Ann. Funct. Anal. 10 (2019), no. 3, 337–349
https://doi.org/10.1215/20088752-2018-0033
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ENDPOINT ESTIMATES FOR MULTILINEAR FRACTIONAL INTEGRAL OPERATORS ON METRIC MEASURE SPACES

YUAN ZHAO,[1] HAIBO LIN,[1] and YAN MENG[2*]

Communicated by A. D. Baranov

ABSTRACT. Let $(\mathcal{X}, d, \mu)$ be a metric measure space such that, for any fixed $x \in \mathcal{X}$, $\mu(B(x, r))$ is a continuous function with respect to $r \in (0, \infty)$. In this paper, we prove endpoint estimates for the multilinear fractional integral operators $I_{m,\alpha}$ from the product of Lebesgue spaces $L^1(\mu) \times \cdots \times L^1(\mu) \times L^{p_{k+1}}(\mu) \times \cdots \times L^{p_m}(\mu)$ into the Lebesgue space $L^q(\mu)$, where $k \in [1, m) \cap \mathbb{N}$, $\alpha \in [k, m)$, $p_i \in (1, \infty)$ for $i \in \{k+1, \ldots, m\}$ and $1/q = k + \sum_{i=k+1}^{m} 1/p_i - \alpha$. We furthermore prove that $I_{m,\alpha}$ is bounded from $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ into $L^\infty(\mu)$, where $p_i \in (1, \infty)$ for $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^{m} 1/p_i = \alpha \in [1, m)$.

1. Introduction

The fractional integral operator is an important tool in the theory of harmonic analysis, especially in the study of the differentiability and smoothness of functions. In the classical Euclidean spaces with Lebesgue measures, Kenig and Stein [3] and Grafakos and Kalton [1] respectively studied the boundedness of multilinear fractional integral operators. Recently, Komori-Furuya [4] established the endpoint estimates for these type of operators.

With the development of the theory, people find that many results in the classical Euclidean spaces still hold true in metric measure spaces (see, e.g., [2], [5]–[9]). In what follows, we always assume that $(\mathcal{X}, d, \mu)$ is a metric measure space with μ being a Radon measure. Sihwaningrum and Sawano [8] established the boundedness of fractional integral operators on Morrey spaces over $(\mathcal{X}, d, \mu)$.

Received Jul. 19, 2018; Accepted Nov. 11, 2018.
First published online Jul. 2, 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47B47; Secondary 42B20, 42B35, 30L99.
Keywords. metric measure space, multilinear fractional integral, endpoint estimate.

The purpose of the present article is to generalize the endpoint estimates for multilinear fractional integral operators on the classical Euclidean spaces with Lebesgue measure, obtained by Komori-Furuya [4], to the metric measure spaces. In what follows, let $\mathbb{N} := \{1, 2, \ldots\}$.

Let $m \in \mathbb{N}$ and $\alpha \in (0, m)$. We define the multilinear fractional integral operator $I_{m,\alpha}$ on $(\mathcal{X}, d, \mu)$ by setting, for any $f_i \in L_b^\infty(\mu)$, $i \in \{1, \ldots, m\}$, $x \in \mathcal{X}$,

$$I_{m,\alpha}(f_1, \ldots, f_m)(x) := \int_{\mathcal{X}^m} \frac{f_1(y_1) \cdots f_m(y_m)}{[\mu(B(x, 6\max\{d(x,y_1), \ldots, d(x,y_m)\}))]^{m-\alpha}} \, d\mu(y_1) \cdots d\mu(y_m), \tag{1.1}$$

where $L_b^\infty(\mu)$ represents the set of $L^\infty(\mu)$ functions with bounded support, and for any $x \in \mathcal{X}$, $r \in (0, \infty)$, $B(x, r) := \{y \in \mathcal{X} : d(y, x) < r\}$. When $m = 1$, we simply denote $I_{1,\alpha}$ by I_α and, moreover, if we replace 6 with 2, then $I_{1,\alpha}$ is the fractional integral operator introduced by Sihwaningrum and Sawano in [8]. In what follows, we always assume that $m \in [2, \infty) \cap \mathbb{N}$.

In this paper, unless otherwise stated, we always assume that $(\mathcal{X}, d, \mu)$ is a metric measure space such that, for any fixed $x \in \mathcal{X}$, $\mu(B(x, r))$ is a continuous function with respect to $r \in (0, \infty)$. Our main results are stated as follows.

Theorem 1.1. *Let $k \in [1, m) \cap \mathbb{N}$, $\alpha \in [k, m)$, and the multilinear fractional integral operator $I_{m,\alpha}$ be as in (1.1). Then, for $p_1 = \cdots = p_k = 1$, $p_{k+1}, \ldots, p_m \in (1, \infty)$, and $1/q = k + \sum_{i=k+1}^m 1/p_i - \alpha$, there exists a positive constant C such that, for all $f_i \in L_b^\infty(\mu)$, $i \in \{1, \ldots, m\}$,*

$$\left\| I_{m,\alpha}(f_1, \ldots, f_m) \right\|_{L^q(\mu)} \le C \prod_{i=1}^k \|f_i\|_{L^1(\mu)} \prod_{i=k+1}^m \|f_i\|_{L^{p_i}(\mu)}. \tag{1.2}$$

Via Theorem 1.1, we can obtain the following corollary.

Corollary 1.2. *Let $k, l \in \mathbb{N}$ with $k + l \in (1, m-1]$, $\alpha \in [k, k+l)$, and the multilinear fractional integral operator $I_{m,\alpha}$ be as in (1.1). Then, for $p_1 = \cdots = p_k = 1$, $p_{k+1}, \ldots, p_{k+l} \in (1, \infty)$, $p_{k+l+1} = \cdots = p_m = \infty$ and $1/q = k + \sum_{i=k+1}^{k+l} 1/p_i - \alpha > 0$, there exists a positive constant C such that, for all $f_i \in L_b^\infty(\mu)$, $i \in \{1, \ldots, m\}$,*

$$\left\| I_{m,\alpha}(f_1, \ldots, f_m) \right\|_{L^q(\mu)} \le C \prod_{i=1}^k \|f_i\|_{L^1(\mu)} \prod_{i=k+1}^{k+l} \|f_i\|_{L^{p_i}(\mu)} \prod_{i=k+l+1}^m \|f_i\|_{L^\infty(\mu)}.$$

Theorem 1.3. *Let $\alpha \in [1, m)$ and the multilinear fractional integral operator $I_{m,\alpha}$ be as in (1.1). Then, for $p_1, \ldots, p_m \in (1, \infty)$, and $\sum_{i=1}^m 1/p_i = \alpha$, there exists a positive constant C such that, for all $f_i \in L_b^\infty(\mu)$, $i \in \{1, \ldots, m\}$,*

$$\left\| I_{m,\alpha}(f_1, \ldots, f_m) \right\|_{L^\infty(\mu)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}.$$

From Theorem 1.3, we can deduce the following corollary.

