Ann. Funct. Anal. 10 (2019), no. 3, 350–356
https://doi.org/10.1215/20088752-2018-0034
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

NONCOMPLEX SYMMETRIC OPERATORS ARE DENSE

TING TING ZHOU[1] and BIN LIANG[2*]

Communicated by S. R. García

Abstract. An operator $T \in \mathcal{B}(\mathcal{H})$ is complex-symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \longrightarrow \mathcal{H}$ so that $CTC = T^*$. In this note, we prove that on finite-dimensional Hilbert space $\mathbb{C}^n$ with $n \geq 3$, noncomplex symmetric operators are dense in $\mathcal{B}(\mathbb{C}^n)$.

1. Introduction

Throughout this article, $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers, respectively. $\mathcal{H}$ will always denote a complex separable Hilbert space. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$.

Generalizing the notion of complex symmetric matrices, García and Putinar [3] initiated a study for complex symmetric operators on Hilbert space which draws many inspirations from function theory, matrix analysis, and other areas.

Definition 1.1. A *conjugation* is a conjugate-linear map $C : \mathcal{H} \longrightarrow \mathcal{H}$ which is both involutive (i.e., $C^2 = I$) and isometric (i.e., $(Cx, Cy) = (y, x)$, $\forall x, y \in \mathcal{H}$).

Definition 1.2. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is *complex-symmetric* if there exists a conjugation C on $\mathcal{H}$ so that $CTC = T^*$.

It is well known that each complex symmetric operator admits a complex symmetric matrix representation with respect to some orthonormal basis of $\mathcal{H}$. Through a series of papers, García, Putinar, and Wogen (see, e.g., [3]–[5]), have

Received Jul. 30, 2018; Accepted Nov. 13, 2018.
First published online Jul. 3, 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A05; Secondary 47B99.
Keywords. complex symmetric operator, finite-dimensional space, small perturbation.

obtained a better understanding of the structure of complex symmetric operators. An effective method of studying complex symmetric operators is to characterize the complex symmetry of special operator classes (see, e.g., [2], [8], [9]). In fact, many operator classes have been proved to be complex-symmetric, such as Hankel operators, truncated Toeplitz operators, normal operators, and binormal operators.

In this paper, we let *CSO* denote the set of *complex symmetric operators* on complex separable infinite-dimensional Hilbert space. In [5], García and Wogen proved that CSO is not closed in the strong operator topology. Moreover, they raised the so-called *norm closure problem* for complex symmetric operators. Later, Zhu, Li, and Ji [10] solved the norm closure problem by proving that CSO is not closed in the norm topology. Shortly thereafter, García and Poole [1] also solved the problem by giving another construction. Furthermore, authors began to consider other approximation problems about complex symmetric operators (see [6], [7]). In particular, Zhu and Li [8] got the following result.

Theorem A ([8, Theorem 1.5]). *Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space. For any $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists a compact operator $K \in \mathcal{B}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is not complex-symmetric.*

To prove this result, the authors used the fact that the Wolf spectrum of each operator on infinite-dimensional space is not empty. For any T and $\varepsilon > 0$, by arbitrarily choosing a λ_0 in the Wolf spectrum of T, we can find a compact operator K with $\|K\| < \varepsilon$ such that $\dim(T + K - \lambda_0) \neq \dim(T + K - \lambda_0)^*$. It follows from [3, Proposition 1] that $T + K$ is not complex-symmetric.

In the finite-dimensional space, as we all know, each operator on $\mathbb{C}^2$ is complex-symmetric (see [3, Example 6]). When $n \geq 3$, although every operator on $\mathbb{C}^n$ is similar to a complex symmetric matrix, not all operators on $\mathbb{C}^n$ are complex-symmetric. And it is easy to prove that the set of complex symmetric operators on the finite-dimensional space is closed in the norm topology. Naturally, one may wonder whether a result similar to that of Theorem A holds for the finite-dimensional case.

As was mentioned, every 2×2 matrix is a complex symmetric operator, so we consider the finite-dimensional Hilbert space with dimension greater than 3. Comparing with Theorem A, we get the following main result.

