Ann. Funct. Anal. 10 (2019), no. 3, 433–445
https://doi.org/10.1215/20088752-2019-0001
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

BIRKHOFF–JAMES ORTHOGONALITY OF OPERATORS IN SEMI-HILBERTIAN SPACES AND ITS APPLICATIONS

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Communicated by J. Chmieliński

Abstract. In the following we generalize the concept of Birkhoff–James orthogonality of operators on a Hilbert space when a semi-inner product is considered. More precisely, for linear operators T and S on a complex Hilbert space $\mathcal{H}$, a new relation $T \perp_A^B S$ is defined if T and S are bounded with respect to the seminorm induced by a positive operator A satisfying $\|T+\gamma S\|_A \geq \|T\|_A$ for all $\gamma \in \mathbb{C}$. We extend a theorem due to Bhatia and Šemrl by proving that $T \perp_A^B S$ if and only if there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$ and $\lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = 0$. In addition, we give some A-distance formulas. Particularly, we prove

$$\inf_{\gamma\in\mathbb{C}} \|T+\gamma S\|_A = \sup\big\{\big|\langle Tx, y\rangle_A\big|; \|x\|_A = \|y\|_A = 1, \langle Sx, y\rangle_A = 0\big\}.$$

Some other related results are also discussed.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. The symbol I stands for the *identity operator* on $\mathcal{H}$. If $T \in \mathbb{B}(\mathcal{H})$, then we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and the kernel of T, respectively, and by $\overline{\mathcal{R}(T)}$ the norm closure of $\mathcal{R}(T)$. Throughout this article, we assume that $A \in \mathbb{B}(\mathcal{H})$ is a positive operator and that P is the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Recall that

Received Oct. 15, 2018; Accepted Jan. 3, 2019.
First published online Jul. 19, 2019.
2010 *Mathematics Subject Classification.* Primary 46C05; Secondary 47B65, 47L05.
Keywords. positive operator, semi-inner product, A-Birkhoff–James orthogonality, A-distance formulas.

A is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Such an A induces a positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ defined by

$$\langle x, y\rangle_A = \langle Ax, y\rangle, \quad x, y \in \mathcal{H}.$$

Denote by $\|\cdot\|_A$ the seminorm induced by $\langle\cdot,\cdot\rangle_A$; that is, $\|x\|_A = \sqrt{\langle x, x\rangle_A}$ for every $x \in \mathcal{H}$. It can be easily seen that $\|\cdot\|_A$ is a norm if and only if A is an injective operator, and that $(\mathcal{H}, \|\cdot\|_A)$ is a complete space if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. For $x, y \in \mathcal{H}$, we say that x and y are A-orthogonal, denoted by $x \perp_A y$, if $\langle x, y\rangle_A = 0$. Note that this definition is a natural extension of the usual notion of orthogonality, which represents the I-orthogonality case. Furthermore, we put

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \big\{T \in \mathbb{B}(\mathcal{H}) : \exists c > 0\ \forall x \in \mathcal{H}; \|Tx\|_A \leq c\|x\|_A\big\}.$$

We consider an operator $T \in \mathbb{B}(\mathcal{H})$ to be A-bounded if T belongs to $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. It can be shown that $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ is a unital subalgebra of $\mathbb{B}(\mathcal{H})$ which, in general, is neither closed nor dense in $\mathbb{B}(\mathcal{H})$ (see [2]). We equip $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ with the seminorm $\|\cdot\|_A$ defined as follows:

$$\|T\|_A = \sup_{x\in\overline{\mathcal{R}(A)}, x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \inf\big\{c > 0; \|Tx\|_A \leq c\|x\|_A, x \in \mathcal{H}\big\} < \infty.$$

In addition, for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\|T\|_A = \sup_{x\in\mathcal{H}, \|x\|_A=1} \|Tx\|_A = \sup\big\{\big|\langle Tx, y\rangle_A\big|; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\big\}.$$

Of course, many difficulties arise. For instance, it may happen that $\|T\|_A = \infty$ for some $T \in \mathbb{B}(\mathcal{H})$. In addition, not any operator admits an adjoint operator for the semi-inner product $\langle\cdot,\cdot\rangle_A$. (For more details about this class of operators, we refer the reader to [2].) In recent years, several results covering some classes of operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ have been extended to $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$ (see [2], [3], and the references therein).

The notion of orthogonality in $\mathbb{B}(\mathcal{H})$ can be introduced in many ways (see, e.g., [13]). When $T, S \in \mathbb{B}(\mathcal{H})$, we say that T is Birkhoff–James orthogonal to S, denoted $T \perp^B S$, if

$$\|T + \gamma S\| \geq \|T\| \quad \text{for all } \gamma \in \mathbb{C}.$$

In Hilbert spaces, this orthogonality is equivalent to the usual notion of orthogonality. This notion of orthogonality plays a very important role in the geometry of Hilbert space operators. For $T, S \in \mathbb{B}(\mathcal{H})$, Bhatia and Šemrl in [4, Remark 3.1] and Paul in [14, Lemma 2] independently proved that $T \perp^B S$ if and only if there exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|Tx_n\| = \|T\| \qquad \text{and} \qquad \lim_{n\to\infty} \langle Tx_n, Sx_n\rangle = 0.$$

It follows then that if the Hilbert space $\mathcal{H}$ is finite-dimensional, $T \perp^B S$ if and only if there is a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$ and $\langle Tx, Sx\rangle = 0$.

A number of authors have recently extended the well-known result of Bhatia and Šemrl (see, e.g., [6], [17], [19]). Moreover, Wójcik [17], [19] showed other ways of proving the Bhatia–Šemrl theorem. Other authors have studied different aspects of orthogonality of operators on various Banach spaces and elements of an arbitrary Hilbert C^*-module (see, e.g., [1], [5], [7], [10], [11], [15], [18], [20]).

Now, let us introduce the notion of A-Birkhoff–James orthogonality of operators in semi-Hilbertian spaces.