Corollary 1.4. *Let $k \in \mathbb{N}$, $l \in [2,\infty) \cap \mathbb{N}$ with $k+l \in (2,m)$, $\alpha \in [k+1,k+l)$, and the multilinear fractional integral operator $I_{m,\alpha}$ be as in (1.1). Then, for $p_1 = \cdots = p_k = 1$, $p_{k+1},\dots,p_{k+l} \in (1,\infty)$, $p_{k+l+1} = \cdots = p_m = \infty$, and $k + \sum_{i=k+1}^{k+l} 1/p_i = \alpha$, there exists a positive constant C such that, for all $f_i \in L_b^\infty(\mu)$, $i \in \{1,\dots,m\}$,*

$$\left\| I_{m,\alpha}(f_1,\dots,f_m) \right\|_{L^\infty(\mu)} \le C \prod_{i=1}^{k} \|f_i\|_{L^1(\mu)} \prod_{i=k+1}^{k+l} \|f_i\|_{L^{p_i}(\mu)} \prod_{i=k+l+1}^{m} \|f_i\|_{L^\infty(\mu)}.$$

Remark 1.5. In the case where $(\mathcal{X},d,\mu) = (\mathbb{R}^n, |\cdot|, dx)$ and $\mu(B(x,r)) = |B(x,r)|$ for any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, Komori-Furuya [4, Theorem 1] proved that Theorem 1.1 holds true for the classical multilinear fractional integral operator $\tilde{I}_{m,\beta}$ defined by setting, for all $f_i \in L_b^\infty(\mathbb{R}^n)$, $i \in \{1,\dots,m\}$, $x \in \mathbb{R}^n$,

$$\tilde{I}_{m,\beta}(f_1,\dots,f_m)(x) := \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{\left(\sum_{i=1}^m |x-y_i|\right)^{mn-\beta}} dy_1 \cdots dy_m$$

with $\beta = \alpha n$. In [4] it was also demonstrated that the range of the indices k and β in the above result cannot be improved by giving counterexamples. Checking the argument used in [4], we see that it is also valid for the operator $I_{m,\alpha}$ in classical Euclidean space $\mathbb{R}^n$ with n-dimensional Lebesgue measure. Thus, the range of the indices k and α in Theorem 1.1 cannot be improved.

We end this section by establishing some notational conventions. Throughout this paper, C denotes a positive constant independent of the main parameters, but which may differ from line to line. The positive constants with subscript such as C_0 do not vary in different situations. The expression $f \lesssim g$ means that there exists a constant C such that $f \le Cg$. For any given $p \in (1,\infty)$, p' represents the conjugate of p. We denote by $L^1_{\mathrm{loc}}(\mu)$ the function space of all μ-locally integrable functions.

2. Proofs of Theorem 1.1 and Corollary 1.2

We begin this section with an important tool of the ρ times modified centered maximal function. Let $\rho \in (1,\infty)$. For all $f \in L^1_{\mathrm{loc}}(\mu)$, the ρ times modified centered maximal function is defined by setting, for all $x \in \mathcal{X}$,

$$M_\rho f(x) := \sup_{r>0} \frac{1}{\mu(B(x,\rho r))} \int_{B(x,r)} \left| f(y) \right| d\mu(y).$$

Nazarov, Treil, and Volberg [6] first introduced the modified centered maximal function with $\rho = 3$, and proved that $M_3 f$ is weak-$(1,1)$ bounded on the separable metric measure space. In the metric space $(\mathcal{X},d,\mu)$ with $\mu(B(x,\cdot))$ being a continuous function in $(0,\infty)$, Terasawa [9] obtained that $M_\rho(f)$ is weak-$(1,1)$ bounded when $\rho \in [2,\infty)$. Sawano in [7, Theorem 1.2] obtained the same result when $(\mathcal{X},d,\mu)$ is separable and in [7, Section 2.3] proved, moreover, that the range of ρ is sharp by giving a counterexample.

Applying the fact that $M_\rho(f)$ is weak-$(1,1)$ bounded when $\rho \in [2,\infty)$ along with the Marcinkiewicz interpolation theorem, one can deduce the following result.

Lemma 2.1. *For any $p \in (1,\infty)$, M_ρ is bounded on $L^p(\mu)$ when $\rho \in [2,\infty)$.*

Via [8, Theorem 1.2], we obtain the following boundedness of the fractional integral operator I_α on the Lebesgue spaces over $(\mathcal{X}, d, \mu)$.

Lemma 2.2. *Let $\alpha \in (0,1)$, $p \in (1, 1/\alpha)$, $1/q = 1/p - \alpha$. Then I_α is bounded from $L^p(\mu)$ to $L^q(\mu)$.*

By Lemma 2.2 and an argument used in [3, Lemma 7], we obtain the following boundedness of the multilinear fractional integral operator $I_{m,\alpha}$ on the product Lebesgue spaces over $(\mathcal{X}, d, \mu)$.

