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THE CONE AND CYLINDER ALGEBRAS

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Dedicated to Professor Anthony To-Ming Lau

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Abstract. In this exposition-type note we present detailed proofs of certain assertions concerning several algebraic properties of the cone and cylinder algebras. These include a determination of the maximal ideals, the solution of the Bézout equation, and a computation of the stable ranks by elementary methods.

0. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathbf{D}$ be its closure. As usual, $C(\mathbf{D}, \mathbb{C})$ denotes the space of continuous, complex-valued functions on $\mathbf{D}$ and $A(\mathbf{D})$ denotes the disk algebra, that is, the algebra of all functions in $C(\mathbf{D}, \mathbb{C})$ which are holomorphic in $\mathbb{D}$. By the Stone–Weierstrass theorem, we have $C(\mathbf{D}, \mathbb{C}) = [z, \overline{z}]_{\text{alg}}$ and $A(\mathbf{D}) = [z]_{\text{alg}}$, the uniformly closed subalgebras generated by z, $\overline{z}$ (resp., z) on $\mathbf{D}$. In this expositional note, we study the uniformly closed subalgebra

$$A_{co} = \big[z, |z|\big]_{\text{alg}} \subseteq C(\mathbf{D}, \mathbb{C})$$

of $C(\mathbf{D}, \mathbb{C})$, which is generated by z and $|z|$, as well as by the algebra

$$\text{Cyl}(\mathbb{D}) = \big\{ f \in C\big(\mathbf{D} \times [0,1], \mathbb{C}\big) : f(\cdot, t) \in A(\mathbf{D}) \text{ for all } t \in [0,1] \big\}.$$

We will call the algebra A_{co} the *cone algebra* and the algebra $\text{Cyl}(\mathbb{D})$ the *cylinder algebra*.

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The reason for choosing these names will become clear later in Theorem 1.3 and Proposition 3.1. The cone algebra appeared first in [15]; the cylinder algebra was around already at the beginning of the development of Gelfand's theory. Their role was mostly reduced to a category of "examples" to illustrate the general Gelfand theory, and no detailed proofs appeared. We think that these algebras deserve a thorough analysis of their interesting algebraic properties and that is the aim of this note. The new main results will be the determination of the stable ranks for the cone algebra, the absence of the corona property (Cn) when (and only when) $n \geq 2$, explicit examples of peak functions, and a simple approach to the calculation of the Bass stable rank for the cylinder algebra.

This paper forms part of an ongoing textbook project of the authors on stable ranks of function algebras. We make this chapter available to the mathematical community, mainly for readers of this special issue of *Annals of Functional Analysis* dedicated to Professor Anthony To-Ming Lau and for master students interested in function theory and function algebras.

1. The cone algebra

We begin with some examples of nontrivial elements in A_{co}. The general situation will be dealt with in Theorem 1.4.

Example 1.1.

(1) For every $\delta > 0$, $q > 0$, and $p \in \mathbb{R}$, $(|z|^q + \delta)^p \in (A_{co})^{-1}$.

(2)

$$f(z) := \begin{cases} \frac{z}{\sqrt{|z|}} & \text{if } 0 < |z| \leq 1 \\ 0 & \text{if } z = 0 \end{cases} \quad \text{belongs to } A_{co}.$$

Proof. (1) Since, by the Weierstrass theorem, $(x^q + \delta)^{\pm p}$ is a uniform limit of polynomials in $x = |z| \in [0, 1]$, we have that $(|z|^q + \delta)^p \in (A_{co})^{-1}$.

(2) First we note that f is continuous on $\mathbf{D}$. For $\delta > 0$, let

$$f_\delta(z) := \frac{z}{\sqrt{|z|} + \delta}.$$

Then, by (1), $f_\delta \in A_{co}$. Since $\|f - f_\delta\|_\infty \leq \delta/(1 + \delta)$ (see below), we conclude that f is a uniform limit of functions in A_{co}; hence f itself belongs to A_{co}. Now we prove our estimate:

$$\Big|\frac{z}{\sqrt{|z|}} - \frac{z}{\sqrt{|z|} + \delta}\Big| = |z|\frac{\delta}{\sqrt{|z|}(\sqrt{|z|} + \delta)} = \frac{\sqrt{|z|}}{\sqrt{|z|} + \delta}\delta =: h\big(\sqrt{|z|}\big).$$

Since the derivative of $h(x) = \frac{\delta x}{x+\delta}$, namely, $h'(x) = \frac{\delta^2}{(x+\delta)^2}$, is strictly positive on $[0, 1]$, we have $h(x) \leq h(1) = \delta/(\delta + 1)$. $\square$

Next, we derive several Banach algebraic properties for A_{co}.

Definition 1.2. Let X be a topological space, and let A be a subalgebra of $C(X, \mathbb{C})$.

(1) The set of invertible n-tuples in A is denoted by $U_n(A)$; that is,

$$U_n(A) = \Big\{(f_1,\dots,f_n) \in A^n : \exists (r_1,\dots,r_n) \in A^n : \sum_{j=1}^n r_j f_j = 1\Big\}.$$

(2) A is said to be *inverse-closed* (on X) if $f \in A$ and $|f| \geq \delta > 0$ on X imply that f is invertible in A.

(3) A satisfies condition (Cn)[3] if

$$U_n(A) = \Big\{(f_1,\dots,f_n) \in A^n : \bigcap_{j=1}^n Z_X(f_j) = \emptyset\Big\},$$

where $Z_X(f) = \{x \in X : f(x) = 0\}$ denotes the zero set of f.

(4) If A is a commutative unital Banach algebra over $\mathbb{C}$, then its spectrum (or set of multiplicative linear functionals on A endowed with the weak-$*$-topology) is denoted by $M(A)$. Moreover, $\widehat{A}$ is the set of Gelfand transforms $\widehat{f}$ of elements in A.

As usual, for a compact set X in $\mathbb{C}$, $P(X)$ is the uniform closure in $C(X,\mathbb{C})$ of the set $\mathbb{C}[z]$ of polynomials.

Theorem 1.3. *Let $A_{co} = [z,|z|]_{\mathrm{alg}} \subseteq C(\mathbf{D},\mathbb{C})$ be the cone algebra. Then we have the following:*

(1) $A(\mathbf{D}) \subseteq A_{co} \subseteq C(\mathbf{D},\mathbb{C})$.

(2) $A_{co}|_{\mathbb{T}} = A(\mathbf{D})|_{\mathbb{T}} = P(\mathbb{T})$.

(3) *For every* $0 < r < 1$, $A_{co}|_{r\mathbb{T}} = P(r\mathbb{T})$.

(4) $M(A_{co})$ *is homeomorphic to the cone*

$$K := \Big\{(x,y,t) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq t, 0 \leq t \leq 1\Big\},$$

a 3-dimensional set (Figure 1).

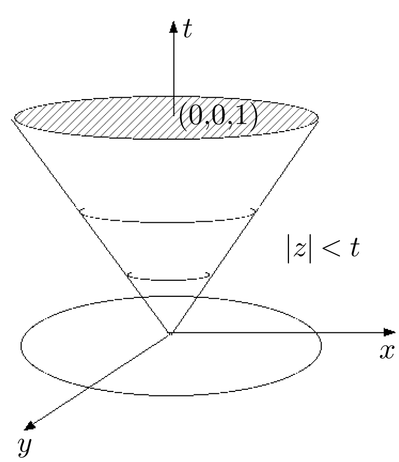

FIGURE 1. The cone as spectrum.

[3] Here (Cn) stands for "Corona condition for n-tuples."

(5) *For $f \in C(\mathbf{D}, \mathbb{C})$ and $0 < r < 1$, let f_r be the dilation of f given by $f_r(z) = f(rz)$. Moreover, let*

$$P_R[f](z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{it} - z|^2} f(Re^{it})\, dt \tag{1.1}$$

be the Poisson integral and $P[f] := P_1[f]$. Then

$$M(A_{co}) = \delta_0 \cup \big\{\psi_{r,a} : 0 < r \leq 1, a \in \mathbf{D} \text{ with } |a| \leq r\big\},$$

where δ_0 is point evaluation at 0, $\psi_{r,a} = \delta_a$ if $|a| = r$ and

$$\psi_{r,a} : \begin{cases} A_{co} \to \mathbb{C}, \\ f \mapsto P[(f_r)|_{\mathbb{T}}](a/r) = P_r[f|_{r\mathbb{T}}](a), \end{cases}$$

if $|a| < r$.

