Ann. Funct. Anal. 10 (2019), no. 4, 562–569
https://doi.org/10.1215/20088752-2019-0014
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A SUFFICIENT CONDITION FOR C^1-SMOOTHNESS OF THE CONJUGATION BETWEEN PIECEWISE SMOOTH CIRCLE HOMEOMORPHISMS

H. AKHADKULOV,[1*] M. S. NOORANI,[2] H. IBRAHIM,[1] and S. AKHATKULOV[3]

Communicated by B. Ricceri

Abstract. Let T_1 and T_2 be two piecewise smooth circle homeomorphisms with countably many break points and identical irrational rotation number. We provide a sufficient condition for C^1-smoothness of the conjugation between T_1 and T_2.

1. Introduction

The first properties of circle homeomorphisms were studied in a classical work of Poincaré in [19]. Every circle homeomorphism $T : S^1 \to S^1$ is given as $T = \pi \circ L_T \circ \pi^{-1}$, where $\pi : \mathbb{R} \to S^1$ is the projection mapping that "winds" a straight line on the circle. The homeomorphism $L_T : \mathbb{R} \to \mathbb{R}$ with property $L_T(x + 1) = L_T(x) + 1$ is the lift of the homeomorphism T of the circle and is defined up to an integer term. The most important arithmetic characteristic of the homeomorphism T is the *rotation number*, which is defined as

$$\rho(T) = \lim_{i\to\infty} \frac{L_T^i(x)}{i} \mod 1,$$

where L_T is the lift of T with S^1 to $\mathbb{R}$ and F^i is ith iteration of F. Poincaré proved that the above limit exists, does not depend on the initial point $x \in \mathbb{R}$ of the lifted trajectory, and, up to addition of an integer, does not depend on the

Received Aug. 12, 2018; Accepted Feb. 23, 2019.
First published online Oct. 29, 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 37E10; Secondary 37C15, 37C40.
Keywords. circle homeomorphism, break point, rotation number, conjugation function.

lift L_T (see [9]). The rotation number $\rho = \rho(T)$ is irrational if and only if the homeomorphism T has no periodic point.

Denjoy [10] proved that if T is an orientation-preserving C^1-diffeomorphism with irrational rotation number ρ, and $\log DT$ is of bounded variation, then T is conjugate to the rigid rotation R_ρ—that is, there exists an essentially unique homeomorphism ψ of the circle such that $T = \psi^{-1} \circ R_\rho \circ \psi$, where DT is the derivative of T. The homeomorphism ψ is called the *conjugation map* between T and R_ρ. At the end of the 1970s the problem of smoothness of the conjugacy of smooth diffeomorphisms was one of the fundamental problems of the theory of circle maps. Nowadays, the problem of smoothness of the conjugacy of smooth diffeomorphisms has come to be very well understood due to several deep results obtained by Katznelson and Ornstein in [15] and [16] and by Sinaĭ and Khanin in [20]. Recently, a remarkable result in this direction was obtained by Akhadkulov, Dzhalilov, and Khanin [5]: it was shown that there exists a subset of irrational numbers of unbounded type such that every circle diffeomorphism satisfying a certain Zygmund condition is absolutely continuously conjugate to the linear rotation, provided that its rotation number belongs to the above set. Natural generalizations of diffeomorphisms are smooth homeomorphisms with breaks, the so-called $\mathcal{P}$*-homeomorphisms*.

Definition 1.1. A homeomorphism T of the circle is called a $\mathcal{P}$*-homeomorphism* if it satisfies the following conditions:

(i) T is differentiable away from countably many points $x_b \in BP(T)$, the so-called *break points* of T, with $BP(T)$ the set of break points of T on S^1, at which left and right derivatives, denoted respectively by DT_- and DT_+, exist, and
$$\frac{DT_-(x_b)}{DT_+(x_b)} \neq 1$$
for all $x_b \in BP(T)$;

(ii) there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 < DT(x) < c_2$ for all $x \in S^1 \backslash BP(T)$, $c_1 < DT_-(x_b) < c_2$, and $c_1 < DT_+(x_b) < c_2$ for all $x_b \in BP(T)$;

(iii) $\log DT$ has bounded variation.