Lemma 2.3. *Let the multilinear fractional integral operators $I_{m,\alpha}$ be as in (1.1). Then, for $p_1, \ldots, p_m \in (1,\infty)$, $\alpha \in (0,m)$, and $1/q = \sum_{i=1}^m 1/p_i - \alpha > 0$, $I_{m,\alpha}$ is bounded from $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ to $L^q(\mu)$.*

Proof. Without loss of generality, we may assume that f_i is a nonnegative function for $i \in \{1, \ldots, m\}$. We note that if there exist positive constants c_i, $i = 1, \ldots, m$ satisfying $\alpha \in (0, \sum_{i=1}^m c_i)$, then one can choose $\alpha_i \in (0, c_i)$ such that $\alpha = \sum_{i=1}^m \alpha_i$. Let $c_i = 1/p_i$ and $1/q_i = 1/p_i - \alpha_i$ with $\alpha_i \in (0, c_i)$. By $\alpha = \sum_{i=1}^m \alpha_i$, we have $1/q = \sum_{i=1}^m 1/q_i$. In view of $\alpha_i \in (0, 1/p_i) \subset (0,1)$ and $m - \alpha = \sum_{i=1}^m (1-\alpha_i)$, we deduce that, for all $x, y_i \in \mathcal{X}$ with $i \in \{1, \ldots, m\}$,

$$\prod_{i=1}^m \big[\mu\big(B\big(x, 6d(x,y_i)\big)\big)\big]^{1-\alpha_i} \le \big[\mu\big(B\big(x, 6\max\{d(x,y_1), \ldots, d(x,y_m)\}\big)\big)\big]^{m-\alpha},$$

which implies that, for any $x \in \mathcal{X}$,

$$I_{m,\alpha}(f_1, \ldots, f_m)(x) \le \prod_{i=1}^m I_{\alpha_i}(f_i)(x).$$

By Lemma 2.2, we know that I_{α_i} is bounded from $L^{p_i}(\mu)$ to $L^{q_i}(\mu)$. This together with Hölder's inequality gives us the desired result of Lemma 2.3. □

Now we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we may assume that f_i is nonnegative for each $i \in \{1, \ldots, m\}$. We consider the following two cases.

Case (I): $\alpha \in (k, m)$. In this case, for any $x \in \mathcal{X}$,

$$\begin{aligned} &I_{m,\alpha}(f_1, \ldots, f_m)(x) \\ &\quad \le \prod_{i=1}^k \|f_i\|_{L^1(\mu)} \int_{\mathcal{X}^{m-k}} \frac{f_{k+1}(y_{k+1}) \cdots f_m(y_m)}{[\mu(B(x, 6\max\{d(x,y_{k+1}), \ldots, d(x,y_m)\}))]^{m-\alpha}} \\ &\quad\quad \times d\mu(y_{k+1}) \cdots d\mu(y_m) \\ &\quad = \prod_{i=1}^k \|f_i\|_{L^1(\mu)} I_{m-k,\alpha-k}(f_{k+1}, \ldots, f_m)(x). \end{aligned}$$

By the fact that $1/q = \sum_{i=k+1}^m 1/p_i - (\alpha - k)$ and Lemma 2.3, we have

$$\left\|I_{m-k,\alpha-k}(f_{k+1},\ldots,f_m)\right\|_{L^q(\mu)} \lesssim \prod_{i=k+1}^{m} \|f_i\|_{L^{p_i}(\mu)}.$$

The above two estimates yield that (1.2) holds true in this case.

Case (II): $\alpha = k$. In this case, we use mathematical induction on m. Consider $m = 2$ at first. It then follows that $\alpha = 1$. Without loss of generality, we assume that $p_1 = 1$ and $p_2 \in (1,\infty)$. For any nonnegative function $g \in L^{p_2'}(\mu)$, we have

$$\begin{aligned}
&\int_{\mathcal{X}} I_{2,1}(f_1,f_2)(x)g(x)\,d\mu(x)\\
&\quad= \int_{\mathcal{X}} g(x)\int_{\mathcal{X}^2} \frac{f_1(y_1)f_2(y_2)}{\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))}\,d\mu(y_1)\,d\mu(y_2)\,d\mu(x)\\
&\quad= \int_{\mathcal{X}} f_1(y_1)\int_{\mathcal{X}^2} \frac{f_2(y_2)g(x)}{\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))}\,d\mu(y_2)\,d\mu(x)\,d\mu(y_1).
\end{aligned}$$

By duality, to prove Theorem 1.1 with $m = 2$, we only need to prove that, for any $y_1 \in \mathcal{X}$,

$$\int_{\mathcal{X}^2} \frac{f_2(y_2)g(x)}{\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))}\,d\mu(y_2)\,d\mu(x) \lesssim \|f_2\|_{L^{p_2}(\mu)}\|g\|_{L^{p_2'}(\mu)}. \tag{2.1}$$

To this end, for any $y_1 \in \mathcal{X}$, write

$$\begin{aligned}
&\int_{\mathcal{X}^2} \frac{f_2(y_2)g(x)}{\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))}\,d\mu(y_2)\,d\mu(x)\\
&\quad\le \int_{d(x,y_2)\le 2d(x,y_1)} \frac{f_2(y_2)g(x)}{\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))}\,d\mu(y_2)\,d\mu(x)\\
&\qquad+ \int_{d(x,y_2)> 2d(x,y_1)} \frac{f_2(y_2)g(x)}{\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))}\,d\mu(y_2)\,d\mu(x)\\
&\quad:= \mathrm{E}(y_1) + \mathrm{F}(y_1).
\end{aligned}$$

For the term $\mathrm{E}(y_1)$, by Hölder's inequality and Lemma 2.1, we obtain that for any $y_1 \in \mathcal{X}$,

$$\begin{aligned}
\mathrm{E}(y_1) &\le \int_{\mathcal{X}} \frac{g(x)}{\mu(B(x,6d(x,y_1)))}\int_{d(x,y_2)\le 2d(x,y_1)} f_2(y_2)\,d\mu(y_2)\,d\mu(x)\\
&\lesssim \|g\|_{L^{p_2'}(\mu)}\left\{\int_{\mathcal{X}} \frac{1}{[\mu(B(x,6d(x,y_1)))]^{p_2}}\right.\\
&\qquad\times\left.\left[\int_{d(x,y_2)\le 2d(x,y_1)} f_2(y_2)\,d\mu(y_2)\right]^{p_2} d\mu(x)\right\}^{1/p_2}\\
&\le \|g\|_{L^{p_2'}(\mu)}\left\|M_3(f_2)\right\|_{L^{p_2}(\mu)} \lesssim \|g\|_{L^{p_2'}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.
\end{aligned}$$

