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NONLINEAR MAPS PRESERVING MIXED LIE TRIPLE PRODUCTS ON FACTOR VON NEUMANN ALGEBRAS

ZHUJUN YANG and JIANHUA ZHANG*

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Abstract. We prove that every bijective map that preserves mixed Lie triple products from a factor von Neumann algebra $\mathcal{M}$ with $\dim \mathcal{M} > 4$ into another factor von Neumann algebra $\mathcal{N}$ is of the form $A \to \epsilon\Psi(A)$, where $\epsilon \in \{1, -1\}$ and $\Psi : \mathcal{M} \to \mathcal{N}$ is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. Also, we give the structure of this map when $\dim \mathcal{M} = 4$.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras over the complex number field $\mathbb{C}$, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a map. We consider that φ preserves mixed Lie triple products if $\varphi([[A, B]_*, C]) = [[\varphi(A), \varphi(B)]_*, \varphi(C)]$ for all $A, B, C \in \mathcal{A}$, where $[A, B] = AB - BA$ is the Lie product and $[A, B]_* = AB - BA^*$ is the skew Lie product of A and B. This kind of map is related to Lie product-preserving maps, skew Lie product-preserving maps, and (skew) commutativity-preserving maps, which have been studied by many authors (see, e.g., [1]–[6], [10], [12]–[15], and the references therein).

Recently, maps preserving the products of the mixture of Lie products and skew Lie products have received a fair amount of attention. For example, Li, Chen, and Wang in [9] proved that a bijective map preserving the Jordan $*$-product ($[[A, B]_*, C]_*$) between two factor von Neumann algebras is either a linear $*$-isomorphism (resp., a conjugate linear $*$-isomorphism) or the negative of

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*Corresponding author.
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a linear $*$-isomorphism (resp., the negative of a conjugate linear $*$-isomorphism). In the present article, we will establish the structure of the nonlinear maps preserving mixed Lie triple products ($[[A, B]_*, C]$) between two factor von Neumann algebras.

Let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Recall that $\mathcal{M}$ is a factor if its center is $\mathbb{C}I$, where I is the identity of $\mathcal{M}$. Let $\mathcal{M}$ be a factor von Neumann algebra. It follows from [7] and [11] that every operator $A \in \mathcal{M}$ can be written as a finite linear combination of projections in $\mathcal{M}$. If $\dim \mathcal{M} < \infty$, then $\mathcal{M}$ is isomorphic to $M_n(\mathbb{C})$, the algebra of all $n \times n$ matrices over $\mathbb{C}$. We assume that the dimensions of the algebras $\mathcal{M}$ and $\mathcal{N}$ are greater than 1 in the following sections.

2. Additivity

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two factor von Neumann algebras, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a bijective map satisfying $\Phi([[A, B]_*, C]) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]$ for all $A, B, C \in \mathcal{M}$. Then Φ is additive.*

Let $P_1 \in \mathcal{M}$ be a nontrivial projection, and let $P_2 = I - P_1$. Write $\mathcal{M}_{ij} = P_i \mathcal{M} P_j$ for $i, j = 1, 2$. Now we will prove Theorem 2.1 using several lemmas.

Lemma 2.2. *We have $\Phi(0) = 0$ and $\Phi(\mathbb{C}I) = \mathbb{C}I$.*

Proof. The surjectivity of Φ implies that there exists $A \in \mathcal{M}$ such that $\Phi(A) = 0$. Thus,

$$\Phi(0) = \Phi\big([[0, 0]_*, A]\big) = \big[\big[\Phi(0), \Phi(0)\big]_*, \Phi(A)\big] = 0.$$

Let $B \in \mathcal{M}$ such that $\Phi(B) = iI$. Then

$$0 = \Phi\big([[B, X]_*, \lambda I]\big) = \big[\big[\Phi(B), \Phi(X)\big]_*, \Phi(\lambda I)\big] = 2i\big[\Phi(X), \Phi(\lambda I)\big]$$

for all $X \in \mathcal{M}$ and $\lambda \in \mathbb{C}$. It follows that $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$. By considering Φ^{-1}, we can obtain that $\Phi(\mathbb{C}I) = \mathbb{C}I$. $\square$

Lemma 2.3. *For any $A, B \in \mathcal{M}$, $[\Phi(A), \Phi(B)] = 0$ if and only if $[A, B] = 0$.*

Proof. It follows from $\Phi(iI) \in \mathbb{C}I$ that

$$\begin{aligned} \Phi\big(2i[A, B]\big) &= \Phi\big([[iI, A]_*, B]\big) = \big[\big[\Phi(iI), \Phi(A)\big]_*, \Phi(B)\big] \\ &= \big(\Phi(iI) - \Phi(iI)^*\big)\big[\Phi(A), \Phi(B)\big] \end{aligned} \tag{2.1}$$

for all $A, B \in \mathcal{M}$. If $\Phi(iI) - \Phi(iI)^* = 0$, then $\Phi(2i[A, B]) = 0 = \Phi(0)$, and so $[A, B] = 0$ for all $A, B \in \mathcal{M}$. This contradiction implies that $\Phi(iI) - \Phi(iI)^* \neq 0$. Hence, by (2.1) and Lemma 2.2, we have that $[\Phi(A), \Phi(B)] = 0$ if and only if $[A, B] = 0$. $\square$