Definition 1.1. An element $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is called *A-Birkhoff–James orthogonal* to another element $S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, denoted by $T \perp_A^B S$, if

$$\|T + \gamma S\|_A \geq \|T\|_A \quad \text{for all } \gamma \in \mathbb{C}.$$

This is a generalization of the notion of Birkhoff–James of Hilbert space operators. Notice that the A-Birkhoff–James orthogonality is homogenous; that is, $T \perp_A^B S \Leftrightarrow (\alpha T) \perp_A^B (\beta S)$ for all $\alpha, \beta \in \mathbb{C}$.

This paper is organized as follows. In Section 2, we obtain characterizations of A-Birkhoff–James orthogonality for bounded linear operators in semi-Hilbertian spaces. In particular, for $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we show that $T \perp_A^B S$ if and only if there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A \qquad \text{and} \qquad \lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = 0.$$

Furthermore, for the finite-dimensional Hilbert space $\mathcal{H}$, we show that $T \perp_A^B S$ if and only if there exists an A-unit vector $x \in \mathcal{H}$ such that $\|Tx\|_A = \|T\|_A$ and $\langle Tx, Sx\rangle_A = 0$. The mentioned property extends the Bhatia–Šemrl theorem.

Finally, in Section 3, some specific formulas for $\inf_{\gamma\in\mathbb{C}} \|T + \gamma S\|_A$, where we have that $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, are given. In particular, we show that

$$\inf_{\gamma\in\mathbb{C}} \|T + \gamma S\|_A = \sup\bigl\{\bigl|\langle Tx, y\rangle_A\bigr|; \|x\|_A = \|y\|_A = 1, Sx \perp_A y\bigr\}.$$

We then apply it to prove that $\inf_{\gamma\in\mathbb{C}} \|T + \gamma S\|_A^2 = \sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x)$, where

$$\Phi_A^{(T,S)}(x) = \begin{cases} \|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\|Sx\|_A^2} & \text{if } \|Sx\|_A \neq 0, \\ \|Tx\|_A^2 & \text{if } \|Sx\|_A = 0. \end{cases}$$

Our results cover and extend the works of Fujii and Nakamoto in [9] and Bhatia and Šemrl in [4].

2. A-Birkhoff–James orthogonality of operators

We first prove a technical lemma that we need in what follows. We use techniques from [3, Theorem 3.2] to prove this result. In fact, the following lemma extends Magajna's lemma in [12].

Lemma 2.1 ([12, Lemma 2.1])**.** *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the set*

$$W_A(T,S) = \bigl\{\xi \in \mathbb{C}; \exists\{x_n\} \subset \mathcal{H}, \|x_n\|_A = 1, \lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A, \text{ and } \lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = \xi\bigr\}$$

is nonempty, compact, and convex.

Proof. Since the seminorm of $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is given by

$$\|T\|_A = \sup\big\{\|Tx\|_A; x \in \overline{\mathcal{R}(A)}, \|x\|_A = 1\big\},$$

there exists a sequence of A-unit vectors $\{x_n\}$ in $\overline{\mathcal{R}(A)}$ such that

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A.$$

Furthermore, using the Cauchy–Schwarz inequality, we have

$$\big|\langle Tx_n, Sx_n\rangle_A\big| \le \|Tx_n\|_A\|Sx_n\|_A \le \|T\|_A\|S\|_A.$$

Hence, $\{\langle Tx_n, Sx_n\rangle_A\}$ is a bounded sequence of complex numbers, so there exists a subsequence $\{\langle Tx_{n_k}, Sx_{n_k}\rangle_A\}$ that converges to some $\xi_0 \in \mathbb{C}$. Thus $\xi_0 \in W_A(T, S)$ and hence $W_A(T, S)$ is nonempty.

On the other hand, considering the definition of $W_A(T, S)$, it follows that

$$W_A(T, S) \subset \big\{\xi \in \mathbb{C}; |\xi| \le \|T\|_A\|S\|_A\big\}.$$

Therefore, to prove that $W_A(T, S)$ is compact, it is enough to show that $W_A(T, S)$ is closed. Let $\xi_n \in W_A(T, S)$, and let $\lim_{n\to+\infty} \xi_n = \xi$. Since $\xi_n \in W_A(T, S)$, there exists a sequence of A-unit vectors $\{x^n_m\}$ in $\mathcal{H}$ such that $\lim_{m\to+\infty}\|Tx^n_m\|_A = \|T\|_A$ and $\lim_{m\to+\infty}\langle Tx^n_m, Sx^n_m\rangle_A = \xi_n$. Now, let $\varepsilon > 0$. Hence

$$\big|\|Tx^n_m\|_A - \|T\|_A\big| < \varepsilon \tag{2.1}$$

and also

$$\big|\langle Tx^n_m, Sx^n_m\rangle_A - \xi_n\big| < \frac{\varepsilon}{2} \tag{2.2}$$

for all sufficiently large m. From (2.1) and (2.2), we get

$$\big|\|Tx^n_m\|_A - \|T\|_A\big| < \varepsilon$$

and

$$\big|\langle Tx^n_m, Sx^n_m\rangle_A - \xi\big| \le \big|\langle Tx^n_m, Sx^n_m\rangle_A - \xi_n\big| + |\xi_n - \xi| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all sufficiently large m. Therefore, we deduce that $\lim_{m\to+\infty}\|Tx^n_m\|_A = \|T\|_A$ and $\lim_{m\to+\infty}\langle Tx^n_m, Sx^n_m\rangle_A = \xi$. Thus $\xi \in W_A(T, S)$ and so $W_A(T, S)$ is closed.