The ratio $\sigma_T(c) := (DT_-(c))/(DT_+(c))$ is called the *jump* of T in c or the T*-jump*. The class of $\mathcal{P}$-homeomorphisms was introduced by Herman [14]. He investigated the invariant measures of piecewise linear circle homeomorphisms with two break points. The existence of the conjugation between R_ρ and a $\mathcal{P}$-homeomorphism with irrational rotation number ρ follows directly from Denjoy's theorem. Since the first paper [11], in which it was shown that the conjugation between R_ρ and a $\mathcal{P}$-homeomorphism with a single break and with irrational rotation number ρ is singular, there have appeared a number of publications proving the singularity of the conjugation measure in different cases (see, e.g., [2], [4]). In this direction, some remarkable results were obtained by two groups of scientists independently in [3] and [12], where the following most general result was shown to hold: if the product of the sizes of all breaks is a nonunit, then the conjugation between R_ρ and a $\mathcal{P}$-homeomorphism with a finite number of breaks is singular.

Next we consider the problem of the regularity of the conjugating map of two $\mathcal{P}$-homeomorphisms with identical irrational rotation numbers. In general, the renormalizations and also rigidity properties of $\mathcal{P}$-homeomorphisms are rather different from those of diffeomorphisms (see [6]–[8]). A remarkable achievement in this direction is due to Adouani [1] and Dzhalilov, Mayer, and Safarov [13], which was obtained independently. They proved that if two $\mathcal{P}$-homeomorphisms with a finite number of breaks with the same irrational rotation number have different products of sizes of breaks, then every conjugacy between them is a singular function. It is well known that for the conjugation map between two $\mathcal{P}$-homeomorphisms there are two possibilities: it is either absolutely continuous or singular. In the case of the absence of $\mathcal{D}$-property (for the definition see [3]), the problem of regularity (absolute continuity and smoothness) of the conjugation between two $\mathcal{P}$-homeomorphisms is one of the complicated problems of this direction. Recently, the problem of C^1-smoothness of the conjugacy between two $C^{2+\epsilon}$ smooth $\mathcal{P}$-homeomorphisms with one break point has been solved by Khanin, Kocić, and Mazzeo [17].

This problem remains open for $\mathcal{P}$-homeomorphisms with several break points. In this paper we provide a sufficient condition for the C^1-smoothness of the conjugating map of two $\mathcal{P}$-homeomorphisms with countably many break points. In order to formulate our main result, let us first recall some necessary notions and facts.

Henceforth, we will always assume that ρ is irrational and we will use its decomposition in an infinite continued fraction (see [18]).

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\cfrac{\cdots}{k_n + \cfrac{1}{\cdots}}}}} := [k_1, k_2, \ldots, k_n, \ldots]. \tag{1.1}$$

The value of a "countable-floor" fraction is the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \ldots, k_n]$. The positive integers $k_n, n \geq 1$ are called *incomplete multiples* and defined uniquely for irrational ρ. The mutually prime positive integers p_n and q_n satisfy the recurrent relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where it is convenient to define $p_{-1} = 0, q_{-1} = 1$ and $p_0 = 1, q_0 = k_1$. Given a circle homeomorphism T with irrational rotation number ρ, one may consider a marked trajectory (i.e., the trajectory of a marked point) $\xi_i = T^i \xi_0 \in S^1$, where $i \geq 0$, and pick out of it the sequence of the dynamical convergents $\xi_{q_n}, n \geq 0$ indexed by the denominators of consecutive rational convergents to ρ. We will also conventionally use $\xi_{q_{-1}} = \xi_0 - 1$. The well-understood arithmetical properties of rational convergents and the combinatorial equivalence between T and rigid rotation $R_\rho : \xi \to \xi + \rho \mod 1$ imply that the dynamical convergents approach the marked point, alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \cdots < \xi_{q_{2m+1}} < \cdots < \xi_{q_0} < \cdots < \xi_{q_{2m}} < \cdots < \xi_{q_2} < \xi_{q_0}.$$

