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SOME SPECTRA PROPERTIES OF UNBOUNDED 2 × 2 UPPER TRIANGULAR OPERATOR MATRICES

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Abstract. Let $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right] : \mathcal{D}(A) \oplus \mathcal{D}(B) \subset \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{H} \oplus \mathcal{K}$ be a closed operator matrix acting in the Hilbert space $\mathcal{H} \oplus \mathcal{K}$. In this paper, we concern ourselves with the completion problems of M_C. That is, we exactly describe the sets $\bigcup_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_*(M_C)$ and $\bigcap_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$, where $\sigma_*(M_C)$ includes the residual spectrum, the continuous spectrum, and the closed range spectrum of M_C, and $\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ denotes the set of closable operators $C : \mathcal{D}(C) \subset \mathcal{K} \longrightarrow \mathcal{H}$ such that $\mathcal{D}(C) \supset \mathcal{D}(B)$ for a given closed operator B acting in $\mathcal{K}$.

1. Introduction and preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be complex infinite-dimensional separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ (resp., $\mathcal{C}(\mathcal{H},\mathcal{K})$, $\mathcal{C}^+(\mathcal{H},\mathcal{K})$) be the set of all bounded (resp., closed, closable) operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, then we use $\mathcal{B}(\mathcal{H})$, $\mathcal{C}(\mathcal{H})$, and $\mathcal{C}^+(\mathcal{H})$ as usual. The range and kernel of $T \in \mathcal{C}(\mathcal{H},\mathcal{K})$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. We denote $\alpha(T) = \dim \mathcal{N}(T)$ and $d(T) = \dim \mathcal{R}(T)^\perp$ and write $P_{\overline{\mathcal{R}(T)}}$ for the orthogonal projection onto $\overline{\mathcal{R}(T)}$ along $\mathcal{R}(T)^\perp$, where $\mathcal{R}(T)^\perp$ is the orthogonal complement of $\mathcal{R}(T)$.

For $T \in \mathcal{C}(\mathcal{H})$, the residual spectrum $\sigma_r(T)$, the continuous spectrum $\sigma_c(T)$, and the closed range spectrum $\sigma_{\mathrm{cr}}(T)$ of T are, respectively, defined by

$$\sigma_r(T) = \big\{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective, but } \overline{\mathcal{R}(T-\lambda I)} \neq \mathcal{K}\big\},$$

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$$\sigma_c(T) = \big\{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective, and } \mathcal{R}(T - \lambda I) \neq \overline{\mathcal{R}(T - \lambda I)} = \mathcal{K}\big\},$$
$$\sigma_{\mathrm{cr}}(T) = \big\{\lambda \in \mathbb{C} : \mathcal{R}(T - \lambda I) \text{ is not closed}\big\}.$$

Recall the definition of the maximal Tseng inverse of a closed operator T.

Definition 1.1 ([2, p. 339]). Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. If there is a linear operator $T^\dagger : \mathcal{D}(T^\dagger) \subset \mathcal{K} \longrightarrow \mathcal{H}$ such that $\mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$, $\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp$, and

$$T^\dagger T x = P_{\overline{\mathcal{R}(T^\dagger)}} x, \quad x \in \mathcal{D}(T),$$
$$T T^\dagger y = P_{\overline{\mathcal{R}(T)}} y, \quad y \in \mathcal{D}(T^\dagger),$$

then $T^\dagger$ is called the *maximal Tseng inverse* of T.

Completion problems of operator matrices play an important role in dilation theory, commutant lifting theory, and interpolation theory (see [4]). Recently, numerous authors have studied completion problems of 2×2 bounded upper triangular operator matrices and obtained several results (see, e.g., [1], [3], [5]–[7], [9]). It is worth mentioning that, in [6], Hai and Chen characterized the sets $\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_C)$ and $\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_c(M_C)$ as follows:

$$\begin{aligned}
&\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_C) \\
&\quad = \Big[\big\{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } d(A - \lambda I) + d(B - \lambda I) > 0\big\} \\
&\qquad \cup \big\{\lambda \in \mathbb{C} : \alpha(B - \lambda I) < d(A - \lambda I) + d(B - \lambda I), \\
&\qquad \alpha(B - \lambda I) \leq d(A - \lambda I)\big\} \\
&\qquad \cup \big\{\lambda \in \mathbb{C} : \alpha(B - \lambda I) = d(A - \lambda I) = \infty\big\}\Big] \backslash \sigma_p(A)
\end{aligned}$$

and

$$\begin{aligned}
&\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_c(M_C) \\
&\quad = \Big[\big\{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I) \text{ is not closed, } d(A - \lambda I) \leq \alpha(B - \lambda I)\big\} \\
&\qquad \cup \big\{\lambda \in \mathbb{C} : \mathcal{R}(B - \lambda I) \text{ is not closed, } d(A - \lambda I) \geq \alpha(B - \lambda I)\big\} \\
&\qquad \cup \big\{\lambda \in \mathbb{C} : d(A - \lambda I) = \alpha(B - \lambda I) = \infty\big\}\Big] \\
&\qquad \backslash \big\{\lambda \in \mathbb{C} : \lambda \in \sigma_p(A) \text{ or } d(B - \lambda I) \neq 0\big\},
\end{aligned}$$

where $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ is a bounded upper triangular operator matrix for given bounded operators A, B. However, these results in the unbounded case have not been considered.

The main goal of this paper is to investigate the properties of the unbounded upper triangular operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(B) \subset \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{H} \oplus \mathcal{K},$$

where A is a bounded or closed operator with dense domain, B is a closed operator with dense domain, and C is a closable operator such that $\mathcal{D}(C) \supset \mathcal{D}(B)$. It is not hard to check that M_C is a closed operator matrix. In this note, applying the space decomposition technique and the maximal Tseng inverse of closed operator, we describe the sets $\bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})}\sigma_r(M_C)$ and $\bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})}\sigma_c(M_C)$ for given bounded operator A and closed operator B, where

$$\mathcal{C}_B^+(\mathcal{K},\mathcal{H})=\{C\in\mathcal{C}^+(\mathcal{K},\mathcal{H}):\mathcal{D}(C)\supset\mathcal{D}(B)\text{ for given closed operator }B\}.$$

These are the extensions of the results in [6]. Moreover, we also obtain the sets $\bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{cr}}(M_C)$ and $\bigcap_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{cr}}(M_C)$ for given closed operators A, B, which extend the results in [5]. It is easy to note that $\sigma_{\mathrm{cr}}(T)=\sigma_M(T)$ for bounded operator T, where $\sigma_M(T)=\{\lambda\in\mathbb{C}:T-\lambda I\text{ is not Moore-Penrose invertible}\}$ is the Moore–Penrose spectrum of T. We conclude with some examples.