Lemma 2.4. *We have $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$ for all $A_{12} \in \mathcal{M}_{12}$ and $A_{21} \in \mathcal{M}_{21}$.*

Proof. Write $T = A_{12}+A_{21}-\Phi^{-1}(\Phi(A_{12})+\Phi(A_{21}))$. For every $B_{ij} \in \mathcal{M}_{ij}$ $(i \neq j)$, it follows from $[[B_{ij}, A_{12}]_*, P_i] = [[B_{ij}, A_{21}]_*, P_i] = 0$ that

$$\begin{aligned}[[\Phi(B_{ij}), \Phi(A_{12}+A_{21})]_*, \Phi(P_i)] &= \Phi\big([[B_{ij}, A_{12}+A_{21}]_*, P_i]\big)\\ &= \Phi\big([[B_{ij}, A_{12}]_*, P_i]\big) + \Phi\big([[B_{ij}, A_{21}]_*, P_i]\big)\\ &= \big[\big[\Phi(B_{ij}), \Phi(A_{12}) + \Phi(A_{21})\big]_*, \Phi(P_i)\big].\end{aligned}$$

Since Φ^{-1} preserves mixed Lie triple products, we have from the above equation that $[[B_{ij}, T]_*, P_i] = 0$. This implies that $P_jTP_j = 0$ for $j = 1, 2$. It follows from $[[A_{12}, P_1]_*, P_1] = [[A_{21}, P_2]_*, P_2] = 0$ that

$$\begin{aligned}[[\Phi(A_{12}+A_{21}), \Phi(P_1)]_*, \Phi(P_1)] &= \Phi\big([[A_{12}+A_{21}, P_1]_*, P_1]\big)\\ &= \Phi\big([[A_{21}, P_1]_*, P_1]\big)\\ &= \Phi\big([[A_{21}, P_1]_*, P_1]\big) + \Phi\big([[A_{12}, P_1]_*, P_1]\big)\\ &= \big[\big[\Phi(A_{12}) + \Phi(A_{21}), \Phi(P_1)\big]_*, \Phi(P_1)\big]\end{aligned}$$

and

$$\begin{aligned}[[\Phi(A_{12}+A_{21}), \Phi(P_2)]_*, \Phi(P_2)] &= \Phi\big([[A_{12}+A_{21}, P_2]_*, P_2]\big)\\ &= \Phi\big([[A_{12}, P_2]_*, P_2]\big)\\ &= \Phi\big([[A_{12}, P_2]_*, P_2]\big) + \Phi\big([[A_{21}, P_2]_*, P_2]\big)\\ &= \big[\big[\Phi(A_{12}) + \Phi(A_{21}), \Phi(P_2)\big]_*, \Phi(P_2)\big].\end{aligned}$$

Then $[[T, P_1]_*, P_1] = [[T, P_2]_*, P_2] = 0$, and so $P_2TP_1 = P_1TP_2 = 0$. Hence $T = 0$. It follows that $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$. □

Lemma 2.5. *We have* $\Phi(\sum_{i,j=1}^{2} A_{ij}) = \sum_{i,j=1}^{2} \Phi(A_{ij})$ *for all* $A_{ij} \in \mathcal{M}_{ij}$.

Proof. Write $T = \sum_{i,j=1}^{2} A_{ij} - \Phi^{-1}(\sum_{i,j=1}^{2} \Phi(A_{ij}))$. It follows from Lemma 2.4 that

$$\begin{aligned}[[\Phi(P_1), \Phi(\sum_{i,j=1}^{2} A_{ij})]_*, \Phi(P_2)] &= \Phi\Big(\Big[\Big[P_1, \sum_{i,j=1}^{2} A_{ij}\Big]_*, P_2\Big]\Big)\\ &= \Phi\big([[P_1, A_{12}+A_{21}]_*, P_2]\big)\\ &= \Phi\big([[P_1, A_{12}+A_{21}]_*, P_2]\big) + \Phi\big([[P_1, A_{11}]_*, P_2]\big)\\ &\quad + \Phi\big([[P_1, A_{22}]_*, P_2]\big)\\ &= \Big[\Big[\Phi(P_1), \sum_{i,j=1}^{2} \Phi(A_{ij})\Big]_*, \Phi(P_2)\Big].\end{aligned}$$

This implies that $[[P_1, T]_*, P_2] = 0$. Thus, $P_1TP_2 = P_2TP_1 = 0$. For every $B_{ij} \in \mathcal{M}_{ij}$ $(i \neq j)$, we have

$$\begin{aligned}[[\Phi(B_{ij}), \Phi(\sum_{k,l=1}^{2} A_{kl})]_*, \Phi(P_i)] &= \Phi\Big(\Big[\Big[B_{ij}, \sum_{k,l=1}^{2} A_{kl}\Big]_*, P_i\Big]\Big)\\ &= \Phi\big([[B_{ij}, A_{jj}]_*, P_i]\big)\end{aligned}$$

$$
\begin{aligned}
&= \Phi\big([[B_{ij}, A_{jj}]_*, P_i]\big) + \Phi\big([[B_{ij}, A_{ji}]_*, P_i]\big) \\
&\quad + \Phi\big([[B_{ij}, A_{ij}]_*, P_i]\big) + \Phi\big([[B_{ij}, A_{ii}]_*, P_i]\big) \\
&= \Big[\Big[\Phi(B_{ij}), \sum_{k,l=1}^{2} \Phi(A_{kl})\Big]_*, \Phi(P_i)\Big].
\end{aligned}
$$

Then $[[B_{ij}, T]_*, P_i] = 0$, and so $P_j T P_j = 0$ for $j = 1, 2$. Hence $T = 0$. It follows that $\Phi(\sum_{i,j=1}^{2} A_{ij}) = \sum_{i,j=1}^{2} \Phi(A_{ij})$. □

Lemma 2.6. *We have* $\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij})$ *for all* $A_{ij}, B_{ij} \in \mathcal{M}_{ij}$, $i \neq j$.

