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INEQUALITIES FOR THE EXTENDED POSITIVE PART OF A VON NEUMANN ALGEBRA RELATED TO OPERATOR-MONOTONE AND OPERATOR-CONVEX FUNCTIONS

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Abstract. We extend inequalities for operator monotone and operator convex functions onto elements of the extended positive part of a von Neumann algebra. In particular, this provides an opportunity to extend the inequalities onto unbounded positive self-adjoint operators.

1. Introduction

Starting with the basic articles by Löwner [9] and Kraus [7], the theory of matrix- and operator-monotone (operator-convex) functions has been intensively developed. More recently, it has led to effective applications in operator theory, quantum information theory, and other areas.

A real-valued function f defined on $K \subset \mathbb{R}$ is said to be *matrix-monotone of order* n (see [9]) if, for any pair of Hermitian matrices $A, B \in \mathbb{M}_n$ with spectra in K,

$$A \leq B \quad \Longrightarrow \quad f(A) \leq f(B).$$

A function f defined on an interval $I \subset \mathbb{R}$ is said to be *matrix-convex of order* n (see [7]) if, for any pair of Hermitian matrices $A, B \in \mathbb{M}_n$ with spectra in I, and

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for all $0 \le \lambda \le 1$,

$$f\big(\lambda A + (1-\lambda)B\big) \le \lambda f(A) + (1-\lambda) f(B).$$

The function f is called *operator-monotone/operator-convex* if it is matrix-monotone/matrix-convex of any order n. The function $f(t) = t^s\,(t \in [0,\infty))$ is operator-monotone for $s \in [0,1]$ and operator-convex for $s \in [-1,0] \cup [1,2]$. Another important example of operator-monotone and operator-convex functions are $\log t$ and $t \log t$, respectively. They appear in the definition of quantum entropy.

In 1980, Kubo and Ando [8] developed a systematic theory of operator means in which each operator mean corresponds to a unique operator-monotone function on $(0,\infty)$. In 1996, Petz [12] studied the theory of monotone metrics and classified them in terms of operator-monotone functions; subsequently, such functions became extremely useful in quantum information theory. In 2005, Osaka and coauthors in [10] studied classes of operator-monotone functions with respect to C^*-algebras. In 2010, using another approach, the first and the second authors in [1] obtained similar results for operator-convex functions with respect to von Neumann algebras and (for C^*-algebras) deduced an appropriate exhaustive analog of the results of [10]. An extension of monotonicity inequalities for certain operator-monotone functions onto unbounded positive self-adjoint operators was fruitfully applied by Pedersen and Takesaki [11] in their study of normal weights on von Neumann algebras. In [1], it was proved that if a continuous nonnegative function f on $[0,\infty)$ is operator-monotone with respect to a von Neumann algebra, then f preserves the natural order on the set of positive self-adjoint operators affiliated with the algebra. In 2016, Bikchentaev in [2] presented monotonocity and convexity criteria for a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ with respect to any C^*-algebra.

The notion of the extended positive part of a von Neumann algebra was introduced by Haagerup [3] in his first work on operator-valued weights in von Neumann algebras. Elements of the extended positive part of a von Neumann algebra can be considered as a dual analog of normal weights on the algebra. The extended positive part of a von Neumann algebra can be identified with a set of operators affiliated with the algebra, which are not necessarily densely defined. Therefore, the authors believe that it is natural to study monotone or convex functions of elements of the extended positive part of a von Neumann algebra, with the expectation that the obtained inequalities could be interesting and useful.

2. Notation and preliminaries

In what follows, $\mathcal{M}$ stands for a von Neumann algebra, $\mathcal{M}_+$ denotes its positive part, and $\mathcal{M}_*^+$ denotes the cone of positive normal functionals on $\mathcal{M}$ [13].

2.1. $\mathcal{M}$-Monotone and $\mathcal{M}$-convex functions.

