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WIDE AND TIGHT SPHERICAL HULLS OF BOUNDED SETS IN BANACH SPACES

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Abstract. Let A be a bounded closed convex set in a Banach space. The boundaries of the wide spherical hull $\eta(A)$ and the tight spherical hull $\theta(A)$ are characterized, the existence of diametral points of these three sets are discussed, and a further relation between these three sets is clarified. Moreover, a new characterization of balls is presented.

1. Introduction

We denote by $\mathcal{H}$ the family of bounded closed sets containing at least two points in a Banach space X, whose *origin* is o, and by $B_X(x,\gamma)$ ($S_X(x,\gamma)$, resp.) the *ball* (*sphere*, resp.) centered at $x \in X$ having radius $\gamma > 0$. The *unit ball* $B_X(o,1)$ (*unit sphere* $S_X(o,1)$, resp.) is simply denoted by B_X (S_X, resp.).

For each $A \in \mathcal{H}$, we put

$$\begin{aligned}
\gamma(A,x) &= \sup\big\{\|x-a\| : a \in A\big\},\\
\gamma'(A,x) &= \inf\big\{\|x-a\| : a \notin A\big\}, \quad \forall x \in X,\\
\gamma(A,B) &= \inf\big\{\gamma(A,x) : x \in B\big\}, \quad \forall B \subseteq X,\\
\gamma(A) = \gamma(A,X), &\qquad \gamma'(A) = \sup\big\{\gamma'(A,x) : x \in A\big\}.
\end{aligned}$$

Here $\gamma(A)$ is called the *radius* of A, and a point $x \in X$ satisfying $\gamma(A) = \gamma(A,x)$ will be called a *center* of A. Note that not every bounded set has a center. Suppose that $x \in A$. We say that x is a *diametral point* of A if $\gamma(A,x) = \delta(A)$,

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and that x is a *strongly diametral point* of A if there exists a point $y \in A$ such that $\|x-y\| = \delta(A)$. A pair of points $x, y \in A$ is called a *diametral pair* if $\|x-y\| = \delta(A)$. The *wide spherical hull* $\eta(A)$ and the *tight spherical hull* $\theta(A)$ of A are defined by

$$\eta(A) = \bigcap_{x\in A} B_X\big(x, \delta(A)\big), \qquad \theta(A) = \bigcap_{x\in \eta(A)} B_X\big(x, \delta(A)\big),$$

where

$$\delta(A) = \sup\big\{\|x-y\| : x, y \in A\big\}$$

is the *diameter* of A. It is known (see the proof of Theorem 5 in [2]) that $\delta(A) = \delta(\theta(A))$ and (see equation (3) in [9]) that

$$\eta(A) = \bigcap_{x\in\theta(A)} B_X\big(x, \delta(A)\big) = \bigcap_{x\in\theta(A)} B_X\big(x, \delta\big(\theta(A)\big)\big) = \eta\big(\theta(A)\big). \tag{1}$$

In the following we use $\operatorname{bd} A$ to denote the *boundary* of A.

The notions of wide spherical hull and tight spherical hull are tightly connected to the notions of complete sets and completions of sets. If the implication

$$x \notin A \Rightarrow \delta\big(A \cup \{x\}\big) > \delta(A)$$

holds, then A is called a *complete set*. If A is a complete set containing a set B and satisfying $\delta(A) = \delta(B)$, then A is called a *completion* of B. It is known (see, e.g., [1, Proposition 3.1] and [3, Theorem 3]) that

(1) $\eta(A)$ and $\theta(A)$ are both bounded closed convex sets,
(2) $A \subseteq \theta(A) \subseteq \eta(A)$,
(3) $\eta(A)$ is the union of all completions of A,
(4) $\theta(A)$ is the intersection of all completions of A,
(5) A is complete if and only if $A = \eta(A)$,
(6) A and $\theta(A)$ have the same centers, the same radius, and the same completions.

Also the following facts are equivalent (see [1, Theorem 3.7], [5, Proposition 2], [4, Theorem 1]):

(1) A has a unique completion,
(2) $\eta(A) = \theta(A)$,
(3) $\theta(A)$ is complete,
(4) $\delta(\eta(A)) = \delta(A)$,
(5) $\eta(A)$ is complete.

There are many problems related to the structures of wide and tight spherical hulls. For example, in [6] Schneider and Moreno mentioned the problem of characterizing spaces X so that the *Maehara set* $\mu(A)$ defined by

$$\mu(A) := \frac{1}{2}\big(\eta(A) + \theta(A)\big)$$

is a completion of A for each $A \in \mathcal{H}$. Note that the original problem is stated for *Minkowski spaces* (i.e., for real finite-dimensional Banach spaces). And they wrote “We do not know whether the Minkowski spaces in which every Maehara set is complete (without necessarily being of constant width) have a simple characterization". This situation indicates that the structure of wide spherical hull and tight spherical hull of bounded sets is still not perfectly known.