Proof. It follows from Lemma 2.5 that

$$
\begin{aligned}
\Phi(A_{ij} + B_{ij}) &= \Phi\Big(\Big[\Big[\frac{i}{2}I, P_i - iA_{ij}\Big]_*, P_j - iB_{ij}\Big]\Big) \\
&= \Big[\Big[\Phi\Big(\frac{i}{2}I\Big), \Phi(P_i - iA_{ij})\Big]_*, \Phi(P_j - iB_{ij})\Big] \\
&= \Big[\Big[\Phi\Big(\frac{i}{2}I\Big), \Phi(P_i) + \Phi(-iA_{ij})\Big]_*, \Phi(P_j) + \Phi(-iB_{ij})\Big] \\
&= \Phi\Big(\Big[\Big[\frac{i}{2}I, P_i\Big]_*, P_j\Big]\Big) + \Phi\Big(\Big[\Big[\frac{i}{2}I, P_i\Big]_*, -iB_{ij}\Big]\Big) \\
&\quad + \Phi\Big(\Big[\Big[\frac{i}{2}I, -iA_{ij}\Big]_*, P_j\Big]\Big) + \Phi\Big(\Big[\Big[\frac{i}{2}I, -iA_{ij}\Big]_*, -iB_{ij}\Big]\Big) \\
&= \Phi(A_{ij}) + \Phi(B_{ij}).
\end{aligned}
$$

□

Lemma 2.7. *We have* $\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$ *for all* $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, $i = 1, 2$.

Proof. Write $T = A_{ii} + B_{ii} - \Phi^{-1}(\Phi(A_{ii}) + \Phi(B_{ii}))$. Let $j \neq i$. It follows from Lemmas 2.5 and 2.6 that, for any $C_{ji} \in \mathcal{M}_{ji}$,

$$
\begin{aligned}
[[\Phi(C_{ji}), \Phi(A_{ii} + B_{ii})]_*, \Phi(P_i)] &= \Phi\big([[C_{ji}, A_{ii} + B_{ii}]_*, P_i]\big) \\
&= \Phi(C_{ji}A_{ii}) + \Phi(C_{ji}B_{ii}) + \Phi(A_{ii}C_{ji}^*) \\
&\quad + \Phi(B_{ii}C_{ji}^*) \\
&= \Phi\big([[C_{ji}, A_{ii}]_*, P_i]\big) + \Phi\big([[C_{ji}, B_{ii}]_*, P_i]\big) \\
&= \big[\big[\Phi(C_{ji}), \Phi(A_{ii})\big]_*, \Phi(P_i)\big] \\
&\quad + \big[\big[\Phi(C_{ji}), \Phi(B_{ii})\big]_*, \Phi(P_i)\big] \\
&= \big[\big[\Phi(C_{ji}), \Phi(A_{ii}) + \Phi(B_{ii})\big]_*, \Phi(P_i)\big]
\end{aligned}
$$

and

$$
\begin{aligned}
[[\Phi(C_{ij}), \Phi(A_{ii} + B_{ii})]_*, \Phi(P_j)] &= \Phi\big([[C_{ij}, A_{ii} + B_{ii}]_*, P_j]\big) \\
&= \Phi\big([[C_{ij}, A_{ii}]_*, P_j]\big) + \Phi\big([[C_{ij}, B_{ii}]_*, P_j]\big) \\
&= \big[\big[\Phi(C_{ji}), \Phi(A_{ii}) + \Phi(B_{ii})\big]_*, \Phi(P_i)\big].
\end{aligned}
$$

Then $[[C_{ji}, T]_*, P_i] = [[C_{ij}, T]_*, P_j] = 0$, and so $P_i T P_i = P_j T P_j = 0$. It is clear that

$$\begin{aligned}[[\Phi(P_j), \Phi(A_{ii}+B_{ii})]_*, \Phi(P_i)] &= \Phi\big([[P_j, A_{ii}+B_{ii}]_*, P_i]\big)\\ &= \Phi\big([[P_j, A_{ii}]_*, P_i]\big) + \Phi\big([[P_j, B_{ii}]_*, P_i]\big)\\ &= \big[\big[\Phi(P_j), \Phi(A_{ii}) + \Phi(B_{ii})\big]_*, \Phi(P_i)\big].\end{aligned}$$

Thus $[[P_j, T]_*, P_i] = 0$, which implies that $P_iTP_j = P_jTP_i = 0$. Then $T = 0$, and so $\Phi(A_{ii}+B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$. □