We define the nth fundamental interval $\Delta^n(\xi_0)$ as the circle arc $[\xi_0, \xi_{q_n}]$ for even n and as $[\xi_{q_n}, \xi_0]$ for odd n. For the marked trajectory, we use the notation $\Delta_0^n = \Delta^n(\xi_0)$, $\Delta_i^n = \Delta^n(\xi_i) = T^i\Delta_0^n$. It is well known that the set $\mathbf{P}_n(\xi_0; T) = \mathbf{P}_n(T)$ of intervals with mutually disjoint interiors defined as

$$\mathbf{P}_n(T) = \{\Delta_i^{n-1}, 0 \le i < q_n; \Delta_j^n, 0 \le j < q_{n-1}\}$$

determines a partition of the circle S^1. The partition $\mathbf{P}_n(T)$ is called the *nth dynamical partition* of the circle. Obviously the partition $\mathbf{P}_{n+1}(T)$ is a refinement of the partition $\mathbf{P}_n(T)$—indeed, the intervals of order n are members of $\mathbf{P}_{n+1}(T)$ and each interval $\Delta_i^{n-1} \in \mathbf{P}_n(T), 0 \le i < q_n$ is partitioned into $k_{n+1} + 1$ intervals belonging to $\mathbf{P}_{n+1}(T)$ such that

$$\Delta_i^{n-1} = \Delta_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^n. \tag{1.2}$$

Let T_1 and T_2 be $\mathcal{P}$-homeomorphisms with identical irrational rotation number ρ, and let ψ be the conjugating homeomorphism between them. Consider dynamical partitions $\mathbf{P}_n(\xi, T_1) = \mathbf{P}_n(T_1)$ and $\mathbf{P}_n(\psi(\xi), T_2) = \mathbf{P}_n(T_2)$ appropriate to the homeomorphisms T_1 and T_2. Denote by $\widehat{\Delta}^n$ the intervals of the partition $\mathbf{P}_n(T_2)$. Since ψ is a conjugacy between T_1 and T_2, we have $\psi(\Delta^n) = \widehat{\Delta}^n$ for any $\Delta^n \in \mathbf{P}_n(T_1)$. Denote by $|A|$ the Lebesgue measure of the corresponding set of $A \subset S^1$. Our main result is the following theorem.

Theorem 1.2. *Let T_1 and T_2 be $\mathcal{P}$-homeomorphisms with identical irrational rotation number. If there exists a sequence (τ_n) such that $\sum_{n=1}^{\infty} \tau_n \le \infty$ and*

$$\left|\log \frac{|\widehat{\Delta}_1|}{|\Delta_1|} - \log \frac{|\widehat{\Delta}_2|}{|\Delta_2|}\right| \le \tau_n \tag{1.3}$$

for each pair of adjacent intervals $\Delta_1, \Delta_2 \in \mathbf{P}_n(T_1)$ or $\Delta_1, \Delta_2 \subset \Delta \in \mathbf{P}_{n-1}(T_1)$ for all $n > 1$, then the conjugation ψ between T_1 and T_2 is C^1 smooth.

2. Proof of main result

Here we first briefly describe the sketch of the proof of our main theorem. To prove the main theorem, we get a sequence of step functions defined on the circle and we show the convergence of this sequence to a continuous function. Then using the limit function, we construct a C^1 smooth conjugation function. The following inequalities will be used in the proof of the main theorem. For any $a, b, c, d > 0$, the following inequalities hold:

$$\min\left\{\frac{a}{b}, \frac{c}{d}\right\} \le \frac{a+c}{b+d} \le \max\left\{\frac{a}{b}, \frac{c}{d}\right\}. \tag{2.1}$$

The proofs of these inequalities are simple; indeed, consider the points $A = (a, b)$, $B = (c, d)$, and $C = (a + c, b + d)$ on the plan xOy. The slope of the ray OC lies between the slopes of the rays OA and OB.