Theorem 1.3. *Assume that $n \in \mathbb{N}$ and $n \geq 3$. For any $T \in \mathcal{B}(\mathbb{C}^n)$ and $\varepsilon > 0$ there exists $A \in \mathcal{B}(\mathbb{C}^n)$ with $\|A\| < \varepsilon$ such that $T + A$ is not complex-symmetric.*

2. Proof of the main result

For each $T \in \mathcal{B}(\mathbb{C}^n)$, we have $\dim\ker(T) = \dim\ker(T^*)$. So the original method for proving Theorem A is useless for proving Theorem 1.3, making it necessary for us to develop other methods. First we offer some preliminaries. For $e, f \in (\mathbb{C}^n)$, we define $e \otimes f \in \mathcal{B}(\mathbb{C}^n)$ as follows:

$$(e \otimes f)(x) = \langle x, f\rangle e \quad \text{for each } x \in \mathbb{C}^n.$$

Hence $e \otimes f$ is a rank 1 operator when $e \neq 0$ and $f \neq 0$. The following lemma confirms that each $n \times n$ matrix can be approximated by another matrix with n many different singular values.

Lemma 2.1 ([8, Proposition 4.3])**.** *Given $T \in \mathcal{B}(\mathbb{C}^n)$ and $\varepsilon > 0$, there exists $A \in \mathcal{B}(\mathbb{C}^n)$ with $\|A\| < \varepsilon$ such that*

$$T + A = \sum_{i=1}^{n} a_i f_i \otimes e_i,$$

where $a_i > 0$ and $a_i \neq a_j$ for all $1 \leq i, j \leq n$ with $i \neq j$, and $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$ are two orthonormal bases of $\mathbb{C}^n$.

Lemma 2.2 ([8, Theorem 1.4])**.** *Assume that $\{e_i\}_{i\in\Lambda}$, $\{f_i\}_{i\in\Lambda}$ are two orthonormal subsets of $\mathcal{H}$ and that $T \in \mathcal{B}(\mathcal{H})$ can be written as*

$$T = \sum_{i\in\Lambda} a_i f_i \otimes e_i,$$

where $a_i > 0$ and $a_i \neq a_j$ for all $i, j \in \Lambda$ with $i \neq j$. Then the following are equivalent:

(1) *T is complex-symmetric;*
(2) *$|\langle e_m, f_n\rangle| = |\langle e_n, f_m\rangle|$ for all $m, n \in \Lambda$ and $\langle e_i, f_j\rangle\langle e_j, f_k\rangle\langle e_k, f_i\rangle = \langle e_i, f_k\rangle\langle e_k, f_j\rangle\langle e_j, f_i\rangle$ for all $i, j, k \in \Lambda$ and $i \leq j \leq k$.*

In fact, what deserves to be mentioned is that García, Poore, and Wyse [2] got this result for the finite-dimensional case in 2011. By Lemma 2.2, one can show that each $n \times n$ matrix can be perturbed to obtain distinct singular values and satisfy some further conditions.

Lemma 2.3. *Given $T \in \mathcal{B}(\mathbb{C}^n)$, $n \geq 3$, and $\varepsilon > 0$, there exists $B \in \mathcal{B}(\mathbb{C}^n)$ with $\|B\| < \varepsilon$ such that*

$$T + B = \sum_{i=1}^{n} a_i f_i \otimes e_i,$$

where $a_i > 0$ and $a_i \neq a_j$ for all $1 \leq i, j \leq n$ with $i \neq j$, where $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$ are two orthonormal bases of $\mathbb{C}^n$, and where one of the two following cases holds:

(1) *$T + B$ is not complex-symmetric,*
(2) *$|\langle e_i, f_i\rangle| \neq |\langle e_j, f_j\rangle|$ for some $1 \leq i, j \leq n$ with $i \neq j$.*

Proof. Given $\varepsilon > 0$, by Lemma 2.1 there exists $B_1 \in \mathcal{B}(\mathbb{C}^n)$ with $\|B_1\| < \frac{\varepsilon}{2}$ such that

$$T + B_1 = \sum_{i=1}^{n} a_i f_i' \otimes e_i,$$

where $a_i > 0$ and $a_i \neq a_j$ for all $1 \leq i, j \leq n$ with $i \neq j$, and $\{e_i\}_{i=1}^n$, $\{f_i'\}_{i=1}^n$ are two orthonormal bases of $\mathbb{C}^n$.

Without loss of generality, we can directly assume that $T + B_1$ is complex-symmetric and that $\langle e_i, f_i'\rangle = \langle e_j, f_j'\rangle$ for all $1 \leq i, j \leq n$. By Lemma 2.2, we have $|\langle e_i, f'_j\rangle| = |\langle e_j, f'_i\rangle|$ for each $1 \leq i, j \leq n$. For $1 \leq i, j \leq n$, we denote $a_{ij} = \langle e_i, f'_j\rangle$. It follows that

$$|a_{ii}| = |a_{jj}| \quad \text{for all } 1 \leq i, j \leq n, \tag{2.1}$$

and

$$|a_{ij}| = |a_{ji}| \quad \text{for all } 1 \leq i, j \leq n. \tag{2.2}$$

To finish the proof, we just need to consider the following two cases.

