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CONE NONNEGATIVITY OF MOORE–PENROSE INVERSES OF UNBOUNDED GRAM OPERATORS

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Abstract. In this article, necessary and sufficient conditions for the cone nonnegativity of Moore–Penrose inverses of unbounded Gram operators are derived. These conditions include statements on acuteness of certain closed convex cones in infinite-dimensional real Hilbert spaces.

1. Introduction

A real square matrix T is called *monotone* if $x \geq 0$ whenever $Tx \geq 0$. Here $x = (x_i) \geq 0$ means that $x_i \geq 0$ for all i. The concept of monotonicity was first proposed by Collatz [4] in connection with the application of finite difference methods for solving elliptic partial differential equations. He showed that a matrix is monotone if and only if it is invertible and the inverse is entrywise nonnegative; hence, the monotonicity of a matrix is equivalent to the nonnegativity of the inverse of a matrix.

The notion of monotonicity has been extended in a great variety of ways. We present a brief review here. Mangasarian [12] considered a rectangular matrix T to be monotone if $Tx \geq 0 \Rightarrow x \geq 0$. He showed, using the duality theorem of linear programming, that T is monotone if and only if T has a nonnegative left inverse. Berman and Plemmons generalized the concept of monotonicity in several ways in a series of articles, where they studied their relationships with nonnegativity of generalized inverses. The book by Berman and Plemmons [2] contains numerous examples of applications of nonnegative generalized inverses

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that include numerical analysis and linear economic models. Sivakumar [14] and several others extended the concept of monotonicity to the infinite-dimensional setting. (We refer the reader to Kammerer and Plemmons [7, Section 6] for applications of nonnegative Moore–Penrose inverses of operators in solving operator equations.)

The monotonicity of Gram matrices and Gram operators has received a lot of attention in recent years. This has been primarily motivated by applications in convex optimization problems. In this connection, there is a well-known result by Cegielski that characterizes nonnegative invertibility of Gram matrices in terms of obtuseness (or acuteness) of certain polyhedral cones (see, for instance, [3, Lemma 1.6]). Later, this characterization of nonnegativity of inverses of Gram matrices was extended to Gram operators over infinite-dimensional real Hilbert spaces (see [10, Theorem 3.6]).

The objective of this article is to obtain results similar to the results of [10] for unbounded Gram operators. In fact, we consider here densely defined closed linear operators (not necessarily bounded) between real Hilbert spaces and obtain the necessary and sufficient conditions for the *cone nonnegativity* (defined in the next section) of Moore–Penrose inverses of Gram operators in terms of acuteness of certain closed convex cones. This is achieved by taking cones in the domain of the Gram operator. It is pertinent to mention that our results are new, and that their proofs involve several results from the theory of Moore–Penrose inverses of unbounded operators. We also mention that our results generalize the existing results due to Kurmayya and Sivakumar [10] and the related results in the literature (see, for instance, [3, Lemma 1.6]).

This article has the following organization. In Section 2, notation, basic definitions, and results are introduced. In Section 3, some preliminary results and the main theorem of this article are presented. Finally, in Section 4 the main theorem is illustrated with examples.

2. Notation and preliminary results

In this section, notation and basic definitions are introduced. Note that, throughout the article, H, H_1, and H_2 denote infinite-dimensional real Hilbert spaces, while $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and the induced norm, respectively. For a closed subspace M of H, P_M denotes the orthogonal projection on H with range M.

Let T be a linear operator with domain $D(T)$, a subspace of H_1, and taking values in H_2. Then the graph $G(T)$ of T is defined by $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$. If $G(T)$ is closed, then T is called a *closed operator*. If $D(T)$ is dense in H_1, then T is called a *densely defined operator*. For a densely defined operator, there exists a unique linear operator $T^* : D(T^*) \to H_1$, where

$$D(T^*) := \big\{ y \in H_2 : \text{ the functional } x \to \langle Tx, y \rangle \text{ for all } x \in D(T) \text{ is continuous} \big\}$$

and $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in D(T)$ and $y \in D(T^*)$. This operator is called the *adjoint* of T. Note that T^* is always closed whether or not T is closed.