In the present article, we study the relation between the sets A, $\eta(A)$, and $\theta(A)$ when $A \in \mathcal{H}$.

2. On the Boundary structure and diametral points

2.1. Characterizations of $\operatorname{bd} \eta(A)$ and $\operatorname{bd} \theta(A)$.

Proposition 2.1. *If $A \in \mathcal{H}$, then $x \in \operatorname{bd} \eta(A)$ if and only if $\gamma(A, x) = \delta(A)$.*

Proof. In [7] it is shown that $\gamma(A, x) = \delta(A)$ holds for each $x \in \operatorname{bd} \eta(A)$. We only need to show that $\gamma(A, x) = \delta(A)$ implies that $x \in \operatorname{bd} \eta(A)$.

Let $\gamma_0 = \gamma(A, x) = \delta(A)$. It is clear that $x \in \eta(A)$. If $x \notin \operatorname{bd} \eta(A)$, then there exists a number $\gamma > 0$ such that $B_X(x, \gamma) \subseteq \eta(A)$. For each $\varepsilon \in (0, \min\{\delta(A), \gamma/2\})$, there exists a point $y \in A$ such that

$$\|x - y\| \geq \delta(A) - \varepsilon.$$

Clearly,

$$z = x + \gamma \frac{x - y}{\|x - y\|} \in B_X(x, \gamma) \subseteq \eta(A).$$

However,

$$\begin{aligned} \|z - y\| &= \left\| x + \gamma \frac{x - y}{\|x - y\|} - y \right\| = \left(1 + \frac{\gamma}{\|x - y\|}\right) \|x - y\| \\ &= \|x - y\| + \gamma \\ &\geq \delta(A) - \varepsilon + \gamma \\ &> \delta(A), \end{aligned}$$

which is a contradiction. Thus $x \in \operatorname{bd} \eta(A)$. □

For $\theta(A)$ we have the following similar result.

Proposition 2.2. *Suppose that $A \in \mathcal{H}$ and that $x \in \theta(A)$. Then $x \in \operatorname{bd} \theta(A)$ if and only if $\gamma(\eta(A), x) = \delta(A)$.*

Proof. First suppose that $x \in \operatorname{bd} \theta(A)$. Suppose to the contrary that $\gamma(\eta(A), x) < \delta(A)$. Then for each point $z \in B_X(x, \delta(A) - \gamma(\eta(A), x))$ and each point $w \in \eta(A)$, we have

$$\|z - w\| \leq \|z - x\| + \|x - w\| \leq \delta(A) - \gamma\big(\eta(A), x\big) + \gamma\big(\eta(A), x\big) = \delta(A).$$

It follows that $B_X(x, \delta(A) - \gamma(\eta(A), x)) \subseteq \theta(A)$, which is a contradiction to the fact that $x \in \operatorname{bd} \theta(A)$.

Now assume that $\gamma(\eta(A), x) = \delta(A)$. If $x \notin \operatorname{bd} \theta(A)$, then there exists a number $\gamma \in (0, \delta(A))$ such that $B_X(x, \gamma) \subseteq \theta(A)$. Since $\gamma(\eta(A), x) = \delta(A)$, there exists a point $y \in \eta(A)$ such that $\|x - y\| > \delta(A) - \gamma$. Then

$$x + \gamma \frac{x - y}{\|x - y\|} \in \theta(A)$$

and

$$\left\| x + \gamma \frac{x-y}{\|x-y\|} - y \right\| = \|x-y\| + \gamma > \delta(A) - \gamma + \gamma = \delta(A),$$

a contradiction. □

2.2. **Diametral points of A, $\eta(A)$, and $\theta(A)$.** As the following example shows, a set $A \in \mathcal{H}$ need not have a diametral point; a set having diametral points does not necessarily have strongly diametral points.

Example 2.3. Let $X = (c_0, \|\cdot\|_\infty)$ and

$$A = \left\{ (\alpha_i)_{i=1}^{\infty} : |\alpha_i| \le 1 - \frac{1}{i}, \forall i \in \mathbb{N} \right\}.$$

Then $\delta(A) = 2$, and A has no diametral point. Now put

$$B = \left\{ (\alpha_i)_{i=1}^{\infty} : 0 \le \alpha_i \le 1 - \frac{1}{i}, \forall i \in \mathbb{N} \right\}.$$

Then $\delta(B) = 1$; each point of B is diametral but not strongly diametral.

Next we characterize the set of diametral points of A and $\theta(A)$.

Proposition 2.4. *For each $A \in \mathcal{H}$, the set of diametral points of A is precisely $A \cap \operatorname{bd} \eta(A)$.*

Proof. If x is a diametral point of A, then $\gamma(A, x) = \delta(A)$. Proposition 2.1 shows that $x \in \operatorname{bd} \eta(A)$. Thus $x \in A \cap \operatorname{bd} \eta(A)$.