We next show that $W_A(T, S)$ is convex. Since $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, every $x \in \mathcal{H}$ can be written in a unique way into $x = y + z$ with $y \in \mathcal{N}(A)$ and $z \in \overline{\mathcal{R}(A)}$. Furthermore, since $A \ge 0$, it follows that $\mathcal{N}(A) = \mathcal{N}(A^{1/2})$ which implies that $\|x\|_A = \|z\|_A$. Thus

$$\begin{aligned} W_A(T, S) = \big\{\xi \in \mathbb{C}; \exists\big\{(y_n, z_n)\big\} \subset \mathcal{N}(A) \times \overline{\mathcal{R}(A)}, \|z_n\|_A = 1,\\ \lim_{n\to+\infty} \big\|T(y_n + z_n)\big\|_A = \|T\|_A, \text{ and}\\ \lim_{n\to+\infty} \langle Ty_n, Sz_n\rangle_A + \langle Tz_n, Sz_n\rangle_A = \xi\big\}. \end{aligned}$$

Since $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$ and $S(\mathcal{N}(A)) \subset \mathcal{N}(A)$. Hence, we get

$$
\begin{aligned}
W_A(T,S) &= \big\{\xi \in \mathbb{C}; \exists \{z_n\} \subset \overline{\mathcal{R}(A)}, \|z_n\|_A = 1, \\
&\qquad \lim_{n\to+\infty} \|Tz_n\|_A = \|T\|_A, \text{ and } \lim_{n\to+\infty} \langle Tz_n, Sz_n\rangle_A = \xi\big\} \\
&= \big\{\xi \in \mathbb{C}; \exists \{z_n\} \subset \overline{\mathcal{R}(A)}, \|z_n\|_A = 1, \\
&\qquad \lim_{n\to+\infty} \|PTz_n\|_A = \|PT|_{\overline{\mathcal{R}(A)}}\|_A, \text{ and } \lim_{n\to+\infty} \langle PTz_n, PSz_n\rangle_A = \xi\big\} \\
&= W_{A_0}(\widetilde{T}, \widetilde{S}),
\end{aligned}
$$

where $A_0 = A|_{\overline{\mathcal{R}(A)}}$, $\widetilde{T} = PT|_{\overline{\mathcal{R}(A)}}$, and $\widetilde{S} = PS|_{\overline{\mathcal{R}(A)}}$. By [12, Lemma 2.1], we conclude that $W_A(T,S)$ is convex. □

Recall that the minimum modulus of $S \in \mathbb{B}(\mathcal{H})$ is defined by

$$m(S) = \inf\big\{\|Sx\| : x \in \mathcal{H}, \|x\| = 1\big\}.$$

This concept is useful in studying linear operators (see [13] and the references therein). The A-minimum modulus of $S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ can be defined by

$$m_A(S) = \inf\big\{\|Sx\|_A : x \in \mathcal{H}, \|x\|_A = 1\big\}.$$

We are now in a position to establish the main result of this section. To establish the following theorem, we use some ideas from [16, Theorem 2].

Theorem 2.2. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) *there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A \qquad \text{and} \qquad \lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = 0,$$

(ii) $\|T+\gamma S\|_A^2 \geq \|T\|_A^2 + |\gamma|^2 m_A^2(S)$ *for all* $\gamma \in \mathbb{C}$,

(iii) $T \perp_A^B S$.

Proof. (i)⇒(ii) Suppose that (i) holds. We have

$$
\begin{aligned}
\|T+\gamma S\|_A^2 &\geq \big\|(T+\gamma S)x_n\big\|_A^2 \\
&= \|Tx_n\|_A^2 + \overline{\gamma}\langle Tx_n, Sx_n\rangle_A + \gamma\langle Sx_n, Tx_n\rangle_A + |\gamma|^2\|Sx_n\|_A^2
\end{aligned}
$$

for all $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$. Thus

$$\|T+\gamma S\|_A^2 \geq \|T\|_A^2 + |\gamma|^2 \lim_{n\to\infty}\sup \|Sx_n\|_A^2 \geq \|T\|_A^2 + |\gamma|^2 m_A^2(S)$$

for all $\gamma \in \mathbb{C}$.

(ii)⇒(iii) This implication is trivial.

(iii)⇒(i) If $\|S\|_A = 0$, then since T is a seminorm, there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. So, the Cauchy–Schwarz inequality implies that

$$\big|\langle Tx_n, Sx_n\rangle_A\big| \leq \|Tx_n\|_A\|Sx_n\|_A \leq \|T\|_A\|S\|_A = 0.$$

Hence, $\lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A = 0$. Now let $\|S\|_A \neq 0$. It is enough to show that $0 \in W_A(T,S)$, where $W_A(T,S)$ is defined as in Lemma 2.1. Let $0 \notin W_A(T,S)$. Lemma 2.1 implies that $W_A(T,S)$ is a nonempty, compact, and convex subset of the complex plane $\mathbb{C}$; hence, because of the rotation, we may suppose that $W_A(T,S)$ is contained in the right half-plane. Therefore there is a line that separates 0 from $W_A(T,S)$. In other words, there exists $\tau > 0$ such that $\operatorname{Re} W_A(T,S) > \tau$. Let

$$\mathcal{H}_\tau = \Big\{x \in \mathcal{H}; \|x\|_A = 1, \text{ and } \operatorname{Re} W_A(T,S) \leq \frac{\tau}{2}\Big\}$$

and

$$\delta = \sup\big\{\|Tx\|_A; x \in \mathcal{H}_\tau\big\}.$$

We first claim that $\delta < \|T\|_A$. Suppose that $\delta \geq \|T\|_A$. Hence $\delta = \|T\|_A$. Thus there exists a sequence of vectors $\{x_n\}$ in $\mathcal{H}_\tau$ such that $\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. As $x_n \in \mathcal{H}_\tau$, so $\|x_n\|_A = 1$ and $\operatorname{Re} W_A(T,S) \leq \frac{\tau}{2}$. Now the sequence $\{\langle Tx_n, Sx_n\rangle_A\}$ is bounded, and hence it has a convergent subsequence; so without loss of generality we can assume that $\{\langle Tx_n, Sx_n\rangle_A\}$ is convergent. If we set $\xi = \lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle_A$, then $\operatorname{Re}(\xi) \leq \frac{\tau}{2}$, and this contradicts the fact that $\operatorname{Re} W_A(T,S) > \frac{\tau}{2}$. Thus $\delta < \|T\|_A$. Let $\gamma_0 = \max\{\frac{-\tau}{2\|S\|_A^2}, \frac{\delta - \|T\|_A}{2\|S\|_A}\}$. Then $\gamma_0 < 0$. We claim that $\|T + \gamma_0 S\|_A < \|T\|_A$. Let x be an A-unit vector in $\mathcal{H}$. If $x \in \mathcal{H}_\tau$, then

$$\begin{aligned}\big\|(T+\gamma_0 S)x\big\|_A &\leq \|Tx\|_A + |\gamma_0|\|Sx\|_A \leq \delta - \gamma_0\|S\|_A \\ &\leq \delta + \frac{\|T\|_A - \delta}{2\|S\|_A}\|S\|_A = \frac{\delta}{2} + \frac{\|T\|_A}{2}\end{aligned}$$

and so $\|(T+\gamma_0 S)x\|_A \leq \frac{\delta}{2} + \frac{\|T\|_A}{2}$.