(6) *$\widehat{A}_{co}$ is the uniform closure of the polynomials $p(z, r)$ on the cone*

$$K = \big\{(z, t) \in \mathbf{D} \times [0, 1] : |z| \leq t\big\}$$

and coincides with

$$\mathcal{A} := \big\{f \in C(K, \mathbb{C}) : f(\cdot, r) \in A(r\mathbf{D})\ \forall r \in]0, 1]\big\}.$$

(7) *A_{co} is inverse-closed; that is, it has property (C1), but A_{co} does not have property (Cn) for any $n \geq 2$.*

(8) *The Shilov boundary of A_{co} coincides with the outer surface*

$$S := \big\{(z, r) \in \mathbb{C} \times \mathbb{R} : 0 \leq r \leq 1, |z| = r\big\}$$

of the cone K (this is the boundary of K without the upper disk $\{(z, 1) \in \mathbb{C} \times \mathbb{R} : |z| < 1\}$). The Bear–Shilov boundary is the closed unit disk.[4]

Proof. (1) This is clear since the polynomials in z are dense in $A(\mathbf{D})$.

(2) Let $f \in A_{co}$. If we choose a sequence of polynomials $p_n \in \mathbb{C}[z, w]$ such that $p_n(z, |z|)$ converges uniformly to f on $\mathbf{D}$, then $p_n(z, 1)$ converges uniformly on $\mathbb{T}$ to $f|_{\mathbb{T}}$. Hence, $f|_{\mathbb{T}} \in P(\mathbb{T}) = A(\mathbf{D})|_{\mathbb{T}}$. Together with (1), we conclude that $A(\mathbf{D})|_{\mathbb{T}} = A_{co}|_{\mathbb{T}}$.

(3) Fix $0 < r < 1$, and let $f \in A_{co}$. Then, for $|z| = r$, $f(z) = \lim p_n(z, r) \in P(r\mathbb{T})$. Hence $A_{co}|_{r\mathbb{T}} \subseteq P(r\mathbb{T})$. Conversely, given $h \in P(r\mathbb{T})$, we let $H := P_r[h]$ be the Poisson extension of h to $r\mathbb{D}$. Then $H \in A(r\mathbb{D})$. Now we define the function f by

$$f(z) = \begin{cases} H(z) & \text{if } |z| \leq r, \\ H(r\frac{z}{|z|}) & \text{if } r \leq |z| \leq 1. \end{cases}$$

Note that f is an extension of H to the unit disk that stays constant on every ray $se^{i\theta}$ beginning at the radius r. Then f is continuous on $\mathbf{D}$. Now $f(z)$ can be

[4]Recall that if X is a compact Hausdorff space and L is a point-separating $\mathbb{K}$-linear subspace of $C(X, \mathbb{K})$ with $\mathbb{K} \subseteq L$, then L admits a smallest closed boundary, which we will call the *Bear–Shilov boundary* (see [2]). The Shilov boundary of a uniform algebra A is the smallest closed boundary of A on its spectrum $M(A)$.

written as $f(z) = H(zg(|z|))$, where g is defined by

$$g(s) = \begin{cases} 1 & \text{if } 0 \le s \le r, \\ \frac{r}{s} & \text{if } r \le s \le 1. \end{cases}$$

Then g is continuous on $[0,1]$. Next, we uniformly approximate on $[0,1]$ the function g by a sequence (q_n) of polynomials in $\mathbb{C}[s]$ and H on $\{|z| \le r\}$ by a sequence of polynomials $(p_n) \in \mathbb{C}[z]$. Let

$$Q_n(s) := \frac{r q_n(s)}{\max_{0\le s\le 1} |s q_n(s)|}.$$

Then also (Q_n) converges uniformly to g on $[0,1]$ because $\max_{0\le s\le 1} |sg(s)| = r$. What we have gained is that $|zQ_n(|z|)| \le r$ for every $z \in \mathbf{D}$. Hence $H(zQ_n(|z|))$ is well defined on $\mathbf{D}$ and $H(zQ_n(|z|))$ converges uniformly on $\mathbf{D}$ to $f(z)$. We claim that $p_n(zQ_n(|z|))$ converges uniformly on $\mathbf{D}$ to $f(z)$, too. In fact,

$$\begin{aligned} \big|p_n\big(zQ_n\big(|z|\big)\big) - f(z)\big| &\le \big|p_n\big(zQ_n\big(|z|\big)\big) - H\big(zQ_n\big(|z|\big)\big)\big| + \big|H\big(zQ_n\big(|z|\big)\big) - f(z)\big| \\ &\le \max_{|w|\le r}\big|p_n(w) - H(w)\big| + \max_{|\xi|\le 1}\big|H\big(\xi Q_n\big(|\xi|\big)\big) - f(\xi)\big|. \end{aligned}$$

Now $Q_n(|z|) \in A_{co}$ implies $zQ_n(|z|) \in A_{co}$ and so $p_n(zQ_n(|z|) \in A_{co}$. We conclude that $f \in A_{co}$. Since $f|_{r\mathbb{T}} = H|_{r\mathbb{T}} = h$, we are done: $P(r\mathbb{T}) \subseteq A_{co}|_{r\mathbb{T}}$.

(4) Here we show that the spectrum of $M(A_{co})$ is homeomorphic to the cone (see Figure 1)

$$K := \big\{(x,y,t) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \le t, 0 \le t \le 1\big\}.$$

To this end, we first note that, with $B := C(\mathbf{D}, \mathbb{C})$,

$$\sigma_B\big(z, |z|\big) = \big\{(a_1, a_2) \in \mathbb{C}^2 : |a_1| \le 1, a_2 = |a_1|\big\}$$

because

$$\big(z - a, |z| - b\big) \in U_2\big(C(\mathbf{D}, \mathbb{C})\big)$$

if and only if the functions $z-a$ and $|z|-b$ have no common zeros in $\mathbf{D}$. Geometrically speaking, $S := \sigma_B(z,|z|)$ is the surface of the cone in Figure 1, without the upper basis $\{(w,1) \in \mathbb{C}\times\mathbb{R} : |w| < 1\}$; we call this the *outer surface* of K. By a general theorem in Banach algebras (see [16]), $\sigma_{A_{co}}(z,|z|)$ is now the polynomial convex hull $\widehat{S}$ of $S = \sigma_B(z,|z|)$, which we are going to determine below. Observe that K can be identified with the following compact subset of $\mathbb{C}^2$,

$$\tilde{C} := \big\{(z_1,z_2) \in \mathbb{C}^2 : |z_1| \le \operatorname{Re} z_2, 0 \le \operatorname{Re} z_2 \le 1, \operatorname{Im} z_2 = 0\big\},$$

and that $S \subseteq \tilde{C} \subseteq \mathbb{R}^3 \times \{0\}$, because

$$S = \big\{(z_1,z_2) \in \mathbb{C}^2 : |z_1| = z_2 = \operatorname{Re} z_2, 0 \le \operatorname{Re} z_2 \le 1, \operatorname{Im} z_2 = 0\big\}.$$

Fix $0 < t \le 1$. We first show that every disk $D_t := \{(w,t) \in \mathbb{C}^2 : |w| \le t\}$ is contained in $\widehat{S}$. To this end, fix $(w,t) \in D_t$ and consider any polynomial $p \in \mathbb{C}[z_1,z_2]$. Then

$$\begin{aligned}\big|p(w,t)\big| &\le \max\big\{\big|p(z_1,t)\big| : |z_1| \le t\big\} = \max\big\{\big|p(z_1,t)\big| : |z_1| = t\big\}\\ &\le \max\big\{\big|p(z_1,z_2)\big| : (z_1,z_2) \in S\big\}.\end{aligned}$$

Hence $(w,t) \in \widehat{S}$ and so $D_t \subseteq \widehat{S}$. Consequently,

$$S \subseteq \tilde{C} = \bigcup_{|t|\le 1} D_t \subseteq \widehat{S}.$$

Since K is a convex set in $\mathbb{R}^3$, $\tilde{C}$ is a convex compact set in $\mathbb{C}^2$, and so $\tilde{C}$ is polynomially convex. Thus $\widehat{S} = \tilde{C}$. We conclude that $(z-a, |z|-b)$ does not belong to $U_2(A_{co})$ if and only if $b \in [0,1]$ and $|a| \le b$.