Conversely, suppose that $x \in A \cap \operatorname{bd} \eta(A)$. Proposition 2.1 shows that $\gamma(A, x) = \delta(A)$. Therefore, x is a diametral of A. □

As we have mentioned, $\eta(\theta(A)) = \eta(A)$. Therefore we have the following.

Corollary 2.5. *For each $A \in \mathcal{H}$, the set of diametral points of $\theta(A)$ is precisely $\theta(A) \cap \operatorname{bd} \eta(A)$.*

With the help of Proposition 2.4 and Corollary 2.5, we can prove the following.

Proposition 2.6. *Let $A \in \mathcal{H}$, $x \in A$. Then x is a diametral point of A if and only if it is a diametral point of $\theta(A)$.*

Proof. First suppose that x is a diametral point of A. By Proposition 2.4,

$$x \in A \cap \operatorname{bd} \eta(A) \subseteq \theta(A) \cap \operatorname{bd} \eta(A).$$

Then Corollary 2.5 shows that x is a diametral point of $\theta(A)$.

Now suppose that $x \in A$ is a diametral point of $\theta(A)$. Then

$$x \in \theta(A) \cap \operatorname{bd} \eta(A),$$

which shows that $\gamma(A, x) = \delta(A)$. Thus x is a diametral point of A. □

The following example shows that $\theta(A)$ may have diametral points while A does not.

Example 2.7. Let X be the linear subspace of $C[0,1]$ that consists of functions $f \in C[0,1]$ satisfying $f(0)=0$, let $\|\cdot\|$ be the norm on X which is given by

$$\|f\| = \|f\|_\infty + \|f\|_1 = \max\big\{\big|f(\alpha)\big| : \alpha \in [0,1]\big\} + \int_0^1 \big|f(\alpha)\big|\,d\alpha, \quad \forall f \in X,$$

and let A be a set defined by

$$A := \big\{f \in X : f(0)=0, f(1)=1, f(\alpha) \in [0,1], \forall \alpha \in (0,1)\big\}.$$

(1) $\delta(A)=2$: Clearly, $\delta(A) \le 2$. For each integer $n \ge 3$, put

$$f_n(\alpha) = \begin{cases} n\alpha, & 0 \le \alpha < \frac{1}{n}, \\ 1, & \frac{1}{n} \le \alpha \le 1, \end{cases} \qquad g_n(\alpha) = \begin{cases} 0, & 0 \le \alpha < 1-\frac{1}{n}, \\ n\alpha + (1-n), & 1-\frac{1}{n} \le \alpha \le 1. \end{cases}$$

Then

$$\begin{aligned} \|f_n - g_n\| &= 1 + \int_0^{\frac{1}{n}} |n\alpha|\,d\alpha + \int_{\frac{1}{n}}^{1-\frac{1}{n}} 1\,d\alpha + \int_{1-\frac{1}{n}}^1 \big|1 - n\alpha - (1-n)\big|\,d\alpha \\ &\ge 1 + 1 - \frac{2}{n} = 2 - \frac{2}{n}. \end{aligned}$$

Thus $\delta(A) \ge 2$. It follows that $\delta(A) = 2$.

(2) A has no diametral point: For each $f \in A$, there exist two numbers γ_1, γ_2 such that $0 < \gamma_1 < \gamma_2 < 1$ and that

$$\frac{1}{4} \le f(\alpha) \le \frac{3}{4}, \quad \forall \alpha \in [\gamma_1, \gamma_2].$$

Then, for each $g \in A$, we have

$$\begin{aligned} \|f-g\| \le 1 &+ \int_0^{\gamma_1} \big|f(\alpha)-g(\alpha)\big|\,d\alpha + \int_{\gamma_1}^{\gamma_2} \big|f(\alpha)-g(\alpha)\big|\,d\alpha \\ &+ \int_{\gamma_2}^1 \big|f(\alpha)-g(\alpha)\big|\,d\alpha \\ &\le 1 + \gamma_1 + 1 - \gamma_2 + \frac{3}{4}(\gamma_2-\gamma_1) = 2 - \frac{1}{4}(\gamma_2-\gamma_1). \end{aligned}$$

Hence f is not diametral.