Proof of Theorem 1.2. Consider the sequence of step functions on S^1 as

$$\varphi_n(x) = \log \frac{|\widehat{\Delta}^n|}{|\Delta^n|}, \quad x \in \Delta^n \setminus \{r^n\}, \tag{2.2}$$

where $\Delta^n \in \mathbf{P}_n(T_1)$ and r^n is the right endpoint of Δ^n. Since the set of intervals $\{\Delta^n \setminus \{r^n\} : \Delta^n \in \mathbf{P}_n(T_1)\}$ are mutually disjoint and cover S^1, the step functions φ_n are well defined on S^1 for all $n \geq 1$. We show that the sequence (φ_n) is a Cauchy sequence. It is clear that

$$\big|\varphi_n(x) - \varphi_{n+m}(x)\big| \leq \sum_{s=n}^{n+m-1} \big|\varphi_s(x) - \varphi_{s+1}(x)\big|. \tag{2.3}$$

Next we estimate $|\varphi_s(x) - \varphi_{s+1}(x)|$. By the property of the dynamical partition, $|\varphi_s(x) - \varphi_{s+1}(x)| = 0$ if $x \in \Delta_j^s \setminus \{r_j^s\}$ for some $0 \leq j < q_{s-1}$ and

$$\big|\varphi_s(x) - \varphi_{s+1}(x)\big| = \Big|\log \frac{|\widehat{\Delta}_i^{s-1}|}{|\Delta_i^{s-1}|} - \log \frac{|\widehat{\Delta}_\star^{s+1}|}{|\Delta_\star^{s+1}|}\Big| \tag{2.4}$$

if $x \in \Delta_i^{s-1} \setminus \{r_i^{s-1}\}$ for some $0 \leq i < q_s$, where $\Delta_\star^{s+1} \in \mathbf{P}_{s+1}(T_1)$ and $\Delta_\star^{s+1} \subset \Delta_i^{s-1}$. By relation (1.2), we have

$$\frac{|\widehat{\Delta}_i^{s-1}|}{|\Delta_i^{s-1}|} = \frac{|\widehat{\Delta}_i^{s+1}| + |\widehat{\Delta}_{i+q_{s-1}}^s| + |\widehat{\Delta}_{i+q_{s-1}+q_s}^s| + \cdots + |\widehat{\Delta}_{i+q_{s-1}+(k_{s+1}-1)q_s}^s|}{|\Delta_i^{s+1}| + |\Delta_{i+q_{s-1}}^s| + |\Delta_{i+q_{s-1}+q_s}^s| + \cdots + |\Delta_{i+q_{s-1}+(k_{s+1}-1)q_s}^s|}.$$

Applying inequality (2.1), we get

$$\frac{|\widehat{\Delta}_\dagger^{s+1}|}{|\Delta_\dagger^{s+1}|} \leq \frac{|\widehat{\Delta}_i^{s-1}|}{|\Delta_i^{s-1}|} \leq \frac{|\widehat{\Delta}_*^{s+1}|}{|\Delta_*^{s+1}|}, \tag{2.5}$$

where

$$\frac{|\widehat{\Delta}_\dagger^{s+1}|}{|\Delta_\dagger^{s+1}|} = \min\Big\{\frac{|\widehat{\Delta}_i^{s+1}|}{|\Delta_i^{s+1}|}, \frac{|\widehat{\Delta}_{i+q_{s-1}+\ell q_s}^s|}{|\Delta_{i+q_{s-1}+\ell q_s}^s|}, 0 \leq \ell < k_{s+1}\Big\},$$
$$\frac{|\widehat{\Delta}_*^{s+1}|}{|\Delta_*^{s+1}|} = \max\Big\{\frac{|\widehat{\Delta}_i^{s+1}|}{|\Delta_i^{s+1}|}, \frac{|\widehat{\Delta}_{i+q_{s-1}+\ell q_s}^s|}{|\Delta_{i+q_{s-1}+\ell q_s}^s|}, 0 \leq \ell < k_{s+1}\Big\}.$$