Definition 2.1. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a Borel function bounded on bounded subsets of $\mathbb{R}^+$. We say that f is *operator-monotone with respect to* $\mathcal{M}$ (or briefly,

$\mathcal{M}$-*monotone*; see [10]) if

$$A, B \in \mathcal{M}_+, \quad A \leq B \quad \Longrightarrow \quad f(A) \leq f(B). \tag{2.1}$$

Similarly, f is said to be $\mathcal{M}$-*convex* (see [1]) if for any $A, B \in \mathcal{M}_+$ and for any $\alpha \in [0, 1]$,

$$f\big(\alpha A + (1-\alpha)B\big) \leq \alpha f(A) + (1-\alpha) f(B). \tag{2.2}$$

Clearly, for any von Neumann algebra $\mathcal{M}$, each $\mathcal{M}$-monotone function non-decreases on $\mathbb{R}^+$ and each $\mathcal{M}$-convex function is convex on $\mathbb{R}^+$.

For $n \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{B}(H_n)$ stand for the algebra of all bounded operators in the n-dimensional Hilbert space H_n. For $n \in \mathbb{N}$, $\mathcal{B}(H_n)$-monotone (-convex functions are matrix-monotone (convex) of order n; $\mathcal{B}(H_\infty)$-monotone (-convex) functions are operator-monotone (-convex).

It is worth noting that, for each von Neumann algebra $\mathcal{M}$, there exists $n \in \mathbb{N} \cup \{\infty\}$ such that the class of $\mathcal{M}$-monotone (-convex) functions coincides with the class of $\mathcal{B}(H_n)$-monotone (-convex) functions (see [1], [10]).

2.2. The extended positive part of a von Neumann algebra. In the remainder of this article, let $\overline{\mathbb{R}}^+$ stand for $[0, +\infty]$.

Definition 2.2 ([3, Definition 1.1]). *The extended positive part* $\widehat{\mathcal{M}}_+$ of a von Neumann algebra $\mathcal{M}$ is the set of maps $m : \mathcal{M}_*^+ \to \overline{\mathbb{R}}^+$ that satisfy:

(i) $m(\lambda\varphi) = \lambda m(\varphi)$, $\varphi \in \mathcal{M}_*^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) = 0$);
(ii) $m(\varphi + \psi) = m(\varphi) + m(\psi)$, φ, $\psi \in \mathcal{M}_*^+$;
(iii) m is lower semicontinuous.

The cone $\mathcal{M}_+$ can be regarded as a subset of $\widehat{\mathcal{M}}_+$. Moreover, the set of positive self-adjoint operators affiliated with $\mathcal{M}$ can be identified as a subset of $\widehat{\mathcal{M}}_+$ (see [14, Example IX.4.5]). The extended positive part $\widehat{\mathcal{M}}_+$ is closed under addition, multiplication by nonnegative scalars and increasing limits (see [3], [14, p. 215]). For $m \in \widehat{\mathcal{M}}_+$ and $C \in \mathcal{M}$, the element C^*mC is defined by $(C^*mC)(\varphi) = m(C\varphi C^*)$, where $C\varphi C^* = \varphi(C^* \cdot C)$.

By [14, Theorem IX.4.8], each $m \in \widehat{\mathcal{M}}_+$ has a unique spectral decomposition of the form

$$m(\varphi) = \int_0^{+\infty} \lambda \, d\varphi\big(e_m(\lambda)\big) + \infty \cdot \varphi(p_m), \quad \varphi \in \mathcal{M}_*^+, \tag{2.3}$$

where $\{e_m(\lambda) : \lambda \in \mathbb{R}^+\}$ is an increasing family of projections in $\mathcal{M}$ strongly continuous from the right, and $p_m = 1 - \lim_{\lambda \to +\infty} e_m(\lambda)$.