(3) $o \in \theta(A)$: We only need to show that $\|f\| \le 2$ holds for each $f \in \eta(A)$. For each sufficiently small $\varepsilon > 0$, there exists an integer $N_0 \ge 2$ such that

$$\max\Big\{\big|f(\alpha)\big| : 0 \le \alpha < 1-\frac{1}{n}\Big\} \ge \|f\|_\infty - \varepsilon \qquad \text{and} \qquad \int_0^{1-\frac{1}{n}} \big|f(\alpha)\big|\,d\alpha \ge \|f\|_1 - \varepsilon$$

holds for each integer $n \ge N_0$. For each $n \ge N_0$, put

$$g_n(\alpha) = \begin{cases} 0, & 0 \le \alpha < 1-\frac{1}{n}, \\ \alpha n + 1 - n, & 1-\frac{1}{n} \le \alpha \le 1. \end{cases}$$

Then $g_n \in A$. Therefore,

$$\begin{aligned} 2 &\geq \|f - g_n\| = \|f - g_n\|_\infty + \|f - g_n\|_1 \\ &\geq \max\Big\{ \big|f(\alpha)\big| : 0 \leq \alpha < 1 - \frac{1}{n} \Big\} + \int_0^{1-\frac{1}{n}} \big|f(\alpha)\big| \, d\alpha \\ &\geq \|f\|_\infty + \|f\|_1 - 2\varepsilon. \end{aligned}$$

It follows that

$$\|f\| = \|f\|_\infty + \|f\|_1 \leq 2.$$

(4) o is a diametral point of $\theta(A)$: For each integer $n \geq 2$, put

$$g_n(\alpha) = \begin{cases} n\alpha, & 0 \leq \alpha < \frac{1}{n}, \\ 1, & \frac{1}{n} \leq \alpha \leq 1. \end{cases}$$

Then $g_n \in A$ and

$$\|g_n\| = 1 + \int_0^{\frac{1}{n}} n\alpha \, d\alpha + \int_{\frac{1}{n}}^1 1 \, d\alpha \geq 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n}.$$

Thus

$$\gamma\big(\theta(A), o\big) \geq \gamma(A, o) \geq 2 - \frac{1}{n}, \quad \forall n \geq 2.$$

It follows that o is a diametral point of $\theta(A)$.

If $A \in \mathcal{H}$ is complete, then $A = \eta(A)$, and by Proposition 2.1 each boundary point of A is a diametral point.

The following example shows that $\eta(A)$ may have no diametral point, even if A has.

Example 2.8. Let X be the subspace of $C[-1, 1]$ consisting of functions f satisfying $f(0) = 0$. Let

$$A = \big\{ f \in X : \alpha \leq f(\alpha) \leq 0, \forall \alpha \in [-1, 0); f(\alpha) \in [0, 1], \forall \alpha \in (0, 1] \big\}.$$

Then $\delta(A) = 1$,

$$\eta(A) = \big\{ f \in X : -1 \leq f(\alpha) \leq 1 + \alpha, \forall \alpha \in [-1, 0); f(\alpha) \in [0, 1], \forall \alpha \in (0, 1] \big\},$$

$\theta(A) = A$, $\delta(\eta(A)) = 2$, $\eta(A)$ has no diametral point while both A and $\theta(A)$ have diametral points.

When the underlying space is finite-dimensional, then both A and $\theta(A)$ have strongly diametral points. As we will show in the next section, both $\theta(A)$ and A can have no diametral point and it is possible that $A \subseteq \operatorname{int} \theta(A)$ for some $A \in \mathcal{H}$.

2.3. **On the distance from A to $\operatorname{bd}\eta(A)$ and $\operatorname{bd}\theta(A)$.**

Proposition 2.9. *For each $A \in \mathcal{H}$, we have*

$$\gamma = \inf\big\{\|x-y\| : x \in A, y \in \operatorname{bd}\eta(A)\big\} = 0.$$

Proof. Otherwise, $\gamma > 0$. Let x be an arbitrary point in A, and y be an arbitrary point in $B_X(x, \gamma/2)$. If $y \notin \operatorname{int}(\eta(A))$, then by the convexity of $\eta(A)$ and the fact that $A \subseteq \eta(A)$, we have

$$B_X(x, \gamma/2) \cap \operatorname{bd}\eta(A) \neq \emptyset,$$

which is in contradiction to the definition of γ. Thus

$$B_X(x, \gamma/2) \subseteq \operatorname{int}\big(\eta(A)\big), \quad \forall x \in A.$$

Clearly, A contains two points u, v such that

$$\|u-v\| \geq \delta(A) - \gamma/4.$$

On the one hand, we have

$$u + \frac{\gamma}{2}\frac{u-v}{\|u-v\|} \in \eta(A),$$

and, on the other hand,

$$\left\|u + \frac{\gamma}{2}\frac{u-v}{\|u-v\|} - v\right\| = \|u-v\| + \frac{\gamma}{2} \geq \delta(A) + \gamma/4 > \delta(A),$$

which is another contradiction. Thus, $\gamma = 0$. □

Corollary 2.10. *For each $A \in \mathcal{H}$,*

$$\gamma = \inf\big\{\|x-y\| : x \in \theta(A), y \in \operatorname{bd}\eta(A)\big\} = 0.$$