Proof of Theorem 2.1. Let $A = \sum_{i,j=1}^2 A_{ij}$, $B = \sum_{i,j=1}^2 B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{M}_{ij}$. It follows from Lemmas 2.5, 2.6, and 2.7 that

$$\begin{aligned}\Phi(A+B) &= \Phi\Big(\sum_{i,j=1}^2 A_{ij} + \sum_{i,j=1}^2 B_{ij}\Big) = \Phi\Big(\sum_{i,j=1}^2 (A_{ij}+B_{ij})\Big)\\ &= \sum_{i,j=1}^2 \Phi(A_{ij}+B_{ij}) = \sum_{i,j=1}^2 \big(\Phi(A_{ij}) + \Phi(B_{ij})\big)\\ &= \Phi\Big(\sum_{i,j=1}^2 A_{ij}\Big) + \Phi\Big(\sum_{i,j=1}^2 B_{ij}\Big) = \Phi(A) + \Phi(B).\end{aligned}$$

Hence Φ is additive. □

3. Structures

In this section, we will prove the following theorem.

Theorem 3.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two factor von Neumann algebras with* $\dim \mathcal{M} > 4$*, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a bijective map satisfying*

$$\Phi\big([[A, B]_*, C]\big) = \big[\big[\Phi(A), \Phi(B)\big]_*, \Phi(C)\big]$$

for all $A, B, C \in \mathcal{M}$. Then there exists $\epsilon \in \{1, -1\}$ such that $\Phi(A) = \epsilon\Psi(A)$ for all $A \in \mathcal{M}$, where $\Psi : \mathcal{M} \to \mathcal{N}$ is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism.

It follows from Theorem 2.1 and Lemma 2.3 that Φ is an additive bijection that preserves commutativity in both directions. Hence by [2, Theorem 3.1],

$$\Phi(A) = a\theta(A) + \xi(A)$$

for all $A \in \mathcal{M}$, where $a \in \mathbb{C}$ is a nonzero scalar, $\theta : \mathcal{M} \to \mathcal{N}$ is an additive Jordan isomorphism, and $\xi : \mathcal{M} \to \mathbb{C}I$ is an additive map. It is easy to check that $\theta(iI) = \pm iI$. Next we will prove Theorem 3.1 by the following lemmas.

Lemma 3.2. *For every $A, B \in \mathcal{M}$, we have*

(1) $\Phi(iA) - \theta(iI)\Phi(A) \in \mathbb{C}I$,
(2) $\Phi([A, B]) = \epsilon[\Phi(A), \Phi(B)]$, *where $\epsilon \in \{1, -1\}$.*

Proof. (1) Let $A \in \mathcal{A}$. Then

$$\begin{aligned}\Phi(iA) - \theta(iI)\Phi(A) &= a\theta(iA) + \xi(iA) - \theta(iI)\Phi(A)\\ &= a\theta(iI)\theta(A) + \xi(iA) - \theta(iI)\Phi(A)\\ &= \theta(iI)\big(a\theta(A) + \xi(A)\big) + \xi(iA) - \theta(iI)\xi(A) - \theta(iI)\Phi(A)\\ &= \xi(iA) - \theta(iI)\xi(A) \in \mathbb{C}I.\end{aligned}$$

(2) It follows from Lemma 2.2 that $\frac{1}{2}(\Phi(iI)^* - \Phi(iI))\theta(iI) = \epsilon I$ for some $\epsilon \in \mathbb{C}$. By (2.1) and assertion (1), we get

$$\begin{aligned}\Phi\big([A,B]\big) = -\Phi\big(i[iA,B]\big) &= \frac{1}{2}\big(\Phi(iI)^* - \Phi(iI)\big)\big[\Phi(iA),\Phi(B)\big]\\ &= \frac{1}{2}\big(\Phi(iI)^* - \Phi(iI)\big)\theta(iI)\big[\Phi(A),\Phi(B)\big] = \epsilon\big[\Phi(A),\Phi(B)\big]\end{aligned}$$

for all $A, B \in \mathcal{M}$. If $A = A^*$, then

$$\begin{aligned}[[\Phi(A),\Phi(B)]_*,\Phi(C)] = \Phi\big(\big[[A,B]_*,C\big]\big) &= \Phi\big(\big[[A,B],C\big]\big)\\ &= \epsilon^2\big[\big[\Phi(A),\Phi(B)\big],\Phi(C)\big]\end{aligned}$$

for all $B, C \in \mathcal{M}$. Thus,

$$(1-\epsilon^2)\Phi(A)\Phi(B) + \Phi(B)\big(\epsilon^2\Phi(A) - \Phi(A)^*\big) \in \mathbb{C}I \tag{3.1}$$

for all $B \in \mathcal{M}$ and $A \in \mathcal{M}$ with $A = A^*$. Let $Q_1 \in \mathcal{N}$ be a nontrivial projection. Then there exists $D \in \mathcal{M}$ such that $\Phi(D) = Q_1$ by the surjectivity of Φ. Taking $B = D$ in (3.1), we have

$$(1-\epsilon^2)\Phi(A)Q_1 + Q_1\big(\epsilon^2\Phi(A) - \Phi(A)^*\big) \in \mathbb{C}I.$$