For a bounded Borel function $f : \overline{\mathbb{R}}^+ \to \mathbb{R}$, the self-adjoint operator $f(m)$ in $\mathcal{M}$ is defined as

$$\varphi\big(f(m)\big) = \int_0^{+\infty} f(\lambda) \, d\varphi\big(e_m(\lambda)\big) + f(+\infty)\varphi(p_m) \qquad (\varphi \in \mathcal{M}_*^+).$$

Similarly, for a Borel function $f:\overline{\mathbb{R}}^+\to\overline{\mathbb{R}}^+$, we determine $f(m)$ by

$$\varphi\big(f(m)\big)=\int_0^{+\infty} f(\lambda)\,d\varphi\big(e_m(\lambda)\big)+f(+\infty)\varphi(p_m) \qquad (\varphi\in\mathcal{M}_*^+).$$

It is easy to see that $f(m)$ is an element of $\widehat{\mathcal{M}}_+$.

The aim of the present article is to show how inequalities of monotonicity (2.1) and convexity (2.2) can be extended from $\mathcal{M}_+$ to $\widehat{\mathcal{M}}_+$ (see Theorem 3.1 and Theorem 4.3). Note that such extensions of certain operator-monotone functions were used in [14] to study properties of the extended positive parts of von Neumann algebras. Note that even formulating of convexity inequalities for unbounded positive self-adjoint operators leads to necessity of the notion of extended positive part since the convex combination of unbounded positive self-adjoint operators cannot be correctly defined as a densely defined operator. Also, we extend Hansen's inequality in [4] for operator-monotone functions (Theorem 3.1 and Corollary 3.2 below).

3. Inequalities related to operator-monotone functions

For $\varepsilon, a>0$, and $\Omega\subset\overline{\mathbb{R}}^+$, we define the following functions on $\overline{\mathbb{R}}^+$:

$$\nu_\varepsilon(\lambda)=\begin{cases}\lambda(1+\varepsilon\lambda)^{-1}, & \lambda\in\mathbb{R}^+,\\ \varepsilon^{-1}, & \lambda=+\infty;\end{cases}$$

$$\eta_a(\lambda)=\begin{cases}\lambda, & \lambda\in[0,a],\\ a, & \lambda\in[a,+\infty];\end{cases}$$

and

$$\chi_\Omega(\lambda)=\begin{cases}1, & \lambda\in\Omega,\\ 0, & \lambda\in\overline{\mathbb{R}}^+\setminus\Omega.\end{cases}$$

The following theorem is a generalization of [1, Theorem 5].

Theorem 3.1. *Let $\mathcal{M}$ be a von Neumann algebra. Let $f:\overline{\mathbb{R}}^+\to\overline{\mathbb{R}}^+$ be such that $f(\mathbb{R}^+)\subset\mathbb{R}^+$ and $f|_{\mathbb{R}^+}$ is $\mathcal{M}$-monotone, and $f(+\infty)\geq\lim_{\lambda\to+\infty}f(\lambda)$. Then for any $m',m''\in\widehat{\mathcal{M}}_+$,*

$$m'\leq m'' \quad\Longrightarrow\quad f(m')\leq f(m'').$$

Proof. Clearly, the case of commutative algebras is trivial. Suppose further that $\mathcal{M}$ is noncommutative. Then it follows from studies in [10] and [1] that f is matrix-monotone of order 2. By [5, Theorem 2.1], this implies that f is continuous on $(0,+\infty)$.

Consider the case $\lim_{\lambda\to+\infty}f(\lambda)=f(+\infty)$. By [14, Lemma IX.4.10(i)], $m'\leq m''$ implies $\nu_\varepsilon(m')\leq\nu_\varepsilon(m'')$ for each $\varepsilon>0$. Since for each $\lambda\in\overline{\mathbb{R}}^+$ the sequence $(\nu_{\frac{1}{n}}(\lambda))$ increasingly converges to λ and f is monotone and continuous on $(0,+\infty]$, we have, for each $\varphi \in \mathcal{M}_*^+$:

$$\varphi\big(f(m')\big) = \lim_{n\to\infty} \varphi\Big(f\big(\nu_{\frac{1}{n}}(m')\big)\Big) \leq \lim_{n\to\infty} \varphi\Big(f\big(\nu_{\frac{1}{n}}(m'')\big)\Big) = \varphi\big(f(m'')\big);$$

that is, $f(m') \leq f(m'')$.