In a similar way, we can prove the following

Proposition 2.11. *For each $A \in \mathcal{H}$,*

$$\gamma = \inf\big\{\|x-y\| : x \in A, y \in \operatorname{bd}\theta(A)\big\} = 0.$$

Proof. Otherwise $\gamma > 0$. In a similar way as in the proof of Proposition 2.9, we have

$$B_X(x, \gamma/2) \subseteq \operatorname{int}\big(\theta(A)\big) \subseteq \operatorname{int}\big(\eta(A)\big), \quad \forall x \in A.$$

Then the distance from each point $x \in A$ to $\operatorname{bd}\eta(A)$ is not less than $\gamma/2$, which is a contradiction to Proposition 2.9. □

One may expect that $\operatorname{bd}\eta(A) \cap A \neq \emptyset$ and $\operatorname{bd}\eta(A) \cap \theta(A) \neq \emptyset$ hold for each $A \in \mathcal{H}$, which is true when the dimension of the underlying space is finite. In the following we show by a concrete example that this is not true in general. We will use the following lemma.

Lemma 2.12. *Suppose that $A \in \mathcal{H}$. Then*

$$\eta(-A) = -\eta(A), \qquad \theta(-A) = -\theta(A).$$

If $A = -A$, then $(\delta(A)/2)B_X$ is a completion of A.

Proof. Let p be an arbitrary point in $\eta(-A)$, and let x be an arbitrary point in A. Then

$$\|-p-x\| = \big\|p-(-x)\big\| \le \delta(A) = \delta(-A).$$

Thus $-p \in \eta(A)$. It follows that $\eta(-A) \subseteq -\eta(A)$. Replacing $-A$ by A, we have $-\eta(A) \subseteq \eta(-A)$. Thus $\eta(-A) = -\eta(A)$. In a similar way we can show that $\theta(-A) = -\theta(A)$.

Now suppose that $-A = A$. Then, for each point $x \in A$, we have

$$\|x\| = \frac{1}{2}\big\|x-(-x)\big\| \le \delta(A)/2,$$

which implies that $x \in (\delta(A)/2)B_X$. Thus, $(\delta(A)/2)B_X$, a ball containing A whose diameter is $\delta(A)$, is a completion of A. □

We denote by X the Banach space $(c_0, \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the usual maximum norm, and by Y the Banach space $(c_0, \|\cdot\|_D)$, where $\|\cdot\|_D$ is *Day's norm*. More precisely, for each $(\alpha_i)_{i=1}^\infty \in c_0$,

$$\big\|(\alpha_i)_{i=1}^\infty\big\|_D = \Big(\sum_{i=1}^\infty \Big(\frac{\beta_i}{2^i}\Big)^2\Big)^{1/2},$$

where $(\beta_i)_{i=1}^\infty$ is a permutation of $(\alpha_i)_{i=1}^\infty$ such that

$$|\beta_i| \ge |\beta_{i+1}|, \quad \forall i \in \mathbb{N}.$$

Each permutation of $(\alpha_i)_{i=1}^\infty$ having this property will be called a *nonincreasing permutation of* $(\alpha_i)_{i=1}^\infty$.

It is not difficult to verify (see, e.g., [11, p. 1774]) that

$$\big\|(\alpha_i)_{i=1}^\infty\big\|_D = \sup\Big\{\Big(\sum_{k=1}^n \Big(\frac{\alpha_{i_k}}{2^k}\Big)^2\Big)^{1/2} : n \in \mathbb{N}, \{i_k : 1 \le k \le n\} \subset \mathbb{N}\Big\}.$$

The space Y is locally uniformly convex and the norm $\|\cdot\|_D$ is equivalent to $\|\cdot\|_\infty$ on c_0 (see [10]). Therefore, if $x \in Y$ and if $\{x_n\}_{n=1}^\infty \subseteq Y$ satisfies

$$\|x\|_D = 1 = \|x_n\|_D, \quad \forall n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n\to\infty} \|x+x_n\|_D = 2,$$

then

$$\lim_{n\to\infty} \|x-x_n\|_D = 0.$$

Now put $A = B_X$. Then A is a bounded closed convex set in Y symmetric with respect to o.

Proposition 2.13. *Let A be the subset of Y defined above. Then*

(i) $\delta(A) = 2\sqrt{3}/3$,
(ii) $B_Y(o, \sqrt{3}/3)$ *is a completion of* A,
(iii) $S_Y(o, \sqrt{3}/3) \cap \operatorname{bd}\eta(A) = \emptyset$,
(iv) $\operatorname{bd}\eta(A) \cap \theta(A) = \emptyset$, *and therefore* $\theta(A)$ *has no diametral point,*
(v) $A \subseteq \operatorname{int}\theta(A)$.