We now turn to estimate $\mathrm{F}(y_1)$. Noting that $d(x,y_2) > 2d(x,y_1)$, we deduce that

$$d(x,y_1) \le d(y_1,y_2) \le 2d(x,y_2).$$

From the fact that $B(y_2,4d(y_2,x)) \subset B(x,5d(x,y_2))$, and by Hölder's inequality and Lemma 2.1, we deduce that, for any $y_1 \in \mathcal{X}$,

$$\begin{aligned}
\mathrm{F}(y_1) &\le \int_{\mathcal{X}} \frac{f_2(y_2)}{\mu(B(y_2, 4d(y_2,x)))} \int_{d(y_2,y_1)\le 2d(y_2,x)} g(x)\,d\mu(x)\,d\mu(y_2) \\
&\le \|f_2\|_{L^{p_2}(\mu)} \Big\{ \int_{\mathcal{X}} \frac{1}{[\mu(B(y_2, 4d(y_2,x)))]^{p_2'}} \\
&\quad \times \Big[\int_{d(y_2,y_1)\le 2d(y_2,x)} g(x)\,d\mu(x) \Big]^{p_2'} d\mu(y_2) \Big\}^{1/p_2'} \\
&\le \|f\|_{L^{p_2}(\mu)} \big\| M_2(g) \big\|_{L^{p_2'}(\mu)} \lesssim \|f\|_{L^{p_2}(\mu)} \|g\|_{L^{p_2'}(\mu)}.
\end{aligned}$$

Therefore, we have proved Theorem 1.1 for $m = 2$.

Now assume that, for any positive integers $m \in [2, \infty)$, $k \in [1, m-1]$ ($k = 1$ when $m = 2$) and $\alpha = k$, Theorem 1.1 holds true. We will prove that Theorem 1.1 remains true when $m + 1$ and $\alpha = k$. We consider the following two cases.

Case (i): $k = m$. In this case, $\alpha = m$, $p_1 = \cdots = p_m = 1$ and $p_{m+1} \in (1, \infty)$, $q = p_{m+1}$. For any nonnegative function $g \in L^{p'_{m+1}}(\mu)$, we have

$$\begin{aligned}
&\int_{\mathcal{X}} I_{m+1,m}(f_1, \ldots, f_{m+1})(x) g(x)\,d\mu(x) \\
&\quad = \int_{\mathcal{X}} g(x) \int_{\mathcal{X}^{m+1}} \frac{\prod_{i=1}^{m+1} f_i(y_i)}{\mu(B(x, 6\max\{d(x,y_1), \ldots, d(x,y_{m+1})\}))} \\
&\qquad \times\, d\mu(y_1)\cdots d\mu(y_{m+1})\,d\mu(x) \\
&\quad = \int_{\mathcal{X}^m} \prod_{i=1}^{m} f_i(y_i) \int_{\mathcal{X}^2} \frac{f_{m+1}(y_{m+1}) g(x)}{\mu(B(x, 6\max\{d(x,y_1), \ldots, d(x,y_{m+1})\}))} \\
&\qquad \times d\mu(y_{m+1})\,d\mu(x)\,d\mu(y_1)\cdots d\mu(y_m).
\end{aligned}$$

Similar to the proof of (2.1), we obtain that, for any $y_1, \ldots, y_m \in \mathcal{X}$,

$$\begin{aligned}
&\int_{\mathcal{X}^2} \frac{f_{m+1}(y_{m+1}) g(x)}{\mu(B(x, 6\max\{d(x,y_1), \ldots, d(x,y_{m+1})\}))}\,d\mu(y_{m+1})\,d\mu(x) \\
&\quad \lesssim \|f_{m+1}\|_{L^{p_{m+1}}(\mu)} \|g\|_{L^{p'_{m+1}}(\mu)},
\end{aligned}$$

which implies that, for any nonnegative function $g \in L^{p'_{m+1}}(\mu)$,

$$\int_{\mathcal{X}} I_{m+1,m}(f_1, \ldots, f_{m+1})(x) g(x)\,d\mu(x) \lesssim \prod_{i=1}^{m} \|f_i\|_{L^1(\mu)} \|f_{m+1}\|_{L^{p_{m+1}}(\mu)} \|g\|_{L^{p'_{m+1}}(\mu)}.$$

It then follows from duality that our desired result holds true for $k = m$.

Case (ii): $k \in [1, m-1]$. For any $x \in \mathcal{X}$, write

$$\begin{aligned}
&I_{m+1,k}(f_1, \ldots, f_{m+1})(x) \\
&\quad \le \int_{d(x,y_m)\le d(x,y_{m+1})} \frac{f_m(y_m)}{[\mu(B(x, 6\max\{d(x,y_1), \ldots, d(x,y_{m+1})\}))]^{m+1-k}}\,d\mu(y_m) \\
&\qquad \times \prod_{i=1}^{m-1} f_i(y_i) f_{m+1}(y_{m+1})\,d\mu(y_1)\cdots d\mu(y_{m-1})\,d\mu(y_{m+1})
\end{aligned}$$

$$+\int_{d(x,y_m)>d(x,y_{m+1})}\frac{f_{m+1}(y_{m+1})}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_{m+1})\}))]^{m+1-k}}$$
$$\times\, d\mu(y_{m+1})\prod_{i=1}^{m} f_i(y_i)\,d\mu(y_1)\cdots d\mu(y_m)$$
$$=: \mathrm{L}(x)+\mathrm{N}(x).$$

For any $x\in\mathcal{X}$,

$$\begin{aligned}\mathrm{L}(x)\le&\int_{\mathcal{X}^m}\frac{\prod_{i=1}^{m-1}f_i(y_i)f_{m+1}(y_{m+1})}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_{m-1}),d(x,y_{m+1})\}))]^{m-k}}\\ &\times\frac{\int_{d(x,y_m)\le d(x,y_{m+1})}f_m(y_m)\,d\mu(y_m)}{\mu(B(x,6d(x,y_{m+1})))}\,d\mu(y_1)\cdots d\mu(y_{m-1})\,d\mu(y_{m+1})\\ \le&\, I_{m,k}(f_1,\ldots,f_{m-1},f_{m+1})(x)M_6(f_m)(x).\end{aligned}$$