If $x \notin \mathcal{H}_\tau$, then we can write $Tx = (r+it)Sx + y$ with $r, t \in \mathbb{R}$ and $Sx \perp_A y$. Thus

$$2r\|S\|_A^2 \geq 2r\|Sx\|_A^2 = 2\operatorname{Re}\langle Tx, Sx\rangle_A > \frac{\tau}{2} \geq -\gamma_0\|S\|_A^2,$$

and hence $2r + \gamma_0 > 0$. Now, let us put

$$\theta := \inf\big\{\|Sx\|_A^2; x \notin \mathcal{H}_\tau, \|x\|_A = 1\big\}.$$

Since $\gamma_0^2 + 2r\gamma_0 < 0$, we obtain

$$\begin{aligned}\big\|(T+\gamma_0 S)x\big\|_A^2 &= \big\langle \big((r+\gamma_0)+it\big)Sx + y, \big((r+\gamma_0)+it\big)Sx + y\big\rangle_A \\ &= \big((r+\gamma_0)^2 + t^2\big)\|Sx\|_A^2 + \|y\|_A^2 \\ &= \|Tx\|_A^2 + (\gamma_0^2 + 2r\gamma_0)\|Sx\|_A^2 \\ &\leq \|Tx\|_A^2 + (\gamma_0^2 + 2r\gamma_0)\inf\big\{\|Sx\|_A^2; x \notin \mathcal{H}_\tau, \|x\|_A = 1\big\} \\ &\leq \|T\|_A^2 + (\gamma_0^2 + 2r\gamma_0)\theta.\end{aligned}$$

Hence $\|(T+\gamma_0 S)x\|_A^2 \leq \|T\|_A^2 + (\gamma_0^2 + 2r\gamma_0)\theta$. Thus in all cases

$$\big\|(T+\gamma_0 S)x\big\|_A^2 \leq \max\Big\{\Big(\frac{\delta}{2} + \frac{\|T\|_A}{2}\Big)^2, \|T\|_A^2 + (\gamma_0^2 + 2r\gamma_0)\theta\Big\},$$

whence

$$\|T+\gamma_0 S\|_A^2 \leq \max\Big\{\Big(\frac{\delta}{2}+\frac{\|T\|_A}{2}\Big)^2, \|T\|_A^2+(\gamma_0^2+2r\gamma_0)\theta\Big\}.$$

Since $\max\{(\frac{\delta}{2}+\frac{\|T\|_A}{2})^2, \|T\|_A^2+(\gamma_0^2+2r\gamma_0)\theta\} < \|T\|_A^2$, we obtain $\|T+\gamma_0 S\|_A < \|T\|_A$. Therefore we deduce that $T \not\perp_A^B S$, which contradicts our hypothesis. The proof is thus completed. ☐

The following corollary gives a direct application of Theorem 2.2 for the case $A = I$.

Corollary 2.3 ([4, Remark 3.1], [14, Lemma 2])**.** *Let $\mathcal{H}$ be a complex Hilbert space, and let $T, S \in \mathbb{B}(\mathcal{H})$. Then the following statements are equivalent:*

(i) $T \perp^B S$,

(ii) *there exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n\to+\infty} \|Tx_n\| = \|T\| \qquad \text{and} \qquad \lim_{n\to+\infty} \langle Tx_n, Sx_n\rangle = 0.$$

In what follows, for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we denote by $\mathbb{M}_A^T$ the set of all A-unit vectors at which T attains the seminorm $\|\cdot\|_A$; that is,

$$\mathbb{M}_A^T = \big\{x \in \mathcal{H} : \|x\|_A = 1, \|Tx\|_A = \|T\|_A\big\}.$$

(For more information on norm-attaining sets, see [8].) In the next theorem, we consider a finite-dimensional Hilbert space and we characterize the A-Birkhoff–James orthogonality of operators in semi-Hilbertian spaces.

Theorem 2.4. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) *there exists $x \in \mathbb{M}_A^T$ such that $Tx \perp_A Sx$,*

(ii) $T \perp_A^B S$.

Proof. (i)⇒(ii) Suppose that (i) holds. Then there exists an A-unit vector $x \in \mathcal{H}$ such that $\|Tx\|_A = \|T\|_A$ and $Tx \perp_A Sx$. Put $x_n = x$ for all $n \in \mathbb{N}$. So, by the equivalence (i)⇔(iii) in Theorem 2.2, we deduce that $T \perp_A^B S$.

(ii)⇒(i) First note that, by using the decomposition $\mathcal{H} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ and letting $A_0 = A|_{\overline{\mathcal{R}(A)}}$, it can be seen that the set $\{x \in \overline{\mathcal{R}(A)}; \|x\|_{A_0} = 1\}$ is homeomorphic to the set $\{x \in \overline{\mathcal{R}(A)}; \|x\| = 1\}$, which is compact since $\overline{\mathcal{R}(A)}$ is finite-dimensional. Thus we get that the set $\{x \in \overline{\mathcal{R}(A)}; \|x\|_{A_0} = 1\}$ is compact.