For the proof of the main results in the next section, we need the following lemmas.

Lemma 1.2. *Let $A\in\mathcal{C}(\mathcal{H})$ and $B\in\mathcal{C}(\mathcal{K})$. Then $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$ for some $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ if and only if $\overline{\mathcal{R}(A)}\neq\mathcal{H}$ or $\overline{\mathcal{R}(B)}\neq\mathcal{K}$.*

Proof. Let $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$ for some $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$. Then for every $x=(x_1x_2)^T\in\mathcal{D}(M_C)$ there exists $0\neq z=(z_1z_2)^T\in\mathcal{R}(M_C)^\perp$ such that

$$(M_Cx,z)=0.$$

That is, $(Ax_1+Cx_2,z_1)=0$ and $(Bx_2,z_2)=0$. If $z_2\neq 0$, then $z_2\in\mathcal{R}(B)^\perp$ and hence $\overline{\mathcal{R}(B)}\neq\mathcal{K}$. If $z_2=0$, then $z_1\neq 0$. Set $x_2=0$; then we get $\overline{\mathcal{R}(A)}\neq\mathcal{H}$. Conversely, if $\overline{\mathcal{R}(B)}\neq\mathcal{H}$, then $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$ for every $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$; if $\overline{\mathcal{R}(A)}\neq\mathcal{K}$, then $\overline{\mathcal{R}(M_0)}\neq\mathcal{H}\oplus\mathcal{K}$, where $C=0$. □

Lemma 1.3. *Let $A\in\mathcal{C}(\mathcal{H})$ and $B\in\mathcal{C}(\mathcal{K})$, and let $\mathcal{R}(A)$ be closed.*

(a) *If $\alpha(B)>d(A)$, then M_C is not injective for every $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$.*

(b) *If $\alpha(B)=d(A)<\infty$, and M_C is injective for every $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$, then $P_{\mathcal{R}(A)^\perp}C\mid_{\mathcal{N}(B)}:\mathcal{N}(B)\longrightarrow\mathcal{R}(A)^\perp$ is invertible for every $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$.*

Proof. Since $\mathcal{R}(A)$ is closed, then M_C admits the representation

$$M_C=\begin{bmatrix}A_1&C_1&C_2\\0&C_3&C_4\\0&0&B_1\end{bmatrix}:\begin{bmatrix}\mathcal{D}(A)\\\mathcal{N}(B)\\\mathcal{N}(B)^\perp\cap\mathcal{D}(B)\end{bmatrix}\longrightarrow\begin{bmatrix}\mathcal{R}(A)\\\mathcal{R}(A)^\perp\\\mathcal{K}\end{bmatrix},$$

and there exists the maximal Tseng inverse $A_1^\dagger$ of A_1 such that $A_1A_1^\dagger=I_{\mathcal{R}(A)}$. Set

$$Q_1=\begin{bmatrix}I&-A_1^\dagger C_1&-A_1^\dagger C_2\\0&I&0\\0&0&I\end{bmatrix}:\begin{bmatrix}\mathcal{D}(A)\\\mathcal{N}(B)\\\mathcal{N}(B)^\perp\cap\mathcal{D}(B)\end{bmatrix}\longrightarrow\begin{bmatrix}\mathcal{D}(A)\\\mathcal{N}(B)\\\mathcal{N}(B)^\perp\cap\mathcal{D}(B)\end{bmatrix}.$$

Then

$$M_CQ_1=\begin{bmatrix}A_1&0&0\\0&C_3&C_4\\0&0&B_1\end{bmatrix}.$$

(a) It follows from $\alpha(B)>d(A)$ that $C_3:\mathcal{N}(B)\longrightarrow\mathcal{R}(A)^\perp$ is not injective. Hence M_C is not injective for every $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ by the injection of Q_1.

(b) If M_C is injective for every $C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})$, then C_3 is injective. It follows from $\alpha(B) = d(A) < \infty$ that $C_3 = P_{\mathcal{R}(A)^\perp} C \mid_{\mathcal{N}(B)}: \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^\perp$ is invertible.

□

Lemma 1.4 ([6, Lemma 2.3]). *Let $A \in \mathcal{B}(\mathcal{H})$, and assume that $\mathcal{R}(A)$ is not closed. Then there is an infinite-dimensional closed subspace $\mathcal{M}$ of $\overline{\mathcal{R}(A)}$ such that $\mathcal{M} \cap \mathcal{R}(A) = \{0\}$.*

Lemma 1.5 ([8, p. 65]). *Let $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right] : \mathcal{D}(A) \oplus \mathcal{D}(B) \subset \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{H} \oplus \mathcal{K}$ be a closed operator matrix with $\mathcal{R}(M_C)$ closed. If $\overline{\mathcal{R}(A)} = \mathcal{H}$, then $\mathcal{R}(B)$ is closed.*

2. Main results

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{C}(\mathcal{K})$ with dense domains. Then*

$$\begin{aligned}\bigcup_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} &\sigma_r(M_C) \\ &= \Big[\big\{\lambda \in \mathbb{C} : \mathcal{R}(A-\lambda I) \text{ is not closed, } d(A-\lambda I) + d(B-\lambda I) > 0\big\} \\ &\quad \cup \big\{\lambda \in \mathbb{C} : \alpha(B-\lambda I) < d(A-\lambda I) + d(B-\lambda I), \\ &\quad \alpha(B-\lambda I) \le d(A-\lambda I)\big\} \\ &\quad \cup \big\{\lambda \in \mathbb{C} : \alpha(B-\lambda I) = d(A-\lambda I) = \infty\big\}\Big] \backslash \sigma_p(A).\end{aligned}$$

Proof. First, we verify that the left-hand side is contained in the right-hand side. Assume without loss of generality that there exists $C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ such that $0 \in \sigma_r(M_C)$; that is, M_C is injective and $\overline{\mathcal{R}(M_C)} \ne \mathcal{H} \oplus \mathcal{K}$. Then we get that A is also injective and that $\overline{\mathcal{R}(A)} \ne \mathcal{H}$ or $\overline{\mathcal{R}(B)} \ne \mathcal{K}$ by Lemma 1.2. That is, $0 \notin \sigma_p(A)$ and $d(A) + d(B) > 0$. Now we consider the following two cases.