This yields

$$(1-\epsilon^2)Q_2\Phi(A)Q_1 = 0 \tag{3.2}$$

for all $A \in \mathcal{M}$ with $A = A^*$, where $Q_2 = I - Q_1$. Then by assertion (1) and (3.2),

$$(1-\epsilon^2)Q_2\Phi(iX)Q_1 = 0 \tag{3.3}$$

for all $X \in \mathcal{M}$ with $X = X^*$. It follows from (3.2) and (3.3) that

$$(1-\epsilon^2)Q_2\Phi(B)Q_1 = 0$$

for all $B \in \mathcal{M}$. Hence $\epsilon \in \{1, -1\}$. □

Remark 3.3. Let ϵ be as above, and let $\Psi = \epsilon\Phi$. It follows from Theorem 2.1 and Lemma 3.2 that $\Psi : \mathcal{M} \to \mathcal{N}$ is an additive bijection preserving mixed Lie triple products and satisfies

$$\Psi\big([A,B]\big) = \big[\Psi(A),\Psi(B)\big]$$

for all $A, B \in \mathcal{M}$. Hence by [15, Theorem 2.1], there exists an additive map $f : \mathcal{M} \to \mathbb{C}I$ with $f([A,B]) = 0$ for all $A, B \in \mathcal{M}$ such that one of the following statements holds:

(1) $\Psi(A) = \varphi(A) + f(A)$ for all $A \in \mathcal{M}$, where $\varphi : \mathcal{M} \to \mathcal{N}$ is an additive isomorphism;

(2) $\Psi(A) = -\varphi(A) + f(A)$ for all $A \in \mathcal{M}$, where $\varphi : \mathcal{M} \to \mathcal{N}$ is an additive anti-isomorphism.

Lemma 3.4. *Statement (2) does not occur; that is, there are no additive anti-isomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ and additive map $f : \mathcal{M} \to \mathbb{C}I$ with $f([A, B]) = 0$ for all $A, B \in \mathcal{M}$ such that $\Psi = -\varphi + f$.*

Proof. If $\Psi = -\varphi + f$, where $\varphi : \mathcal{M} \to \mathcal{N}$ is an additive anti-isomorphism and $f : \mathcal{M} \to \mathbb{C}I$ is an additive map with $f([A, B]) = 0$ for all $A, B \in \mathcal{M}$, then

$$\Psi\big([[A, B]_*, C]\big) = -\varphi\big([[A, B]_*, C]\big) = \big[\varphi(B)\varphi(A) - \varphi(A^*)\varphi(B), \varphi(C)\big]$$

for all $A, B, C \in \mathcal{M}$. On the other hand, we have

$$\begin{aligned}\Psi\big([[A, B]_*, C]\big) &= \big[\big[\Psi(A), \Psi(B)\big]_*, \Psi(C)\big]\\ &= \big[\big[-\varphi(A) + f(A), -\varphi(B) + f(B)\big]_*, -\varphi(C) + f(C)\big]\\ &= \big[\big[\varphi(A) - f(A), -\varphi(B) + f(B)\big]_*, \varphi(C)\big]\\ &= \big[\big[\varphi(A), -\varphi(B)\big]_* + \big[\varphi(A), f(B)\big]_* + \big[f(A), \varphi(B)\big]_*, \varphi(C)\big].\end{aligned}$$

It follows from the surjectivity of φ that

$$\begin{aligned}&\big(\varphi(A^*) - \varphi(A)\big)\varphi(B) + \big(\varphi(B) - f(B)\big)\big(\varphi(A)^* - \varphi(A)\big)\\ &\quad + \big(f(A) - f(A)^*\big)\varphi(B) \in \mathbb{C}I \end{aligned} \tag{3.4}$$

for all $A, B \in \mathcal{M}$. Let $P \in \mathcal{M}$ be a nontrivial projection. Then $\varphi(P)$ is a nontrivial idempotent in $\mathcal{N}$. Taking $B = P$ in (3.4), we have

$$\begin{aligned}&\big(\varphi(A^*) - \varphi(A)\big)\varphi(P) + \big(\varphi(P) - f(P)\big)\big(\varphi(A)^* - \varphi(A)\big)\\ &\quad + \big(f(A) - f(A)^*\big)\varphi(P) \in \mathbb{C}I. \end{aligned} \tag{3.5}$$

Multiplying (3.5) on the right-hand side by $\varphi(P^\perp)$ and on the left-hand side by $\varphi(P)$, we get

$$\big(I - f(P)\big)\varphi(P)\big(\varphi(A)^* - \varphi(A)\big)\varphi(P^\perp) = 0 \tag{3.6}$$

for all $A \in \mathcal{M}$. Replacing $\varphi(A)$ by $i\varphi(A)$ in (3.6), we have

$$\big(I - f(P)\big)\varphi(P)\big(\varphi(A)^* + \varphi(A)\big)\varphi(P^\perp) = 0. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\big(I - f(P)\big)\varphi(P)\varphi(A)\varphi(P^\perp) = 0$$

for all $A \in \mathcal{M}$. Hence $f(P) = I$ for any nontrivial projection $P \in \mathcal{M}$, and so by (3.5)

$$\varphi(P^\perp)\big(\varphi(A)^* - \varphi(A)\big)\varphi(P^\perp) \in \mathbb{C}\varphi(P^\perp) \tag{3.8}$$