The set of all closed operators between H_1 and H_2 is denoted by $\mathcal{C}(H_1, H_2)$ and $\mathcal{C}(H) := \mathcal{C}(H, H)$. By the closed graph theorem [13, Theorem 2.15], an everywhere-defined closed operator is bounded; hence, the domain of an unbounded closed operator is a proper subspace of a Hilbert space. For $T \in \mathcal{C}(H_1, H_2)$, the null space and the range space of T are denoted by $N(T)$ and $R(T)$, respectively, and the space $C(T) := D(T) \cap N(T)^{\perp}$ is called the *carrier* of T. In fact, $D(T) = N(T) \oplus^{\perp} C(T)$. The following properties of a densely defined $T \in \mathcal{C}(H_1, H_2)$ will be used in the sequel: $N(T) = R(T^*)^{\perp}$; $N(T^*) = R(T)^{\perp}$; $N(T^*T) = N(T)$; and $\overline{R(T^*T)} = \overline{R(T^*)}$ (see [1] for a detailed study of these properties).

Some more properties of a densely defined $T \in \mathcal{C}(H_1, H_2)$ are collected in the following lemma.

Proposition 2.1 ([1, Exercise 9, p. 336])**.** *For a densely defined $T \in \mathcal{C}(H_1, H_2)$, the following statements are equivalent:*

(1) *$R(T)$ is closed,*
(2) *$R(T^*)$is closed,*
(3) *$R(T^*T)$ is closed, and in this case, $R(T^*T) = R(T^*)$,*
(4) *$R(TT^*)$ is closed, and in this case, $R(TT^*) = R(T)$.*

(For further results on unbounded operators, we refer the reader to [5].)

Now, we move on to the definition of the Moore–Penrose inverse of a densely defined closed linear operator between real Hilbert spaces.

Definition 2.2. ([1, Definition 2, p. 339]) Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then there exists a unique densely defined operator $T^{\dagger} \in \mathcal{C}(H_2, H_1)$ with domain $D(T^{\dagger}) = R(T) \oplus^{\perp} R(T)^{\perp}$ that has the following properties:

(1) $TT^{\dagger}y = P_{\overline{R(T)}}y$, for all $y \in D(T^{\dagger})$,
(2) $T^{\dagger}Tx = P_{N(T)^{\perp}}x$, for all $x \in D(T)$,
(3) $N(T^{\dagger}) = R(T)^{\perp}$.

This unique operator $T^{\dagger}$ is called the *Moore–Penrose inverse* of T.

The next lemma presents well-known properties of $T^{\dagger}$ that will be used in proving the main result of this paper.

Theorem 2.3. *([1, Theorem 2, p. 341]) Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

(1) *$D(T^{\dagger}) = R(T) \oplus^{\perp} R(T)^{\perp}$,*
(2) *$R(T^{\dagger}) = C(T)$,*
(3) *$T^{\dagger}$ is densely defined and $T^{\dagger} \in \mathcal{C}(H_2, H_1)$,*
(4) *$T^{\dagger}$ is continuous if and only $R(T)$ is closed,*
(5) *$T^{\dagger\dagger} = T$,*
(6) *$T^{*\dagger} = T^{\dagger *}$,*
(7) *$N(T^{*\dagger}) = N(T)$,*
(8) *$(T^*T)^{\dagger} = T^{\dagger}T^{*\dagger}$,*
(9) *$(TT^*)^{\dagger} = T^{*\dagger}T^{\dagger}$.*

The notion of a least square solution is adopted next.

Definition 2.4. Let $P := P_{\overline{R(T)}}$. If $y \in R(T) \oplus^{\perp} R(T)^{\perp}$, then the equation

$$Tx = Py \tag{2.1}$$

always has a solution. This solution is called a *least square solution*. If $x \in D(T)$ is a least square solution, then

$$\|Tx - y\|^2 = \|Py - y\|^2 = \min_{z \in D(T)} \|Tz - y\|^2.$$

The unique vector with the minimal norm among all least square solutions is called the *least square solution of minimal norm* of equation (2.1) and is given $x = T^{\dagger}y$.