It follows from inequalities (2.4) and (2.5) and the assumptions of Theorem 1.2 that

$$\big|\varphi_s(x) - \varphi_{s+1}(x)\big| \leq \Big|\log \frac{|\widehat{\Delta}_*^{s+1}|}{|\Delta_*^{s+1}|} - \log \frac{|\widehat{\Delta}_\dagger^{s+1}|}{|\Delta_\dagger^{s+1}|}\Big| \leq \tau_{s+1}.$$

Therefore, the right-hand side of (2.3) does not exceed the following:

$$\big|\varphi_n(x) - \varphi_{n+m}(x)\big| \leq \sum_{s=n}^{n+m-1} \tau_{s+1}. \tag{2.6}$$

Since the series $\sum_{n=1}^{\infty} \tau_n$ is convergent, we have

$$\sum_{s=n}^{n+m-1} \tau_{s+1} \to 0, \quad n, m \to \infty.$$

Hence, the sequence (φ_n) is a Cauchy sequence. Let

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x).$$

We now show that φ is continuous. For any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $3\sum_{s=n_0}^{\infty} \tau_s \le \epsilon$. Choose $\delta := \delta_{n_0}(\epsilon) = \min_{x\in S^1} |\Delta^{n_0}(x)|/2$. Consider any $x, y \in S^1$ with $|x-y| \le \delta$. Since $|x-y| \le \delta$, there can be two cases: either $x, y \in \Delta^{n_0} \setminus \{r^{n_0}\}$ or x and y lie on the two adjacent half-open intervals $\Delta_1^{n_0} \setminus \{r_1^{n_0}\}$ and $\Delta_2^{n_0} \setminus \{r_2^{n_0}\}$ of $\mathbf{P}_{n_0}(T_1)$, respectively. It is obvious that

$$\big|\varphi(x) - \varphi(y)\big| \le \big|\varphi(x) - \varphi_{n_0}(x)\big| + \big|\varphi_{n_0}(x) - \varphi_{n_0}(y)\big| + \big|\varphi_{n_0}(y) - \varphi(y)\big|. \tag{2.7}$$

If $x, y \in \Delta^{n_0} \setminus \{r^{n_0}\}$, then

$$\big|\varphi_{n_0}(x) - \varphi_{n_0}(y)\big| = \left|\frac{|\widehat{\Delta}^{n_0}|}{|\Delta^{n_0}|} - \frac{|\widehat{\Delta}^{n_0}|}{|\Delta^{n_0}|}\right| = 0.$$

The first and third differences on the right-hand side of the inequality (2.7) can be estimated as

$$\big|\varphi(x) - \varphi_{n_0}(x)\big| \le \sum_{s=n_0+1}^{\infty} \tau_s, \qquad \big|\varphi(y) - \varphi_{n_0}(y)\big| \le \sum_{s=n_0+1}^{\infty} \tau_s. \tag{2.8}$$

Thus, the relation (2.7) takes the form

$$\big|\varphi(x) - \varphi(y)\big| \le 2\sum_{s=n_0+1}^{\infty} \tau_s \le \frac{2\epsilon}{3}. \tag{2.9}$$

Now, if x and y lie on the two adjacent half-open intervals $\Delta_1^{n_0} \setminus \{r_1^{n_0}\}$ and $\Delta_2^{n_0} \setminus \{r_2^{n_0}\}$ of $\mathbf{P}_{n_0}(T_1)$, respectively, then by the condition of the main theorem,

$$\big|\varphi_{n_0}(x) - \varphi_{n_0}(y)\big| = \left|\log\frac{|\widehat{\Delta}_1^{n_0}|}{|\Delta_1^{n_0}|} - \log\frac{|\widehat{\Delta}_2^{n_0}|}{|\Delta_2^{n_0}|}\right| \le \tau_{n_0} \le \frac{\epsilon}{3}. \tag{2.10}$$