for all $A \in \mathcal{M}$ and any nontrivial projection $P \in \mathcal{M}$. Replacing $\varphi(A)$ by $i\varphi(A)$ in (3.8), we can obtain that

$$\varphi(P^\perp A P^\perp) = \varphi(P^\perp)\varphi(A)\varphi(P^\perp) \in \mathbb{C}\varphi(P^\perp) = \varphi(\mathbb{C}P^\perp)$$

for all $A \in \mathcal{M}$ and any nontrivial projection $P \in \mathcal{M}$. This implies that

$$P^{\perp}\mathcal{M}P^{\perp} = \mathbb{C}P^{\perp} \qquad \text{and} \qquad P\mathcal{M}P = \mathbb{C}P$$

for any nontrivial projection $P \in \mathcal{M}$. It follows that $\mathcal{M}$ is isomorphic to $M_2(\mathbb{C})$, the algebra of all 2×2 matrices over $\mathbb{C}$, which contradicts the assumption that $\dim \mathcal{M} > 4$. □

Lemma 3.5. *We have that Ψ is an additive $*$-isomorphism.*

Proof. It follows from Remark 3.3 and Lemma 3.4 that $\Psi = \varphi + f$, where $\varphi : \mathcal{M} \to \mathcal{N}$ is an additive isomorphism and $f : \mathcal{M} \to \mathbb{C}I$ is an additive map with $f([A, B]) = 0$ for all $A, B \in \mathcal{M}$. Thus,

$$\Psi\big([[A, B]_*, C]\big) = \varphi\big([[A, B]_*, C]\big) = \big[\varphi(A)\varphi(B) - \varphi(B)\varphi(A^*), \varphi(C)\big]$$

for all $A, B, C \in \mathcal{M}$. On the other hand, we have

$$\begin{aligned}\Psi\big([[A, B]_*, C]\big) &= \big[\big[\Psi(A), \Psi(B)\big]_*, \Psi(C)\big] \\ &= \big[\big[\varphi(A) + f(A), \varphi(B) + f(B)\big]_*, \varphi(C) + f(C)\big] \\ &= \big[\big[\varphi(A) + f(A), \varphi(B) + f(B)\big]_*, \varphi(C)\big] \\ &= \big[\big[\varphi(A), \varphi(B)\big]_* + \big[f(A), \varphi(B)\big]_* + \big[\varphi(A), f(B)\big]_*, \varphi(C)\big].\end{aligned}$$

It follows from the surjectivity of φ that

$$\varphi(B)\big(\varphi(A)^* - \varphi(A^*)\big) + \varphi(B)\big(f(A)^* - f(A)\big) + f(B)\big(\varphi(A)^* - \varphi(A)\big) \in \mathbb{C}I \tag{3.9}$$

for all $A, B \in \mathcal{M}$. Let $\lambda \in \mathbb{C}$, and let $P \in \mathcal{M}$ be a nontrivial projection. Multiplying (3.9) on the left-hand side by $\varphi(P^{\perp})$ and on the right-hand side by $\varphi(P)$, and then taking $B = \lambda P$, we have

$$f(\lambda P)\varphi(P^{\perp})\big(\varphi(A)^* - \varphi(A)\big)\varphi(P) = 0 \tag{3.10}$$

for all $A \in \mathcal{M}$. Similarly, we can obtain from (3.10) that

$$f(\lambda P)\varphi(P^{\perp})\varphi(A)\varphi(P) = 0$$

for all $A \in \mathcal{M}$. Then $f(\lambda P) = 0$ for all $\lambda \in \mathbb{C}$ and any nontrivial projection $P \in \mathcal{M}$. This yields that

$$f(\lambda I) = f(\lambda P) + f(\lambda P^{\perp}) = 0$$

for all $\lambda \in \mathbb{C}$. Since every $A \in \mathcal{M}$ can be written as a finite linear combination of projections in $\mathcal{M}$, it follows that $f(A) = 0$ for all $A \in \mathcal{M}$. Now (3.9) becomes

$$\varphi(B)\big(\varphi(A)^* - \varphi(A^*)\big) \in \mathbb{C}I \tag{3.11}$$

for all $A, B \in \mathcal{M}$. In particular, $\varphi(A)^* - \varphi(A^*) \in \mathbb{C}I$ for all $A \in \mathcal{M}$. If $\varphi(A)^* - \varphi(A^*) \neq 0$ for some $A \in \mathcal{M}$, then by (3.11), $\varphi(B) \in \mathbb{C}I$ for all $B \in \mathcal{M}$. This contradiction implies that $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{M}$. Hence $\Psi = \varphi$ is an additive $*$-isomorphism. □

Proof of Theorem 3.1. It follows from Remark 3.3 and Lemma 3.5 that $\Phi = \epsilon\Psi$ and $\Psi : \mathcal{M} \to \mathcal{N}$ is an additive $*$-isomorphism. Thus $\Psi(iI) = \pm iI$, $\Psi(bI) = bI$ for any rational number b, and Ψ is an order-preserving map on the collection of all positive elements. Let $r \in \mathbb{R}$ be any real number. Then there exist two sequences of rational numbers $\{a_n\}$ and $\{b_n\}$ such that $a_n \le r \le b_n$ and $\lim a_n = \lim b_n = r$. Hence

$$a_n I = \Psi(a_n I) \le \Psi(rI) \le \Psi(b_n I) = b_n I.$$

Letting $n \to \infty$, we have $\Psi(rI) = rI$ for all $r \in \mathbb{R}$. This yields that, for any $\lambda = a + ib \in \mathbb{C}$,

$$\Psi(\lambda I) = \Psi(aI) + \Psi(ibI) = (a \pm ib)I = \lambda I \text{ or } \bar{\lambda} I.$$