(5) Let $m \in M(A_{co})$, and put $a := m(z)$ and $r := m(|z|)$. Then $(a,r) \in \sigma_{A_{co}}(z,|z|)$. Hence, by (4), $r \in [0,1]$, $a \in \mathbf{D}$, and $|a| \le r$. That is, $|m(z)| \le m(|z|)$. Let $f \in A_{co}$ and $p_n \in \mathbb{C}[z,w]$ be a sequence of polynomials such that $p_n(z,|z|)$ converges uniformly on $\mathbf{D}$ to $f(z)$. By (3), $f|_{r\mathbb{T}} \in P(r\mathbb{T})$. Now

$$\lim p_n(z,r) = f(z) \quad \text{(uniformly on } |z| = r). \tag{1.2}$$

By the maximum principle, $p_n(\xi,r)$ converges uniformly on $r\mathbf{D} = \{|\xi| \le r\}$ to a function $\check{f}$ with $\check{f} = f|_{r\mathbb{T}}$. Moreover,

$$\check{f}(w) = P\big[(f_r)|_{\mathbb{T}}\big](w/r) = P_r[f|_{r\mathbb{T}}](w) \quad \text{for } |w| < r.$$

On the other hand, because $m(z) = a$ and $m(|z|) = r$,

$$m(f) = \lim m(p_n) = \lim p_n(a,r).$$

Hence, if $|a| < r$, we conclude that $m(f) = \check{f}(a)$. In other words, $m = \psi_{r,a}$. If $|a| = r$, then this limit $m(f)$ coincides with $f(a)$ by (1.2); that is, $m = \delta_a$.

It remains to show the converse; that is, $\psi_{r,a} = \delta_a \in M(A_{co})$ when $|a| = r$ (which is clear) and $\psi_{r,a} \in M(A_{co})$ for every $(a,r) \in K$ with $|a| < r$ and $0 < r \le 1$ (see Figure 2). To this end, let $T_r : A_{co} \to A(\mathbf{D})$ be the map given by $T_r(f) = P[(f_r)|_{\mathbb{T}}]$. Since $(f_r)|_{\mathbb{T}} \in P(\mathbb{T})$ and since the Poisson operator is multiplicative on $P(\mathbb{T})$, we deduce that T_r is an algebra homomorphism. Hence, for every $\xi \in \mathbb{D}$, the map m given by $m := \delta_\xi \circ T_r$ is a homomorphism of A_{co} into

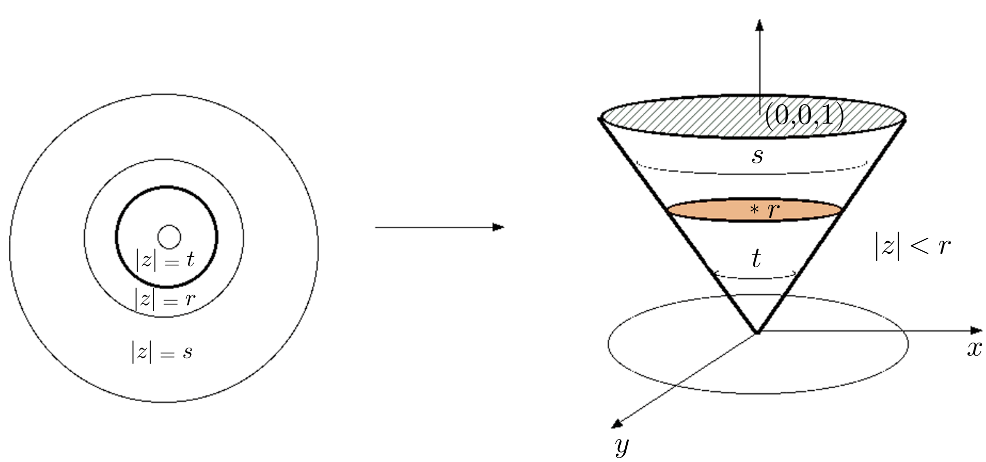

FIGURE 2. Functionals on the disk correspond to functionals on the surface of the cone.

$\mathbb{C}$. Since $m(\mathbf{1}) = 1$, we conclude that $m \in M(A_{co})$. If $\xi \in \mathbb{D}$ is chosen so that $a = r\xi$, then $m = \psi_{r,a}$.

(6) From (5) we conclude that the Gelfand transform $\widehat{f}$ of f has the following properties: $\widehat{f}(0 + i0, 0) = f(0)$ and, if $0 < r \leq 1$, then $\widehat{f}(w, r) = f(w)$ whenever $|w| = r$, $w \in \mathbb{C}$, and $\widehat{f}(w, r) = P_r[f|_{r\mathbb{T}}](w)$ whenever $|w| < r$. Since $f|_{r\mathbb{T}} \in P(r\mathbb{T})$ is the boundary function of a holomorphic function on $|w| < r$, we conclude from (3) that $\widehat{f}(\cdot, r) \in A(r\mathbf{D})$. Hence

$$\widehat{A}_{co} \subseteq \mathcal{A} = \big\{f \in C(K, \mathbb{C}) : f(\cdot, r) \in A(r\mathbf{D}) \ \forall r \in]0, 1]\big\}.$$

Next we observe that $\widehat{z} = z$ and $\widehat{|z|} = t$ on $K = \{(z, t) \in \mathbf{D} \times [0, 1], |z| \leq t\}$. Hence, if $p \in \mathbb{C}[z, w]$, then with $P(z) := p(z, |z|)$, $z \in \mathbf{D}$, we see that

$$\widehat{P}(z, r) = p(z, r).$$

Since A_{co} is a uniform algebra, $(\widehat{A}_{co}, \|\cdot\|_{M(A)})$ is isomorphic isometric to A_{co}. Thus $\widehat{A}_{co}$ is the closure $\mathcal{P}(K)$ of the polynomials of the form $p(z, r)$ within $C(K, \mathbb{C})$. That $\mathcal{P}(K) = \mathcal{A}$ follows immediately from Bishop's antisymmetric decomposition theorem [10, p. 60] and the fact that the maximal antisymmetric sets for $\mathcal{P}(K)$ as well as $\mathcal{A}$[5] are the disks

$$D_t = \big\{(w, t) \in \mathbf{D} : |w| \leq t\big\},$$

with $0 < t \leq 1$ and the singleton $\{(0 + i0, 0)\}$. An elementary proof can be given along the same lines as in Theorem 1.4.

(7) Suppose that $f \in A_{co}$ has no zeros on $\mathbf{D}$. By a theorem of Borsuk (see [3, p. 99]) f has a continuous logarithm on $\mathbf{D}$, say, $f = e^g$ for some $g \in C(\mathbf{D}, \mathbb{C})$. Let

$$N := \operatorname{ind}(f_r)|_{\mathbb{T}} = n\big(\widehat{f}(\cdot, r), 0\big)$$

be the index (or winding number) of $h := (f_r)|_{\mathbb{T}}$ (see [3, p. 84]). Note that h has no zeros on $\mathbb{T}$. Hence, N is well defined. This number, though, coincides with the number of zeros of the holomorphic function $\widehat{f}(\cdot, r)$ in $r\mathbb{D}$. But

$$\operatorname{ind}(f|_{r\mathbb{T}}) = \operatorname{ind}(e^g|_{r\mathbb{T}}) = \operatorname{ind}(e^{g_r}|_{\mathbb{T}}) = 0.$$

Thus the Gelfand transform $\widehat{f}$ of f does not vanish on $M(A)$. Hence f is invertible in A_{co}. In other words, property (C1) is satisfied.