It remains to study the case $\lim_{\lambda\to+\infty} f(\lambda) > f(+\infty)$. Clearly, it suffices to consider the function $\chi_{\{+\infty\}}$. Since $\chi_{\{+\infty\}}(m) = p_m$ for $m \in \widehat{\mathcal{M}}_+$ (see (2.3)) and $m' \leq m''$ implies that $p_{m'} \leq p_{m''}$ by construction (see [14, Theorem IX.4.8]), we get the desired inequality for the function $\chi_{\{+\infty\}}$. □

Corollary 3.2. *Let $f : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ be such that $f(\mathbb{R}^+) \subset \mathbb{R}^+$ and $f|_{\mathbb{R}^+}$ is an operator-monotone function with respect to $\mathcal{M} \otimes \mathcal{B}(H_2)$, and let $f(+\infty) \geq \lim_{\lambda\to+\infty} f(\lambda)$. Then for any $m \in \widehat{\mathcal{M}}_+$ and $C \in \mathcal{M}$ with $\|C\| \leq 1$,*

$$C^* f(m) C \leq f(C^* m C). \tag{3.1}$$

Proof. Note, first, that f is continuous on $(0, +\infty)$ since the restriction of f is supposed to be $\mathcal{M} \otimes \mathcal{B}(H_2)$-monotone.

Analyzing the proof of Hansen's theorem [4] we see that the inequality

$$C^* h(X) C \leq h(C^* X C) \tag{3.2}$$

was proved, in fact, under supposition that $X \in \mathcal{M}_+$, $C \in \mathcal{M}$, $\|C\| \leq 1$, and a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\mathcal{M} \otimes \mathcal{B}(H_2)$-monotone.

In addition to the hypothesis of the lemma, let f be continuous on $\overline{\mathbb{R}}^+$. Then for each $\varepsilon > 0$ it follows from (3.2) that

$$C^* f\big(\nu_\varepsilon(m)\big) C \leq f\big(C^* \nu_\varepsilon(m) C\big). \tag{3.3}$$

Since $f(\nu_{\frac{1}{n}}(m)) \nearrow f(m)$ as $n \to \infty$, we have

$$C^* f\big(\nu_{\frac{1}{n}}(m)\big) C \nearrow C^* f(m) C \quad (n \to \infty). \tag{3.4}$$

Using the fact that $\nu_\varepsilon(m) \leq m$ implies that $C^* \nu_\varepsilon(m) C \leq C^* m C$, from Theorem 3.1 we get

$$f\big(C^* \nu_\varepsilon(m) C\big) \leq f(C^* m C) \quad (\varepsilon > 0). \tag{3.5}$$

From (3.3), (3.4), and (3.5), we conclude that

$$C^* f(m) C \leq f(C^* m C).$$

It remains to exclude the continuity supposition. For this purpose, it suffices to observe that (3.1) holds for the functions $\chi_{(0,\infty]}$ and $\chi_{\{+\infty\}}$ since $\varepsilon \nu_\varepsilon \nearrow \chi_{(0,\infty]}$ as $\varepsilon \nearrow +\infty$ and $\varepsilon \nu_\varepsilon \searrow \chi_{\{+\infty\}}$ as $\varepsilon \searrow 0$, pointwise on $\overline{\mathbb{R}}^+$. □

4. Convexity inequalities

Let $\mathrm{C}[0, +\infty]$ stand for the space of $\mathbb{R}$-valued continuous functions on $\overline{\mathbb{R}}^+$ and let $\mathrm{C}^+[0, +\infty]$ denote the cone of positive functions. We will use the symbol $\xrightarrow{s}$ to denote convergence in the strong operator topology in $\mathcal{M}$.