(For further results related to generalized inverses of unbounded densely defined closed operators, we refer the reader to [6] and [11].)

The following result from [9] will be used in proving the main theorem.

Theorem 2.5. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Assume that $R(T)$ is closed. Then*

$$(T^*T)^{\dagger}T^* \subseteq T^*(TT^*)^{\dagger} = T^{\dagger}.$$

Here, by $A \subseteq B$ we mean $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$.

Next, we introduce the definition of the Gram operator that plays a key role in our discussions.

Definition 2.6. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then the operator T^*T is called the *Gram operator* of T.

Now, we briefly describe the notion of a cone. A subset K of a Hilbert space H is called a *cone* if (i) $x, y \in K \Rightarrow x + y \in K$ and (ii) $x \in K$, $\alpha \in \mathbb{R}$, $\alpha \geq 0 \Rightarrow \alpha x \in K$. For a subset K of a Hilbert space H, the dual of K is denoted by K^* and is defined by $K^* = \{x \in H \colon \langle x, t \rangle \geq 0, \text{ for all } t \in K\}$ and $K^{**} = (K^*)^*$. Note that, in general, $K^{**} = \overline{K}$, where the bar denotes the closure of K. If $H = \ell^2$, the Hilbert space of all square summable real sequences, and $K = \ell^2_+ = \{x \in \ell^2 \colon x_i \geq 0, \forall i\}$, then $K^* = \ell^2_+$, and hence $K^{**} = \ell^2_+$. A cone C is said to be acute if $\langle x, y \rangle \geq 0$, for all $x, y \in C$.

Finally, we end up this section with the definition of cone nonnegativity of an operator.

Definition 2.7. Let K_1 and K_2 be two closed convex cones in Hilbert spaces H_1 and H_2, respectively. Then a linear operator $T : H_1 \to H_2$ is said to be (K_1, K_2)-nonnegative (or simply cone nonnegative with respect to K_1, K_2) if $T(K_1) \subseteq K_2$.

Suppose that T is a real matrix of order $m \times n$. Then $T \geq 0$ (or simply T nonnegative) is equivalent to $T(\mathbb{R}^n_+) \subseteq \mathbb{R}^m_+$, where $\mathbb{R}^n_+$ and $\mathbb{R}^m_+$ denote nonnegative orthants in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Thus, we emphasize that usual entrywise nonnegativity of matrices is equivalent to cone nonnegativity of matrices with respect to nonnegative orthants as cones.

3. Main results

As mentioned in the Introduction, nonnegative invertibility (or monotonicity) of Gram matrices has applications in the convex optimization problems. Also, it has applications in solving equations of the form $T^*Tx = y$, where T is a matrix. So far, many results have been derived on characterization of nonnegative invertibility of Gram matrices and Gram operators. Some of these results have been generalized for bounded operators which are not invertible. These results motivated us to prove Theorem 3.4 for densely defined closed operators (not necessarily bounded). First, we prove some preliminary results.

Lemma 3.1. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed range, and let K be a closed convex cone in $D(T^*T)$ such that $K^* \subset D(T^*T)$. Let $C = T(K)$. If $u \in C^* \cap D(T^*)$, then $T^*u \in K^*$.*

Proof. Let $u \in C^* \cap D(T^*)$, and let $r \in K$. Then $0 \leq \langle u, Tr\rangle = \langle T^*u, r\rangle$. □

Lemma 3.2. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed range. Let K be a closed convex cone in $D(T^*T)$ such that $K^* \subset D(T^*T)$. Let $C = T(K)$, and let $D = (T^\dagger)^*(K^*)$. Then the following are equivalent:*

(1) *$C^* \cap D(T^*) \cap R(T)$ is acute;*
(2) *for all $x, y \in D(T^*T)$ with $T^*Tx \in K^*, T^*Ty \in K^*$, the inequality $\langle T^*Tx, y\rangle \geq 0$ holds.*

Proof. (1) $\Longrightarrow$ (2): Let $x, y \in D(T^*T)$ satisfy $T^*Tx \in K^*$ and $T^*Ty \in K^*$. For $r \in K$, we have $Tr \in C$; hence,

$$\langle Tx, Tr\rangle = \langle T^*Tx, r\rangle \geq 0,$$

so $Tx \in C^*$. Similarly, we can show that $Ty \in C^*$. Since $C^* \cap D(T^*) \cap R(T)$ is acute, we have $0 \leq \langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$.