Proof. (i) Let $(\alpha_i)_{i=1}^\infty$ and $(\beta_i)_{i=1}^\infty$ be two arbitrary points in A, and let $(\gamma_i)_{i=1}^\infty$ be a nonincreasing permutation of $(\alpha_i-\beta_i)_{i=1}^\infty$. On the one hand, we have

$$\left\|(\alpha_i-\beta_i)_{i=1}^\infty\right\|_D=\left(\sum_{i=1}^\infty\left(\frac{\gamma_i}{2^i}\right)^2\right)^{1/2}\le\left(\sum_{i=1}^\infty\frac{4}{4^i}\right)^{1/2}=\frac{2\sqrt{3}}{3}.$$

Therefore,

$$\delta(A)\le\frac{2\sqrt{3}}{3}.$$

On the other hand, for each $n\in\mathbb{N}$, let $(\alpha_i^n)_{i=1}^\infty$ and $(\beta_i^n)_{i=1}^\infty$ be two points in A such that

$$\begin{aligned}\alpha_i^n&=-\beta_i^n=1,\quad\forall i\le n\\ \alpha_i^n&=\beta_i^n=0,\quad\forall i>n.\end{aligned}$$

Then

$$\left\|(\alpha_i^n-\beta_i^n)_{i=1}^\infty\right\|_D=\left(\sum_{i=1}^n\frac{4}{4^i}\right)^{1/2}=\frac{2\sqrt{3}}{3}\cdot\left(1-\frac{1}{4^n}\right)^{1/2}.$$

It follows that $\delta(A)\ge 2\sqrt{3}/3$. Thus (i) follows.

(ii) This follows from (i), Lemma 2.12, and the fact that $A=-A$.

(iii) It suffices to show that

$$\gamma(A,x)<\delta(A),\quad\forall x\in S_Y\left(o,\frac{\sqrt{3}}{3}\right).\tag{2}$$

Let $x=(\alpha_i)_{i=1}^\infty$ be an arbitrary point in $S_Y(o,\sqrt{3}/3)$, and let $(\eta_i)_{i=1}^\infty$ be a nonincreasing permutation of $(\alpha_i)_{i=1}^\infty$. Then

$$\sum_{i=1}^\infty\frac{|\eta_i|^2}{4^i}=\frac{1}{3}.$$

Since

$$\lim_{i\to\infty}\eta_i=0,$$

there exists a number $N\in\mathbb{N}$ such that

$$|\eta_i|\le\frac{1}{2},\quad\forall i>N.$$

Therefore,

$$\begin{aligned}\frac{1}{3}=\sum_{i=1}^\infty\frac{|\eta_i|^2}{4^i}&=\sum_{i=1}^N\frac{|\eta_i|^2}{4^i}+\sum_{i=N+1}^\infty\frac{|\eta_i|^2}{4^i}\\ &\le\sum_{i=1}^N\frac{|\eta_1|^2}{4^i}+\sum_{i=N+1}^\infty\frac{1}{4^{i+1}}\\ &=\frac{1}{3}\left(\eta_1^2-\frac{\eta_1^2}{4^N}\right)+\frac{1}{3\cdot 4^{N+1}}.\end{aligned}$$

Hence,

$$\eta_1^2 \geq \frac{1-\frac{1}{4^{N+1}}}{1-\frac{1}{4^N}} = \frac{4^{N+1}-1}{4^{N+1}-4} > 1.$$

This shows that $x \notin A$.

Suppose the contrary, namely, that $\gamma(A,x) = \delta(A)$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ such that

$$\delta(A) \geq \|x - x_n\|_D \geq \delta(A) - \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

By the triangle inequality, we have

$$\delta(A)/2 \geq \|x_n\|_D \geq \|x - x_n\|_D - \|x\|_D \geq \delta(A)/2 - \frac{1}{n}.$$

Put

$$u = \frac{x}{\|x\|_D}, \qquad u_n = -\frac{x_n}{\|x_n\|_D}, \quad \forall n \in \mathbb{N}.$$

Then

$$\begin{aligned}\|u + u_n\|_D &= \left\| \frac{x}{\|x\|_D} - \frac{x_n}{\|x_n\|_D} \right\|_D \\ &= \left\| \frac{x}{\|x\|_D} - \frac{x_n}{\|x\|_D} + \left(1 - \frac{\|x\|_D}{\|x_n\|_D}\right) \frac{x_n}{\|x\|_D} \right\|_D,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|u + u_n\|_D = \lim_{n\to\infty} \frac{\|x - x_n\|_D}{\|x\|_D} = 2.$$

Since $(c_0, \|\cdot\|_D)$ is locally uniformly convex,

$$\lim_{n\to\infty} u_n = u.$$

Therefore

$$\lim_{n\to\infty} x_n = -\lim_{n\to\infty} \big(\|x_n\|_D \cdot u_n\big) = -\frac{\delta(A)}{2} u = -x.$$

Since $A = -A$ is closed and $x \notin A$, this is impossible. Hence $\gamma(A,x) < \delta(A)$.