Proposition 4.1. *For a net $\{m_\alpha\} \subset \widehat{\mathcal{M}}_+$ and $m \in \widehat{\mathcal{M}}_+$, the following statements are equivalent:*

(i) $\nu_1(m_\alpha) \xrightarrow{s} \nu_1(m)$,
(ii) $f(m_\alpha) \xrightarrow{s} f(m)$ *for each* $f \in \mathrm{C}[0,+\infty]$.

Proof. Clearly, it suffices to prove $(i) \implies (ii)$. Let $f \in \mathrm{C}[0,+\infty]$. Since $f = (f \circ \nu_1^{-1}) \circ \nu_1$ and the function $f \circ \nu_1^{-1} : [0,1] \to \mathbb{R}$ is continuous, the implication follows from Kaplansky's results on s-continuity of operator functions (see [6]; see also [13, Section II.4]). □

Lemma 4.2. *Let* $f_n \in \mathrm{C}^+[0,+\infty]$ $(n \in \mathbb{N})$ *such that for any* $\lambda \in \overline{\mathbb{R}}^+$, $f_n(\lambda) \nearrow f(\lambda)$ $(n \to \infty)$. *Let* $\{m_\alpha\} \subset \widehat{\mathcal{M}}_+$, $m \in \widehat{\mathcal{M}}_+$ *satisfying* $m_\alpha \nearrow m$. *Then for any* $\varphi \in \mathcal{M}_*^+$,

$$\varphi\big(f(m)\big) \le \varliminf_{\alpha} \varphi\big(f(m_\alpha)\big). \tag{4.1}$$

Proof. We apply lower semi-continuity arguments as follows. Note, first, that $\varphi(f_n(m)) \nearrow \varphi(f(m))$ $(n \to \infty)$ and $\varphi(f_n(m_\alpha)) \nearrow \varphi(f(m_\alpha))$ $(n \to \infty)$ for each α. By [14, Lemma IX.4.10(ii)], $m_\alpha \nearrow m$ implies that $\nu_1(m_\alpha) \nearrow \nu_1(m)$. Hence $f_n(m_\alpha) \xrightarrow{s} f_n(m)$ each $n \in \mathbb{N}$, by Proposition 4.1. Since the net $\{f_n(m_\alpha)\}$ is uniformly bounded for each $n \in \mathbb{N}$, it follows that $\lim_\alpha \varphi(f_n(m_\alpha)) = \varphi(f_n(m))$ for each $n \in \mathbb{N}$.

Let $\varphi(f(m)) < +\infty$. For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\varphi(f_{n_0}(m)) > \varphi(f(m)) - \varepsilon$. Then there exists α_0 such that $\varphi(f_{n_0}(m_\alpha)) > \varphi(f(m)) - \varepsilon$ when $\alpha \ge \alpha_0$. All the more, $\varphi(f(m_\alpha)) > \varphi(f(m)) - \varepsilon$ when $\alpha \ge \alpha_0$. Therefore, (4.1) holds true. Similar arguments can be applied for the case $\varphi(f(m)) = +\infty$. □

Theorem 4.3. *Let* $f : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ *be such that* $f(\mathbb{R}^+) \subset \mathbb{R}^+$, $f|_{\mathbb{R}^+}$ *is* $\mathcal{M}$*-convex, and* $f(+\infty) = \lim_{\lambda\to+\infty} f(\lambda)$. *Then, for* $m', m'' \in \widehat{\mathcal{M}}_+$ *and* $\alpha \in [0,1]$,

$$f\big(\alpha m' + (1-\alpha)m''\big) \le \alpha f(m') + (1-\alpha) f(m''). \tag{4.2}$$