It follows that $\Psi(\lambda A) = \lambda\Psi(A)$ or $\Psi(\lambda A) = \bar{\lambda}\Psi(A)$ for all $A \in \mathcal{M}$ and all $\lambda \in \mathbb{C}$. Hence Ψ is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. □

Corollary 3.6 ([8, Theorem 10.5.1])**.** *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} > 2$, and let $\Phi : \mathrm{B}(\mathcal{H}) \to \mathrm{B}(\mathcal{H})$ be a bijective map satisfying $\Phi([[A, B]_*, C]) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]$ for all $A, B, C \in \mathrm{B}(\mathcal{H})$. Then there exists $\epsilon \in \{1, -1\}$ such that $\Phi(A) = \epsilon UAU^*$ for all $A \in \mathrm{B}(\mathcal{H})$, where U is a unitary or conjugate unitary operator.*

4. The case for $\dim \mathcal{M} = 4$

Let $\mathcal{M}$ and $\mathcal{N}$ be two factor von Neumann algebras, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a bijection preserving mixed Lie triple products. If $\dim \mathcal{M} = 4$, then we can assume that $\dim \mathcal{N} = 4$ by Theorem 3.1. Therefore, without loss of generality, we can assume that $\mathcal{M} = \mathcal{N} = M_2(\mathbb{C})$. Let $E_{ij} \in M_2(\mathbb{C})$ be the matrix unit whose (i, j) position is 1 and all other positions are 0. For any $A = (a_{ij}) \in M_2(\mathbb{C})$, $\overline{A} = (\overline{a}_{ij})$, $A^t = (a_{ji})$, and $A^* = (\overline{a}_{ji})$. In this section, we will prove the following theorem.

Theorem 4.1. *Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a bijection satisfying*

$$\Phi\big([[A, B]_*, C]\big) = \big[\big[\Phi(A), \Phi(B)\big]_*, \Phi(C)\big]$$

for all $A, B, C \in M_2(\mathbb{C})$. Then there exist $\epsilon \in \{1, -1\}$ and a unitary matrix $U \in M_2(\mathbb{C})$ such that one of the following statements holds:

(1) $\Phi(A) = \epsilon UAU^*$ *for all* $A \in M_2(\mathbb{C})$*;*
(2) $\Phi(A) = \epsilon U\bar{A}U^*$ *for all* $A \in M_2(\mathbb{C})$*;*
(3) $\Phi(A) = -\epsilon UA^tU^* + \epsilon\, \mathrm{tr}(A)I$ *for all* $A \in M_2(\mathbb{C})$*;*
(4) $\Phi(A) = -\epsilon UA^*U^* + \epsilon\overline{\mathrm{tr}(A)}I$ *for all* $A \in M_2(\mathbb{C})$*.*

Proof. We see that the condition $\dim \mathcal{M} > 4$ appears only in the proof of Lemma 3.4. Hence there exist $\epsilon \in \{1, -1\}$ such that $\Phi = \epsilon\Psi$ and Ψ satisfies one of the following statements:

(a) $\Psi(A) = \varphi(A) + f(A)$ for all $A \in M_2(\mathbb{C})$, where $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is an additive isomorphism and $f : M_2(\mathbb{C}) \to \mathbb{C}I$ is an additive map with $f([A, B]) = 0$ for all $A, B \in M_2(\mathbb{C})$;
(b) $\Psi(A) = -\varphi(A) + f(A)$ for all $A \in M_2(\mathbb{C})$, where $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is an additive anti-isomorphism and $f : M_2(\mathbb{C}) \to \mathbb{C}I$ is an additive map with $f([A, B]) = 0$ for all $A, B \in M_2(\mathbb{C})$.

If statement (a) holds, with the same argument as in the proof of Theorem 3.1, then $\Psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. Hence there exists a unitary matrix $U \in M_2(\mathbb{C})$ such that $\Psi(A) = UAU^*$ for all $A \in M_2(\mathbb{C})$, or $\Psi(A) = U\bar{A}U^*$ for all $A \in M_2(\mathbb{C})$.

If statement (b) holds, by the same argument as in the proof of Lemma 3.4, then

$$\begin{aligned}\big(\varphi(A^*) - \varphi(A)\big)\varphi(B) + \big(\varphi(B) - f(B)\big)\big(\varphi(A)^* - \varphi(A)\big)\\ + \big(f(A) - f(A)^*\big)\varphi(B) \in \mathbb{C}I \end{aligned} \tag{4.1}$$

for all $A, B \in M_2(\mathbb{C})$, and $f(E) = I$ for any nontrivial idempotent $E \in M_2(\mathbb{C})$. Thus,

$$f(I) = f(E) + f(E^\perp) = 2I.$$

Taking $B = I$ in (4.1), we have $\varphi(A^*) - \varphi(A)^* \in \mathbb{C}I$ for all $A \in M_2(\mathbb{C})$. This implies that

$$\varphi(E_{11}) = \varphi(E_{11})^*, \qquad \varphi(E_{22}) = \varphi(E_{22})^*, \qquad \varphi(E_{12}) = \varphi(E_{21})^*.$$