Now, suppose that (ii) holds. Put $\widetilde{T} = PT|_{\overline{\mathcal{R}(A)}}$ and $\widetilde{S} = PS|_{\overline{\mathcal{R}(A)}}$. Therefore, by the equivalence (i)⇔(iii) in Theorem 2.2, there exists a sequence of A_0-unit vectors $\{x_n\}$ in $\overline{\mathcal{R}(A)}$ such that

$$\lim_{n\to+\infty} \|\widetilde{T}x_n\|_{A_0} = \|\widetilde{T}\|_{A_0} \qquad \text{and} \qquad \lim_{n\to+\infty} \langle \widetilde{T}x_n, \widetilde{S}x_n\rangle_{A_0} = 0.$$

Since the set $\{x \in \overline{\mathcal{R}(A)}; \|x\|_{A_0} = 1\}$ is compact, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $x \in \overline{\mathcal{R}(A)}$ with $\|x\|_{A_0} = 1$. This yields $\|\widetilde{T}x\|_{A_0} = \lim_{k\to+\infty}\|\widetilde{T}x_{n_k}\|_{A_0} = \|\widetilde{T}\|_{A_0}$ and $\langle \widetilde{T}x, \widetilde{S}x\rangle_{A_0} = \lim_{k\to+\infty}\langle \widetilde{T}x_{n_k}, \widetilde{S}x_{n_k}\rangle_{A_0} = 0$. From this it follows that $x \in \mathbb{M}_A^T$ and $Tx \perp_A Sx$. ☐

As an immediate consequence of Theorem 2.4, we have the following result.

Corollary 2.5. *Let $\mathcal{H}$ be finite-dimensional, and let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following statements are equivalent:*

(i) $T \perp_A^B S$;

(ii) *there exists $x \in \mathbb{M}_A^T$ such that, for every $\gamma \in \mathbb{C}$,*

$$\|Tx + \gamma Sx\|_A^2 = \|Tx\|_A^2 + |\gamma|^2 \|Sx\|_A^2.$$

3. Some A-distance formulas

In this section, we give some formulas for the A-distance of an operator to the class of multiple scalars of another operator in semi-Hilbertian spaces. For $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have, by definition, $d_A(T, \mathbb{C}S) := \inf_{\gamma \in \mathbb{C}} \|T + \gamma S\|_A$. The following auxiliary lemma is needed for next results.

Lemma 3.1. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then there exists $\zeta_0 \in \mathbb{C}$ such that*

$$d_A(T, \mathbb{C}S) = \|T + \zeta_0 S\|_A.$$

Proof. If $\|S\|_A = 0$, then

$$\|T + \gamma S\|_A \geq \|T\|_A - |\gamma| \|S\|_A = \|T\|_A$$

for all $\gamma \in \mathbb{C}$. It is therefore enough to put $\zeta_0 = 0$. If $\|S\|_A \neq 0$, then put $\mathbb{D} := \{\gamma \in \mathbb{C}; |\gamma| \leq \frac{2\|T\|_A}{\|S\|_A}\}$ and define $f : \mathbb{D} \to \mathbb{R}$ by the formula $f(\gamma) = \|T + \gamma S\|_A$. Clearly, f is continuous and attains its minimum at, say, $\zeta_0 \in \mathbb{D}$ (of course, there may be many such points). Then $\|T + \gamma S\|_A \geq \|T + \zeta_0 S\|_A$ for all $\gamma \in \mathbb{D}$. If $\gamma \notin \mathbb{D}$, then $|\gamma| > \frac{2\|T\|_A}{\|S\|_A}$. Since $0 \in \mathbb{D}$, we obtain

$$\|T + \gamma S\|_A \geq |\gamma| \|S\|_A - \|T\|_A > 2\|T\|_A - \|T\|_A = \|T\|_A \geq \|T + \zeta_0 S\|_A.$$

Thus $\|T + \gamma S\|_A \geq \|T + \zeta_0 S\|_A$ for all $\gamma \notin \mathbb{D}$. Therefore, $\|T + \gamma S\|_A \geq \|T + \zeta_0 S\|_A$ for all $\gamma \in \mathbb{C}$. So, we conclude that $\inf_{\gamma \in \mathbb{C}} \|T + \gamma S\|_A = \|T + \zeta_0 S\|_A$ and hence $d_A(T, \mathbb{C}S) = \|T + \zeta_0 S\|_A$. □

The following result is a kind of Pythagorean relation for bounded operators in semi-Hilbertian spaces.

Theorem 3.2. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ with $m_A(S) > 0$. Then there exists a unique $\zeta_0 \in \mathbb{C}$ such that*

$$\big\|(T + \zeta_0 S) + \gamma S\big\|_A^2 \geq \|T + \zeta_0 S\|_A^2 + |\gamma|^2 m_A^2(S)$$

for every $\gamma \in \mathbb{C}$.

Proof. By Lemma 3.1, there exists $\zeta_0 \in \mathbb{C}$ such that

$$\inf_{\gamma \in \mathbb{C}} \|T + \gamma S\|_A = \|T + \zeta_0 S\|_A;$$

equivalently,

$$\inf_{\xi \in \mathbb{C}} \big\|(T + \zeta_0 S) + \xi S\big\|_A = \|T + \zeta_0 S\|_A.$$

Thus $(T+\zeta_0 S)\perp_A^B S$. So, by the equivalence (i)⇔(ii) in Theorem 2.2, for every $\gamma\in\mathbb{C}$, we have

$$\big\|(T+\zeta_0 S)+\gamma S\big\|_A^2 \geq \|T+\zeta_0 S\|_A^2 + |\gamma|^2 m_A^2(S).$$

Now, suppose that ζ_1 is another point satisfying the inequality

$$\big\|(T+\zeta_1 S)+\gamma S\big\|_A^2 \geq \|T+\zeta_1 S\|_A^2 + |\gamma|^2 m_A^2(S) \quad (\gamma\in\mathbb{C}).$$

Choose $\gamma=\zeta_0-\zeta_1$ to get

$$\begin{aligned}\|T+\zeta_0 S\|_A^2 &= \big\|(T+\zeta_1 S)+(\zeta_0-\zeta_1)S\big\|_A^2\\ &\geq \|T+\zeta_1 S\|_A^2 + |\zeta_0-\zeta_1|^2 m_A^2(S)\\ &\geq \|T+\zeta_0 S\|_A^2 + |\zeta_0-\zeta_1|^2 m_A^2(S).\end{aligned}$$

Hence $0\geq |\zeta_0-\zeta_1|^2 m_A^2(S)$. Since $m_A^2(S)>0$, we get $|\zeta_0-\zeta_1|^2=0$; equivalently, $\zeta_0=\zeta_1$. This shows that ζ_0 is unique. □

We now establish one of our main results. In fact, in what follows, we provide a version of the Bhatia–Šemrl theorem (see [4, p. 84]) in the setting of operators in semi-Hilbertian spaces.