Next we show that (C2) is not satisfied. In fact, consider the pair $(z, 1 - |z|^2)$. Although this pair is invertible in $C(\mathbf{D}, \mathbb{C})$, it is not invertible in A_{co}. To see this, we assume the contrary. Thus, there exist $a, b \in A_{co}$ such that

$$a(z)z + b(z)\big(1 - |z|^2\big) = 1.$$

In particular, $a(z)z = 1$ for $|z| = 1$. In other words, $a(z) = \overline{z}$. But by (2), $A_{co}|_{\mathbb{T}} = P(\mathbb{T})$. Since $\overline{z} \notin P(\mathbb{T})$ (otherwise $P(\mathbb{T})$ would coincide with $C(\mathbb{T}, \mathbb{C})$), we have obtained a contradiction. Thus we have found a pair (f, g) of functions in A_{co} without common zeros on $\mathbf{D}$ but for which $(f, g) \notin U_2(A_{co})$. This implies that A_{co} does not have property (Cn) for any $n \geq 2$.

[5]Recall that a closed subset E of K is said to be a *set of antisymmetry* for a function algebra $A \subseteq C(K, \mathbb{C})$ if every function in A which is real valued on E is already constant on E.

Here is another way to see that $(z, 1-|z|^2)$ is not in $U_2(A_{co})$. Let $\psi_{1,0} \in M(A)$ be the functional in (5), where $r = 1$ and $a = 0$. Then $\psi_{1,0}(f) = P[f|_{\mathbb{T}}](0)$. But if $f(z) = 1 - |z|^2$, then $f \equiv 0$ on $\mathbb{T}$ and so $\psi_{1,0}(1 - |z|^2) = 0$. But $\psi_{1,0}(z) = 0$, too, since $P[e^{it}](w) = w$ for every $w \in \mathbb{D}$. Thus z and $1 - |z|^2$ both belong to the kernel of a multiplicative linear functional on A_{co}.

(8) We first determine the Bear–Shilov boundary of A_{co}. To this end, it is sufficient to show that every $a \in \mathbf{D}$ is a peak point for A_{co}. So let $a \in \mathbf{D}$. If $a = 0$, then we take the peak functions $f(z) = \frac{1}{1+|z|}$ or $f(z) = 1 - |z|$. If $a \neq 0$, then we first choose a peak function $p(x) \in \mathbb{C}([0,1], \mathbb{R}^+)$ with $p(x) \leq Cx$, $p(|a|) = 1$ and $0 \leq p(x) < 1$ for $x \in [0,1] \setminus \{|a|\}$. Now let

$$f(z) := \begin{cases} (1 + \frac{z}{|z|} e^{-i \arg a}) \frac{p(|z|)}{2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then $f \in C(\mathbf{D}, \mathbb{C})$, $|f| \leq 1$, and if $|z| \neq |a|$, then $|f(z)| \leq p(|z|) < 1$. If $|z| = |a|$, then, with $z = |a|e^{it}$,

$$\Big|1 + \frac{z}{|z|} e^{-i \arg a}\Big| = |1 + e^{i(t - \arg a)}| < 2 \quad \text{if } t \neq \arg a \mod 2\pi.$$

Hence $|f(z)| < 1$ for all $z \in \mathbf{D} \setminus \{a\}$.

It remains to show that $f \in A_{co}$. To this end, it suffices to prove that $zp(|z|)/|z| \in A_{co}$. According to the Weierstrass theorem, let $P_n \in \mathbb{C}[x]$ be a sequence of polynomials uniformly converging on $[0,1]$ to $p(x)$. Then $P_n(|z|)$ converges uniformly to $p(|z|)$ on $\mathbf{D}$; since $P_n(|z|) \in A_{co}$, its limit $p(|z|) \in A_{co}$ does too. Now $\delta + |z| \in (A_{co})^{-1}$ for every $\delta > 0$ by Example 1.1. Hence $q_\delta(z) := zp(|z|)/(\delta + |z|) \in A_{co}$. But $\lim_{\delta \to 0} q_\delta(z) = zp(|z|)/|z|$ uniformly on $\mathbf{D}$ because

$$\Big|\frac{zp(|z|)}{\delta + |z|} - \frac{zp(|z|)}{|z|}\Big| = p\big(|z|\big) \frac{\delta}{\delta + |z|} \leq \frac{C|z|}{\delta + |z|} \delta \leq C\delta \to 0.$$

Thus we have shown that f is a peak function at a in A_{co}. Here is a different example. For $0 < |a| \leq 1$, let

$$f(z) = a + z e^{-\frac{|z|}{|a|}}.$$

Then $f \in A_{co}$, and f takes its maximum modulus in $\mathbf{D}$ only at $z = a$. In fact, since the function xe^{-x} takes its maximum on $[0, \infty[$ at $x = e^{-1}$,

$$\big|f(z)\big| \leq |a| + |z| e^{-\frac{|z|}{|a|}} = |a| \Big(1 + \frac{|z|}{|a|} e^{-\frac{|z|}{|a|}}\Big) \leq |a|(1 + e^{-1}),$$

with equality at $z = a$. Now the last inequality is strict for $|z| \neq |a|$. If $z = |a|e^{it}$ and $a = |a|e^{i \arg a}$, then

$$\big|f(z)\big| = |a||e^{i \arg a} + e^{it} e^{-1}| < |a|(1 + e^{-1}) \iff \arg a \neq t \mod 2\pi.$$

Hence f takes its maximum modulus only at a and so a is a peak point for A_{co}. We conclude that the Bear–Shilov boundary is $\mathbf{D}$. Moving to $\widehat{A}_{co}$, we get that

$$\widehat{f}(w, r) = \begin{cases} (1 + \frac{w}{r} e^{-i \arg a}) \frac{p(r)}{2} & \text{if } (w, r) \in K,\ r \neq 0, \\ 0 & \text{if } w = r = 0 \end{cases}$$

is a peak function at $(a, |a|) \in S \subseteq \partial K$. Since $\widehat{g}(\cdot, r) \in A(r\mathbf{D})$ for every $g \in A_{co}$, we just need to apply the maximum principle for holomorphic functions to conclude that no point in the interior of the cone and on its upper surface $\{(w, 1) : |w| < 1\}$ is a peak point for $\widehat{A}_{co}$. Hence the Shilov boundary coincides with the outer surface of the cone. $\square$

Results on the peak sets for A_{co} can be found in [11].

Theorem 1.4. *We have the following identity:*

$$\big[z, |z|\big]_{\text{alg}} = \big\{f \in C(\mathbf{D}, \mathbb{C}) : f|_{r\mathbb{T}} \in P(r\mathbb{T})\ \forall r \in]0, 1]\big\}.$$

Proof. (i) This follows from Bishop's antisymmetric decomposition theorem [10, p. 60] and the fact that the maximal antisymmetric sets for A_{co} are the circles $\{|z| = r\}$, $0 \le r \le 1$. We would like to present the following elementary proof, too.

(ii) Let $A^* = \{f \in C(\mathbf{D}, \mathbb{C}) : f|_{r\mathbb{T}} \in P(r\mathbb{T})\}$. We already know that $[z, |z|]_{\text{alg}} \subseteq A^*$. Observe that every $h \in P(r\mathbb{T})$ is the trace of a function H that is continuous on $\{|z| \le r\}$ and holomorphic in $\{|z| < r\}$. Hence H writes as $H(z) = \sum_{n=0}^{\infty} h_n z^n$, where h_n are the Taylor coefficients of H. They are given by the formula

$$\begin{aligned} h_n = \frac{1}{n!} H^{(n)}(0) &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{H(\zeta)}{\zeta^{n+1}}\, d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{h(re^{it})}{r^n} e^{-int}\, dt. \end{aligned}$$

The Fourier series associated with h then has the form

$$h(re^{it}) \sim \sum_{n=0}^{\infty} h_n r^n e^{int}.$$

Now fix $f \in A^*$. Since $f|_{r\mathbb{T}} \in P(r\mathbb{T})$, the preceding lines imply that, for every $0 < r \le 1$, the Fourier series of the family of functions f_r are[6] given by

$$f_r(e^{it}) = f(re^{it}) \sim \sum_{n=0}^{\infty} a_n(r) r^n e^{int}.$$

Here

$$r^n \big|a_n(r)\big| \le \big\|(f_r)|_{\mathbb{T}}\big\|_1 \le \|f\|_\infty, \tag{1.3}$$

where $\|\cdot\|_1$ is the L_1-norm on $\mathbb{T}$ and $\|\cdot\|_\infty$ is the supremum norm on $\mathbf{D}$, and

$$\lim_{r\to 0} r^n a_n(r) = 0 \quad \text{for every } n = 1, 2, \ldots \tag{1.4}$$

because

$$2\pi r^n a_n(r) = \int_0^{2\pi} f(re^{it}) e^{-int}\, dt \underset{r\to 0}{\to} f(0) \int_0^{2\pi} e^{-int}\, dt = 0.$$

[6]Note that the Fourier series $\sum_{n=0}^{\infty} b_n(r) e^{int}$ for $(f_r)|_{\mathbb{T}}$ would not be useful to our problem here, since at a later stage of the proof we really need the factor r^n.