(2) $\Longrightarrow$ (1): Let $u, v \in C^* \cap D(T^*) \cap R(T)$. Let $u = Tx$ for some $x \in D(T)$. Since $u \in D(T^*)$, T^*u is defined; that is, $x \in D(T^*T)$. Similarly, $v = Ty$ for some $y \in D(T^*T)$.

Next we show that $\langle u, v\rangle \geq 0$. Since $u \in C^*$, for $r \in K$ we have

$$0 \leq \langle Tx, Tr\rangle = \langle T^*Tx, r\rangle.$$

Thus, $T^*Tx \in K^*$. With a similar argument, we can conclude that $T^*Ty \in K^*$. By assumption,

$$\langle u, v\rangle = \langle Tx, Ty\rangle = \langle T^*Tx, y\rangle \geq 0;$$

hence $C^* \cap D(T^*) \cap R(T)$ is acute. □

Lemma 3.3. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed range. Let K be a closed convex cone in $D(T^*T)$ such that $K^* \subset D(T^*T)$. Let $D = (T^\dagger)^*(K^*)$. Then D is acute if and only if $\langle r, (T^*T)^\dagger s\rangle \geq 0$, for every $r, s \in K^*$.*

Proof. Let $x, y \in D$. Then $x = (T^\dagger)^*r, y = (T^\dagger)^*s$ for some $r, s \in K^*$. Then D is acute if and only if

$$0 \leq \langle x, y\rangle = \left\langle (T^\dagger)^*r, (T^\dagger)^*s\right\rangle = \left\langle r, T^\dagger(T^\dagger)^*s\right\rangle = \left\langle r, (T^*T)^\dagger s\right\rangle$$

by (8) of Theorem 2.3. □

Next, we prove the main result of this paper.

Theorem 3.4. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed range. Let K be a closed convex cone in $D(T^*T)$ with $T^\dagger T(K) \subseteq K$. Let $C = T(K)$, and let $D = (T^\dagger)^*(K^*)$. Then the following conditions are equivalent:*

(1) $(T^*T)^\dagger(K^*) \subseteq K$,
(2) $C^* \cap D(T^*) \cap R(T) \subseteq C$,
(3) D *is acute,*
(4) $C^* \cap D(T^*) \cap R(T)$ *is acute,*
(5) $T^*Tx \in P_{R(T^*)}(K^*) \Longrightarrow x \in K$,
(6) $T^*Tx \in K^* \Longrightarrow x \in K$.

Proof. (1) $\Longrightarrow$ (2): Let $u \in C^* \cap D(T^*) \cap R(T)$. Then $u = Tp$ for some $p \in C(T)$. Then $T^\dagger u = T^\dagger Tp = P_{N(T)^\perp} p = p$. Since $u \in D(T^*)$, by Theorem 2.5, $T^\dagger u = (T^*T)^\dagger T^* u$. Set $z = T^\dagger u$. Then $Tz = TT^\dagger u = P_{R(T)} u = u$. Also, $T^*u \in K^*$ by Lemma 3.1 so that, by the assumption, $z = (T^*T)^\dagger T^* u \in K$. Thus, $u \in C$.