(iv) (iii) implies that

$$\begin{aligned}\theta(A) \cap \operatorname{bd} \eta(A) &\subseteq B_Y\Big(o, \frac{\sqrt{3}}{3}\Big) \cap \operatorname{bd} \eta(A) \\ &\subseteq S_Y\Big(o, \frac{\sqrt{3}}{3}\Big) \cap \operatorname{bd} \eta(A) = \emptyset,\end{aligned}$$

where the second inclusion holds since each interior point of $B_Y(o, \sqrt{3}/3)$ is also an interior point of $\eta(A)$. Corollary 2.5 shows that $\theta(A)$ has no diametral point.

(v) Let $(\alpha_i)_{i=1}^{\infty}$ be an arbitrary point in $\operatorname{bd} A$, and let N_0 be the number of elements in this sequence whose absolute value is not less than $1/2$. Let $(\beta_i)_{i=1}^{\infty}$ be an arbitrary point in $\eta(A)$. We show that

$$\big\|(\alpha_i - \beta_i)_{i=1}^{\infty}\big\|_D^2 < \delta(A)^2 - \frac{3}{4^{N_0+2}}. \tag{3}$$

For each $k \in \mathbb{N}$, let $N_k \geq \max\{k+1, N_0+1\}$ be an integer such that

$$|\alpha_i - \beta_i| \leq \frac{1}{2^k}, \quad \forall i \geq N_k.$$

Therefore there are at most $N_k - 1$ numbers in the sequence $(\alpha_i - \beta_i)_{i=1}^{\infty}$ whose absolute value is strictly greater than $1/2^k$.

For each $i < N_k$, put

$$\gamma_i^k = \begin{cases} 1, & \beta_i \leq 0; \\ -1, & \beta_i > 0, \end{cases}$$

and, for each $i \geq N_k$, put $\gamma_i^k = \alpha_i$. Then $(\gamma_i^k)_{i=1}^{\infty} \in \operatorname{bd} A$. Suppose that $\pi(\cdot)$ is a permutation of natural numbers such that the sequence $(\eta_i)_{i=1}^{\infty}$ defined by

$$\eta_i = \alpha_{\pi(i)} - \beta_{\pi(i)}, \quad \forall i \in \mathbb{N},$$

is a non-increasing permutation of $(\alpha_i - \beta_i)_{i=1}^{\infty}$. Then

$$\begin{aligned}
&\delta(A)^2 - \left\|(\alpha_i - \beta_i)_{i=1}^{\infty}\right\|_D^2 \\
&\quad \geq \left\|(\gamma_i^k - \beta_i)_{i=1}^{\infty}\right\|_D^2 - \left\|(\alpha_i - \beta_i)_{i=1}^{\infty}\right\|_D^2 \\
&\quad \geq \sum_{i=1}^{N_k-1} \frac{(\gamma_{\pi(i)}^k - \beta_{\pi(i)})^2}{4^i} - \sum_{i=1}^{N_k-1} \frac{(\alpha_{\pi(i)} - \beta_{\pi(i)})^2}{4^i} - \sum_{i=N_k}^{\infty} \frac{(\alpha_{\pi(i)} - \beta_{\pi(i)})^2}{4^i} \\
&\quad \geq \sum_{i=1}^{N_k-1} \frac{(\gamma_{\pi(i)}^k - \beta_{\pi(i)})^2}{4^i} - \sum_{i=1}^{N_k-1} \frac{(\alpha_{\pi(i)} - \beta_{\pi(i)})^2}{4^i} - \sum_{i=N_k}^{\infty} \frac{(\frac{1}{2^k})^2}{4^i} \\
&\quad \geq \sum_{i=1}^{N_k-1} \frac{(\gamma_{\pi(i)}^k - \beta_{\pi(i)})^2}{4^i} - \sum_{i=1}^{N_k-1} \frac{(\alpha_{\pi(i)} - \beta_{\pi(i)})^2}{4^i} - \frac{1}{3 \cdot 16^k}. \qquad (4)
\end{aligned}$$

Clearly,

$$(\gamma_{\pi(i)}^k - \beta_{\pi(i)})^2 - (\alpha_{\pi(i)} - \beta_{\pi(i)})^2 \geq 0, \quad \forall i \in \mathbb{N}.$$

First suppose that there exists an integer $i_0 \leq N_0$ such that $|\alpha_{\pi(i_0)}| < 1/2$. If $|\gamma_{\pi(i_0)}^k - \beta_{\pi(i_0)}| = 1 + |\beta_{\pi(i_0)}|$, then (4) implies that

$$\begin{aligned}
\delta(A)^2 - \left\|(\alpha_i - \beta_i)_{i=1}^{\infty}\right\|_D^2 &\geq \frac{(1 + |\beta_{\pi(i_0)}|)^2 - (\alpha_{\pi(i_0)} - \beta_{\pi(i_0)})^2}{4^{i_0}} - \frac{1}{3 \cdot 16^k} \\
&\geq \frac{3}{4^{i_0+1}} - \frac{1}{3 \cdot 16^k} > \frac{3}{4^{N_0+2}} - \frac{1}{3 \cdot 16^k}.
\end{aligned}$$