Case I: Assume that $\mathcal{R}(A)$ is not closed. We still get that $0 \notin \sigma_p(A)$ and $d(A) + d(B) > 0$.

Case II: Assume that $\mathcal{R}(A)$ is closed. Then we get $\alpha(B) \le d(A)$ from Lemma 1.3(a). If $\alpha(B) = \infty$, then $d(A) = \infty$; that is, $\alpha(B) = d(A) = \infty$. If $\alpha(B) < \infty$, then we obtain $d(A) + d(B) > \alpha(B)$. Otherwise, suppose that $d(A) + d(B) \le \alpha(B)$. Then from $\alpha(B) \le d(A)$, we have that $d(B) = 0$ and $d(A) = \alpha(B) < \infty$. Hence C_3 is invertible by Lemma 1.3(b). Set

$$Q_2 = \begin{bmatrix} I & -A_1^\dagger C_1 & -A_1^\dagger C_2 + -A_1^\dagger C_1 C_3^{-1} C_4 \\ 0 & I & -C_3^{-1} C_4 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \cap \mathcal{D}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \cap \mathcal{D}(B) \end{bmatrix}.$$

Then $M_C Q_2 = \left[\begin{smallmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{smallmatrix}\right]$. Clearly, $\overline{\mathcal{R}(M_C Q_2)} = \mathcal{H} \oplus \mathcal{K}$, and hence $\overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$ by $\mathcal{R}(M_C Q_2) \subset \mathcal{R}(M_C)$. This leads to a contradiction.

Next to prove the opposite inclusion, we consider the following three cases.

Case I: Suppose that A is injective, $\mathcal{R}(A)$ is not closed, and $d(A)+d(B)>0$. Then $\dim\mathcal{R}(A)=\infty$, and hence we get an infinite-dimensional closed subspace $\mathcal{M}$ of $\overline{\mathcal{R}(A)}$ by Lemma 1.4. Set $C=\left[\begin{smallmatrix} J_1 & 0\\ 0 & 0\end{smallmatrix}\right]:\mathcal{N}(B)\oplus\mathcal{N}(B)^\perp\longrightarrow\overline{\mathcal{R}(A)}\oplus\mathcal{R}(A)^\perp$, where $J_1:\mathcal{N}(B)\longrightarrow\mathcal{M}$ is a unitary operator. Then M_C has an operator matrix representation

$$M_C=\begin{bmatrix} A_1 & J_1 & 0\\ 0 & 0 & 0\\ 0 & 0 & B_1\end{bmatrix}:\begin{bmatrix}\mathcal{D}(A)\\ \mathcal{N}(B)\\ \mathcal{N}(B)^\perp\cap\mathcal{D}(B)\end{bmatrix}\longrightarrow\begin{bmatrix}\overline{\mathcal{R}(A)}\\ \mathcal{R}(A)^\perp\\ \mathcal{K}\end{bmatrix}.$$

It is not hard to see that M_C is injective by the definition of J_1 and the injection of A, B_1. We also get $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$ from $d(A)+d(B)>0$. That is, $0\in\sigma_r(M_C)$.

Case II: Suppose that A is injective and that $\alpha(B)\leq d(A)$ and $\alpha(B)<d(A)+d(B)$. Then there exists an injection $J_2:\mathcal{N}(B)\longrightarrow\mathcal{R}(A)^\perp$. Hence M_C is injective, where

$$M_C=\begin{bmatrix} A_1 & 0 & 0\\ 0 & J_2 & 0\\ 0 & 0 & B_1\end{bmatrix}:\begin{bmatrix}\mathcal{D}(A)\\ \mathcal{N}(B)\\ \mathcal{N}(B)^\perp\cap\mathcal{D}(B)\end{bmatrix}\longrightarrow\begin{bmatrix}\overline{\mathcal{R}(A)}\\ \mathcal{R}(A)^\perp\\ \mathcal{K}\end{bmatrix}.$$

If $d(B)>0$, then $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$. If $d(B)=0$, then $d(A)>\alpha(B)$ since $\alpha(B)<d(A)+d(B)$. Hence $\overline{\mathcal{R}(J_2)}=\mathcal{R}(J_2)\neq\mathcal{R}(A)^\perp$. Thus $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$. Therefore $0\in\sigma_r(M_C)$.

Case III: Suppose that A is injective and that $\alpha(B)=d(A)=\infty$. Then there exists an injection $J_3:\mathcal{N}(B)\longrightarrow\mathcal{R}(A)^\perp$ such that $\overline{\mathcal{R}(J_3)}\neq\mathcal{R}(A)^\perp$. Hence M_C is injective and $\overline{\mathcal{R}(M_C)}\neq\mathcal{H}\oplus\mathcal{K}$, where

$$M_C=\begin{bmatrix} A_1 & 0 & 0\\ 0 & J_3 & 0\\ 0 & 0 & B_1\end{bmatrix}:\begin{bmatrix}\mathcal{D}(A)\\ \mathcal{N}(B)\\ \mathcal{N}(B)^\perp\cap\mathcal{D}(B)\end{bmatrix}\longrightarrow\begin{bmatrix}\overline{\mathcal{R}(A)}\\ \mathcal{R}(A)^\perp\\ \mathcal{K}\end{bmatrix}.$$