(2) $\Longrightarrow$ (3): Let $x = (T^\dagger)^* u$ and $y = (T^\dagger)^* v$ with $u, v \in K^*$. Since

$$\begin{aligned} R\big((T^\dagger)^*\big) = R\big((T^*)^\dagger\big) = C(T^*) &= D(T^*) \cap N(T^*)^\perp \\ &= D(T^*) \cap \overline{R(T)} \\ &= D(T^*) \cap R(T), \end{aligned}$$

$x, y \in D(T^*) \cap R(T)$. Let $r \in K$. We have $r' = T^\dagger Tr \in K$ (as $T^\dagger T(K) \subseteq K$). Then

$$\langle x, Tr \rangle = \big\langle (T^\dagger)^* u, Tr \big\rangle = \langle u, T^\dagger Tr \rangle = \langle u, r' \rangle \geq 0.$$

Thus, $x \in C^*$. Since $C^* \cap D(T^*) \cap R(T) \subseteq C$, we have $x \in C$. Thus, $x = Tp$ for some $p \in K$.

Finally, with $p' = T^\dagger Tp \in K$, we have

$$\langle x, y \rangle = \big\langle Tp, (T^\dagger)^* v \big\rangle = \langle T^\dagger Tp, v \rangle = \langle p', v \rangle \geq 0;$$

hence D is acute.

(3) $\Longrightarrow$ (4): Let x, y be such that $r = T^*Tx \in K^*$ and $s = T^*Ty \in K^*$. Since D is acute, by Lemma 3.3,

$$\begin{aligned} 0 \leq \big\langle r, (T^*T)^\dagger s \big\rangle &= \big\langle T^*Tx, (T^*T)^\dagger T^*Ty \big\rangle \\ &= \big\langle x, (T^*T)(T^*T)^\dagger (T^*T)y \big\rangle \\ &= \big\langle x, (T^*T)y \big\rangle \\ &= \langle T^*Tx, y \rangle. \end{aligned}$$

By Lemma 3.2, $C^* \cap D(T^*) \cap R(T)$ is acute.

(4) $\Longrightarrow$ (5): Let $T^*Tx = P_{R(T^*)} w$ for some $w \in K^*$. Since $R(T^*T) = R(T^*)$, we have $T^*Tx = P_{R(T^*T)} w$; hence $x = (T^*T)^\dagger w$ (by Definition 2.4).

Let $r \in K^*$. Then

$$\langle x, r \rangle = \big\langle (T^*T)^\dagger w, r \big\rangle = \big\langle T^\dagger (T^\dagger)^* w, r \big\rangle = \big\langle (T^\dagger)^* w, (T^\dagger)^* r \big\rangle.$$

Set $u = (T^\dagger)^* w, v = (T^\dagger)^* r$. Then, as was shown earlier, $u, v \in R(T) \cap D(T^*)$. For $t \in K$, with $t' = T^\dagger T t \in K$, we have

$$\langle u, Tt\rangle = \big\langle (T^\dagger)^* w, Tt\big\rangle = \langle w, T^\dagger T t\rangle = \langle w, t'\rangle \geq 0,$$

so $u \in C^*$. Along similar lines, it can be shown that $v \in C^*$. Thus, for all $r \in K^*, \langle x, r\rangle = \langle u, v\rangle \geq 0$ so that $x \in (K^*)^* = K$.

(5) $\implies$ (6): Choose x such that $T^*Tx \in K^*$. We have

$$T^*Tx = P_{R(T^*T)}(T^*Tx) = P_{R(T^*)}(T^*Tx) \in P_{R(T^*)}(K^*);$$

hence, by (5), $x \in K$.

(6) $\implies$ (1): Let $u = (T^*T)^\dagger v$ with $v \in K^*$.

Then $T^*Tu = T^*T(T^*T)^\dagger v = P_{R(T^*)} v = T^\dagger T v$. Then, for $r \in K$ with $r' = T^\dagger T r \in K$, we have

$$\langle T^*Tu, r\rangle = \langle T^\dagger T v, r\rangle = \langle v, T^\dagger T r\rangle = \langle v, r'\rangle \geq 0.$$

Thus, $T^*Tu \in K^*$. As (6) holds, $u \in K$. Thus, $(T^*T)^\dagger(K^*) \subseteq K$.

This completes the proof of the theorem. $\square$

The following result gives equivalent conditions, under which $(T^*T)^\dagger$ leaves a cone invariant if K is self-dual.

Corollary 3.5. *In addition to the conditions of Theorem 3.4, suppose that K is self-dual (i.e., $K^* = K$). Then the conditions (2)–(6) are equivalent to $(T^*T)^\dagger(K) \subseteq K$.*

Remark 3.6.