And again by (2.8) and (2.10), the relation (2.7) takes the form

$$\big|\varphi(x) - \varphi(y)\big| \le \epsilon. \tag{2.11}$$

Hence, the inequalities (2.9) and (2.11) give the continuity of the function φ. Next we define a new function h on S^1 as

$$h(x) = \int_0^x e^{\varphi(t)}\, dt.$$

It is clear that $h \in C^1$ and $h'(x) = e^{\varphi(x)}$. On the other hand, for any $x \in S^1$ there is a sequence of intervals (Δ_n), $\Delta_n \in \mathbf{P}_n$ such that $x \in \Delta_n$ for all $n \in \mathbb{N}$. By (2.2), for such intervals we have

$$\frac{|\psi(\Delta^n)|}{|\Delta^n|} = \frac{|\widehat{\Delta}^n|}{|\Delta^n|} = e^{\varphi_n(x)}. \tag{2.12}$$

Taking the limit from (2.12), we get

$$\psi'(x) = e^{\varphi(x)} = h'(x)$$

for all $x \in S^1$. Integrating this, we obtain the equality

$$\psi(x) = h(x) + \psi(0) \tag{2.13}$$

for all $x \in S^1$. Hence, since h is C^1 and by the equality (2.13), the conjugation ψ is also C^1. Theorem 1.2 is completely proved. □

Acknowledgments. The authors are grateful to the editor-in-chief and the referees for their careful reading and useful suggestions.

H. Akhadkulov and Ibrahim's work was partially supported by Ministry of Higher Education of Malaysia Fundamental Research Grant Scheme FRGS/1/2017 /STG06/UKM/01/1, code S/O 14192. Noorani's work was partially supported by grant DIP-2017-011.

References

1. A. Adouani, *Conjugation between circle maps with several break points*, Ergodic Theory Dynam. Systems **36** (2016), no. 8, 2351–2383. Zbl 1370.37080. MR3570015. DOI 10.1017/etds.2015.32. 564
2. A. Adouani and H. Marzougui, *Sur les homéomorphismes du cercle de classe PC^r par morceaux ($r \geq 1$) qui sont conjugués C^r par morceaux aux rotations irrationnelles*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 3, 755–775. Zbl 1147.37024. MR2427509. 563
3. A. Adouani and H. Marzougui, *Singular measures for class P-circle homeomorphisms with several break points*, Ergodic Theory Dynam. Systems **34** (2014), no. 2, 423–456. Zbl 1300.37029. MR3233699. DOI 10.1017/etds.2012.133. 563, 564
4. H. Akhadkulov, *Some circle homeomorphisms with break-type singularities* (in Russian), Uspekhi Mat. Nauk **61** (2006), no. 5, 183–184; English translation in Russian Math. Surveys **61** (2006), no. 5, 981–983. Zbl 1142.37340. MR2328263. DOI 10.1070/RM2006v061n05ABEH004362. 563
5. H. Akhadkulov, A. Dzhalilov, and K. Khanin, *Notes on a theorem of Katznelson and Orstein*, Discrete Contin. Dyn. Syst. **37** (2017), no. 9, 4587–4609. Zbl 1368.37046. MR3661811. DOI 10.3934/dcds.2017197. 563
6. H. Akhadkulov, A. Dzhalilov, and D. Mayer, *On conjugations of circle homeomorphisms with two break points*, Ergodic Theory Dynam. Systems **34** (2014), no. 3, 725–741. Zbl 1350.37047. MR3199790. DOI 10.1017/etds.2012.159. 564
7. H. Akhadkulov, A. Dzhalilov, and M. S. Md. Noorani, *On conjugacies between piecewise-smooth circle maps*, Nonlinear Anal. **99** (2014), 1–15. Zbl 1347.37046. MR3160522. DOI 10.1016/j.na.2013.12.013. 564
8. H. Akhadkulov, M. S. Md. Noorani, and S. Akhatkulov, *Renormalization of circle diffeomorphisms with a break-type singularity*, Nonlinearity **30** (2017), no. 7, 2687–2717. Zbl 06759512. MR3670003. DOI 10.1088/1361-6544/aa6d94. 564
9. I. P. Cornfeld, S. V. Fomin, and Y. G. Sinaĭ, *Ergodic Theory*, Grundlehren Math. Wiss. **245**, Springer, New York, 1982. Zbl 0493.28007. MR0832433. DOI 10.1007/978-1-4615-6927-5. 563
10. A. Denjoy, *Sur les courbes définies par les équations différentielles à la surface du tore*, J. Math. Pures Appl. (9) **11** (1932), 333–375. Zbl 0006.30501. 563
11. A. Dzhalilov and K. Khanin, *On an invariant measure for homeomorphisms of the circle with one break point* (in Russian), Uspekhi Mat. Nauk **51** (1996), no. 6, 201–202; English translation in Russian Math. Surveys **51** (1996), no. 6, 1198–1199. Zbl 0894.58033. MR1440163. 563