Therefore $0\in\sigma_r(M_C)$. This proof is complete. □

Theorem 2.2. *Let $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{C}(\mathcal{K})$ with dense domains. Then*

$$\begin{aligned}&\bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})}\sigma_c(M_C)\\ &\quad=\big[\{\lambda\in\mathbb{C}:\mathcal{R}(A-\lambda I)\text{ is not closed},\ d(A-\lambda I)\leq\alpha(B-\lambda I)\}\\ &\qquad\cup\{\lambda\in\mathbb{C}:\mathcal{R}(B-\lambda I)\text{ is not closed},\ d(A-\lambda I)\geq\alpha(B-\lambda I)\}\\ &\qquad\cup\{\lambda\in\mathbb{C}:d(A-\lambda I)=\alpha(B-\lambda I)=\infty\}\big]\\ &\qquad\setminus\{\lambda\in\mathbb{C}:\lambda\in\sigma_p(A)\text{ or }d(B-\lambda I)\neq 0\}.\end{aligned}$$

Proof. First, we prove that the right-hand side contains the left-hand side. Without loss of generality, we suppose that there exists $C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ such that $0\in\sigma_c(M_C)$; that is, M_C is injective and $\mathcal{R}(M_C)\neq\overline{\mathcal{R}(M_C)}=\mathcal{H}\oplus\mathcal{K}$. Then A is injective and $d(B)=0$. That is, $0\notin\{\lambda\in\mathbb{C}:\lambda\in\sigma_p(A)\text{ or }d(B-\lambda I)\neq 0\}$. There are three cases to be considered.

Case I: Suppose that $\mathcal{R}(B)$ is not closed; then $\mathcal{R}(A)$ is not closed or $d(A) \geq \alpha(B)$. Otherwise, assume that $\mathcal{R}(A)$ is closed and that $d(A) < \alpha(B)$. It follows from Lemma 1.3(a) that M_C is not injective. This is a contradiction.

Case II: Suppose that $\mathcal{R}(A)$ is not closed; then we get that $\mathcal{R}(B)$ is not closed or $d(A) \leq \alpha(B)$. Otherwise, assume that $\mathcal{R}(B)$ is closed and that $d(A) > \alpha(B)$. Then $\mathcal{R}(B) = \mathcal{K}$, and hence M_C has the following representation:

$$M_C = \begin{bmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \cap \mathcal{D}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{bmatrix}.$$

Set $P_1 = \left[\begin{smallmatrix} I & 0 & -B_1^{-1}C_2 \\ 0 & I & -B_1^{-1}C_4 \\ 0 & 0 & I \end{smallmatrix}\right] : \left[\begin{smallmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{smallmatrix}\right]$. Then

$$P_1 M_C = \begin{bmatrix} A_1 & C_1 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{bmatrix}.$$

It follows from $\overline{\mathcal{R}(M_C)} = \mathcal{H} \oplus \mathcal{K}$ and the bijection of P_1 that $\overline{\mathcal{R}(P_1 M_C)} = \mathcal{H} \oplus \mathcal{K}$. On the other hand, we get $\mathcal{R}(C_3) = \overline{\mathcal{R}(C_3)} \neq \mathcal{R}(A)^\perp$ from $d(A) > \alpha(B)$, and hence $\overline{\mathcal{R}(P_1 M_C)} \neq \mathcal{H} \oplus \mathcal{K}$, which is a contradiction.

Case III: Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Then M_C has the following representation:

$$M_C = \begin{bmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \cap \mathcal{D}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{bmatrix}.$$

Set $P_2 = \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & -B_1^{-1}C_4 \\ 0 & 0 & I \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{smallmatrix}\right]$. Then we get

$$P_2 M_C Q_1 = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{bmatrix}.$$

It follows from the closedness of $\mathcal{R}(M_C)$ and the bijection of P_2, Q_1 that $\mathcal{R}(P_2 M_C Q_1)$ is not closed. Then $\mathcal{R}(C_3) \neq \overline{\mathcal{R}(C_3)}$, and thus $d(A) = \alpha(B) = \infty$.

Next we verify that the reverse case. For this we will consider three cases.

Case I: Assume that A is injective, that $d(B) = 0$, that $\mathcal{R}(A)$ is not closed, and that $d(A) \leq \alpha(B)$. Then there exist two closed subspaces $\triangle_1, \triangle_2$ of $\mathcal{N}(B)$ such that $\mathcal{N}(B) = \triangle_1 \oplus \triangle_2$ and $\dim \triangle_2 = d(A)$. Set $C = \left[\begin{smallmatrix} J_1 & 0 & 0 \\ 0 & U_1 & 0 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \triangle_1 \\ \triangle_2 \\ \mathcal{N}(B)^\perp \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{smallmatrix}\right]$, where $J_1 : \triangle_1 \longrightarrow \mathcal{M} \subset \overline{\mathcal{R}(A)}$ is a unitary operator by Lemma 1.4 and $U_1 : \triangle_2 \longrightarrow \mathcal{R}(A)^\perp$ is also a unitary operator. It follows from the definitions of J_1, U_1 and the bijection of A_1, B_1 that M_C is injective, where

$$M_C = \begin{bmatrix} A_1 & J_1 & 0 & 0 \\ 0 & 0 & U_1 & 0 \\ 0 & 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \triangle_1 \\ \triangle_2 \\ \mathcal{N}(B)^{\perp} \cap \mathcal{D}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \\ \mathcal{K} \end{bmatrix}.$$

We also see that $d(M_C) = 0$ and that $\mathcal{R}(M_C)$ is not closed from $d(B) = 0$ and the unclosedness of $\mathcal{R}(A)$. That is, $0 \in \sigma_c(M_C)$.

Case II: Assume that A is injective, $d(B) = 0$, that $\mathcal{R}(B)$ is not closed, and that $d(A) \geq \alpha(B)$.

If $d(A) = \alpha(B)$, set $C = \left[\begin{smallmatrix} 0 & 0 \\ U & 0 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{smallmatrix}\right]$, where $U : \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^{\perp}$ is a unitary operator. It is easy to see that $0 \in \sigma_c(M_C)$.