(1) If $K \subseteq C(T^*T)$, then the condition $T^\dagger T(K) \subseteq K$ is satisfied automatically.

(2) If T is one-to-one, then $T^\dagger T = I$, and hence, in this case, $T^\dagger T(K) \subseteq K$ holds for any cone in $D(T)$.

4. Examples

In this section, we illustrate Theorem 3.4 with examples.

Example 4.1. Let $H = \ell^2$, and let $D(T) = \{(x_1, x_2, \dots) \in H \colon \sum_{j=1}^\infty |jx_j|^2 < \infty\}$. Define $T : D(T) \to H$ by

$$\begin{aligned} &T(x_1, x_2, x_3, \dots, x_n, \dots) \\ &\quad = (x_1, 2x_2, 3x_3, \dots, nx_n, \dots) \quad \text{for all } (x_1, x_2, \dots) \in D(T). \end{aligned}$$

Since $D(T)$ contains c_{00}, the space of all sequences having at most finitely many nonzero terms, we have $\overline{D(T)} = H$. Clearly, T is unbounded and closed since $T^* = T$. Also, $R(T)$ is closed. In fact, T^{-1} exists, and

$$T^{-1}(y_1, y_2, y_3, \dots, y_n, \dots) = \Big(y_1, \frac{y_2}{2}, \frac{y_3}{3}, \dots, \frac{y_n}{n}, \dots\Big), \quad \text{for all } (y_n) \in H.$$

Note that $D(T^*T) = \{(x_n) \in H : \sum_{n=1}^\infty n^4 |x_n|^4 < \infty\}$. Let

$$K = \big\{(x_n) \in D(T^2) : x_n \geq 0 \text{ for all } n \in \mathbb{N}\big\}.$$

Clearly, $K^* = K$ and $T^\dagger T(K) = K$; hence K satisfies the hypothesis of Theorem 3.4. In this case, $D = T^{\dagger^*}(K^*) = T^{-1}(K)$. Let $x, y \in D$. Then $x = T^{-1}u, y = T^{-1}v$ for some $u, v \in H$. Let $u = \sum_{n=1}^\infty \langle u, e_n\rangle e_n$ and $v = \sum_{n=1}^\infty \langle v, e_n\rangle e_n$. (Here $\{e_n : n \in \mathbb{N}\}$ is the standard orthonormal basis for H.) Then

$$\begin{aligned}\langle x, y\rangle &= \langle T^{-2}u, v\rangle \\ &= \sum_{n=1}^\infty \frac{1}{n^2}\langle u, e_n\rangle\langle v, e_n\rangle \\ &\geq 0 \quad \big(\text{since } \langle u, e_n\rangle, \langle v, e_n\rangle \geq 0\big).\end{aligned}$$

Therefore, D is acute. Hence, by Theorem 3.4, $(T^*T)^\dagger$ is nonnegative with respect to the cone K. This can be easily verified independently by using the definition.

Example 4.2. Let $H = \ell^2$ and $D(T) = \{(x_1, x_2, \dots, x_n, \dots)\colon \sum_{j=2}^\infty |jx_j|^2 < \infty\}$. Define $T : D(T) \to H$ by

$$T(x_1, x_2, \dots, x_n, \dots) = (0, 2x_2, 3x_3, 4x_4, \dots) \quad \text{for all } (x_1, x_2, \dots) \in H.$$

Observe that T is densely defined, $T = T^*$, and $N(T) = \{(x_1, 0, 0, \dots)\colon x_1 \in \mathbb{C}\}$; hence $C(T) = \{(0, x_2, x_3, \dots)\colon \sum_{j=2}^\infty |jx_j|^2 < \infty\}$. We can show that $R(T)$ is closed and

$$T^\dagger(y_1, y_2, y_3, \dots,) = \Big(0, \frac{y_2}{2}, \frac{y_3}{3}, \dots\Big), \quad (y_n) \in \ell^2.$$

