$

**Theorem 2.5.** *Let $\psi \in \Psi$, and let $\psi \geq \psi_r$ $(1 \leq r \leq 2)$. If the function $\frac{\psi(s)}{\psi_r(s)}$ attains its maximum at $s = 1/2$ and $r \leq t \leq \infty$, then*

$$S_{X,t}(1)\big(\|\cdot\|_\psi\big) = \frac{1}{\psi(1/2)}.$$

*Proof.* By Proposition 1.3, we have $\|\cdot\|_r \leq \|\cdot\|_\psi \leq M_2\|\cdot\|_r$. Let $x, y \in X$, $(x,y) \neq (0,0)$, where $X = \mathbb{R}^2$. Then

$$\begin{aligned}\|x+\tau y\|_\psi^t + \|x-\tau y\|_\psi^t &\geq \|x+\tau y\|_r^t + \|x-\tau y\|_r^t \\ &\geq 2S_{X,t}^t(\tau)\big(\|\cdot\|_r\big)\min\big\{\|x\|_r^t, \|y\|_r^t\big\} \\ &\geq \frac{2}{M_2^t}S_{X,t}^t(\tau)\big(\|\cdot\|_r\big)\min\big\{\|x\|_\psi^t, \|y\|_\psi^t\big\}\end{aligned}$$

from the definition of $S_{X,t}(\tau)$ implies that

$$S_{X,t}(\tau)\big(\|\cdot\|_\psi\big) \geq \frac{1}{M_2}S_{X,t}(\tau)\big(\|\cdot\|_r\big).$$

Note that $r \le t \le \infty$ and the function $\frac{\psi(s)}{\psi_r(s)}$ attains its maximum at $s = 1/2$; that is, $M_2 = \frac{\psi(1/2)}{\psi_r(1/2)}$. Proposition 2.2 implies that

$$S_{X,t}(\tau)\big(\|\cdot\|_\psi\big) \ge \frac{1}{M_2} S_{X,t}(\tau)\big(\|\cdot\|_r\big) = \frac{1}{\psi(1/2)}. \tag{2.1}$$

On the other hand, let us put $x = (a,a), y = (a,-a)$, where $a = \frac{1}{2\psi(1/2)}$; hence, $\|x\|_\psi = \|y\|_\psi = 1$, and

$$\Big(\frac{\|x+y\|_\psi^t + \|x-y\|_\psi^t}{2}\Big)^{1/t} = 2a = \frac{1}{\psi(1/2)}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$S_{X,t}(1)\big(\|\cdot\|_\psi\big) = \frac{1}{\psi(1/2)}.$$

□

**Theorem 2.6.** *Let $\psi \in \Psi$, and let $\psi \le \psi_r$ $(2 \le r < \infty)$. If the function $\frac{\psi_r(s)}{\psi(s)}$ attains its maximum at $s = 1/2$ and $r' \le t \le \infty$, then*

$$S_{X,t}(1)\big(\|\cdot\|_\psi\big) = 2\psi(1/2).$$

*Proof.* By Proposition 1.3, we have $\|\cdot\|_\psi \le \|\cdot\|_r \le M_1\|\cdot\|_\psi$. Let $x, y \in X$, $(x,y) \ne (0,0)$, where $X = \mathbb{R}^2$. Then

$$\begin{aligned}\|x+\tau y\|_\psi^t + \|x-\tau y\|_\psi^t &\ge \frac{1}{M_1^t}\big(\|x+\tau y\|_r^t + \|x-\tau y\|_r^t\big)\\ &\ge \frac{2}{M_1^t} S_{X,t}^t(\tau)\big(\|\cdot\|_r\big)\min\big\{\|x\|_r^t, \|y\|_r^t\big\}\\ &\ge \frac{2}{M_1^t} S_{X,t}^t(\tau)\big(\|\cdot\|_r\big)\min\big\{\|x\|_\psi^t, \|y\|_\psi^t\big\}\end{aligned}$$

from the definition of $S_{X,t}(\tau)$ implies that

$$S_{X,t}(\tau)\big(\|\cdot\|_\psi\big) \ge \frac{1}{M_1} S_{X,t}(\tau)\big(\|\cdot\|_r\big).$$

Note that $r' \le t \le \infty$ and the function $\frac{\psi_r(s)}{\psi(s)}$ attains its maximum at $s = 1/2$; that is, $M_1 = \frac{\psi_r(1/2)}{\psi(1/2)}$. Proposition 2.2 implies that

$$S_{X,t}(1)\big(\|\cdot\|_\psi\big) \ge \frac{1}{M_1} S_{X,t}(1)\big(\|\cdot\|_r\big) = 2\psi(1/2). \tag{2.3}$$