Case 1. There exist some $1 \leq i_1, j_1 \leq n$ *such that* $|a_{i_1 j_1}| \neq \frac{1}{\sqrt{n}}$. Since $\{f'_j\}_{j=1}^n$ is an orthonormal basis of $\mathbb{C}^n$, we have

$$\sum_{j=1}^{n} |a_{ij}|^2 = \sum_{j=1}^{n} \left|\langle e_i, f'_j\rangle\right|^2 = \|e_i\|^2 = 1, \quad \text{for each } 1 \leq i \leq n. \tag{2.3}$$

It follows that not all $\{a_{ij}\}_{i,j=1}^n$ have the same absolute value. Notice that $n \geq 3$ and (2.1) hold for $\{a_{ii}\}_{i=1}^n$, and there exist $1 \leq i_0, j_0, p \leq n$ with $i_0 \neq j_0$, $i_0 \neq p$, and $j_0 \neq p$ such that $|a_{pp}| = |a_{i_0 i_0}| \neq |a_{i_0 j_0}|$. We define

$$f_{i_0} = \cos(t) f'_{i_0} + \sin(t) f'_{j_0} \qquad \text{and} \qquad f_{j_0} = -\sin(t) f'_{i_0} + \cos(t) f'_{j_0},$$

where $t > 0$ and t is small enough, and will be fixed later. Moreover, we denote $f_k = f'_k$ for $1 \leq k \leq n$ with $k \neq i_0$ and $k \neq j_0$. It is easy to see that $\{f_k\}_{k=1}^n$ is an orthonormal basis of $\mathbb{C}^n$.

We define $T_1 = \sum_{i=1}^n a_i f_i \otimes e_i$. We will show that $|\langle e_{i_0}, f_{i_0}\rangle| \neq |\langle e_p, f_p\rangle|$ for some sufficiently small $t > 0$. Otherwise, there exists $\delta > 0$ such that

$$\left|\langle e_{i_0}, f_{i_0}\rangle\right| = \left|\langle e_p, f_p\rangle\right| \quad \text{for all } 0 < t < \delta. \tag{2.4}$$

It is easy to see that $|\langle e_{i_0}, f_{i_0}\rangle|^2 - |\langle e_p, f_p\rangle|^2$ is a real analytic function of t. If it vanishes on $(0, \delta)$, then it is identically zero. In particular, it vanishes at $t = \frac{\pi}{2}$. In fact, when $t = \frac{\pi}{2}$, we have $f_{i_0} = f'_{j_0}$ and $f_p = f'_p$. It leads to

$$|a_{i_0 j_0}|^2 - |a_{pp}|^2 = \left|\langle e_{i_0}, f'_{j_0}\rangle\right|^2 - \left|\langle e_p, f'_p\rangle\right|^2 = \left|\langle e_{i_0}, f_{i_0}\rangle\right|^2 - \left|\langle e_p, f_p\rangle\right|^2 = 0.$$

This is contradicted by the fact that $|a_{i_0 j_0}| \neq |a_{pp}|$.

So we can choose a sufficiently small t such that

$$\left|\langle e_{i_0}, f_{i_0}\rangle\right| \neq \left|\langle e_p, f_p\rangle\right| \qquad \text{and} \qquad \left\|T_1 - (T + B_1)\right\| < \frac{\varepsilon}{2}.$$

Set $B = T_1 - T$. Then $\|B\| < \varepsilon$ and $T + B = T_1$. This completes the proof of Case 1.

Case 2. We have $|a_{ij}| = \frac{1}{\sqrt{n}}$ *for all* $1 \leq i, j \leq n$. We claim that there exists an arbitrarily small positive number t such that one of the following three numbers $|\cos(t) a_{11} + \sin(t) a_{12}|$, $|\cos(t) a_{12} + \sin(t) a_{13}|$, and $|\cos(t) a_{11} + \sin(t) a_{13}|$ is not $\frac{1}{\sqrt{n}}$. Otherwise, there exists $\delta > 0$ such that

$$\begin{aligned} \left|\cos(t) a_{11} + \sin(t) a_{12}\right| &= \left|\cos(t) a_{12} + \sin(t) a_{13}\right| \\ &= \left|\cos(t) a_{11} + \sin(t) a_{13}\right| = \frac{1}{\sqrt{n}} \end{aligned} \tag{2.5}$$

for all $0 < t < \delta$. We denote $a_{1j} = \frac{1}{\sqrt{n}} e^{2\pi i \theta_j}$ for $1 \le j \le 3$. By (2.5) and a direct calculation, we have

$$Re(a_{11}\overline{a_{12}}) = Re(a_{12}\overline{a_{13}}) = Re(a_{11}\overline{a_{13}}) = 0.$$