Theorem 3.3. *Let $T,S\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$d_A(T,\mathbb{C}S)=\sup\big\{\big|\langle Tx,y\rangle_A\big|;\|x\|_A=\|y\|_A=1, Sx\perp_A y\big\}.$$

Proof. Let $x,y\in\mathcal{H}$, $\|x\|_A=\|y\|_A=1$, and let $Sx\perp_A y$. The Cauchy–Schwarz inequality implies that

$$\big|\langle Tx,y\rangle_A\big| = \big|\big\langle (T+\gamma S)x,y\big\rangle_A\big| \leq \big\|(T+\gamma S)x\big\|_A\|y\|_A \leq \|T+\gamma S\|_A$$

for all $\gamma\in\mathbb{C}$. Thus

$$\sup\big\{\big|\langle Tx,y\rangle_A\big|;\|x\|_A=\|y\|_A=1, Sx\perp_A y\big\} \leq \|T+\gamma S\|_A$$

for all $\gamma\in\mathbb{C}$ and so

$$\sup\big\{\big|\langle Tx,y\rangle_A\big|;\|x\|_A=\|y\|_A=1, Sx\perp_A y\big\} \leq \inf_{\gamma\in\mathbb{C}}\|T+\gamma S\|_A.$$

Hence

$$\sup\big\{\big|\langle Tx,y\rangle_A\big|;\|x\|_A=\|y\|_A=1, Sx\perp_A y\big\} \leq d_A(T,\mathbb{C}S). \tag{3.1}$$

On the other hand, by Lemma 3.1, there exists $\zeta_0\in\mathbb{C}$ such that $d_A(T,\mathbb{C}S)=\|T+\zeta_0 S\|_A$. We assume that $\zeta_0=0$ (otherwise, we just replace T by $T+\zeta_0 S$). Thus $d_A(T,\mathbb{C}S)=\|T\|_A$; equivalently, $T\perp_A^B S$. Then, by the equivalence (i)⇔(iii) in Theorem 2.2, there exists a sequence of A-unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to+\infty}\|Tx_n\|_A=\|T\|_A$ and $\lim_{n\to+\infty}\langle Tx_n,Sx_n\rangle_A=0$. Now, let $Tx_n=\alpha_n Sx_n+\beta_n y_n$ with $Sx_n\perp_A y_n$, $\|y_n\|_A=1$, and $\alpha_n,\beta_n\in\mathbb{C}$. Then we have

$$\begin{aligned}
d_A^2(T,\mathbb{C}S) &= \|T\|_A^2 = \lim_{n\to+\infty}\|Tx_n\|_A^2 \\
&= \lim_{n\to+\infty}\langle \alpha_n Sx_n+\beta_n y_n, \alpha_n Sx_n+\beta_n y_n\rangle_A \\
&= \lim_{n\to+\infty}\langle \alpha_n Sx_n, \alpha_n Sx_n\rangle_A + |\beta_n|^2 \\
&= \lim_{n\to+\infty}\langle Tx_n-\beta_n y_n, \alpha_n Sx_n\rangle_A + |\beta_n|^2 \\
&= \lim_{n\to+\infty}\alpha_n\langle Tx_n, Sx_n\rangle_A - \overline{\alpha_n}\beta_n\langle y_n, Sx_n\rangle_A + |\beta_n|^2 = \lim_{n\to+\infty}|\beta_n|^2.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
d_A(T,\mathbb{C}S) &= \lim_{n\to+\infty}|\beta_n| = \lim_{n\to+\infty}\big|\langle \beta_n y_n, y_n\rangle_A\big| \\
&= \lim_{n\to+\infty}\big|\langle Tx_n-\alpha_n Sx_n, y_n\rangle_A\big| = \lim_{n\to+\infty}\big|\langle Tx_n, y_n\rangle_A\big| \\
&\leq \sup\big\{\big|\langle Tx,y\rangle_A\big|;\, \|x\|_A=\|y\|_A=1, Sx\perp_A y\big\},
\end{aligned}$$

whence

$$d_A(T,\mathbb{C}S) \leq \sup\big\{\big|\langle Tx,y\rangle_A\big|;\, \|x\|_A=\|y\|_A=1, Sx\perp_A y\big\}. \tag{3.2}$$

From (3.1) and (3.2), we conclude that

$$d_A(T,\mathbb{C}S) = \sup\big\{\big|\langle Tx,y\rangle_A\big|;\, \|x\|_A=\|y\|_A=1, Sx\perp_A y\big\}. \qquad \square$$

For $T\in\mathbb{B}(\mathcal{H})$, Fujii and Nakamoto in [9] proved that $d_A(T,\mathbb{C}I)$ can be written in the form

$$d(T,\mathbb{C}I) = \Big(\sup_{\|x\|=1}\big(\|Tx\|^2-\big|\langle Tx,x\rangle\big|^2\big)\Big)^{1/2} = \sup_{\|x\|=1}\big\|Tx-\langle Tx,x\rangle x\big\|, \tag{3.3}$$

which shows that $d_A(T,\mathbb{C}I)$ is the supremum over the lengths of all perpendiculars from Tx to x, where x passes over the set of unit vectors. In the following theorem, for $T,S\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$, we show that $d_A(T,\mathbb{C}S)$ can also be expressed in the form generalizing (3.3).