On the other hand, let us put $x = (1,0), y = (0,1)$. Then $\|x\|_\psi = \|y\|_\psi = 1$ and

$$\Big(\frac{\|x+y\|_\psi^t + \|x-y\|_\psi^t}{2}\Big)^{1/t} = 2\psi(1/2). \tag{2.4}$$

From (2.3) and (2.4), we have

$$S_{X,t}(1)\big(\|\cdot\|_\psi\big) = 2\psi(1/2).$$

□

*Example* 2.7. Let $X = \mathbb{R}^2$ with the norm

$$\|x\| = \max\big\{\|x\|_2, \lambda\|x\|_1\big\} \quad (1/\sqrt{2} \le \lambda \le 1).$$

Then

$$S_{X,t}(1)\big(\|\cdot\|\big) = \frac{1}{\lambda} \quad (2 \le t \le \infty).$$

*Proof.* It is very easy to check that $\|x\| = \max\{\|x\|_2, \lambda\|x\|_1\} \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(s) = \big\|(1-s, s)\big\| = \max\big\{\psi_2(s), \lambda\big\} \ge \psi_2(s).$$

Therefore,

$$\frac{\psi(s)}{\psi_2(s)} = \max\Big\{1, \frac{\lambda}{\psi_2(s)}\Big\}.$$

Since $\psi_2(s)$ attains its minimum at $s = 1/2$, $\frac{\psi(s)}{\psi_2(s)}$ attains its maximum at $s = 1/2$. Therefore, from Theorem 2.5, we have

$$S_{X,t}(1)\big(\|\cdot\|_\psi\big) = \frac{1}{\psi(1/2)} = \frac{1}{\lambda}. \qquad \square$$

*Example* 2.8. Let $X$ be the 2-dimensional Cesàro space $\mathrm{ces}_2^{(2)}$. Then

$$S_{X,t}(1)(\mathrm{ces}_2^{(2)}) = \sqrt{\frac{10}{5+\sqrt{5}}} \quad (2 \le t \le \infty).$$

*Proof.* We first define

$$|x, y| = \Big\|\Big(\frac{2x}{\sqrt{5}}, 2y\Big)\Big\|_{\mathrm{ces}_2^{(2)}}$$

for $(x, y) \in \mathbb{R}^2$. It follows that $\mathrm{ces}_2^{(2)}$ is isometrically isomorphic to $(\mathbb{R}^2, |\cdot|)$ and $|\cdot|$ is an absolute and normalized norm, and the corresponding convex function is given by

$$\psi(s) = \Big[\frac{4(1-s)^2}{5} + \Big(\frac{1-s}{\sqrt{5}} + s\Big)^2\Big]^{1/2}.$$

Indeed, $T : \mathrm{ces}_2^{(2)} \to (\mathbb{R}^2, |\cdot|)$ defined by $T(x, y) = (\frac{x}{\sqrt{5}}, 2y)$ is an isometric isomorphism. In the sequence, we prove that $\psi(s) \ge \psi_2(s)$. Note that

$$\Big(\frac{1-s}{\sqrt{5}} + s\Big)^2 \ge \Big(\frac{1-s}{\sqrt{5}}\Big)^2 + s^2.$$

Consequently,

$$\psi(s) \ge \big((1-s)^2 + s^2\big)^{1/2} = \psi_2(s).$$

Some elementary computation shows that $\frac{\psi(s)}{\psi_2(s)}$ attains its maximum at $s = 1/2$. Therefore, from Theorem 2.5, we have

$$S_{X,t}(1)(\mathrm{ces}_2^{(2)}) = \frac{1}{\psi(1/2)} = \sqrt{\frac{10}{5+\sqrt{5}}}. \qquad \square$$

*Example* 2.9. Let $X = \mathbb{R}^2$ with the norm

$$\|x\| = \max\big\{\|x\|_2, \lambda\|x\|_\infty\big\} \quad (1 \le \lambda \le \sqrt{2}).$$

Then

$$S_{X,t}(1)\big(\|\cdot\|\big) = \frac{\sqrt{2}}{\lambda} \quad (2 \le t \le \infty).$$

*Proof.* It is obvious to check that the norm $\|x\| = \max\{\|x\|_2, \lambda\|x\|_\infty\}$ is absolute, but not normalized, since $\|(1,0)\| = \|(0,1)\| = \lambda$. Let us put

$$|\cdot| = \frac{\|\cdot\|}{\lambda} = \max\Big\{\frac{\|\cdot\|_2}{\lambda}, \|\cdot\|_\infty\Big\}.$$