If $\alpha(B) < d(A) < \infty$, then there exist two finite-dimensional subspaces Ω_1, Ω_2 of $\mathcal{R}(A)^{\perp}$ such that $\mathcal{R}(A)^{\perp} = \Omega_1 \oplus \Omega_2$ and $\dim \Omega_1 = \alpha(B)$. Hence there exists a finite-dimensional subspace Ω_2' of $\mathcal{R}(B)$ such that $\dim \Omega_2 = \dim \Omega_2'$ and thus there is a unitary operator $U : \Omega_2' \longrightarrow \Omega_2$. We define $J_2 : \mathcal{K} \longrightarrow \Omega_2$ as

$$J_2 x = \begin{cases} Ux, & x \in \Omega_2', \\ 0, & x \in \mathcal{K} \backslash \Omega_2'. \end{cases}$$

Clearly, J_2 is surjective. Set $C = \left[\begin{smallmatrix} 0 & 0 \\ U_2 & 0 \\ 0 & J_2 B_1 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \cap \mathcal{D}(B) \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \overline{\mathcal{R}(A)} \\ \Omega_1 \\ \Omega_2 \end{smallmatrix}\right]$, where $U_2 : \mathcal{N}(B) \longrightarrow \Omega_1$ is a unitary operator. It is not hard to see that M_C is injective and $d(M_C) = 0$, where

$$M_C = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & J_2 B_1 \\ 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \cap \mathcal{D}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \Omega_1 \\ \Omega_2 \\ \mathcal{K} \end{bmatrix}.$$

We also obtain that $\mathcal{R}(M_C)$ is not closed since $\mathcal{R}(B)$ is not closed. Therefore $0 \in \sigma_c(M_C)$.

If $\alpha(B) < d(A) = \infty$, then there exist two closed subspaces Ω_1, Ω_2 of $\mathcal{R}(A)^{\perp}$ such that $\mathcal{R}(A)^{\perp} = \Omega_1 \oplus \Omega_2$ and $\dim \Omega_1 = \alpha(B) < \infty$ and $\dim \Omega_2 = \infty$. Set $C = \left[\begin{smallmatrix} 0 & 0 \\ U_2 & 0 \\ 0 & U_3 B_1 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \cap \mathcal{D}(B) \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \overline{\mathcal{R}(A)} \\ \Omega_1 \\ \Omega_2 \end{smallmatrix}\right]$, where $U_2 : \mathcal{N}(B) \longrightarrow \Omega_1$ and $U_3 : \mathcal{K} \longrightarrow \Omega_2$ are unitary operators. It is not hard to verify that $0 \in \sigma_c(M_C)$.

Case III: Suppose that A is injective, $d(B) = 0$, and $d(A) = \alpha(B) = \infty$. Then there is an injective operator $J : \mathcal{N}(B) \longrightarrow \mathcal{R}(A)^{\perp}$ such that $\mathcal{R}(J) \neq \overline{\mathcal{R}(J)} = \mathcal{R}(A)^{\perp}$. Set

$$M_C = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \cap \mathcal{D}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \\ \mathcal{K} \end{bmatrix}.$$

Then $0 \in \sigma_c(M_C)$. This proof is complete. □

Remark 2.3. Clearly, Theorems 2.1 and 2.2 extend the result of [6].

The next two main results are about the closed range spectrum completion problems of M_C for given closed operators A and B.

Theorem 2.4. *Let $A \in \mathcal{C}(\mathcal{H})$ and $B \in \mathcal{C}(\mathcal{K})$ with dense domains. Then*

$$\bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C) = \sigma_{\mathrm{cr}}(A) \cup \sigma_{\mathrm{cr}}(B) \cup \big\{\lambda \in \mathbb{C} : \alpha(B-\lambda I) = d(A-\lambda I) = \infty\big\}.$$

Proof. First, we verify that the right-hand side contains the left-hand side. Let $\lambda \in \bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$. Then there exists some $C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ such that $\mathcal{R}(M_C-\lambda I)$ is not closed. Suppose that $\lambda \notin \sigma_{\mathrm{cr}}(A)\cup\sigma_{\mathrm{cr}}(B)\cup\{\lambda \in \mathbb{C} : \alpha(B-\lambda I) = d(A-\lambda I) = \infty\}$; that is, $\lambda \in \{\lambda \in \mathbb{C} : \mathcal{R}(A-\lambda I)$ and $\mathcal{R}(B-\lambda I)$ are closed and $\min\{\alpha(B-\lambda I), d(A-\lambda I)\} < \infty\}$. Then $M_C - \lambda I$ admits the following decomposition:

$$M_C - \lambda I = \begin{bmatrix} A_1(\lambda) & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1(\lambda) \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A-\lambda I) \\ \mathcal{N}(B-\lambda I) \\ \mathcal{N}(B-\lambda I)^\perp \cap \mathcal{D}(B-\lambda I) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A-\lambda I) \\ \mathcal{R}(A-\lambda I)^\perp \\ \mathcal{R}(B-\lambda I) \\ \mathcal{R}(B-\lambda I)^\perp \end{bmatrix}.$$

Set $U = \begin{bmatrix} I & I & -C_2B_1^{-1}(\lambda) & 0 \\ 0 & I & -C_4B_1^{-1}(\lambda) & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A-\lambda I) \\ \mathcal{R}(A-\lambda I)^\perp \\ \mathcal{R}(B-\lambda I) \\ \mathcal{R}(B-\lambda I)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A-\lambda I) \\ \mathcal{R}(A-\lambda I)^\perp \\ \mathcal{R}(B-\lambda I) \\ \mathcal{R}(B-\lambda I)^\perp \end{bmatrix}$ and $V = \begin{bmatrix} I & -A_1^\dagger(\lambda)C_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A-\lambda I) \\ \mathcal{N}(B-\lambda I) \\ \mathcal{N}(B-\lambda I)^\perp\cap\mathcal{D}(B-\lambda I) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{D}(A-\lambda I) \\ \mathcal{N}(B-\lambda I) \\ \mathcal{N}(B-\lambda I)^\perp\cap\mathcal{D}(B-\lambda I) \end{bmatrix}$. Then

$$U(M_C - \lambda I)V = \begin{bmatrix} A_1(\lambda) & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1(\lambda) \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $\mathcal{R}(C_3)$ is closed by $\min\{\alpha(B-\lambda I), d(A-\lambda I)\} < \infty$, and then $\mathcal{R}(C_3)$ is closed from the bijection of U, V. This is a contradiction. Therefore $\lambda \in \sigma_{\mathrm{cr}}(A) \cup \sigma_{\mathrm{cr}}(B) \cup \{\lambda \in \mathbb{C} : \alpha(B-\lambda I) = d(A-\lambda I) = \infty\}$.