Theorem 3.4. *Let $T,S\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$d_A^2(T,\mathbb{C}S) = \sup_{\|x\|_A=1}\Phi_A^{(T,S)}(x),$$

where

$$\Phi_A^{(T,S)}(x) = \begin{cases} \|Tx\|_A^2 - \frac{|\langle Tx,Sx\rangle_A|^2}{\|Sx\|_A^2} & \text{if } \|Sx\|_A\neq 0,\\ \|Tx\|_A^2 & \text{if } \|Sx\|_A=0.\end{cases}$$

Proof. For every $\gamma\in\mathbb{C}$ and every A-unit vector $x\in\mathcal{H}$ such that $\|Sx\|_A\neq 0$, we have

$$\begin{aligned}
&\|Tx+\gamma Sx\|_A^2 - \frac{|\langle Tx+\gamma Sx, Sx\rangle_A|^2}{\|Sx\|_A^2} \\
&\quad= \|Tx\|_A^2 + |\gamma|^2\|Sx\|_A^2 + 2\,\mathrm{Re}\,\langle Tx,\gamma Sx\rangle_A \\
&\qquad - \frac{|\langle Tx,Sx\rangle_A|^2 + |\gamma|^2\|Sx\|_A^4 + 2\|Sx\|_A^2\,\mathrm{Re}\,\langle Tx,\gamma Sx\rangle_A}{\|Sx\|_A^2}
\end{aligned}$$

$$= \|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\|Sx\|_A^2}.$$

Thus

$$\Phi_A^{(T,S)}(x) = \|Tx + \gamma Sx\|_A^2 - \frac{|\langle Tx + \gamma Sx, Sx\rangle_A|^2}{\|Sx\|_A^2} \leq \|Tx + \gamma Sx\|_A^2 \leq \|T + \gamma S\|_A^2.$$

Also, in the case $\|Sx\|_A = 0$ we have

$$\Phi_A^{(T,S)}(x) = \|Tx\|_A^2 \leq \big(\|Tx + \gamma Sx\|_A + \|\gamma Sx\|_A\big)^2 = \|Tx + \gamma Sx\|_A^2 \leq \|T + \gamma S\|_A^2.$$

Hence we obtain $\Phi_A^{(T,S)}(x) \leq \|T + \gamma S\|_A^2$ for every A-unit vector $x \in \mathcal{H}$ and every $\gamma \in \mathbb{C}$. Therefore, $\sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x) \leq \|T + \gamma S\|_A^2$ for every $\gamma \in \mathbb{C}$ and consequently,

$$\sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x) \leq \inf_{\gamma\in\mathbb{C}} \|T + \gamma S\|_A^2.$$

Thus

$$\sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x) \leq d_A^2(T, \mathbb{C}S). \tag{3.4}$$

Now, take A-unit vectors $x, y \in \mathcal{H}$ such that $Sx \perp_A y$. If $\|Sx\|_A = 0$, then

$$\big|\langle Tx, y\rangle_A\big|^2 \leq \|Tx\|_A^2 \|y\|_A^2 = \Phi_A^{(T,S)}(x) \leq \sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x).$$

If $\|Sx\|_A \neq 0$, then

$$\begin{aligned}\big|\langle Tx, y\rangle_A\big|^2 &= \left|\left\langle Tx - \frac{\langle Tx, Sx\rangle_A}{\|Sx\|_A^2} Sx, y\right\rangle_A\right|^2 \\ &\leq \left\langle Tx - \frac{\langle Tx, Sx\rangle_A}{\|Sx\|_A^2} Sx, Tx - \frac{\langle Tx, Sx\rangle_A}{\|Sx\|_A^2} Sx\right\rangle_A \\ &= \|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\|Sx\|_A^2} = \Phi_A^{(T,S)}(x) \leq \sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x).\end{aligned}$$

So, we conclude that $|\langle Tx, y\rangle_A|^2 \leq \sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x)$ for all A-unit vectors $x, y \in \mathcal{H}$ such that $Sx \perp_A y$. Therefore, Theorem 3.3 implies that

$$d_A^2(T, \mathbb{C}S) \leq \sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x). \tag{3.5}$$

Now, the result follows from (3.4) and (3.5). □

We close this paper with the following inf-sup equality in semi-Hilbertian spaces.

Theorem 3.5. *Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$\inf_{\gamma\in\mathbb{C}} \sup_{\|x\|_A=1} \big\|(T + \gamma S)x\big\|_A^2 = \sup_{\|x\|_A=1} \inf_{\gamma\in\mathbb{C}} \big\|(T + \gamma S)x\big\|_A^2.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. If $\|Sx\|_A = 0$, then

$$\big\|(T + \gamma S)x\big\|_A \geq \|Tx\|_A - |\gamma|\|Sx\|_A = \|Tx\|_A$$

for all $\gamma \in \mathbb{C}$. Thus

$$\|Tx\|_A^2 \geq \inf_{\gamma\in\mathbb{C}} \big\|(T + \gamma S)x\big\|_A^2 \geq \|Tx\|_A^2,$$

whence $\inf_{\gamma\in\mathbb{C}} \|(T + \gamma S)x\|_A^2 = \|Tx\|_A^2$. Hence $\inf_{\gamma\in\mathbb{C}} \|(T + \gamma S)x\|_A^2 = \Phi_A^{(T,S)}(x)$.

If $\|Sx\|_A \neq 0$, then simple computations show that

$$\big\|(T + \gamma S)x\big\|_A^2 = \|Sx\|_A^2 \Big|\frac{\langle Tx, Sx\rangle_A}{\|Sx\|_A^2} + \gamma\Big|^2 + \|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\|Sx\|_A^2}.$$

Thus $\|(T + \gamma S)x\|_A^2$ achieves its minimum at $-\frac{\langle Tx,Sx\rangle_A}{\|Sx\|_A^2}$ and the minimum value is $\|Tx\|_A^2 - \frac{|\langle Tx,Sx\rangle_A|^2}{\|Sx\|_A^2}$. Hence $\inf_{\gamma\in\mathbb{C}} \|(T + \gamma S)x\|_A^2 = \Phi_A^{(T,S)}(x)$ for every A-unit vector $x \in \mathcal{H}$. From this, by Theorem 3.4, we conclude that

$$\begin{aligned}\sup_{\|x\|_A=1} \inf_{\gamma\in\mathbb{C}} \big\|(T + \gamma S)x\big\|_A^2 &= \sup_{\|x\|_A=1} \Phi_A^{(T,S)}(x) \\ &= d_A^2(T, \mathbb{C}S) \\ &= \inf_{\gamma\in\mathbb{C}} \|T + \gamma S\|_A^2 = \inf_{\gamma\in\mathbb{C}} \sup_{\|x\|_A=1} \big\|(T + \gamma S)x\big\|_A^2.\end{aligned}$$

□

Acknowledgments. The author would like to thank the referees for their valuable comments, which helped to improve the exposition.