Then $|\cdot| \in \mathbb{N}_\alpha$, and its corresponding function is

$$\psi(s) = \big\|(1-s, s)\big\| = \max\Big\{\frac{\psi_2(s)}{\lambda}, \psi_\infty(s)\Big\} \le \psi_2(s).$$

Thus

$$\frac{\psi_2(s)}{\psi(s)} = \min\Big\{\lambda, \frac{\psi_2(s)}{\psi_\infty(s)}\Big\}.$$

Consider the increasing continuous function $g(s) = \frac{\psi_2(s)}{\psi(s)}$ $(0 \le s \le 1/2)$. Because $g(0) = 1$ and $g(1/2) = \sqrt{2}$, there exists a unique $0 \le a \le 1$ such that $g(a) = \lambda$. In fact, $g(s)$ is symmetric with respect to $s = 1/2$. Then we have

$$g(s) = \begin{cases} \frac{\psi_2(s)}{\psi(s)}, & s \in [0,a] \cup [1-a, a], \\ \lambda, & s \in [a, 1-a]. \end{cases}$$

Obviously, $g(s)$ attains its maximum at $s = 1/2$. Hence, from Lemma 2.4 and Theorem 2.6, we have

$$S_{X,t}(1)\big(\|\cdot\|\big) = S_{X,t}(1)\big(|\cdot|\big) = 2\psi(1/2) = \frac{\sqrt{2}}{\lambda}. \qquad \square$$

*Example* 2.10 (Lorentz sequence spaces). Let $\omega_1 \ge \omega_2 > 0, 2 \le r < \infty$. Thus, if we have a 2-dimensional Lorentz sequence space (i.e., $\mathbb{R}^2$ with the norm)

$$\big\|(z,\omega)\big\|_{\omega,r} = \big(\omega_1|x_1^*|^r + \omega_2|x_2^*|^r\big)^{1/r},$$

where $(x_1^*, x_2^*)$ is the rearrangement of $(|z|, |\omega|)$ satisfying $x_1^* \ge x_2^*$, then

$$S_{X,t}(1)\big(\big\|(z,\omega)\big\|_{\omega,r}\big) = \Big(\frac{\omega_1 + \omega_2}{\omega_1}\Big)^{1/r} \quad (r' \le t \le \infty).$$

*Proof.* It is obvious that $|\cdot| = (\|(z,\omega)\|_{\omega,r})/\omega_1^{1/q} \in \mathbb{N}_\alpha$, and the corresponding convex function is given by

$$\psi(s) = \begin{cases} [(1-s)^r + (\omega_2/\omega_1)s^r]^{1/r}, & s \in [0, 1/2], \\ [s^r + (\omega_2/\omega_1)(1-s)^r]^{1/r}, & s \in [1/2, 1]. \end{cases}$$

Obviously, $\psi(s) \le \psi_r(s)$ and $\Phi(s) = \frac{\psi_r(s)}{\psi(s)}$. It suffices to consider $\Phi(s)$ for $s \in [0, 1/2]$ since $\Phi(s)$ is symmetric with respect to $s = 1/2$. Note that, for $s \in [0, 1/2]$,

$$\Phi^r(s) = \frac{\psi_r^r(s)}{\psi^r(s)} = \frac{(1-s)^r + s^r}{(1-s)^r + (\omega_2/\omega_1)s^r} = \frac{u(s)}{v(s)}.$$

Some elementary computation shows that $u(s) - v(s) = (1 - (\omega_2/\omega_1))s^r$ attains its maximum and $v(s)$ attains its minimum at $s = 1/2$. Hence

$$\Phi^r(s) = \frac{u(s) - v(s)}{v(s)} + 1$$

attains its maximum at $s = 1/2$, and so does $\Phi(s)$. Then from Lemma 2.4 and Theorem 2.6, we have

$$S_{X,t}(1)\big(\|\cdot\|\big) = S_{X,t}(1)\big(|\cdot|\big) = 2\psi(1/2) = \Big(\frac{\omega_1 + \omega_2}{\omega_1}\Big)^{1/r}. \qquad \square$$

*Example* 2.11. Let $X_\lambda = \mathbb{R}^2$ with the norm

$$\|x\|_\lambda = \big(\|x\|_p^2 + \lambda\|x\|_q^2\big) \quad (\lambda \ge 0).$$

(i) If $1 \le p \le q \le 2$, then $S_{X,t}(1)(\|\cdot\|) = \sqrt{\frac{4(1+\lambda)}{2^{\frac{2}{q}}\lambda + 2^{\frac{2}{p}}}}$.

(ii) If $2 \le p \le q \le \infty$, then $S_{X,t}(1)(\|\cdot\|) = \sqrt{\frac{2^{\frac{2}{q}}\lambda + 2^{\frac{2}{p}}}{1+\lambda}}$.