Next, we prove the converse conclusion. Let $\lambda \in \sigma_{\mathrm{cr}}(A) \cup \sigma_{\mathrm{cr}}(B)$. Then $\lambda \in \sigma_{\mathrm{cr}}(M_{C_0})$, where $C_0 = 0$. Thus

$$\lambda \in \bigcup_{C\in\mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C).$$

Let $\lambda \in \{\lambda \in \mathbb{C} : \alpha(B-\lambda I) = d(A-\lambda I)\} \setminus (\sigma_{\mathrm{cr}}(A) \cup \sigma_{\mathrm{cr}}(B)) = \{\lambda \in \mathbb{C} : \mathcal{R}(A-\lambda I)$ and $\mathcal{R}(B-\lambda I)$ are closed and $\alpha(B-\lambda I) = d(A-\lambda I) = \infty\}$. Set $S : \mathcal{N}(B-\lambda I) \longrightarrow \mathcal{R}(A-\lambda I)^\perp$ such that

$$S(f_i) = \frac{1}{i} g_i, \quad i = 1, 2, \ldots,$$

where $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are the standard orthogonal bases of $\mathcal{N}(B-\lambda I)$ and $\mathcal{R}(A-\lambda I)^{\perp}$, respectively. It is easy to prove that $\mathcal{R}(S)$ is not closed. Set

$$C=\begin{bmatrix}0 & 0\\ S & 0\end{bmatrix}:\begin{bmatrix}\mathcal{N}(B-\lambda I)\\ \mathcal{N}(B-\lambda I)^{\perp}\end{bmatrix}\longrightarrow\begin{bmatrix}\mathcal{R}(A-\lambda I)\\ \mathcal{R}(A-\lambda I)^{\perp}\end{bmatrix}.$$

Then $\mathcal{R}(M_C-\lambda I)$ also is not closed, that is, $\lambda\in\sigma_{\mathrm{cr}}(M_C)$, where $M_C-\lambda I=\begin{bmatrix}A_1(\lambda) & 0 & 0\\ 0 & S & 0\\ 0 & 0 & B_1(\lambda)\\ 0 & 0 & 0\end{bmatrix}:\begin{bmatrix}\mathcal{D}(A-\lambda I)\\ \mathcal{N}(B-\lambda I)\\ \mathcal{N}(B-\lambda I)^{\perp}\cap\mathcal{D}(B-\lambda I)\end{bmatrix}\longrightarrow\begin{bmatrix}\mathcal{R}(A-\lambda I)\\ \mathcal{R}(A-\lambda I)^{\perp}\\ \mathcal{R}(B-\lambda I)\\ \mathcal{R}(B-\lambda I)^{\perp}\end{bmatrix}$. Hence

$$\lambda\in\bigcup_{C\in\mathcal{C}_B^{+}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{cr}}(M_C).$$

The proof is complete. $\square$

We immediately obtain the following corollary which extends the result of [5].

Corollary 2.5. *Let $A\in\mathcal{C}(\mathcal{H})$ and $B\in\mathcal{C}(\mathcal{K})$ with dense domains. Then*

$$\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{cr}}(M_C)=\sigma_{\mathrm{cr}}(A)\cup\sigma_{\mathrm{cr}}(B)\cup\big\{\lambda\in\mathbb{C}:\alpha(B-\lambda I)=d(A-\lambda I)=\infty\big\}.$$

Remark 2.6. For given $A\in\mathcal{C}(\mathcal{H})$ and $B\in\mathcal{C}(\mathcal{K})$ with dense domains, we have $\sigma_{\mathrm{cr}}(M_C)\nsubseteq\sigma_{\mathrm{cr}}(A)\cup\sigma_{\mathrm{cr}}(B)$. In fact, assume that $\sigma_{\mathrm{cr}}(M_C)\subset\sigma_{\mathrm{cr}}(A)\cup\sigma_{\mathrm{cr}}(B)$ for every $C\in\mathcal{C}_B^{+}(\mathcal{K},\mathcal{H})$. Then $\bigcup_{C\in\mathcal{C}_B^{+}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{cr}}(M_C)\subset\sigma_{\mathrm{cr}}(A)\cup\sigma_{\mathrm{cr}}(B)$. This contradicts the result of Theorem 2.4.

Theorem 2.7. *Let $A\in\mathcal{C}(\mathcal{H})$ and $B\in\mathcal{C}(\mathcal{K})$ with dense domains. Then*

$$\bigcap_{C\in\mathcal{C}_B^{+}(\mathcal{K},\mathcal{H})}\sigma_{\mathrm{cr}}(M_C)=\big\{\lambda\in\sigma_{\mathrm{cr}}(A):\alpha(B-\lambda I)<\infty\big\}\cup\big\{\lambda\in\sigma_{\mathrm{cr}}(B):d(A-\lambda I)<\infty\big\}.$$

Proof. First, we prove that the left-hand side contains the right-hand side.