Let r be a positive constant satisfying that $1/r=\sum_{i=k+1}^{m-1}1/p_i+1/p_{m+1}$. It then follows that $1/q=1/p_m+1/r$. This, together with the above estimate, Hölder's inequality, Lemma 2.1, and the hypothesis that Theorem 1.1 is true for any $m,k\in[1,m-1]$ and $\alpha=k$, shows that

$$\begin{aligned}\big\|\mathrm{L}(\cdot)\big\|_{L^q(\mu)}&\lesssim\big\|M_6(f_m)\big\|_{L^{p_m}(\mu)}\big\|I_{m,k}(f_1,\ldots,f_{m-1},f_{m+1})\big\|_{L^r(\mu)}\\ &\lesssim\|f_m\|_{L^{p_m}(\mu)}\prod_{i=1}^{k}\|f_i\|_{L^1(\mu)}\prod_{i=k+1}^{m-1}\|f_i\|_{L^{p_i}(\mu)}\|f_{m+1}\|_{L^{p_{m+1}}(\mu)}.\end{aligned}$$

Similarly,

$$\big\|\mathrm{N}(\cdot)\big\|_{L^q(\mu)}\lesssim\prod_{i=1}^{k}\|f_i\|_{L^1(\mu)}\prod_{i=k+1}^{m+1}\|f_i\|_{L^{p_i}(\mu)}.$$

Combining the estimates for $\mathrm{L}(x)$ and $\mathrm{N}(x)$ yields that Theorem 1.1 holds true in this case, which completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. We still assume that f_i is a nonnegative function for each $i\in\{1,\ldots,m\}$. It is easy to see that there exists $i_0\in\{1,\ldots,k+l\}$ such that $d(x,y_{i_0})=\max\{d(x,y_1,),\ldots,d(x,y_{k+l})\}$. Let $B_0:=B(x,6d(x,y_{i_0}))$, $R_0:=6d(x,y_{i_0})$. For $s\in\mathbb{N}$, set

$$R_s:=\sup\big\{R>0:\mu\big(B(x,5R)\big)<2^s\mu(6B_0)\big\}>0$$

and

$$A_0:=B_0,\qquad A_s:=B(x,R_s)\setminus B(x,R_{s-1}).$$

For $s\in\mathbb{N}$, we have $\mu(A_s)\le\mu(B(x,R_s))<2^s\mu(6B_0)$, and if $y\in A_s$, then $d(x,y)\ge R_{s-1}$, which implies that $\mu(B(x,6(x,y)))\ge\mu(B(x,6R_{s-1}))>2^{s-1}\mu(6B_0)$. It then follows that, for all $s\in\mathbb{N}$ and $y\in A_s$,

$$\frac{\mu(A_s)}{\mu(B(x,6d(x,y)))}\le\frac{2^s\mu(6B_0)}{2^{s-1}\mu(6B_0)},$$

from which we can deduce that, for any $x\in\mathcal{X}$,

$$\begin{aligned}
&\int_{\mathcal{X}^{m-k-l}} \frac{1}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_m)\}))]^{m-\alpha}}\,d\mu(y_{k+l+1})\cdots d\mu(y_m)\\
&\quad\lesssim \prod_{j=k+l+1}^{m}\int_{\mathcal{X}} \frac{1}{[\mu(B(x,6d(x,y_{i_0})))+\mu(B(x,6d(x,y_j)))]^{1+\frac{k+l-\alpha}{m-k-l}}}\,d\mu(y_j)\\
&\quad\le \prod_{j=k+l+1}^{m}\Big\{\int_{A_0}\frac{1}{[\mu(B_0)]^{1+\frac{k+l-\alpha}{m-k-l}}}\,d\mu(y_j)\\
&\qquad+\sum_{s=1}^{\infty}\int_{A_s}\frac{1}{[\mu(B(x,6d(x,y_j)))]^{1+\frac{k+l-\alpha}{m-k-l}}}\,d\mu(y_j)\Big\}\\
&\quad\le \prod_{j=k+l+1}^{m}\Big\{\frac{1}{[\mu(B_0)]^{\frac{k+l-\alpha}{m-k-l}}}+\sum_{s=1}^{\infty}\frac{2}{[2^{s-1}\mu(6B_0)]^{\frac{k+l-\alpha}{m-k-l}}}\Big\}\\
&\quad\lesssim \frac{1}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_{k+l})\}))]^{k+l-\alpha}}. \qquad (2.2)
\end{aligned}$$

Applying (2.2), we have that, for any $x\in\mathcal{X}$,

$$\begin{aligned}
&I_{m,\alpha}(f_1,\ldots,f_m)(x)\\
&\quad\lesssim \prod_{i=k+l}^{m}\|f_i\|_{L^\infty(\mu)}\int_{\mathcal{X}^{k+l}}\frac{\prod_{i=1}^{k+l}f_i(y_i)}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_{k+l})\}))]^{k+l-\alpha}}\\
&\qquad\times d\mu(y_1)\cdots d\mu(y_{k+l})\\
&\quad\lesssim \prod_{i=k+l}^{m}\|f_i\|_{L^\infty(\mu)}I_{k+l,\alpha}(f_1,\ldots,f_{k+l})(x).
\end{aligned}$$

Then, by the assumption that $p_1=\cdots=p_k=1$, $p_{k+1},\ldots,p_{k+l}\in(1,\infty)$, $\alpha\in[k,k+l)$, and $1/q=k+\sum_{i=k+1}^{k+l}1/p_i-\alpha>0$, and Theorem 1.1, we deduce that

$$\big\|I_{k+l,\alpha}(f_1,\ldots,f_m)\big\|_{L^q(\mu)}\lesssim\prod_{i=1}^{k}\|f_i\|_{L^1(\mu)}\prod_{i=k+1}^{k+l}\|f_i\|_{L^{p_i}(\mu)}.$$