It follows that

$$\cos(\theta_1 - \theta_2) = \cos(\theta_2 - \theta_3) = \cos(\theta_1 - \theta_3) = 0.$$

This means that

$$\theta_1 - \theta_2 = k\pi + \frac{\pi}{2}, \qquad \theta_1 - \theta_3 = m\pi + \frac{\pi}{2}, \qquad \theta_2 - \theta_3 = l\pi + \frac{\pi}{2}$$

for some integers k, m, l. Since this is impossible, the claim holds.

Without loss of generality, we assume that $|\cos(t)a_{11} + \sin(t)a_{12}| \ne \frac{1}{\sqrt{n}}$. We let

$$h_1 = \cos(t)f_1' + \sin(t)f_2', \qquad h_2 = -\sin(t)f_1' + \cos(t)f_2',$$

and $h_j = f_j'$ for $3 \le j \le n$. Then $\{h_j\}_{j=1}^n$ is an orthonormal basis of $\mathbb{C}^n$.

We denote

$$H = \sum_{i=1}^{n} a_i h_i \otimes e_i.$$

It is easy to see that $\|T + B_1 - H\| < \frac{\varepsilon}{2}$ when $t > 0$ is small enough. Also we have

$$\big|\langle e_1, h_1\rangle\big| = \big|\big\langle e_1, \cos(t)f_1' + \sin(t)f_2'\big\rangle\big| = \big|\cos(t)a_{11} + \sin(t)a_{12}\big| \ne \frac{1}{\sqrt{n}}.$$

This reduces the proof to Case 1. The proof of Lemma 2.3 is thus complete. □

Now we will prove Theorem 1.3.

Proof of Theorem 1.3. For any $T \in \mathcal{B}(\mathbb{C}^n)$ and $\varepsilon > 0$, by Lemma 2.3 there exists $B \in \mathcal{B}(\mathbb{C}^n)$ with $\|B\| < \frac{\varepsilon}{2}$ such that either $T + B$ is not complex-symmetric, or $T + B$ can be written as the following form:

$$T + B = \sum_{i=1}^{n} a_i f_i \otimes e_i, \tag{2.6}$$

where $a_i > 0$ and $a_i \ne a_j$ for all $1 \le i, j \le n$ with $i \ne j$, and $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$ are two orthonormal bases of $\mathbb{C}^n$. Furthermore, there exist $1 \le i_0, j_0 \le n$ with $i_0 \ne j_0$ such that $|\langle e_{i_0}, f_{i_0}\rangle| \ne |\langle e_{j_0}, f_{j_0}\rangle|$.

If $T+B$ is not complex-symmetric, then the proof is finished. So we can directly assume that $T + B$ is complex-symmetric with form (2.6) and that $|\langle e_{i_0}, f_{i_0}\rangle| \ne |\langle e_{j_0}, f_{j_0}\rangle|$. We let

$$f_{i_0}' = \cos(t)f_{i_0} + \sin(t)f_{j_0}, \qquad f_{j_0}' = -\sin(t)f_{i_0} + \cos(t)f_{j_0},$$

and $f_j' = f_j$ for $1 \le j \le n$ with $j \ne i_0$ and $j \ne j_0$. Then $\{f_j'\}_{j=1}^n$ is an orthonormal basis of $\mathbb{C}^n$. We let $S_t = \sum_{i=1}^n a_i f_i' \otimes e_i$, where t will be determined later.

Claim. *There exists $t > 0$ where t is small enough such that S_t is not complex-symmetric.*

Otherwise, there exists $0 < \delta < \frac{\pi}{2}$ such that S_t is complex-symmetric for all $0 < t < \delta$. By Lemma 2.2, we have $|\langle e_{i_0}, f'_{j_0}\rangle| = |\langle e_{j_0}, f'_{i_0}\rangle|$ for all $0 < t < \delta$.