*Proof.* It is obvious to check that the norm $\|x\|_\lambda = (\|x\|_p^2 + \lambda\|x\|_q^2)^{1/2}$ is absolute, but not normalized. Let us put

$$|\cdot|_\lambda^0 = \frac{\|\cdot\|}{\sqrt{1+\lambda}}.$$

Therefore, $|\cdot|_\lambda^0 \in \mathbb{N}_\alpha$, and its corresponding function is $\psi_\lambda(t) = \|(1-t, t)\|_\lambda$.

(i) Suppose that $1 \le p \le q \le 2$. Since $\psi_\lambda(t) \ge \psi_2(t)$, $\frac{\psi_\lambda(s)}{\psi_2(s)}$ attains its maximum at $s = 1/2$ and $2 = r < t \le \infty$. By Theorem 2.5, we get that $S_{X,t}(1)(\|\cdot\|_\psi) = \frac{1}{\psi(1/2)} = \sqrt{\frac{4(1+\lambda)}{2^{\frac{2}{q}}\lambda + 2^{\frac{2}{p}}}}$.

(ii) Suppose that $2 \le p \le q \le \infty$. Since $\psi_\lambda(t) \le \psi_2(t)$, $\frac{\psi_2(s)}{\psi_\lambda(s)}$ attains its maximum at $s = 1/2$ and $2 = r' \le t \le \infty$. By Theorem 2.6, we get that $S_{X,t}(1)(\|\cdot\|_\psi) = 2\psi(1/2) = \sqrt{\frac{2^{\frac{2}{q}}\lambda + 2^{\frac{2}{p}}}{1+\lambda}}$. $\square$

## 3. Schäffer-type constant and uniform normal structure

In this section, some relationships among the Schäffer-type constant $S_{X,t}(\tau)$ and parameters $J(\epsilon, X)$ were given where $J(\epsilon, X)$ are nonsquareness parameters defined by

$$J(\epsilon, X) = \sup\big\{\min\big\{\|x + \epsilon y\|, \|x + \epsilon y\|\big\} : x, y \in S_X\big\}.$$

Using the same method as in [5], we give some sufficient conditions for which a Banach space $X$ has uniform normal structure in terms of the Schäffer-type constant.

In [9], Saejung proved the following theorem.

**Theorem 3.1.** *A Banach space $X$ has uniform normal structure if $J(\epsilon, X) < \frac{\epsilon+\sqrt{4+\epsilon^2}}{2}$ for some $0 < \epsilon \leq 1$.*

The proof of the following theorem is similar to Theorem 10 in [5], and so we omit the proof.

**Theorem 3.2.** *Let $X$ be a Banach space. Then*

$$2\big[S_{X,t}(\tau)\big]^t\big[J(\epsilon, X)\big]^t \leq \big[1+\tau+\epsilon(1-\tau)\big]^t + \big[1-\tau+\epsilon(1+\tau)\big]^t$$

*for all $0 \leq \tau, \epsilon \leq 1$, and $1 < t < \infty$. In particular,*

$$2\big[S_{X,t}(\tau)\big]^t\big[J(\tau, X)\big]^t \leq (1+2\tau-\tau^2)^t + (1+\tau^2)^t.$$

**Corollary 3.3.** *Let $X$ be a Banach space with*

$$S_{X,t}(\tau) > \frac{1}{g(\tau)}\Big(\frac{(1+2\tau-\tau^2)^t + (1+\tau^2)^t}{2}\Big)^{1/t}$$

*for some $\tau \in (0, 1]$, where $g(\tau) = \frac{\tau+\sqrt{4+\tau^2}}{2}$. Then $X$ has uniform normal structure.*

*Proof.* It follows from Theorem 3.2 that

$$\begin{aligned}\frac{1}{g^t(\tau)}\big[(1+2\tau-\tau^2)^t + (1+\tau^2)^t\big]\big[J(\tau, X)\big]^t &< 2\big[S_{X,t}(\tau)\big]^t\big[J(\tau, X)\big]^t \\ &\leq (1+2\tau-\tau^2)^t + (1+\tau^2)^t.\end{aligned}$$

Then $J(\tau, X) < g(\tau) = \frac{\tau+\sqrt{4+\tau^2}}{2}$, and the proof is complete from Theorem 3.1. □

*Remark* 3.4. Letting $\tau = 1$ and $t = 2$, we can easily get the results of $f(x) > 12 - 4\sqrt{5}$. Then $X$ has uniform normal structure. This is an extension and an improvement of the results in [4]. Letting $\tau = 1/2$, we can get some new result which is complementary to [14, Theorem 28].

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