Combining the above two estimates, we obtain

$$\big\|I_{m,\alpha}(f_1,\ldots,f_m)\big\|_{L^q(\mu)}\lesssim\prod_{i=1}^{k}\|f_i\|_{L^1(\mu)}\prod_{i=k+1}^{k+l}\|f_i\|_{L^{p_i}(\mu)}\prod_{i=k+l+1}^{m}\|f_i\|_{L^\infty(\mu)},$$

which completes the proof of Corollary 1.2. □

3. Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. Without loss of generality, we assume that f_i is nonnegative for $i\in\{1,\ldots,m\}$. To prove this theorem, we will adopt induction on m. We first prove that Theorem 1.3 is valid when $m=2$. For any $x\in\mathcal{X}$,

$$
\begin{aligned}
&I_{2,\alpha}(f_1,f_2)(x)\\
&\quad\le \int_{d(x,y_2)\le d(x,y_1)} \frac{f_1(y_1)f_2(y_2)}{[\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))]^{2-\alpha}}\,d\mu(y_1)\,d\mu(y_2)\\
&\qquad+\int_{d(x,y_2)>d(x,y_1)} \frac{f_1(y_1)f_2(y_2)}{[\mu(B(x,6\max\{d(x,y_1),d(x,y_2)\}))]^{2-\alpha}}\,d\mu(y_1)\,d\mu(y_2)\\
&\quad=: \mathrm{O}(x)+\mathrm{P}(x).
\end{aligned}
$$

To deal with $\mathrm{O}(x)$, notice that, for any $x\in\mathcal{X}$, and $y_1,y_2\in\mathcal{X}$ satisfying $d(x,y_2)\le d(x,y_1)$, we have $d(y_1,y_2)\le 2d(x,y_1)$ and $B(y_1,4d(y_1,x))\subset B(x,5d(x,y_1))$. Then, for any $x,y_1\in\mathcal{X}$,

$$
\begin{aligned}
&\frac{1}{\mu(B(x,6d(x,y_1)))}\int_{d(x,y_2)\le d(x,y_1)} f_2(y_2)\,d\mu(y_2)\\
&\quad\le \frac{1}{\mu(B(y_1,4d(y_1,x)))}\int_{d(y_1,y_2)\le 2d(y_1,x)} f_2(y_2)\,d\mu(y_2)\le M_2(f_2)(y_1). \qquad (3.1)
\end{aligned}
$$

Due to $\alpha\in[1,2)$, we have $1/p_1+1/p_2=\alpha\ge 1$, which means that $p_1'\ge p_2$. It then follows from Hölder's inequality, (3.1), and Lemma 2.1 that, for any $x\in\mathcal{X}$,

$$
\begin{aligned}
\mathrm{O}(x)&\le \int_{\mathcal{X}} \frac{f_1(y_1)}{[\mu(B(x,6d(x,y_1)))]^{2-\alpha}}\int_{d(x,y_2)\le d(x,y_1)} f_2(y_2)\,d\mu(y_2)\,d\mu(y_1)\\
&\le \|f_1\|_{L^{p_1}(\mu)}\Big\{\int_{\mathcal{X}} \frac{[\int_{d(x,y_2)\le d(x,y_1)} f_2(y_2)\,d\mu(y_2)]^{p_1'}}{[\mu(B(x,6d(x,y_1)))]^{(2-\alpha)p_1'}}\,d\mu(y_1)\Big\}^{1/p_1'}\\
&\le \|f_1\|_{L^{p_1}(\mu)}\Big\{\int_{\mathcal{X}} \frac{1}{[\mu(B(y_1,4d(y_1,x)))]^{(2-\alpha)p_1'-p_2}}\\
&\qquad\times\Big[\frac{\int_{d(y_1,y_2)\le 2d(y_1,x)} f_2(y_2)\,d\mu(y_2)}{\mu(B(y_1,4d(y_1,x)))}\Big]^{p_2}\\
&\qquad\times\Big[\int_{d(y_1,y_2)\le 2d(y_1,x)} f_2(y_2)\,d\mu(y_2)\Big]^{p_1'-p_2}\,d\mu(y_1)\Big\}^{1/p_1'}\\
&\lesssim \|f_1\|_{L^{p_1}(\mu)}\Big\{\int_{\mathcal{X}} \frac{1}{[\mu(B(y_1,4d(y_1,x)))]^{(2-\alpha)p_1'-p_2}}\big[M_2(f_2)(y_1)\big]^{p_2}\|f_2\|_{L^{p_2}(\mu)}^{(p_1'-p_2)}\\
&\qquad\times\big[\mu\big(B\big(y_1,4d(y_1,x)\big)\big)\big]^{(1-1/p_2)(p_1'-p_2)}\,d\mu(y_1)\Big\}^{1/p_1'}\\
&\lesssim \|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}^{(p_1'-p_2)/p_1'}\Big\{\int_{\mathcal{X}}\big[M_2(f_2)(y_1)\big]^{p_2}\,d\mu(y_1)\Big\}^{1/p_1'}\\
&\lesssim \|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.
\end{aligned}
$$

Similarly, for any $x\in\mathcal{X}$,

$$\mathrm{P}(x)\lesssim \|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Combining the estimates for $\mathrm{O}(x)$ and $\mathrm{P}(x)$ yields that

$$\big\|I_{2,\alpha}(f_1,f_2)\big\|_{L^\infty(\mu)}\lesssim \|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Now assuming that Theorem 1.3 holds true for any positive integer $m \in [2,\infty)$ and $\alpha \in [1,m)$, we will prove that Theorem 1.3 is valid when $m+1$ and $\alpha \in [1,m+1)$. To this end, we consider the following two cases.