It is easy to see that $|\langle e_{i_0}, f'_{j_0}\rangle|^2 - |\langle e_{j_0}, f'_{i_0}\rangle|^2$ is a real analytic function of t. If it vanishes on $(0, \delta)$, then it is identically zero. In particular, it vanishes at $t = \frac{\pi}{2}$. In fact, when $t = \frac{\pi}{2}$, we have $f'_{i_0} = f_{j_0}$ and $f'_{j_0} = -f_{i_0}$. It leads to

$$\left|\langle e_{i_0}, f_{i_0}\rangle\right|^2 - \left|\langle e_{j_0}, f_{j_0}\rangle\right|^2 = \left|\langle e_{i_0}, f'_{j_0}\rangle\right|^2 - \left|\langle e_{j_0}, f'_{i_0}\rangle\right|^2 = 0,$$

which is a contradiction. So the claim holds.

Hence one can choose a suitable $t > 0$ such that

$$\left|\langle e_{i_0}, f'_{j_0}\rangle\right| \neq \left|\langle e_{j_0}, f'_{i_0}\rangle\right| \tag{2.7}$$

and

$$\left|a_{i_0}(f'_{i_0} - f_{i_0})\right| + \left|a_{j_0}(f'_{j_0} - f_{j_0})\right| < \frac{\varepsilon}{2}. \tag{2.8}$$

By (2.7) and Lemma 2.2, S_t is not complex-symmetric.

Denote $C = S_t - (T + B)$. It follows from (2.8) that $\|C\| < \frac{\varepsilon}{2}$. Set $A = B + C = S_t - T$. Then $\|A\| < \varepsilon$ and $T + A$ is not complex-symmetric. This completes the proof. □

Acknowledgments. The authors wish to thank the reviewers for a careful reading and valuable comments which greatly improved our paper.

Liang's work was partially supported by Fundamental Research Funds for the Central Universities grant BLX201826. Zhou's work was partially supported by National Natural Science Foundation of China grants 11401088 and 11271150.

References

1. S. R. García and D. E. Poore, *On the norm closure problem for complex symmetric operators*, Proc. Amer. Math. Soc. **141** (2013), no. 2, art. ID 549. Zbl 1264.47003. MR2996959. DOI 10.1090/S0002-9939-2012-11347-4. 351
2. S. R. García, D. E. Poore, and M. K. Wyse, *Unitary equivalence to a complex symmetric matrix: A modulus criterion*, Oper. Matrices **5** (2011), no. 2, 273–287. Zbl 1221.15056. MR2830598. DOI 10.7153/oam-05-19. 351, 352
3. S. R. García and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315. Zbl 1087.30031. MR2187654. DOI 10.1090/S0002-9947-05-03742-6. 350, 351
4. S. R. García and M. Putinar, *Complex symmetric operators and applications, II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931. Zbl 1123.47030. MR2302518. DOI 10.1090/S0002-9947-07-04213-4. 350
5. S. R. García and W. R. Wogen, *Complex symmetric partial isometries*, J. Funct. Anal. **257** (2009), no. 4, 1251–1260. Zbl 1166.47023. MR2535469. DOI 10.1016/j.jfa.2009.04.005. 350, 351
6. K. Guo, Y. Ji, and S. Zhu, *A C^*-algebra approach to complex symmetric operators*, Trans. Amer. Math. Soc. **367** (2015), no. 10, 6903–6942. Zbl 1393.47018. MR3378818. DOI 10.1090/S0002-9947-2015-06215-1. 351
7. S. Zhu, *Approximation of complex symmetric operators*, Math. Ann. **364** (2016), no. 1–2, 373–399. Zbl 1336.47015. MR3451391. DOI 10.1007/s00208-015-1221-0. 351
8. S. Zhu and C. G. Li, *Complex symmetry of a dense class of operators*, Integral Equations Operator Theory **73** (2012), no. 2, 255–272. Zbl 1284.47014. MR2921067. DOI 10.1007/s00020-012-1957-9. 351, 352

9. S. Zhu and C. G. Li, *Complex symmetric weighted shifts*, Trans. Amer. Math. Soc. **365** (2013), no. 1, 511–530. Zbl 1282.47045. MR2984066. DOI 10.1090/S0002-9947-2012-05642-X. 351
10. S. Zhu, C. G. Li, and Y. Q. Ji, *The class of complex symmetric operators is not norm closed*, Proc. Amer. Math. Soc. **140** (2012), no. 5, 1705–1708. Zbl 1251.47004. MR2869154. DOI 10.1090/S0002-9939-2011-11345-5. 351

[1]Department of Mathematics, Jilin University, Changchun 130012, People's Republic of China.
E-mail address: zhoutingting@jlu.edu.cn

[2]College of Science, Beijing Forestry University, Beijing, 100083, People's Republic of China.
E-mail address: liangbinmath@163.com