If $\lambda\in\{\lambda\in\sigma_{\mathrm{cr}}(B):d(A-\lambda I)<\infty\}$, then $\mathcal{R}(M_C-\lambda I)$ is not closed for every $C\in\mathcal{C}_B^{+}(\mathcal{K},\mathcal{H})$. In fact, assume that there exists $C\in\mathcal{C}_B^{+}(\mathcal{K},\mathcal{H})$ such that $\mathcal{R}(M_C-\lambda I)$ is closed. Then from

$$M_C-\lambda I=\begin{bmatrix}0 & A_1(\lambda) & C_1 & C_2\\ 0 & 0 & C_3 & C_4\\ 0 & 0 & 0 & B_1(\lambda)\\ 0 & 0 & 0 & 0\end{bmatrix}:\begin{bmatrix}\mathcal{N}(A-\lambda I)\\ \mathcal{N}(A-\lambda I)^{\perp}\cap\mathcal{D}(A-\lambda I)\\ \mathcal{N}(B-\lambda I)\\ \mathcal{N}(B-\lambda I)^{\perp}\cap\mathcal{D}(B-\lambda I)\end{bmatrix}\longrightarrow\begin{bmatrix}\overline{\mathcal{R}(A-\lambda I)}\\ \mathcal{R}(A-\lambda I)^{\perp}\\ \overline{\mathcal{R}(A-\lambda I)}\\ \mathcal{R}(A-\lambda I)^{\perp}\end{bmatrix},$$

we see that $\mathcal{R}(M'(\lambda))$ is closed, where $M'(\lambda)=\begin{bmatrix}A_1(\lambda) & C_1 & C_2\\ 0 & C_3 & C_4\\ 0 & 0 & B_1(\lambda)\end{bmatrix}$. Thus by Lemma 1.5, we have that $\mathcal{R}\left(\begin{bmatrix}C_3 & C_4\\ 0 & B_1(\lambda)\end{bmatrix}\right)$ is closed. It follows from $d(A-\lambda I)<\infty$

that C_3, C_4 are compact. Hence $\mathcal{R}\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & B_1(\lambda) \end{smallmatrix}\right]\right)$ is closed; that is, $\mathcal{R}(B_1(\lambda)$ is closed which contradicts $\lambda \in \sigma_{\mathrm{cr}}(B)$. Therefore

$$\lambda \in \bigcap_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C).$$

Let $\lambda \in \{\lambda \in \sigma_{\mathrm{cr}}(A) : \alpha(B-\lambda I) < \infty\}$. Suppose that there exists $C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})$ such that $\mathcal{R}(M_C - \lambda I)$ is closed. It follows from the above decomposition that $\mathcal{R}(M'(\lambda))$ is closed. Also, from $\alpha(B - \lambda I) < \infty$ we get that C_1, C_3 are compact. Then $\mathcal{R}(\widetilde{M}(\lambda))$ is closed, where $\widetilde{M}(\lambda) = \left[\begin{smallmatrix} A_1(\lambda) & 0 & C_2 \\ 0 & 0 & C_4 \\ 0 & 0 & B_1(\lambda) \end{smallmatrix}\right]$. Since $\alpha(\widetilde{M}(\lambda)) = \alpha(B - \lambda I) < \infty$, then $\widetilde{M}(\lambda)$ is left Fredholm. So $A_1(\lambda)$ is also left Fredholm. Then $\mathcal{R}(A_1(\lambda)) = \mathcal{R}(A - \lambda I)$ is closed, which leads to a contradiction. Hence

$$\lambda \in \bigcap_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C).$$

Next we verify the converse conclusion. Let $\lambda \in \bigcap_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$, but $\lambda \notin \{\lambda \in \sigma_{\mathrm{cr}}(A) : \alpha(B - \lambda I) < \infty\} \cup \{\lambda \in \sigma_{\mathrm{cr}}(B) : d(A - \lambda I) < \infty\}$. Clearly, the following four cases will be considered.

Case I: Assume that $\lambda \in \{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I)$ and $\mathcal{R}(B - \lambda I)$ are closed$\}$. Set $C = 0$. Then $\mathcal{R}(M_C - \lambda I)$ is closed. Hence $\lambda \notin \bigcap_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$, which leads to a contradiction.

Case II: Assume that $\lambda \in \{\lambda \in \mathbb{C} : \mathcal{R}(A - \lambda I)$ is closed, $d(A - \lambda I) = \infty\}$. If $\mathcal{R}(B - \lambda I)$ is closed, then the proof is the same as that of case I. If $\mathcal{R}(B - \lambda I)$ is not closed, then $\mathcal{R}(B^* - \overline{\lambda} I)$ is also not closed. Thus $\dim \mathcal{N}(B - \lambda I)^\perp = \infty$. Hence there exists a unitary operator $U : \mathcal{N}(B - \lambda I)^\perp \longrightarrow \mathcal{R}(A - \lambda I)^\perp$. Set

$$C = \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B - \lambda I) \\ \mathcal{N}(B - \lambda I)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A - \lambda I) \\ \mathcal{R}(A - \lambda I)^\perp \end{bmatrix}.$$

We claim that $\mathcal{R}(M_C - \lambda I)$ is closed. In fact, we only need to prove that $\mathcal{R}\left(\left[\begin{smallmatrix} U \\ B_1(\lambda) \end{smallmatrix}\right]\right)$ is closed. For this, let $\left[\begin{smallmatrix} y_1^n \\ y_2^n \end{smallmatrix}\right] \in \mathcal{R}\left(\left[\begin{smallmatrix} U \\ B_1(\lambda) \end{smallmatrix}\right]\right)$ and $\left[\begin{smallmatrix} y_1^n \\ y_2^n \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] (n \longrightarrow \infty)$. Then there exists $x_n \in \mathcal{N}(B-\lambda I)^\perp \cap \mathcal{D}(B-\lambda I)$ such that $Ux_n \longrightarrow y_1$ as $n \longrightarrow \infty$ and $B_1(\lambda)x_n \longrightarrow y_2$ as $n \longrightarrow \infty$. From the definition of U and the closedness of $B_1(\lambda)$, we get $B_1(\lambda)U^{-1}y_1 = y_2$ and $U^{-1}y_1 \in \mathcal{N}(B - \lambda I)^\perp \cap \mathcal{D}(B - \lambda I)$. This implies that

$$\begin{bmatrix} U \\ B_1(\lambda) \end{bmatrix} U^{-1} y_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

so $\mathcal{R}\left(\left[\begin{smallmatrix} U \\ B_1(\lambda) \end{smallmatrix}\right]\right)$ is closed, which contradicts $\lambda \in \bigcap_{C \in \mathcal{C}_B^+(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$.