Case (I): $\alpha \in [(m+1)/m, m+1)$. In this case, an argument similar to that of estimate (2.2) together with Hölder's inequality gives us that, for any $x \in \mathcal{X}$,

$$\begin{aligned}
&I_{m+1,\alpha}(f_1,\ldots,f_m,f_{m+1})(x)\\
&\quad\le \int_{\mathcal{X}^m}\bigg[\int_{\mathcal{X}}\frac{1}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_{m+1})\}))]^{(m+1-\alpha)p'_{m+1}}}\\
&\quad\quad\times d\mu(y_{m+1})\bigg]^{1/p'_{m+1}}\\
&\quad\quad\times \|f_{m+1}\|_{L^{p_{m+1}}(\mu)}\prod_{i=1}^m f_i(y_i)\,d\mu(y_1)\cdots d\mu(y_m)\\
&\quad\lesssim \int_{\mathcal{X}^m}\frac{\prod_{i=1}^m f_i(y_i)}{[\mu(B(x,6\max\{d(x,y_1),\ldots,d(x,y_m)\}))]^{(m+1)-\alpha-\frac{1}{p'_{m+1}}}}\\
&\quad\quad\times d\mu(y_1)\cdots d\mu(y_m)\\
&\quad\quad\times \|f_{m+1}\|_{L^{p_{m+1}}(\mu)}\\
&\quad= \|f_{m+1}\|_{L^{p_{m+1}}(\mu)} I_{m,\alpha-1/p_{m+1}}(f_1,\ldots,f_m)(x).
\end{aligned}$$

We may assume that $1 < p_1 \le p_2 \le \cdots \le p_m \le p_{m+1} < \infty$. From this and $\sum_{i=1}^{m+1} 1/p_i = \alpha$, we deduce that

$$\alpha - \frac{1}{p_{m+1}} = \sum_{i=1}^m \frac{1}{p_i} \ge \frac{m}{p_{m+1}},$$

which implies that $1/p_{m+1} \le \alpha/(m+1)$. By $\alpha \in [(m+1)/m, m+1)$, we further have

$$\alpha - \frac{1}{p_{m+1}} \ge \alpha - \frac{\alpha}{m+1} \ge 1. \tag{3.2}$$

On the other hand, since $p_1,\ldots,p_m \in (1,\infty)$, we have

$$\alpha - \frac{1}{p_{m+1}} = \sum_{i=1}^m \frac{1}{p_i} < m,$$

which, together with (3.2), shows that $\alpha - \frac{1}{p_{m+1}} \in [1,m)$. Thus, from the hypothesis of induction, we conclude that

$$\begin{aligned}
\big\|I_{m+1,\alpha}(f_1,\ldots,f_m,f_{m+1})\big\|_{L^\infty(\mu)} &\lesssim \|f_{m+1}\|_{L^{p_{m+1}}(\mu)}\big\|I_{m,\alpha-1/p_{m+1}}(f_1,\ldots,f_m)\big\|_{L^\infty(\mu)}\\
&\lesssim \prod_{i=1}^{m+1}\|f_i\|_{L^{p_i}(\mu)}.
\end{aligned}$$

Case (II): $\alpha \in [1,(m+1)/m)$. For any $x \in \mathcal{X}$, write

$$
\begin{aligned}
&I_{m+1,\alpha}(f_1,\dots,f_m,f_{m+1})(x)\\
&\quad\le \int_{d(x,y_{m+1})\le d(x,y_m)} \frac{\prod_{i=1}^{m+1} f_i(y_i)}{[\mu(B(x,6\max\{d(x,y_1),\dots,d(x,y_{m+1})\}))]^{m+1-\alpha}}\\
&\qquad\times d\mu(y_1)\cdots d\mu(y_m)\,d\mu(y_{m+1})\\
&\qquad+\int_{d(x,y_{m+1})> d(x,y_m)} \frac{\prod_{i=1}^{m+1} f_i(y_i)}{[\mu(B(x,6\max\{d(x,y_1),\dots,d(x,y_{m+1})\}))]^{m+1-\alpha}}\\
&\qquad\times d\mu(y_1)\cdots d\mu(y_m)\,d\mu(y_{m+1})\\
&\quad=: \mathrm{Q}(x)+\mathrm{R}(x).
\end{aligned}
$$

We first deal with $\mathrm{Q}(x)$. By using an argument similar to that of estimate (3.1), we have that, for any $x\in\mathcal{X}$,

$$
\begin{aligned}
\mathrm{Q}(x) &\le \int_{\mathcal{X}^m} \frac{\prod_{i=1}^{m} f_i(y_i)}{[\mu(B(x,6\max\{d(x,y_1),\dots,d(x,y_m)\}))]^{m-\alpha}}\frac{1}{\mu(B(x,6d(x,y_m)))}\\
&\quad\times\int_{d(x,y_{m+1})\le d(x,y_m)} f_{m+1}(y_{m+1})\,d\mu(y_{m+1})\,d\mu(y_1)\cdots d\mu(y_m)\\
&\le \int_{\mathcal{X}^m} \frac{\prod_{i=1}^{m} f_i(y_i)}{[\mu(B(x,6\max\{d(x,y_1),\dots,d(x,y_m)\}))]^{m-\alpha}}\frac{1}{\mu(B(y_m,4d(y_m,x)))}\\
&\quad\times\int_{d(y_m,y_{m+1})\le 2d(y_m,x)} f_{m+1}(y_{m+1})\,d\mu(y_{m+1})\,d\mu(y_1)\cdots d\mu(y_m)\\
&\le \int_{\mathcal{X}^m} \frac{\prod_{i=1}^{m} f_i(y_i)M_2(f_{m+1})(y_m)}{[\mu(B(x,6\max\{d(x,y_1),\dots,d(x,y_m)\}))]^{m-\alpha}}\,d\mu(y_1)\cdots d\mu(y_m)\\
&\lesssim I_{m,\alpha}\big(f_1,\dots,f_{m-1},f_mM_2(f_{m+1})\big)(x).
\end{aligned}
$$

We may assume that $1<p_1\le p_2\le\cdots\le p_m\le p_{m+1}<\infty$. Then

$$\frac{1}{p_m}+\frac{1}{p_{m+1}}\le\frac{2}{m-1}\Big(\frac{1}{p_1}+\cdots+\frac{1}{p_{m-1}}\Big).$$

From $1/p_m+1/p_{m+1}=\alpha-(1/p_1+\cdots+1/p_{m-1})$, we deduce that

$$\alpha-\Big(\frac{1}{p_1}+\cdots+\frac{1}{p_{m-1}}\Big)\le\frac{2}{m-1}\Big(\frac{1}{p_1}+\cdots+\frac{1}{p_{m-1}}\Big).$$

It then follows that

$$\frac{1}{p_1}+\cdots+\frac{1}{p_{m-1}}\ge\alpha\frac{m-1}{m+1}.$$

Taking $\alpha\in[1,(m+1)/m)$ into account, we have

$$\frac{1}{p_m}+\frac{1}{p_{m+1}}\le\alpha-\alpha\frac{m-1}{m+1}<\frac{2}{m}\le 1.$$