Note also that the map $r \mapsto r^n a_n(r)$, $n \in \mathbb{N}$, is a continuous function on $[0,1]$; in fact, since f is uniformly continuous on $\mathbf{D}$,

$$2\pi\left|r^n a_n(r) - r'^n a_n(r')\right| = \left|\int_0^{2\pi} e^{-int}\big(f(re^{it}) - f(r'e^{it})\big)\,dt\right| \\ \leq \int_0^{2\pi}\left|f(re^{it}) - f(r'e^{it})\right|dt \leq 2\pi\varepsilon \tag{1.5}$$

if $|r - r'| < \delta$. Consider now, for the parameter $r \in]0,1]$ and $0 < \rho \leq 1$, the polynomial (in $z \in \mathbb{C}$)

$$p_N(z,r) = \sum_{n=0}^{N} a_n(r)\rho^n z^n.$$

We show that

$$\max_{|z|=r}\left|f(z) - p_N(z,r)\right| < \varepsilon \tag{1.6}$$

for suitably chosen ρ, ρ close to 1, and some $N \in \mathbb{N}$, with N and ρ independent of r.

Let us start the proof of (1.6). Since f is uniformly continuous on $\mathbf{D}$, we may choose $\eta \in]0,1]$, independent of r, so that $|f(re^{it}) - f(re^{i\theta})| < \varepsilon$ for $|t - \theta| < \eta$ and every $r \in [0,1]$. Fix $t \in [0, 2\pi[$. Let $I = I(t) \subseteq \mathbb{T}$ be the arc centered at t and with arc length 2η. Then

$$\begin{aligned}
&\left|f_r(e^{it}) - P\big[(f_r)|_{\mathbb{T}}\big](\rho e^{it})\right| \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{1-|\rho|^2}{|e^{i\theta} - \rho e^{it}|^2}\big(f_r(e^{it}) - f_r(e^{i\theta})\big)\,d\theta\right| \\
&\leq \frac{2\|f\|_\infty}{2\pi}\int_{\{\theta: e^{i\theta}\in\mathbb{T}\setminus I\}}\frac{1-|\rho|^2}{|e^{i\theta} - \rho e^{it}|^2}\,d\theta + \frac{\varepsilon}{2\pi}\int_{\{\theta: e^{i\theta}\in I\}}\frac{1-|\rho|^2}{|e^{i\theta} - \rho e^{it}|^2}\,d\theta.
\end{aligned}$$

Note that the second integral is less than ε, because the integral $\int_0^{2\pi} P\,d\theta$ of the Poisson kernel is 1.

Since, for $\eta \leq |\theta - t| \leq \pi$ and ρ close to 1,

$$\begin{aligned}
|e^{i\theta} - \rho e^{it}| &= |e^{i(\theta-t)} - \rho| \geq |e^{i(\theta-t)} - 1| - (1-\rho) \\
&= 2\left|\sin\Big(\frac{\theta - t}{2}\Big)\right| - (1-\rho) \\
&\geq \frac{2\eta}{\pi} - (1-\rho) \geq \frac{\eta}{\pi},
\end{aligned}$$

we see that the first integral tends to 0 as $\rho \to 1$. Hence, for all t,

$$\sup_{0<r\leq 1}\left|f_r(e^{it}) - P\big[(f_r)|_{\mathbb{T}}\big](\rho e^{it})\right| < c\varepsilon \tag{1.7}$$

for some ρ sufficiently close to 1 and independent of r. Now

$$\check{f}(t) := P\big[(f_r)|_{\mathbb{T}}\big](\rho e^{it}) = \sum_{n=0}^{\infty} a_n(r) r^n \rho^n e^{int}.$$

Because by (1.3)

$$L := \Big|\sum_{n=N+1}^{\infty} a_n(r) r^n \rho^n e^{int}\Big| \le \|f\|_\infty \sum_{n=N+1}^{\infty} \rho^n,$$

we see that $L < \varepsilon$ whenever N is sufficiently large. Note that N is independent of r. Hence

$$\Big|\check{f}(t) - \sum_{n=0}^{N} a_n(r) r^n \rho^n e^{int}\Big| < \varepsilon.$$

We conclude from (1.7) that

$$\Big|f_r(e^{it}) - \sum_{n=0}^{N} a_n(r) \rho^n r^n e^{int}\Big| < c\varepsilon + \varepsilon = \tilde{c}\varepsilon \tag{1.8}$$

for every $r \in]0,1]$. This proves our claim (1.6).

Now the coefficients $a_n(r)$ of the polynomial $p_N(z) := \sum_{n=0}^{N} a_n(r)\rho^n z^n$ are continuous functions for $r \in]0,1]$[7] and $a_0(r)$ is continuous on $[0,1]$. In order to be able to use the Weierstrass approximation theorem, we need to modify the $a_n(r)$ a little bit near the origin for $n \neq 0$ by multiplying them with r^N, r close to 0. According to (1.4), for $\varepsilon > 0$, there exists $\delta > 0$ such that, for $0 \le r \le \delta$,

$$\sum_{n=1}^{N} \big|a_n(r) r^n\big| < \varepsilon/2.$$

Let $\kappa \in C([0,1],[0,1])$ be defined as $\kappa(r) = r^N$ whenever $0 \le r \le \delta/2$ and $\kappa(r) = 1$ for $\delta \le r \le 1$. Consider the functions

$$p_N^*(z) := a_0(r) + \sum_{n=1}^{N} \kappa(r) a_n(r) \rho^n z^n.$$

Note that the new coefficients, $\kappa(r)a_n(r)$, are continuous on $[0,1]$ due to (1.4) and (1.5). Now, for $\delta \le r \le 1$ and $|z| = r$,

$$\big|p_N(z) - p_N^*(z)\big| \le \big|1 - \kappa(r)\big| \Big|\sum_{n=1}^{N} a_n(r) \rho^n z^n\Big| = 0.$$

If $0 \le r \le \delta$ and $|z| = r$, then

$$\begin{aligned}\big|p_N(z) - p_N^*(z)\big| &= \big|1 - \kappa(r)\big| \Big|\sum_{n=1}^{N} \big(a_n(r) r^n\big) \rho^n e^{int}\Big| \\ &\le 2 \sum_{n=1}^{N} \big|a_n(r)\big| r^n \le \varepsilon.\end{aligned}$$

[7]Note that, in general, $a_n(r)$ is not continuous at $r = 0$ for $n \ge 0$, as the function $f(z) = z/|\sqrt{|z|} = z/\sqrt{r} \in A_{co}$ shows.