The author's work was partially supported by Shanghai Municipal Science and Technology Commission grant 18590745200.

References

1. L. Arambašić and R. Rajić, *The Birkhoff-James orthogonality in Hilbert C^*-modules*, Linear Algebra Appl. **437** (2012), no. 7, 1913–1929. Zbl 1257.46025. MR2946368. DOI 10.1016/j.laa.2012.05.011. 435
2. M. L. Arias, G. Corach, and M. C. Gonzalez, *Metric properties of projections in semi-Hilbertian spaces*, Integral Equations Operator Theory **62** (2008), no. 1, 11–28. Zbl 1181.46018. MR2442900. DOI 10.1007/s00020-008-1613-6. 434
3. H. Baklouti, K. Feki, and O. A. M. Sid Ahmed, *Joint numerical ranges of operators in semi-Hilbertian spaces*, Linear Algebra Appl. **555** (2018), 266–284. Zbl 06914727. MR3834203. DOI 10.1016/j.laa.2018.06.021. 434, 435
4. R. Bhatia and P. Šemrl, *Orthogonality of matrices and some distance problems*, Linear Algebra Appl. **287** (1999), no. 1–3, 77–85. Zbl 0937.15023. MR1662861. DOI 10.1016/S0024-3795(98)10134-9. 434, 435, 439, 441
5. T. Bottazzi, C. Conde, M. S. Moslehian, P. Wójcik, and A. Zamani, *Orthogonality and parallelism of operators on various Banach spaces*, J. Aust. Math. Soc. **106** (2019), no. 2, 160–183. Zbl 07039555. MR3919376. DOI 10.1017/S1446788718000150. 435
6. J. Chmieliński, T. Stypuła, and P. Wójcik, *Approximate orthogonality in normed spaces and its applications*, Linear Algebra Appl. **531** (2017), 305–317. Zbl 1383.46013. MR3682706. DOI 10.1016/j.laa.2017.06.001. 434

7. J. Chmieliński and P. Wójcik, *Approximate symmetry of Birkhoff orthogonality*, J. Math. Anal. Appl. **461** (2018), no. 1, 625–640. Zbl 1402.46009. MR3759561. DOI 10.1016/j.jmaa.2018.01.031. 435
8. J. Falcó, D. García, M. Maestre, and P. Rueda, *Spaceability in norm-attaining sets*, Banach J. Math. Anal. **11** (2017), no. 1, 90–107. Zbl 1366.46032. MR3571146. DOI 10.1215/17358787-3750182. 439
9. M. Fujii and R. Nakamoto, *An estimation of the transcendental radius of an operator*, Math. Japon. **27** (1982), no. 5, 637–638. Zbl 0496.47005. MR0675564. 435, 442
10. P. Ghosh, D. Sain, and K. Paul, *On symmetry of Birkhoff-James orthogonality of linear operators*, Adv. Oper. Theory **2** (2017), no. 4, 428–434. Zbl 1386.46017. MR3730038. DOI 10.22034/aot.1703-1137. 435
11. D. J. Keckić, *Orthogonality in* $\mathcal{C}_1$ *and* $\mathcal{C}_\infty$ *spaces and normal derivations*, J. Operator Theory **51** (2004), no. 1, 89–104. Zbl 1068.46024. MR2055806. 435
12. B. Magajna, *On the distance to finite-dimensional subspaces in operator algebras*, J. Lond. Math. Soc. (2) **47** (1993), no. 3, 516–532. Zbl 0742.47010. MR1214913. DOI 10.1112/jlms/s2-47.3.516. 435, 437
13. M. S. Moslehian and A. Zamani, *Characterizations of operator Birkhoff-James orthogonality*, Canad. Math. Bull. **60** (2017), no. 4, 816–829. Zbl 1387.46019. MR3710664. DOI 10.4153/CMB-2017-004-5. 434, 437
14. K. Paul, *Translatable radii of an operator in the direction of another operator*, Sci. Math. **2** (1999), no. 1, 119–122. Zbl 0952.47032. MR1688391. 434, 439
15. K. Paul, D. Sain, A. Mal, and K. Mandal, *Orthogonality of bounded linear operators on complex Banach spaces*, Adv. Oper. Theory **3** (2018), no. 3, 699–709. Zbl 1404.46015. MR3795110. DOI 10.15352/aot.1712-1268. 435
16. J. G. Stampfli, *The norm of a derivation*, Pacific J. Math. **33** (1970), 737–747. Zbl 0197.10501. MR0265952. 437
17. P. Wójcik, *The Birkhoff orthogonality in pre-Hilbert* C^**-modules*, Oper. Matrices **10** (2016), no. 3, 713–729. Zbl 1361.46016. MR3568310. DOI 10.7153/oam-10-44. 434
18. P. Wójcik, *Gateaux derivative of the norm in* $\mathcal{K}(X;Y)$, Ann. Funct. Anal. **7** (2016), no. 4, 678–685. Zbl 1366.46010. MR3555759. DOI 10.1215/20088752-3661179. 435
19. P. Wójcik, *Birkhoff orthogonality in classical* M*-ideals*, J. Aust. Math. Soc. **103** (2017), no. 2, 279–288. Zbl 1383.46018. MR3703927. DOI 10.1017/S1446788716000537. 434
20. A. Zamani, M. S. Moslehian, M. T. Chien, and H. Nakazato, *Norm-parallelism and the Davis-Wielandt radius of Hilbert space operators*, Linear Multilinear Algebra, published online 26 July 2018. DOI 10.1080/03081087.2018.1484422. 435

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