Let r be a positive constant satisfying that $1/r=1/p_m+1/p_{m+1}$. Then $r\in(1,\infty)$, which together with the hypothesis of induction, Hölder's inequality, and Lemma 2.1, implies that

$$\|Q(\cdot)\|_{L^\infty(\mu)} \lesssim \prod_{i=1}^{m-1} \|f_i\|_{L^{p_i}(\mu)} \|f_m M_2(f_{m+1})\|_{L^r(\mu)} \le \prod_{i=1}^{m+1} \|f_i\|_{L^{p_i}(\mu)}.$$

Similarly,

$$\|R(\cdot)\|_{L^\infty(\mu)} \lesssim \prod_{i=1}^{m+1} \|f_i\|_{L^{p_i}(\mu)}.$$

Combining the estimates for $Q(x)$ and $R(x)$, we obtain

$$\|I_{m+1,\alpha}(f_1, \ldots, f_m, f_{m+1})\|_{L^\infty(\mu)} \lesssim \prod_{i=1}^{m+1} \|f_i\|_{L^{p_i}(\mu)},$$

which completes the proof of Theorem 1.3. □

Proof of Corollary 1.4. We still suppose that f_i is nonnegative for $i \in \{1, \ldots, m\}$. From (2.2), we know that, for any $x \in \mathcal{X}$,

$$\begin{aligned} &I_{m,\alpha}(f_1, \ldots, f_m)(x) \\ &\quad\le \prod_{i=1}^{k} \|f_i\|_{L^1(\mu)} \prod_{i=k+l+1}^{m} \|f_i\|_{L^\infty(\mu)} \\ &\qquad\times \int_{\mathcal{X}^{m-k-l}} \frac{\prod_{i=k+1}^{k+l} f_i(y_i)}{[\mu(B(x, 6\max\{d(x,y_{k+1}), \ldots, d(x,y_{k+l})\}))]^{k+l-\alpha}} \\ &\qquad\times d\mu(y_{k+1}) \cdots d\mu(y_{k+l}) \\ &\quad= \prod_{i=1}^{k} \|f_i\|_{L^1(\mu)} \prod_{i=k+l+1}^{m} \|f_i\|_{L^\infty(\mu)} I_{l,\alpha-k}(f_{k+1}, \ldots, f_{k+l})(x). \end{aligned}$$

Notice that $\alpha \in [k+1, k+l)$, that is, $\alpha - k \in [1, l)$. This together with $\sum_{i=k+1}^{k+l} 1/p_i = (\alpha - k)$ and Theorem 1.3 shows that

$$\|I_{l,\alpha-k}(f_1, \ldots, f_m)\|_{L^\infty(\mu)} \lesssim \prod_{i=k+1}^{k+l} \|f_i\|_{L^{p_i}(\mu)}.$$

Therefore,

$$\|I_{m,\alpha}(f_1, \ldots, f_m)\|_{L^\infty(\mu)} \lesssim \prod_{i=1}^{k} \|f_i\|_{L^1(\mu)} \prod_{i=k+1}^{k+l} \|f_i\|_{L^{p_i}(\mu)} \prod_{i=k+l+1}^{m} \|f_i\|_{L^\infty(\mu)},$$

which completes the proof of Corollary 1.4. □

Acknowledgments. The authors wish to express their sincere thanks to the referees for a careful reading of the manuscript and for many valuable suggestions and remarks that improved the presentation of this article. Indeed, it was because of one such suggestion that we were able to obtain (2.2) without any extra assumptions, allowing us to prove that the ranges of all the indices in this work are the same as those in the classical Euclidean spaces.

References

1. L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), no. 1, 151–180. Zbl 0982.46018. MR1812822. DOI 10.1007/PL00004426. 337
2. Y. Han, D. Müller, and D. Yang, *A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces*, Abstr. Appl. Anal. **2008**, no. 893409. Zbl 1193.46018. MR2485404. DOI 10.1155/2008/893409. 337
3. C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), no. 1, 1–15. Zbl 0952.42005. MR1682725. DOI 10.4310/MRL.1999.v6.n1.a1. 337, 340
4. Y. Komori-Furuya, *Notes on endpoint estimates for multilinear fractional integral operators*, Proc. Amer. Math. Soc. **145** (2017), no. 4, 1515–1526. Zbl 1362.42031. MR3601544. DOI 10.1090/proc/13389. 337, 338, 339
5. F. Nazarov, S. Treil, and A. Volberg, *Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces*, Int. Math. Res. Not. IMRN **1997**, no. 15, 703–726. Zbl 0889.42013. MR1470373. DOI 10.1155/S1073792897000469. 337
6. F. Nazarov, S. Treil, and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces*, Int. Math. Res. Not. IMRN **1998**, no. 9, 463–487. Zbl 0918.42009. MR1626935. DOI 10.1155/S1073792898000312. 337, 339
7. Y. Sawano, *Sharp estimates of the modified Hardy-Littlewood maximal operator on the nonhomogeneous space via covering lemmas*, Hokkaido Math. J. **34** (2005), no. 2, 435–458. Zbl 1088.42010. MR2159006. DOI 10.14492/hokmj/1285766231. 337, 339
8. I. Sihwaningrum and Y. Sawano, *Weak and strong type estimates for fractional integral operators on Morrey spaces over metric measure spaces*, Eurasian Math. J. **4** (2013), no. 1, 76–81. Zbl 1277.42019. MR3118893. 337, 338, 340
9. Y. Terasawa, *Outer measures and weak type (1,1) estimates of Hardy-Littlewood maximal operators*, J. Inequal. Appl. **2006**, no. 15063. Zbl 1090.42012. MR2215476. DOI 10.1155/JIA/2006/15063. 337, 339

[1]College of Science, China Agricultural University, Beijing 100083, People's Republic of China.

E-mail address: zhaoyuancau@126.com; haibolincau@126.com

[2]School of Mathematics, Renmin University of China, Beijing 100872, People's Republic of China.

E-mail address: mengyan@ruc.edu.cn