Hence p_N^* is uniformly close to p_N. Now, for every $n \geq 1$, there is a polynomial $q_n(r) := \sum_{j=0}^{M(n)} b_j(n) r^j$ such that

$$\max_{0 \leq r \leq 1} \left| q_n(r) - \kappa(r) a_n(r) \right| < \varepsilon/(N+1).$$

For $n = 0$ we choose $q_0 \in \mathbb{C}[x]$ such that $|a_0(r) - q_0(r)| < \varepsilon/(N+1)$ on $[0,1]$. Consequently, with $z = re^{it}$,

$$\begin{aligned} &\left| f(re^{it}) - \sum_{n=0}^{N} q_n(r) \rho^n r^n e^{int} \right| \\ &\leq \left| f(re^{it}) - \sum_{n=0}^{N} a_n(r) \rho^n r^n e^{int} \right| + \left| p_N(z) - p_N^*(z) \right| + \sum_{n=0}^{N} \varepsilon/(N+1) \\ &\overset{(1.8)}{\leq} \tilde{c}\varepsilon + \varepsilon + \sum_{n=0}^{N} \varepsilon/(N+1) = c^* \varepsilon. \end{aligned}$$

In other words, for any $z \in \mathbf{D}$,

$$\left| f(z) - \sum_{n=0}^{N} q_n\big(|z|\big) \rho^n z^n \right| < c^* \varepsilon.$$

Thus $A^* = A_{co}$. We also deduce that

$$\widehat{A}_{co} = \big\{ f \in C(K, \mathbb{C}) : f(\cdot, r) \in A(r\mathbf{D}) \ \forall r \in]0,1] \big\}. \qquad \square$$

2. The stable ranks of the cone algebra

The following concepts were originally introduced by H. Bass [1] and M. Rieffel [14].

Definition 2.1. Let A be a commutative unital Banach algebra over $\mathbb{R}$ or $\mathbb{C}$.

(1) An $(n+1)$-tuple $(f_1, \dots, f_n, g) \in U_{n+1}(A)$ is called *reducible* (in A) if there exists $(a_1, \dots, a_n) \in A^n$ such that $(f_1 + a_1 g, \dots, f_n + a_n g) \in U_n(A)$.

(2) The *Bass stable rank* of A, denoted by $\operatorname{bsr} A$, is the smallest integer n such that every element in $U_{n+1}(A)$ is reducible. If no such n exists, then $\operatorname{bsr} A = \infty$.

(3) The *topological stable rank* of A, $\operatorname{tsr} A$, is the least integer n for which $U_n(A)$ is dense in A^n, or infinite if no such n exists.

We refer the reader to the work of L. Vasershtein [17]; G. Corach and A. Larotonda [4], [5]; G. Corach and D. Suárez [6], [7], [8], [9]; and the authors dealing with numerous aspects of these notions in the realm of function algebras. The computation of the stable rank of our algebras above will be based on the following three results from a higher analysis course.

Theorem (A). *Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$ be a map. Suppose that $E \subseteq U$ has n-dimensional Lebesgue measure zero. Let $0 < \alpha \leq 1$. Then, under each of the following conditions, $f(E)$ has n-dimensional Lebesgue measure zero, too:*

(1) *f satisfies a Hölder–Lipschitz condition (of order α) on U; that is, there is $M > 0$ such that*

$$\big\|f(x) - f(y)\big\| \leq M\|x - y\|^\alpha \quad \text{for every } x, y \in U.$$

(2) $f \in C^1(U, \mathbb{R}^n)$.

A proof of the following version of Rouché's theorem (continuous-holomorphic pairs) is in [13, Theorem 20].

Theorem (B). *Let $K \subseteq \mathbb{C}$ be compact, and let $f \in C(K, \mathbb{C})$ and $g \in A(K)$, where $A(K)$ is the set of all functions continuous on K and holomorphic in the interior K° of K. Suppose that, on ∂K,*

$$|f + g| < |f| + |g|.$$

Then f has a zero on K° whenever g has a zero on K°. The converse does not hold, in general.

Theorem (C) ([3, p. 97])**.** *Let $K \subseteq \mathbb{C}$ be compact, and let C be a bounded component of $\mathbb{C} \setminus K$ and $\beta \in C$. Then the function $f(z) = z - \beta$ defined on K is zero-free on K but does not admit a zero-free extension to $K \cup C$.*

Theorem 2.2. *We have* $\operatorname{bsr} A_{co} = 2$ *and* $\operatorname{tsr} A_{co} = 2$.

Proof. We first show that $\operatorname{tsr} A_{co} \leq 2$. Let $(f, g) \in (A_{co})^2$. Choose polynomials $p(z, w)$ and $q(z, w)$ in $\mathbb{C}[z, w]$ such that

$$\Big|f(z) - p\big(z, |z|\big)\Big| + \Big|g(z) - q\big(z, |z|\big)\Big| < \varepsilon \quad \text{for every } z \in \mathbf{D}.$$

Let $P(z) = p(z, |z|)$ and $Q(z) = q(z, |z|)$. By the proof of assertion (6) of Theorem 1.3, the Gelfand transforms of P and Q are polynomials, too, such that

$$\widehat{P}(z, r) = p(z, r) \qquad \text{and} \qquad \widehat{Q}(z, r) = q(z, r).$$

We shall now use Theorem (A). To this end, we observe that the functions $\widehat{P}$ and $\widehat{Q}$ satisfy a Lipschitz condition on K. Let

$$\tilde{K} = \Big\{(x, y, t, v) \in \mathbb{R}^4 : v = 0, 0 \leq t \leq 1, \sqrt{x^2 + y^2} \leq t\Big\},$$

which is of course nothing else than our cone K, respectively, $\tilde{C}$ (but viewed as a set in $\mathbb{R}^4$). Then $\tilde{K}$ has 4-dimensional Lebesgue measure zero. Now we look at the map

$$\mu : \begin{cases} \tilde{K} \subseteq \mathbb{R}^4 \to \mathbb{R}^4, \\ (x, y, t, v) \mapsto (\operatorname{Re} p(x + iy, t), \operatorname{Im} p(x + iy, t), \operatorname{Re} q(x + iy, t), \operatorname{Im} q(x + iy, t)). \end{cases}$$

Then μ satisfies a Lipschitz condition on $\tilde{K}$, too. Hence, by Theorem (A), $\mu(\tilde{K})$ has measure zero in $\mathbb{R}^4$. Thus there exists a null sequence $(\varepsilon_n, \varepsilon'_n)$ in $\mathbb{C}^2$ such that $p(z, t) - \varepsilon_n$ and $q(z, t) - \varepsilon'_n$ have no common zero on

$$K = \big\{(z, t) \in \mathbf{D} \times [0, 1] : |z| \leq t\big\}.$$

Since $K = M(\widehat{A}_{co}) \sim M(A_{co})$, these pairs are invertible in $\widehat{A}_{co}$ and so

$$\big(p\big(z, |z|\big) - \varepsilon_n, q\big(z, |z|\big) - \varepsilon'_n\big) \in U_2(A_{co}).$$

Hence $\operatorname{tsr} A_{co} \leq 2$.

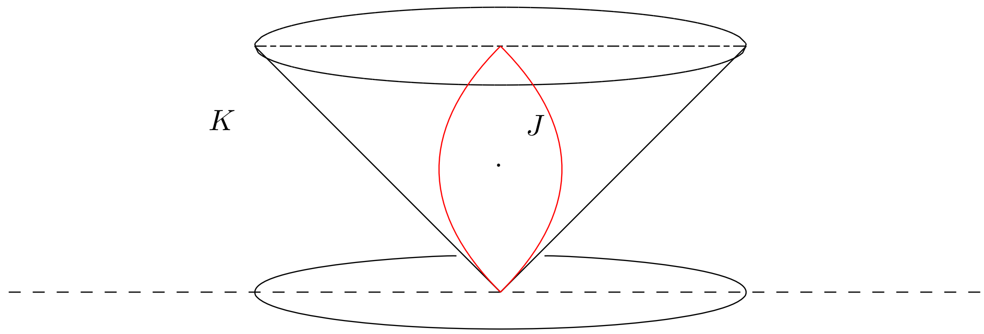

FIGURE 3. The spectrum of the cone algebra and $Z(\widehat{g}) = J$.

Next we prove that tsr $A_{co} \geq 2$. Let $f(z) = z$. If we suppose that there exists $u \in (A_{co})^{-1}$ such that $\|u - z\|_\infty < 1/2$, then, on $\mathbb{T}$,

$$\big|u(z) - z\big| < \frac{1}{2} < 1 \leq |z| + \big|u(z)\big|.$$

Hence, by Rouché's Theorem (B), u has a zero in $\mathbf{D}$. Thus, u is not invertible in A_{co}. In sum, we showed that tsr $A_{co} = 2$.