It can be seen that $T = T^*$ and $D(T^2) = \{(x_n) \in H : \sum_{n=2}^\infty n^4|x_n|^4 < \infty\}$. Take

$$K = \big\{(x_n) \in D(T^2) : x_n \geq 0 \text{ for all } n = 2, 3, \dots\big\}.$$

It is easy to verify that $K^* = K$ and $T^\dagger T(K) \subseteq K$. Also, $D = T^{\dagger^*}(K^*) = T^\dagger(K)$. Let $x, y \in D$. Then $x = T^\dagger u, y = T^\dagger v$ for some $u, v \in H$. Let $u = \sum_{n=1}^\infty \langle u, e_n\rangle e_n$ and $v = \sum_{n=1}^\infty \langle v, e_n\rangle e_n$. Then

$$\begin{aligned}\langle x, y\rangle &= \langle T^\dagger u, T^\dagger v\rangle \\ &= \sum_{n=2}^\infty \frac{1}{n^2}\langle u, e_n\rangle\langle v, e_n\rangle \\ &\geq 0 \quad \big(\text{since } \langle u, e_n\rangle, \langle v, e_n\rangle \geq 0\big).\end{aligned}$$

Therefore, D is acute. Hence, by Theorem 3.4, $(T^*T)^\dagger$ is positive with respect to the cone K.

Example 4.3. Let $\mathcal{AC}[0, \pi]$ denote the space of all absolutely continuous functions on $[0, \pi]$. Let

$$\begin{aligned}H &:= \text{the real space } L^2[0, \pi] \text{ of real-valued functions,} \\ H' &:= \big\{\phi \in \mathcal{AC}[0, \pi] : \phi' \in H\big\}, \\ H'' &:= \{\phi \in H' : \phi' \in H'\}.\end{aligned}$$

Let $L := \frac{d}{dt}$ with $D(L) = \{x \in H' : \phi(0) = \phi(\pi) = 0\}$.

It can be shown using the fundamental theorem of integral calculus that $L \in \mathcal{C}(H)$. Let $\phi_n = \sin(nt), n \in \mathbb{N}$. Then $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis for H and is contained in $D(L)$; hence L is densely defined. Also, $C(L) = D(L)$; that is, L is one-to-one. It can be shown that $R(L) = \{y \in H : \int_0^\pi y(t)\,dt = 0\} = \operatorname{span}\{1\}^\perp$; hence, in this case, $D(L^\dagger) = H$. Let $\psi_n = \sqrt{\frac{2}{\pi}}\cos(nt), t \in [0,\pi], n \in \mathbb{N}$. Then $\{\psi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $R(L)$.

We have $L^*L = -\frac{d^2}{dt^2}$ with $D(L^*L) = \{\phi \in H'' : \phi(0) = 0 = \phi(\pi)\}$ (see [1, p. 349]). By using the projection method, we can show that

$$L^\dagger(y) = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, \psi_n\rangle \phi_n \tag{4.1}$$

(see [8, Example 3.5]). Let $K = \{\phi \in D(L^*L) : \langle \phi, \phi_n\rangle \geq 0, \text{ for all } n \in \mathbb{N}\}$. Then K is a cone and $K^* = K$. We verify condition (1) of Theorem 3.4. First note that, by equation 4.1, we have

$$L^{\dagger *}\phi = \sum_{n=1}^{\infty} \frac{1}{n}\langle \phi, \phi_n\rangle \psi_n, \quad \text{for all } \phi \in H. \tag{4.2}$$

Now, let $f \in K$. Then

$$\begin{aligned}(L^*L)^\dagger(f) &= L^\dagger(L^\dagger)^*(f)\\ &= \sum_{n=1}^{\infty} \frac{1}{n^2}\langle f, \phi_n\rangle \phi_n.\end{aligned}$$

Since $f \in K$, we have $\langle f, \phi_n\rangle \geq 0$ for all $n \in \mathbb{N}$ and so $\frac{1}{n^2}\langle f, \phi_n\rangle \geq 0$ for all $n \in \mathbb{N}$. This concludes that $(L^*L)^\dagger(K^*) \subseteq K$.

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