Case III: Assume that $\lambda \in \{\lambda \in \mathbb{C} : \mathcal{R}(B - \lambda I)$ is closed, $\alpha(B - \lambda I) = \infty\}$. If $\mathcal{R}(A - \lambda I)$ is closed, then the proof is similar to that of case I. If $\mathcal{R}(A - \lambda I)$ is not closed, then $\dim \overline{\mathcal{R}(A - \lambda I)} = \infty$. Thus there exists a unitary operator $U : \mathcal{N}(B - \lambda I) \longrightarrow \overline{\mathcal{R}(A - \lambda I)}$. Set

$$C = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B - \lambda I) \\ \mathcal{N}(B - \lambda I)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A - \lambda I)} \\ \mathcal{R}(A - \lambda I)^\perp \end{bmatrix}.$$

We claim that $\mathcal{R}(M_C - \lambda I)$ is closed. Indeed, we only need to verify that $\mathcal{R}([A_1(\lambda)\ U])$ is closed. Suppose that $y_n \in \mathcal{R}([A_1(\lambda)\ U]) \subset \overline{\mathcal{R}(A-\lambda I)}$, and $y_n \longrightarrow y (n \longrightarrow \infty)$. Then there exist $x_n \in \mathcal{N}(B-\lambda I)$ $(n = 1, 2, \dots)$ such that $Ux_n = y_n \longrightarrow y(n \longrightarrow \infty)$. Set $x = \left[\begin{smallmatrix} 0 \\ U^{-1}y \end{smallmatrix}\right]$. Then

$$\begin{bmatrix} A_1(\lambda) & U \end{bmatrix} x = y.$$

Hence $\mathcal{R}([A_1(\lambda)\ U])$ is closed, which contradicts $\lambda \in \bigcap_{C \in \mathcal{C}^+_B(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$.

Case IV: Assume that $\lambda \in \{\lambda \in \mathbb{C} : \alpha(B-\lambda I) = d(A-\lambda I) = \infty\}$. If $\mathcal{R}(A-\lambda I)$ or $\mathcal{R}(B-\lambda I)$ is closed, then the proof is similar to the above cases. If $\mathcal{R}(A-\lambda I)$ and $\mathcal{R}(B-\lambda I)$ are not closed, then $\dim \mathcal{N}(B-\lambda I)^\perp = \dim \overline{\mathcal{R}(A-\lambda I)} = \infty$. Thus there exist unitary operators $U_1 : \mathcal{N}(B-\lambda I) \longrightarrow \overline{\mathcal{R}(A-\lambda I)}$ and $U_2 : \mathcal{N}(B-\lambda I)^\perp \longrightarrow \mathcal{R}(A-\lambda I)^\perp$. Set

$$C = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix},$$
$$C = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B-\lambda I) \\ \mathcal{N}(B-\lambda I)^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{\mathcal{R}(A-\lambda I)} \\ \mathcal{R}(A-\lambda I)^\perp \end{bmatrix}.$$

We can easily see that $\mathcal{R}([A_1(\lambda)\ U_1])$ and $\mathcal{R}\left(\left[\begin{smallmatrix} U_2 \\ B_1(\lambda) \end{smallmatrix}\right]\right)$ are closed by the proofs of cases II and III. Hence $\mathcal{R}(M_C - \lambda I)$ is closed. This contradicts $\lambda \in \bigcap_{C \in \mathcal{C}^+_B(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C)$. The proof is complete. □

From the proof of Theorem 2.7 we have the next corollary, which is an extension of the result of [5].

Corollary 2.8. *Let $A \in \mathcal{C}(\mathcal{H})$ and $B \in \mathcal{C}(\mathcal{K})$ with dense domains. Then*

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\mathrm{cr}}(M_C) = \left\{\lambda \in \sigma_{\mathrm{cr}}(A) : \alpha(B-\lambda I) < \infty\right\} \cup \left\{\lambda \in \sigma_{\mathrm{cr}}(B) : d(A-\lambda I) < \infty\right\}.$$

3. Examples

In this section, we give a couple of examples.

Example 3.1. Let $\mathcal{H} = C[0,1]$, let $\mathcal{K} = L_2[0,1]$, and let the entries A, B of the upper triangular operator matrix M_C be defined by

$$Au(t) = tu(t), \quad u \in \mathcal{H}, t \in [0,1]$$

and

$$Bx = x'', \quad x \in \mathcal{D}(B),$$

where $\mathcal{D}(B) = \{x \in \mathcal{K} : x, x' \in AC[0,1], x'' \in \mathcal{K}, x(0) = x(1) = 0\}$. By calculation, we get $0 \in \sigma_r(A)$, that is, $\alpha(A) = 0, d(A) > 0$. Then $\alpha(B) < d(A)$ and $\alpha(B) < d(A) + d(B)$. From Theorem 2.1, there is a closable operator C such that $0 \in \sigma_r(M_C)$.

On the other hand, set $C = 0$. Then $0 \in \sigma_r(M_C)$.

Example 3.2. Let $\mathcal{H} = l^2(-\infty, +\infty)$ and $\mathcal{K} = L^2(-\infty, +\infty)$. For $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathcal{H}$ define operator A as

$$y = Ax = (\ldots, y_{-1}, y_0, y_1, \ldots), \qquad y_k = x_{k+1}, \quad k = \ldots, -1, 0, 1, \ldots.$$

Next we define operator B as

$$Bx = \frac{dx}{dt}, \quad x \in \mathcal{D}(B),$$
$$\mathcal{D}(B) = \big\{x \in \mathcal{K} : x \in AC(-\infty, +\infty), x' \in \mathcal{K}\big\}.$$

By calculation, we get $0 \in \rho(A) \cap \sigma_c(B)$. Then $\alpha(B) = 0 = d(A)$, and hence there is a closable operator C such that $0 \in \sigma_c(M_C)$ from Theorem 2.2.

On the other hand, set $C = 0$. Then $0 \in \sigma_c(M_C)$.

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