Since A_{co} is a Banach algebra, we have $1 \leq \operatorname{bsr} A_{co} \leq \operatorname{tsr} A_{co} \leq 2$. It remains to prove that bsr $A_{co} \geq 2$. The idea is to unveil a function $g \in A_{co}$ such that the zero set of $\widehat{g}$ on

$$\begin{aligned} K &= \big\{(z,t) \in \mathbb{C} \times [0,1] : |z| \leq t, 0 \leq t \leq 1\big\} \\ &= \big\{(x,y,t) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq t, 0 \leq t \leq 1\big\} \end{aligned}$$

is a Jordan curve J contained in the plane $y = 0$ (see Figure 3) and a function $f \in A_{co}$ satisfying $Z(\widehat{f}) \cap Z(\widehat{g}) = \emptyset$ such that $\widehat{f}$ is a translation of the identity map on J.

So let

$$f(z) = z + i\Big(|z| - \frac{1}{2}\Big) \qquad \text{and} \qquad g(z) = z^2 - |z|^2\big(1-|z|\big)^2.$$

Then f and g belong to A_{co}. Their Gelfand transforms are given by

$$\widehat{f}(z,t) = z + i\Big(t - \frac{1}{2}\Big) \qquad \text{and} \qquad \widehat{g}(z,t) = z^2 - t^2(1-t)^2.$$

Since it is more convenient to work with $\mathbb{R}^2$-valued functions (instead of $\mathbb{C}$-valued ones) when they are defined on K (K viewed as a subset of $\mathbb{R}^3$ instead of $\mathbb{C}\times\mathbb{R}$), we put

$$F(x,y,t) := \big(\operatorname{Re}\widehat{f}(x+iy,t), \operatorname{Im}\widehat{f}(x+iy,t)\big)$$

and deduce the following representation of the zero set of $\widehat{g}$ and the associated action of $\widehat{f}$:

$$Z(\widehat{g}) = \big\{\big(\pm t(1-t), t\big) \in \mathbb{C}\times\mathbb{R} : 0 \leq t \leq 1\big\} = \big\{\big(\pm t(1-t), 0, t\big) \in \mathbb{R}^3 : 0 \leq t \leq 1\big\},$$

and

$$\begin{aligned} F\big(\pm t(1-t), 0, t\big) &= \big(\operatorname{Re}\widehat{f}\big(\pm t(1-t), t\big), \operatorname{Im}\widehat{f}\big(\pm t(1-t), t\big)\big) \\ &= \Big(\pm t(1-t), t - \frac{1}{2}\Big). \end{aligned}$$

Then $J := Z(\widehat{g})$ is a Jordan curve contained in K and J does not meet $Z(\widehat{f})$. Hence $(f,g) \in U_2(A_{co})$. Moreover, we see that $F|_J$ is a translation map; in fact, using complex coordinates in the plane $\{(x,y,t) \in \mathbb{R}^3 : y = 0\}$, and putting $w = x + it$, the action of F on J can be written as $\tilde{F}(w) = w - i/2$, because with $x = \pm t(1-t)$, $\tilde{F}(x+it) = x + i(t - 1/2) = w - i/2$.

In view of achieving a contradiction, suppose now that (f,g) is reducible in A_{co}. Then

$$\widehat{u} := \widehat{f} + \widehat{a}\,\widehat{g}$$

is a zero-free function on K for some $a, u \in A_{co}$. Restricting $\widehat{f}$ to $Z(\widehat{g})$, we find that the translated identity mapping on the Jordan curve $Z(\widehat{g})$ has a zero-free extension to the interior of that curve in the plane $y = 0$. Since $(0,0,1/2)$ is surrounded by that curve, we get a contradiction to Theorem (C). We conclude that the pair (f,g) is not reducible in A_{co} and so bsr $A_{co} \geq 2$. Putting all together, bsr $A_{co} = 2$. $\square$

3. The cylinder algebra

Suppose that $\{(f_t, g_t) : t \in [0,1]\}$ is a family of functions in $A(\mathbf{D})$ such that

$$Z(f_t) \cap Z(g_t) = \emptyset$$

for every t. By the Nullstellensatz for the disk algebra, for each parameter t, there is a solution $(x_t, y_t) \in A(\mathbf{D})^2$ to the Bézout equation $x_t f_t + y_t g_t = 1$. If the family $\{(f_t, g_t) : t \in [0,1]\}$ depends continuously on t, do there exist solutions to the Bézout equation that also depend continuously on t? This problem has an affirmative answer and is best described by introducing the *cylinder algebra*:

$$\mathrm{Cyl}(\mathbb{D}) = \big\{ f \in C\big(\mathbf{D} \times [0,1], \mathbb{C}\big) : f(\cdot, t) \in A(\mathbf{D}) \text{ for all } t \in [0,1] \big\}.$$

Proposition 3.1. *Let* $\mathrm{Cyl}(\mathbb{D})$ *be the cylinder algebra. Then*

(1) $\mathrm{Cyl}(\mathbb{D})$ *is a uniformly closed subalgebra of* $C(\mathbf{D} \times [0,1], \mathbb{C})$.

(2) *The set* $\mathbb{C}[z,t]$ *of polynomials of the form*

$$\sum_{j,k=0}^{N} a_{j,k} z^j t^k, \quad a_{j,k} \in \mathbb{C}, N \in \mathbb{N}$$

is dense in $\mathrm{Cyl}(\mathbb{D})$.

(3) $M(\mathrm{Cyl}(\mathbb{D})) = \{\delta_{(a,t)} : (a,t) \in \mathbf{D} \times [0,1]\}$, *where*

$$\delta_{(a,t)} : \begin{cases} \mathrm{Cyl}(\mathbb{D}) \to \mathbb{C}, \\ f \mapsto f(a,t). \end{cases}$$

(4) *An ideal* M *in* $\mathrm{Cyl}(\mathbb{D})$ *is maximal if and only if*

$$M = M(z_0, t_0) := \big\{ f \in \mathrm{Cyl}(\mathbb{D}) : f(z_0, t_0) = 0 \big\}$$

for some $(z_0, t_0) \in \mathbf{D} \times [0,1]$. *In particular,* $\mathrm{Cyl}(\mathbb{D})$ *is natural on* $\mathbf{D} \times [0,1]$.

(5) *Let* $f_j \in \mathrm{Cyl}(\mathbb{D})$, $j = 1, \ldots, n$. *Then the Bézout equation* $\sum_{j=1}^n x_j f_j = 1$ *admits a solution in* $\mathrm{Cyl}(\mathbb{D})$ *if and only if the functions* f_j *do not have a common zero on the cylinder* $\mathbf{D} \times [0,1]$.

Proof. (1) This is clear.

(2) Let $f \in \mathrm{Cyl}(\mathbf{D})$. Then, for every fixed $t \in [0,1]$, $f(\cdot,t) \in A(\mathbf{D})$ and so $f(\cdot,t)$ admits a Taylor series $\sum_{n=0}^{\infty} a_n(t) z^n$, where the Taylor coefficients are given by

$$a_n(t) = \frac{1}{n!}\frac{\partial^n f}{\partial^n z}(0,t) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi,t)}{\xi^{n+1}}\,d\xi.$$

The uniform continuity of f on $\mathbf{D}\times[0,1]$ now implies that $t \mapsto a_n(t)$ is a continuous function on $[0,1]$ because

$$\begin{aligned}\big|a_n(t)-a_n(s)\big| &\le \frac{1}{2\pi}\int_{|\xi|=1}\frac{|f(\xi,t)-f(\xi,s)|}{|\xi|^{n+1}}|d\xi| \\ &\le \varepsilon \quad \text{for } |s-t|<\delta.\end{aligned}$$

In particular, $|a_n(t)| \le \|f\|_\infty$ for all $t\in[0,1]$ and $n\in\mathbb{N}$. The Weierstrass theorem now yields polynomials $p_n \in \mathbb{C}[t]$ such that

$$\big|p_n(t)-a_n(t)\big| < \varepsilon 2^{-n} \quad \text{for every } t\in[0,1].$$