Proof. For each $n \in \mathbb{N}$, since $f|_{\mathbb{R}^+}$ is $\mathcal{M}$-convex, we have

$$f\big(\alpha\eta_n(m') + (1-\alpha)\eta_n(m'')\big) \le \alpha f\big(\eta_n(m')\big) + (1-\alpha) f\big(\eta_n(m'')\big). \tag{4.3}$$

Also,

$$\alpha\eta_n(m') + (1-\alpha)\eta_n(m'') \nearrow \alpha m' + (1-\alpha)m'' \quad (n \to \infty). \tag{4.4}$$

Therefore, by [14, Lemma IX.4.10(ii)],

$$\nu_1\big(\alpha\eta_n(m') + (1-\alpha)\eta_n(m'')\big) \nearrow \nu_1\big(\alpha m' + (1-\alpha)m''\big) \quad (n \to \infty). \tag{4.5}$$

Now let us consider the case $f(0) = \lim_{\lambda\to+0} f(\lambda)$ (i.e., f is continuous on $\overline{\mathbb{R}}^+$). Let $f(+\infty) \in \mathbb{R}^+$ (i.e., $f \in \mathrm{C}[0,+\infty]$). Then, by Proposition 4.1, from (4.5) this implies that

$$f\big(\alpha\eta_n(m') + (1-\alpha)\eta_n(m'')\big) \xrightarrow{s} f\big(\alpha m' + (1-\alpha)m''\big). \tag{4.6}$$

Clearly,

$$\alpha f\big(\eta_n(m')\big) + (1-\alpha) f\big(\eta_n(m'')\big) \xrightarrow{s} \alpha f(m') + (1-\alpha) f(m''). \tag{4.7}$$

Combining (4.6), (4.7) and (4.3) we obtain (4.2).

Now, let $f(+\infty) = +\infty$. Then there exists $n_0 \in \mathbb{N}$ such that $f(\lambda)$ monotonically increases on $[n_0, +\infty]$. Consequently, $f \circ \eta_{n+1} \geq f \circ \eta_n$ for $n \geq n_0$. Clearly, $f \circ \eta_n \in \mathrm{C}^+[0, +\infty]$ for each $n \in \mathbb{N}$ and $(f \circ \eta_n)(\lambda) \to f(\lambda)$ $(n \to \infty)$ for each $\lambda \in \overline{\mathbb{R}}^+$. Then, for $\varphi \in \mathcal{M}_*^+$, applying (4.4) and Lemma 4.2, we obtain

$$\begin{aligned}\varphi\big(f\big(\alpha m' + (1-\alpha)m''\big)\big) &\leq \varliminf_{n\to\infty} \varphi\big(f\big(\alpha\eta_n(m') + (1-\alpha)\eta_n(m'')\big)\big)\\ &\leq \lim_{n\to\infty} \varphi\big(\alpha f\big(\eta_n(m')\big) + (1-\alpha) f\big(\eta_n(m'')\big)\big)\\ &= \varphi\big(\alpha f(m') + (1-\alpha) f(m'')\big).\end{aligned}$$

Therefore, (4.2) holds true. To complete the proof, it suffices to note that (4.2) holds true for the function $\chi_{\{0\}}$, as is easy to check. $\square$

Remark 4.4. Certainly, the presented results do not exhaust all inequalities for the extended positive part of a von Neumann algebra, which involve operator-monotone and operator-convex functions. For example, we can consider the function $h(\lambda) = \lambda \log \lambda$ $(0 < \lambda < +\infty)$, $h(0) = 0$, $h(+\infty) = +\infty$, correctly define $h(m)$ for $m \in \widehat{\mathcal{M}}_+$, and prove the inequality

$$h\big(\alpha m' + (1-\alpha)m''\big) \leq \alpha h(m') + (1-\alpha)h(m'') \quad (m', m'' \in \widehat{\mathcal{M}}_+,\ 0 \leq \alpha \leq 1).$$

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