12. A. Dzhalilov, D. Mayer, and U. Safarov, *Piecewise-smooth circle homeomorphisms with several break points* (in Russian), Izv. Ross. Akad. Nauk Ser. Mat. **76** (2012), no. 1, 101–120; English translation in Izv. Math. **76** (2012), no. 1, 94–112. Zbl 1242.37030. MR2951817. DOI 10.1070/IM2012v076n01ABEH002576. 563
13. A. Dzhalilov, D. Mayer, and U. Safarov, *On the conjugation of piecewise smooth circle homeomorphisms with a finite number of break points*, Nonlinearity **28** (2015), no. 7, 2441–2461. MR3366651. DOI 10.1088/0951-7715/28/7/2441. 564
14. M.-R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. Inst. Hautes Études Sci. **49** (1979), 5–233. Zbl 0448.58019. MR0538680. 563
15. Y. Katznelson and D. Ornstein, *The differentiability of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems **9** (1989), no. 4, 643–680. Zbl 0819.58033. MR1036902. DOI 10.1017/S0143385700005277. 563
16. Y. Katznelson and D. Ornstein. *The absolute continuity of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems **9** (1989), no. 4, 681–690. MR1036903. DOI 10.1017/S0143385700005289. 563
17. K. Khanin, S. Kocić, and E. Mazzeo, *C^1-rigidity of circle maps with breaks for almost all rotation numbers*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 5, 1163–1203. Zbl 1388.37050. MR3720027. DOI 10.24033/asens.2642. 564
18. A. Khinchin, *Continued Fractions*, University of Chicago Press, Chicago, 1964. Zbl 0117.28601. MR0161833. 564
19. H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle (I)*, J. Math. Pures Appl. (9) **7** (1881), 375–422. 562
20. Y. Sinaĭ and K. Khanin, *Smoothness of conjugacies of diffeomorphisms of the circle with rotations* (in Russian), Uspekhi Mat. Nauk **44** (1989), no. 1, 57–82; English translation in Russian Math. Surveys **44** (1989), no. 1, 69–99. Zbl 0701.58053. MR0997684. DOI 10.1070/RM1989v044n01ABEH002008. 563

[1]School of Quantitative Sciences, Universiti Utara Malaysia, CAS 06010, UUM Sintok Kedah Darul Aman, Malaysia.

E-mail address: akhadkulov@yahoo.com; linda@uum.edu.my

[2]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor Darul Ehsan, Malaysia.

E-mail address: msn@ukm.edu.my

[3]Faculty of Applied Mathematics and Informatics, Samarkand State University, Boulevard 15, 140104 Samarkand, Uzbekistan.

E-mail address: akhatkulov@yahoo.com