For any nontrivial idempotent $E \in M_2(\mathbb{C})$ and $\lambda \in \mathbb{C}$, taking $B = \lambda E$ in (4.1) and then multiplying on the left-hand side by $\varphi(E)$ and on the right-hand side by $\varphi(E^\perp)$, we have

$$\big(\varphi(\lambda E) - f(\lambda E)\big)\varphi(E)\big(\varphi(A)^* - \varphi(A)\big)\varphi(E^\perp) = 0$$

for all $A \in M_2(\mathbb{C})$. It follows that

$$\varphi(\lambda E) = f(\lambda E)\varphi(E). \tag{4.2}$$

Since φ is an additive anti-isomorphism and $\varphi(\mathbb{C}I) \subseteq \mathbb{C}I$, there exists an additive isomorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that $\varphi(\lambda I) = \tau(\lambda)I$ for all $\lambda \in \mathbb{C}$. Thus,

$$\varphi(\lambda E) = \varphi(\lambda I)\varphi(E) = \tau(\lambda)\varphi(E). \tag{4.3}$$

This together with (4.2) gives us that $f(\lambda E) = \tau(\lambda)I$ for any nontrivial idempotent $E \in M_2(\mathbb{C})$ and $\lambda \in \mathbb{C}$. It follows from $f(\lambda E_{12}) = f(\lambda E_{21}) = 0$ that

$$f(A) = \tau\big(\mathrm{tr}(A)\big)I \tag{4.4}$$

for all $A \in M_2(\mathbb{C})$. Since $\varphi(E_{11})$, $\varphi(E_{22})$ are nontrivial projections in $M_2(\mathbb{C})$ and $\varphi(E_{11}) + \varphi(E_{22}) = I$, there exists a unitary matrix $V \in M_2(\mathbb{C})$ such that

$$V^*\varphi(E_{11})V = E_{11} \qquad \text{and} \qquad V^*\varphi(E_{22})V = E_{22}. \tag{4.5}$$

This and the fact that $\varphi(E_{12}) = \varphi(E_{22})\varphi(E_{12})\varphi(E_{11})$ and $\varphi(E_{21}) = \varphi(E_{12})^*$ yield

$$V^*\varphi(E_{12})V = aE_{21} \qquad \text{and} \qquad V^*\varphi(E_{21})V = \bar{a}E_{12} \tag{4.6}$$

for some $a \in \mathbb{C}$ with $|a| = 1$. From (4.3), (4.5), and (4.6), we have for any $\lambda \in \mathbb{C}$,

$$\begin{aligned}\varphi(\lambda E_{11}) &= \tau(\lambda)VE_{11}V^*, \qquad \varphi(\lambda E_{22}) = \tau(\lambda)VE_{22}V^*,\\ \varphi(\lambda E_{12}) &= a\tau(\lambda)VE_{21}V^*, \qquad \varphi(\lambda E_{21}) = \bar{a}\tau(\lambda)VE_{12}V^*.\end{aligned}$$

It follows that

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = V \begin{bmatrix} \tau(a_{11}) & \bar{a}\tau(a_{21}) \\ a\tau(a_{12}) & \tau(a_{22}) \end{bmatrix} V^* = U \begin{bmatrix} \tau(a_{11}) & \tau(a_{21}) \\ \tau(a_{12}) & \tau(a_{22}) \end{bmatrix} U^* \tag{4.7}$$

for all $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \in M_2(\mathbb{C})$, where $U = V\left[\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right]$ is a unitary matrix in $M_2(\mathbb{C})$.

Taking $A = \left[\begin{smallmatrix} 0 & \lambda \\ \lambda & \lambda \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ in (4.1), we obtain the following from (4.4) and (4.7) that:

$$\begin{bmatrix} 0 & \tau(\bar{\lambda}) - \tau(\lambda) \\ \tau(\bar{\lambda}) - \tau(\lambda) & \tau(\bar{\lambda}) - \tau(\lambda) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \overline{\tau(\lambda)} - \tau(\lambda) \\ \overline{\tau(\lambda)} - \tau(\lambda) & \overline{\tau(\lambda)} - \tau(\lambda) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & \tau(\lambda) - \overline{\tau(\lambda)} \end{bmatrix} \in \mathbb{C}I.$$

It follows that $\tau(\bar{\lambda}) = \overline{\tau(\lambda)}$ for all $\lambda \in \mathbb{C}$. Hence $\tau(\lambda) = \lambda$ for all $\lambda \in \mathbb{C}$, or $\tau(\lambda) = \bar{\lambda}$ for all $\lambda \in \mathbb{C}$. By (4.4) and (4.7), we have $\Psi(A) = -UA^tU^* + \mathrm{tr}(A)I$ for all $A \in M_2(\mathbb{C})$, or $\Psi(A) = -UA^*U^* + \overline{\mathrm{tr}(A)}I$ for all $A \in M_2(\mathbb{C})$. □

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School of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710062, People's Republic of China.
E-mail address: zjyang2100@163.com; jhzhang@snnu.edu.cn