Otherwise, $\gamma_{\pi(i_0)}^k = \alpha_{\pi(i_0)}$ and $|\gamma_{\pi(i_0)}^k - \beta_{\pi(i_0)}| = |\alpha_{\pi(i_0)} - \beta_{\pi(i_0)}| \leq 1/2^k$. The construction of $(\gamma_i^k)_{i=1}^{\infty}$ shows that there are at least $N_k - 1$ integers i such that $|\gamma_i^k - \beta_i| = 1 + |\beta_i|$. Thus there exists an $j_0 \geq N_k$ such that

$$|\gamma_{\pi(j_0)}^k - \beta_{\pi(j_0)}| = 1 + |\beta_{\pi(j_0)}|.$$

By replacing, in (4), $(\gamma^k_{\pi(i_0)} - \beta_{\pi(i_0)})^2$ with $(\gamma^k_{\pi(j_0)} - \beta_{\pi(j_0)})^2$ we obtain

$$\begin{aligned}\delta(A)^2 - \big\|(\alpha_i - \beta_i)_{i=1}^\infty\big\|_D^2 &\geq \frac{(1+|\beta_{\pi(j_0)}|)^2 - (\alpha_{\pi(i_0)} - \beta_{\pi(i_0)})^2}{4^{i_0}} - \frac{1}{3\cdot 16^k} \\ &\geq \frac{3}{4^{i_0+1}} - \frac{1}{3\cdot 16^k} > \frac{3}{4^{N_0+2}} - \frac{1}{3\cdot 16^k}.\end{aligned}$$

Now suppose that, for each $i_0 \leq N_0$, we have $|\alpha_{\pi(i_0)}| \geq 1/2$. Then $|\alpha_{\pi(N_0+1)}| < 1/2$. In a similar way as above, we can show that

$$\delta(A)^2 - \big\|(\alpha_i - \beta_i)_{i=1}^\infty\big\|_D^2 \geq \frac{3}{4^{N_0+2}} - \frac{1}{3\cdot 16^k}.$$

Since k is arbitrary, (3) holds. Thus $\gamma(\eta(A), (\alpha_i)_{i=1}^\infty) < \delta(A)$, which implies that $(\alpha_i)_{i=1}^\infty \in \operatorname{int}\theta(A)$. Therefore $A \subseteq \operatorname{int}\theta(A)$. □

3. A characterization of balls

In this short section we prove a simple characterization of the case when $\eta(A)$ is a ball.

Theorem 3.1. *Let $A \in \mathcal{H}$. Then $\eta(A)$ is a ball if and only if*

$$\gamma\big(\eta(A)\big) = \delta\big(\eta(A)\big)/2 = \delta(A)/2. \tag{5}$$

Proof. First suppose that $\eta(A)$ is a ball. Then $\eta(A)$ is complete and, therefore, is the unique completion of A. It follows that

$$\delta\big(\eta(A)\big) = \delta(A). \tag{6}$$

Since $\eta(A)$ is a ball,

$$\gamma\big(\eta(A)\big) = \gamma\big(\eta(A), \eta(A)\big) = \delta\big(\eta(A)\big)/2. \tag{7}$$

Then (5) follows from (6) and (7).

Conversely, suppose that (5) holds. Then $\eta(A)$ is the unique completion of A and, therefore, a complete set whose diameter is $\delta(A)$. Thus (see [8])

$$\gamma\big(\eta(A)\big) + \gamma'\big(\eta(A)\big) = \delta\big(\eta(A)\big) = \delta(A). \tag{8}$$

From (8) and (5) it follows that

$$\gamma\big(\eta(A)\big) = \gamma'\big(\eta(A)\big) = \delta\big(\eta(A)\big)/2.$$

Applying Theorem 3.1 in [8] we have that $\eta(A)$ is a ball. □

Remark 3.2. Note that $\delta(\eta(A)) = 2\gamma(\eta(A))$ does not imply that $\eta(A)$ is a ball. For example, let $X = l_\infty^2$, $x = o$, $y = (1,0)$, $A = [x,y]$. Then $\delta(A) = 1$, $\gamma(A) = \frac{1}{2}$, $\delta(\eta(A)) = 2$, $\gamma(\eta(A)) = 1$. But $\eta(A)$ is not a ball.

Corollary 3.3. *A set $A \in \mathcal{H}$ is a ball if and only if it is the unique completion of a set $B \in \mathcal{H}$ and $\delta(A) = 2\gamma(A)$.*

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