We claim that, for $\rho \in]0,1[$ sufficiently close to 1 and N sufficiently large, the polynomial q given by

$$q(z,t) = \sum_{n=0}^{N} p_n(t)\rho^n z^n$$

is uniformly close to $f(z,t)$. In fact, due to uniform continuity again, we may choose $\rho \in]0,1[$ so that $|f(z,t)-f(\rho z,t)| < \varepsilon$ for every $(z,t)\in\mathbf{D}\times[0,1]$. Hence

$$\begin{aligned}\big|f(z,t)-q(z,t)\big| &\le \big|f(z,t)-f(\rho z,t)\big| + \big|f(\rho z,t)-q(z,t)\big| \\ &\le \varepsilon + \sum_{n=0}^{N}\big|p_n(t)-a_n(t)\big|\rho^n|z|^n + \sum_{n=N+1}^{\infty}\big|a_n(t)\big|\rho^n|z|^n \\ &\le \varepsilon + \varepsilon\sum_{n=0}^{N}2^{-n} + \|f\|_\infty\sum_{n=N+1}^{\infty}\rho^n \\ &\le 3\varepsilon + \|f\|_\infty\frac{\rho^{N+1}}{1-\rho} \le 4\varepsilon\end{aligned}$$

for N large.

(3) Let $m \in M(\mathrm{Cyl}(\mathbb{D}))$, and denote by $\boldsymbol{c}$ the coordinate function $\boldsymbol{c}(z,t) := z$ and by $\boldsymbol{r}$ the coordinate function $\boldsymbol{r}(z,t)=t$. Note that $\boldsymbol{c},\boldsymbol{r}\in\mathrm{Cyl}(\mathbb{D})$. Let

$$(z_0,t_0) := \big(m(\boldsymbol{c}), m(\boldsymbol{r})\big).$$

Then $z_0 \in \mathbf{D}$ because $|m(\boldsymbol{c})| \le \|\boldsymbol{c}\|_\infty = 1$. Now $t_0 \in \sigma(\boldsymbol{r})$, the spectrum of $\boldsymbol{r}$ in $\mathrm{Cyl}(\mathbb{D})$. Because for $\lambda\in\mathbb{C}$ the function $\boldsymbol{r}-\lambda \in \mathrm{Cyl}(\mathbb{D})^{-1}$ if and only if $\lambda \notin [0,1]$, we see that $t_0 = m(\boldsymbol{r}) \in [0,1]$. Consequently, $(z_0,t_0) \in \mathbf{D}\times[0,1]$.

Given $f\in\mathrm{Cyl}(\mathbb{D})$, let (p_n) be a sequence of polynomials in $\mathbb{C}[z,t]$ converging uniformly on $\mathbf{D}\times[0,1]$ to f. Then

$$m(p_n) = p_n(z_0,t_0) \to f(z_0,t_0).$$

Hence $m = \delta_{(z_0,t_0)}$.

(4) and (5) These assertions follow from Gelfand's theory. $\square$

Recall that the cylinder algebra was defined as

$$\mathrm{Cyl}(\mathbb{D}) = \big\{f \in C\big(\mathbf{D}\times[0,1],\mathbb{C}\big) : f(\cdot,t) \in A(\mathbf{D})\big\}.$$

For technical reasons, we let t vary now in the interval $[-1,1]$. In this subsection we determine the Bass and topological stable ranks of $\mathrm{Cyl}(\mathbb{D})$.[8] The original question that led us to consider this algebra was the following: Let

$$\mathscr{F} := \big\{(f_t, g_t) : t \in [-1,1]\big\}$$

be a family of disk-algebra functions with $Z(f_t)\cap Z(g_t) = \emptyset$. Then, by the Jones–Marshall–Wolff theorem (see [12]), for each parameter t, there is $(u_t, y_t) \in A(\mathbf{D})^2$, u_t invertible, such that $u_t f_t + y_t g_t = 1$. If the family $\mathscr{F}$ depends continuously on t, do there exist solutions to this type of the Bézout equation that also depend continuously on t? Quite surprisingly, this is no longer the case. This stays in contrast to the unrestricted Bézout equation $x_t f_t + y_t g_t = 1$ dealt with in Proposition 3.1. Here is the outcome.

Theorem 3.2. *If* $\mathrm{Cyl}(\mathbb{D})$ *is the cylinder algebra, then* $\operatorname{bsr}\mathrm{Cyl}(\mathbb{D}) = \operatorname{tsr}\mathrm{Cyl}(\mathbb{D}) = 2$.

Proof. We first show that $\operatorname{bsr}\mathrm{Cyl}(\mathbb{D}) \geq 2$. Let $f(z,t) = z + it$ and $g(z,t) = z^2 - (1-t^2)$. Then $(f,g) \in U_2(\mathrm{Cyl}(\mathbb{D}))$ because

$$(z+it)(z-it) - g(z,t) = 1.$$

Suppose that (f,g) is reducible. Then $u := f + ag$ is a zero-free function on $\mathbf{D}\times[-1,1]$ for some $a \in \mathrm{Cyl}(\mathbb{D})$. Now the zero set

$$Z(g) = \big\{(\pm\sqrt{1-t^2}, t) : -1 \leq t \leq 1\big\} = \big\{(x,y,t) \in \mathbb{R}^3 : y = 0, x^2 + t^2 = 1\big\}$$

is a (vertical) circle (Figure 4). Restricting f to $Z(g)$ and using complex coordinates w on the disk D formed by $Z(g)$, we obtain, with $w = \pm\sqrt{1-t^2} + it$, that

$$F(w) := f(\pm\sqrt{1-t^2}, t) = \pm\sqrt{1-t^2} + it = w.$$

Thus f is the identity mapping on the circle $Z(g)$ and $u|_D$ is a zero-free extension of $f|_{Z(g)}$. This is a contradiction to Theorem (C).

Next we prove that $\operatorname{tsr}\mathrm{Cyl}(\mathbb{D}) \leq 2$. Let $(f,g) \in \mathrm{Cyl}(\mathbb{D})^2$. According to Proposition 3.1, let $\boldsymbol{F} := (p,q) \in (\mathbb{C}[z,t])^2$ be chosen so that

$$\|p - f\|_\infty + \|q - g\|_\infty < \varepsilon.$$

By Theorem (A), $\boldsymbol{F}(\mathbb{R}^3) \subseteq \mathbb{C}^2$ has 4-dimensional Lebesgue measure zero. Hence there is a null sequence (ε_n, η_n) in $\mathbb{C}^2$ such that

$$(\varepsilon_n, \eta_n) \notin \boldsymbol{F}(\mathbb{R}^3).$$

Consequently, the pairs

$$(p - \varepsilon_n, q - \eta_n)$$

[8] Corach and Suárez determined in [6, p. 5] the Bass stable rank of $C([0,1], A(\mathbb{D}))$, which coincides with $\mathrm{Cyl}(\mathbb{D})$, by using advanced methods from algebraic topology as well as the Arens–Royden theorem.

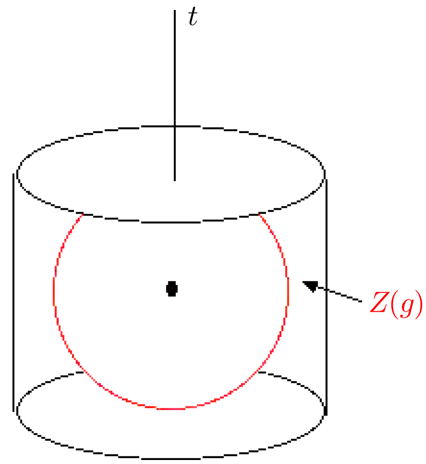

FIGURE 4. The spectrum of the cylinder algebra.

are invertible in $\mathrm{Cyl}(\mathbb{D})$ by Proposition 3.1(5). Thus $\operatorname{tsr}\mathrm{Cyl}(\mathbb{D}) \leq 2$.

Combining both facts, we deduce that $\operatorname{bsr}\mathrm{Cyl}(\mathbb{D}) = \operatorname{tsr}\mathrm{Cyl}(\mathbb{D}) = 2$. □

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